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GLOBAL ATTRACTORS OF INFINITE-DIMENSIONAL NONAUTONOMOUS DYNAMICAL SYSTEMS. I

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Abstract. The article is devoted to the infinite-dimensional abstract nonautonomous dynamical systems, which admit the compact global attractor. It is shown, that nonautonomous dynamical system, which has the bounded absorbing (weakly absorbing) set, also has a compact global attractor, if its operators of translation along the trajectories are compact (asymptotically compact; satisfies the condition of Ladyzhenskaya). This results are precised and strengthened for the nonautonomous dynamical systems with minimal basis. The conditions of existence of the compact global attractor for the skew-product dynamical systems (cocycles) are presented. The necessary and sufficient conditions of the existence of compact global attractor are given in terms of Lyapunov functions. The applications of obtained results for the different classes of the evolutionary equations are given.

During last years the ideas and methods developed in theory of finite-dimensional dynamical systems are actively used in theory of infinite-dimensional systems [1-9] and in functional-differential equations which generate them [2-3] and also in differential equations with partial derivatives [1,4]. In the works of the author [5,6] many important facts are gathered and systematize,which deal with abstract infinite-dimensional dynamical systems,which admit a compact global attractor. The aim of the work is using for abstract nonautonomous dynamical systems with infinite-dimensional phase spaces some results, which were earlier established for autonomous infinite-dimensional systems or for nonautonomous finite-dimensional systems [7,8]. Our point of view [7] in studying nonautonomous dissipative differential equations is such that some abstract nonautonomous dynamical system which has a compact global attractor is naturally put in correspondence to every nonautonomous differential equation. Such method permits to solve a lot of questions, which appear during studying dissipative differential equations,using the general theory of dynamical systems. Let us notice,that there is another point of view in studying this problem: with every nonautonomous differential equation some double-parametric family of mappings of phase space is connected (look,for example,at [10-13]). We consider the first point of view to be better,as it permits to use the ideas,methods and results of the theory of dynamical systems while studying different classes of nonautonomous evolutional equations .But there is sufficiently strong connection between the mentioned above methods of studying nonautonomous equations . More precisely this question is discussed at the end of this article .

§ 1. Global attractors of autonomous dynamical systems.

Let (X,ρ) be the full metric space , $\mathbb{R}(\mathbb{Z})$ is a group of real numbers, $S=\mathbb{R}$ or $\mathbb{Z}$, $S_+=\{s|s\in S, s\geq 0\}$ and $T(S_+\subseteq T)$ is subsemigroup of group S. By (X,T,π) define a dynamical system on X and let W is some family of subsets of X. A dynamical system (X,T,π) is called W-dissipative , if for any $\varepsilon>0$ and $M\in W$ there is $L(\varepsilon,M)>0$ such that $\pi^t M\subseteq B(K,\varepsilon)$ for all $t\geq L(\varepsilon,M)$, where K is some fixed subset from X, which depends on W only; $B(K,\varepsilon)$ is open ε-neighborhood K and $\pi^t M=\{\pi(x,t)=xt|x\in M\}$. Then the set K let us call by attractor for W. The most interesting for applications are cases, when K is bounded or compact and $W=\{\{x\}|x\in X\}$, $W=C(X)$ (where $C(X)$ is the family of all compact subsets of X), $W=\{B(x,\delta_x): x\in X, \delta_x>0$ is fixed $\}$ or $W=\mathbb{B}(X)$ (where $\mathbb{B}(X)$ is the family of all bounded subsets of X).

The system (X,T,π) is called [1-5]:

- point-wise dissipative ,if there is $K\subseteq X$ such that for all $x\in X$

$$\lim_{t\to+\infty}\rho(x\cdot t,K)=0; \tag{1.1}$$

- compactly dissipative ,if the equality (1.1) takes place uniformly in x on compacts from X ;

- locally dissipative ,if for any point $p\in X$ there is $\delta_p>0$ such that the equality (1.1) takes place uniformly in $x\in B(p,\delta_p)$;

- boundedly dissipative ,if the equality (1.1) takes place uniformly in x on every bounded subset from X.

During studying dissipative systems we distinguish two cases , when K is compact or bounded (but is not compact). According to this the system (X,T,π) is called point-wise k (b)-dissipative ,if (X,T,π) is point-wise dissipative and the set K, mentioned in (1.1), is compact (bounded). Analogically are defined definitions of a compactly k (b)-dissipative system and the other types of dissipativity . Let (X,T,π) is compactly k- dissipative and K is a compact set , which is attractor of all compact subsets of X. Suppose

$$J=\Omega(K), \tag{1.2}$$

where $\Omega(K)=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi^\tau K}$. We can show [2-3,7-8],that the set J, defined by the equality (1.2), does not depend on selection of the attractor K, and it is characterized by the properties of the dynamical system (X,T,π) itself only . The set J is called the Levinson centre of the compact dissipative system (X,T,π). Let us mention some facts, which we will need below .

Theorem 1.1 [2-3,7-8]. *If (X,T,π) is compactly dissipative dynamical system and J is its Levinson centre , then:*

1. *J is invariant,that is $\pi^t J=J$ for all $t\in T$;*
2. *J is orbitally stable , that is for any $\varepsilon>0$ there is $\delta(\varepsilon)>0$ such that from $\rho(x,J)<\delta$ it follows $\rho(x\cdot t,J)<\varepsilon$ for all $t\geq 0$;*
3. *J is an attractor of the family of all compact subsets from X;*
4. *J is the maximal compact invariant set of (X,T,π).*

The dynamical system (X,T,π) is called [5-8]:

- locally completely continuous ,if for any $p \in X$ there are $\delta = \delta_p > 0$ and $l = l_p > 0$ such that $\pi^l B(x,\delta)$ is relatively compact ;

- weakly dissipative ,if there is a nonempty compact $K \subseteq X$ such that $\omega_x \cap K \neq \emptyset$ for any $x \in X$. Then the compact K is called a weak attractor of the system (X,T,π).

Theorem 1.2 [6-7]. *If the dynamical system (X,T,π) is weakly dissipative and locally completely continuous , then (X,T,π) is locally k - dissipative.*

Lemma 1.3[1,5]. *Let $B \in \mathbb{B}(X)$, then the next conditions are equivalent :*

1. *for any $\{x_k\} \subseteq B$ and $t_k \to +\infty$ the sequence $\{x_k \cdot t_k\}$ is relatively compact;*
2. *a.$\Omega(B) \neq \emptyset$ and is compact;*
   b.$\Omega(B)$ is invariant and
$$\lim_{t\to+\infty} \sup_{x\in B} \rho(x\cdot t, \Omega(B)) = 0. \tag{1.3}$$
3. *there is a nonempty compact $K \subseteq X$ such that*
$$\lim_{t\to+\infty} \sup_{x\in B} \rho(x\cdot t, K) = 0. \tag{1.4}$$

Remark 1.1. *From theorem 1.1 and lemma 1.3 it follows , that the dynamical system (X,T,π) is boundedly k-dissipative then and only then ,when it is compactly k-dissipative and its Levinson centre J is the attractor of the family of all bounded subsets from X. In this case the set J is called by the global attractor of the dynamical system (X,T,π).*

According to [9], we will say that the dynamical system (X,T,π) satisfies the condition of Ladyzhenskaya ,if for any set $M \in \mathbb{B}(X)$ it is carrying out one of the conditions 1.- 3.of lemma 1.1.

Theorem 1.4 [5,9]. *Let (X,T,π) satisfies the condition of Ladyzhenskaya , then the next conditions are equivalent :*

1. *there is a bounded set $B_0 \subseteq X$ such that for any $x \in X$ there will be $\tau(x) > 0$ such that $x \cdot t \in B_0$ for all $t \geq \tau$;*
2. *there is a bounded set $B_0 \subseteq X$ such that for any $x \in X$ there will be $\tau(x) \geq 0$ such that $x \cdot \tau \in B_0$;*
3. *there is a nonempty compact $K_1 \subseteq X$ such that $\omega_x \subseteq K_1$ for all $x \in X$;*
4. *there is a nonempty compact $K_2 \subseteq X$ such that $\omega_x \cap K_2 \neq \emptyset$ for all $x \in X$;*
5. *there is a nonempty compact set $K_3 \subseteq X$ such that for any bounded set $B \subseteq X$ takes place the equality*
$$\lim_{t\to+\infty} \sup_{x\in B} \rho(x\cdot t, K_3) = 0. \tag{1.5}$$
6. *there is a bounded set B_0 such that $\pi^t B \subseteq B_0$ for all $t \geq L(B)$.*

Theorem 1.5 [5]. *Let (X,T,π) is pointwisely k-dissipative. In order to (X,T,π) were locally dissipative ,it is necessary and sufficiently that for any $p \in X$ there will be $\delta_p > 0$ and a compact K_p such that*
$$\lim_{t\to+\infty} \sup_{x\in B(p,\delta_p)} \rho(x\cdot t, K_p) = 0. \tag{1.6}$$

§ 2. Global attractors of nonautonomous dynamical systems.

Let Y be a compact topological space , (E, h, Y) is locally trivial banach stratification [14] and $|\cdot|$ is the norm on (E, h, Y) co-ordinate with the metric ρ on E (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$). Let us remember [7,15],that the three $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ is called by a nonautonomous dynamical system ,if $h : E \to Y$ is an homomorphism of the dynamical system (E, T_1, π) on (Y, T_2, σ) .

A nonautonomous dynamical system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ we will call pointwisely (compactly, locally, boundedly) dissipative, if (E, T_1, π) is so.

By Levinson centre of the compactly dissipative system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ we will call Levinson centre of (E, T_1, π).

Theorem 2.1. *Let $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ is a nonautonomous dynamical system and for any bounded set $M \in \mathbb{B}(X)$ there is $l = l(M) > 0$ such that $\pi^l(M)$ is relatively compact (that is the dynamical system (E, T_1, π) is completely continuous), then the next conditions are equivalent :*

1. *there is a positive number r such that for any $x \in X$ there will be $\tau = \tau(x) \geq 0$ for which $|x \cdot \tau| < r$;*
2. *the dynamical system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ is compactly dissipative and*
$$\lim_{t \to +\infty} \sup_{|x| \leq R} \rho(x \cdot t, J) = 0 \tag{2.1}$$
for any $R > 0$,where J is Levinson centre (E, T_1, π), that is the nonautonomous system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ admits the compact global attractor .

Proof. Evidently,from 2. it follows 1.. Let us show that in conditions of theorem 2.1 takes place also the opposite implication . Suppose $A(r) = \{x \in E | \; |x| \leq r\}$, where $r > 0$ is the number figuring in condition 1.. As Y is compact and the banach stratification (E, h, Y) is locally trivial, then its null section $\Theta = \{\theta_y | y \in Y$,where θ_y is the null element of the layer $E_y = h^{-1}(y)\}$ is compact and ,hence, the set $A(r)$ is bounded ,as $A(r) \subseteq S(\theta, r) = \{x \in E | \; |\rho(x, \theta) \leq r\}$. According to the condition of the theorem for bounded set M there is a positive number l such that $\pi^l M$ is relatively compact.Let $x \in M$ and $\tau = \tau(x) \geq 0$ such that $x \cdot \tau \in M$, then $x \cdot (\tau + l) \in K = \overline{\pi^l M}$. Thus the nonempty compact K is a weak attractor of the system (E, T_1, π) and according to theorem 1.2 the dynamical system (E, T_1, π) is compactly dissipative. Let J is Levinson centre of (E, T_1, π) and $R > 0$, then the set $A(R) = \{x \in E | \; |x| \leq R\}$, as it was noticed above, is bounded ,and for it there will be a number $l > 0$ such that $\pi^l A(R)$ is relatively compact and as (E, T_1, π) is compactly dissipative, then its Levinson centre J, according to theorem 1.1 , attracts the set $\pi^l A(R)$, and ,hence , the equality (2.1) takes place. Theorem is proved .

Corollary 2.1. *Let $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ be a nonautonomous dynamical system and vector stratification of (E, T_1, π) is finite-dimensional, then the conditions 1. and 2. of theorem 2.1 are equivalent .*

This assertion follows from theorem 2.1 as for any $r > 0$ the set $\{x \in E | \; |x| \leq r\}$ is compact,if vector stratification of (E, h, Y) is finite-dimensional,and ,hence,the dynamical system (E, T_1, π) is completely continuous.

Remark 2.1. *For finite-dimensional systems (that is the stratification of* (E, h, Y) *is finite-dimensional) theorem 2.1 was earlier proved in* [16] *.*

Theorem 2.2. *Let* $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ *be a nonautonomous dynamical system and* (E, T_1, π) *satisfies the condition of Ladyzhenskaya ,then the conditions 1. and 2. of theorem 2.1 are equivalent.*

Proof. As Y is compact and (E, h, Y) is locally trivial then for any $R > 0$ the set $\{x \in E | \, |x| \leq R\}$ is bounded. According to the condition 1. of theorem 2.1 for any $x \in E$ there is $\tau = \tau(x) \geq 0$ such that $x \cdot \tau \in A(r) = \{x \in E| \, |x| \leq r\}$. According to theorem 1.4 the dynamical system (E, T_1, π) is compactly dissipative. Let J is Levinson centre of (E, T_1, π) and $R > 0$. As the set $M = A(R) = \{x \in E| \, |x| \leq R\}$ is bounded , then according to the condition of the theorem and lemma 1.3 the set $\Omega(M) \neq \emptyset$, is compact, invariant and the equality (1.3)takes place. As J is the maximal compact invariant set in (E, T_1, π) (look at theorem 1.1),then $\Omega(M) \subseteq J$ and, hence, the equality (2.1) takes place. Theorem is proved .

The dynamical system (E, T_1, π) is called [1-2] asymptotically compact, if for any bounded close positively invariant set $M \in \mathbb{B}(E)$ there is a nonempty compact, such that the equality (1.4) takes place .

Remark 2.2. *Let us notice that a dynamical system is asymptotically compact, , if it satisfies one of the following two conditions : the dynamical system* (E, T_1, π) *is completely continuous or it satisfies the condition of Ladyzhenskaya . It is evident ,that the opposite assertion does not take place .*

Theorem 2.3. *Let* $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ *be a nonautonomous dynamical system and* (E, T_1, π) *is asymptotically compact, then the next conditions are equivalent :*

1. *there is a positive number* R_0 *and for any* $R > 0$ *there will be* $l(R) > 0$ *such that*
$$|\pi^t x| \leq R_0 \tag{2.2}$$
for all $t \geq l(R)$ *and* $|x| \leq R$ *;*
2. *the dynamical system* $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ *admit the compact global attractor, that is it is compactly dissipative and for its Levinson centre* J *the equality (2.1) takes place for any* $R > 0$ *.*

Proof. Evidently from 2. it follows 1. , that is why for proving the theorem it is sufficiently to show , that from 1. it follow 2. Let $M_0 \in \mathbb{B}(E)$, then there is $R > 0$ such that $M_0 \subseteq A(R) = \{x \in E| \, |x| \leq R\}$. According to the condition 1. for the given number R there will be $l = l(R) > 0$ such that (2.2) takes place and, in particular, the set $M = \bigcup\{\pi^t M_0 | t \geq l(R)\}$ is bounded and positively invariant. As (E, T_1, π) is asymptotically compact ,for the set M there will be a nonempty compact K for which the equality (1.4)takes place. For ending the proof of the theorem it is sufficiently to cite theorem 2.2 . Theorem is proved .

Theorem 2.4. *Let* $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ *be a nonautonomous dynamical system and the mappings* $\pi^t = \pi(\cdot, t) : E \to E (t \in T_1)$ *are represented like a sum* $\pi(x, t) = \varphi(x, t) + \psi(x, t)$ *for all* $t \in T_1$ *and*
$x \in E$

and the conditions are fulfilled :

1. $|\varphi(x,t)| \leq m(t,r)$ *for all* $t \in T_1, r > 0$ *and* $|x| \leq r$ *, where* $m : T_1 \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $m(t,r) \to 0$ *for* $t \to +\infty$ *;*
2. *mappings* $\psi(\cdot,t) : E \to E(t > 0)$ *are conditionally completely continuous , that is* $\psi(A,t)$ *is relatively compact for any* $t > 0$ *and a bounded positively invariant set* $A \subseteq E$ *.*

Then the dynamical system (E, T_1, π) *is asymptotically compact.*

Proof. Let $A \subseteq E$ is a bounded set such that $\Sigma^+(A) = \bigcup\{\pi^t A | t \geq 0\}$ is also bounded , $r > 0$ and $A \subseteq \{x \in E | \ |x| \leq r\}$. Let us show , that for any $\{x_k\} \subseteq A$ and $t_k \to +\infty$, the sequence $\{x_k \cdot t_k\}$ is relatively compact. We will convinced , that the set $M = \{x_k \cdot t_k\}$ may be covered by a compact ε net for any $\varepsilon > 0$. Let $\varepsilon > 0$ and $l > 0$ such that $m(l,r) < \varepsilon/2$ and let us represent M in the form of unification $M_1 \cup M_2$, where $M_1 = \{x_k \cdot t_k\}_{k=1}^{k_1}$, $M_2 = \{x_k \cdot t_k\}_{k=k_1+1}^{+\infty}$ and $k_1 = \max\{k | t_k < l\}$. The set M_2 is the subset of the set $\pi^l(\Sigma^+(A))$ the elements of which we can represent in the form of $\varphi(x,l) + \psi(x,l)(x \in \Sigma^+(A))$. As the set $\psi(\Sigma^+(A), l)$ is relatively compact, then it may be covered by a finite $\varepsilon/2$ net . Let us notice that for any $y \in \varphi(\Sigma^+(A), l)$ there is $x \in \Sigma^+(A)$ such that $y = \varphi(x,l)$ and $|y| = |\varphi(x,l)| \leq m(l,r) < \varepsilon/2$. that is why the null section Θ of the stratification of (E,h,Y) is an $\varepsilon/2$ net of the set $\varphi(\Sigma^+(A), l)$. Thus M_2, and ,hence, M is covering by a compact ε net and as the space E is full ,then the set $M = \{x_k \cdot t_k\}$is relatively compact. Now for ending the proof of the theorem is sufficiently to cite the lemma 1.3 . Theorem is proved .

Remark 2.3. *a. Theorem 2.4 generalizes on nonautonomous systems, and in autonomous case it defines more precisely a well-known for autonomous systems fact (look,for example,at* $[1, 17-19]$ *).*

b. For finite-dimensional systems (that is when vector stratification of (E,h,Y) *is finite-dimensional), theorems 2.1-2.3 are proved in [7,16], for infinite-dimensional systems partial results are contained in [20].*

v. The assertion ,close to theorem 2.1 is contained in the work [21].

§ 3.Global attractors of nonautonomous dynamical systems with minimal base .

Everywhere in this paragraph we suppose that $< (E, T_1, \pi), (Y, T_2, \sigma) >$ is the nonautonomous dynamical system, Y is a compact minimal set and (E,h,Y) is a locally trivial banach stratification .

Theorem 3.1. *Let the next conditions are fulfilled :*

1. (E, T_1, π) *is completely continuous , that is for any bounded set* $A \subseteq E$ *there is* $l = l(a) > O$ *such that* $\pi^l(A)$ *is relatively compact;*
2. *all motions* (E, T_1, π) *are bounded on* T_+*, that is* $\sup\{|x \cdot t| \ |t \in T_+\} < +\infty$ *for any* $x \in E$ *;*
3. *there are* y_0 *and* $R_0 > 0$ *such that for any* $x \in E_{y_0}$ *there will be* $\tau = \tau(x) \geq 0$ *such that*

$$|x \cdot \tau| < R_0. \tag{3.1}$$

Then the nonautonomous dynamical system $< (E, T_1, \pi), (Y, T_2, \sigma) >$ *admit the compact global attractor.*

Proof. Let $R > R_0$, then for any $x \in E$ there is $\tau = \tau(x) \geq 0$ such that $|x \cdot \tau| < R$. If it were not so,then there will be $R' > R_0$ any $x'_0 \in E$ such that

$$|x'_0 \cdot \tau| > R' \tag{3.2}$$

for all $\tau \geq 0$. As the dynamical system (E, T_1, π) is completely continuous and as it takes place the boundedness on T_+ of the motion $\pi(x^1, t)$ the point x^1 is stable L^+ and as Y is minimal ,then the set $\omega_{x^1} \cap E_{y_0}$ is nonempty ,and according to condition (3.2) we have

$$|x \cdot t| \geq R' \tag{3.3}$$

for all $x \in \omega_{x^1} \cap E_{y_0}$ and $t \geq 0$. Inequality (3.3) contradicts (3.1). This contradiction proves the assertion we need. Now for ending the proof of the theorem it is sufficiently to cite theorem 2.1 .

Remark 3.2. *1.For finite-dimensional systems (that is vector stratification* (E, h, Y) *is finite-dimensional) theorem 3.1 increases theorem 2.6.1 from* [22]*, exactly the condition of uniform boundedness is changed for ordinary boundedness of trajectories of* (E, T_1, π)*.*

2.If the condition of minimality of Y *in theorem 3.1 is taken away,then it is not true even in the class of linear nonautonomous systems.*

This is proved by the following example .

Example 3.3. Let us consider the linear differential equation

$$x' = a(t)x, \tag{3.4}$$

where $a \in C(R, R)$ is defined by the equality $a(t) = -1 + \sin t^{\frac{1}{3}}$. Let us remark the next properties of the function a and the equation (3.4):

1. $a'(t) \to 0$ for $t \to +\infty$;
2. $a(t) \in [-2, 0]$ for all $t \in R$;
3. $\{a_\tau | \tau \geq 0\}$ is relatively compact in $C(R, R)$, где $a_\tau(t) = a(t + \tau)(t \in R)$;
4. $\omega_a \neq \emptyset$ and is compact ;
5. all functions from ω_a are constant and $b(t) = c \in [-2, 0](t \in R)$ for any $b \in \omega_a$;
6. $a(t_n) = 0$ then and only then , when $t_n = -1 + (\frac{\pi}{2} + 2\pi n)^2$ $(n \in \mathbb{Z})$;
7. there is $\{t_{n_k}\} \subseteq \{t_n\}$ such that $a(t + t_{n_k}) \to b(t)$ and $b(t) = 0$ for all $t \in R$;
8. for any $b \in H^+(a) = \overline{\{a_\tau | \tau \in R_+\}}$ the inequality

$$|\varphi(t, x, b)| \leq |x| \tag{3.5}$$

   takes place for all $x \in R$ and $t \in R_+$, where $\varphi(t, x, b)$ is the solution of the equation

$$y'(t) = b(t)y, \tag{3.6}$$

   going through the point $x \in R$ for $t = 0$;

9. if $b \in \omega_a \setminus \{0\}$, then $b(t) = c < 0 (t \in R)$ and ,hence,

$$\lim_{t \to +\infty} |\varphi(t, x, b)| = 0 \tag{3.7}$$

for all $x \in R$;

10. if $b = 0 (b \in \omega_a)$, then $\varphi(t, x, b) = x$ for all $t \in R$.

Suppose $Y = H^+(a)$ and define by $(Y, \mathbb{R}_+, \sigma)$ the dynamical system of displacements on Y. Let $X = \mathbb{R} \times Y$ and $(X, \mathbb{R}_+, \pi)$ is a semigroup dynamical system on X, where $\pi = (\varphi, \sigma)$ (that is $\pi((x, b), t) = (\varphi(t, x, b), b_t)$ for all $(x, b) \in X$ and $t \in \mathbb{R}_+$). Then $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma) >$ is a nonautonomous dynamical system, generated by the equation (3.4), where $h = pr_2 : X \to Y$. From the properties 1.-10.it follows , that for the nonautonomous dynamical system $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma) >$, generated by the equation (3.4), all the conditions of theorem 3.1 are carried out ,except the minimality of Y, and it has no the compact global attractor.

Corollary 3.4. *Let (E, T_1, π) be completely continuous and for any $y \in Y$ there is $R(y) \geq 0$ such that*

$$\overline{\lim_{t \to +\infty}} |x \cdot t| \leq R(y) \tag{3.8}$$

for any $x \in E_y$, then the nonautonomous dynamical system $< (X, T_1, \pi), (Y, T_2, \sigma), h >$ admits the compact global attractor.

This assertion follows from theorem 3.1, if we will notice , that from condition (3.8) it follows the boundedness on T_+ of every motion from (X, T_1, π) .

Theorem 3.5. *Let the next conditions are carrying out :*

1. *(E, T_1, π) is asymptotically compact, that is for any bounded semi-continuous set $A \subset E$ there is a nonempty compact K_A such that*

$$\lim_{t \to +\infty} \beta(\pi^t A, K_A) = 0; \tag{3.9}$$

2. *(E, T_1, π) is asymptotically bounded , that is for any bounded set $A \subset E$ there is $l = l(A) \geq 0$ such that $\cup \{\pi^t A | t \geq l\}$ is bounded ;*
3. *there are $y_0 \in Y$ and $R_0 > 0$ such that (3.1) is fulfilled.*

Then the nonautonomous dynamical system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ admits the maximal compact attractor.

Proof. First, let us notice, that in conditions of theorem 3.5 the dynamical system (E, T_1, π) satisfies the condition of Ladyzhenskaya .Let $R > R_0$, then for any $x \in E$ there will be $\tau = \tau(x) \geq 0$ such that $|x \cdot \tau| < R$. If we suppose that it is not so ,then there will be $x^1 \in E$ and $R' > R_0$ such that

$$|x^1 \cdot \tau| \geq R' > R_0 \tag{3.10}$$

for all $\tau \geq 0$ and, hence , $\omega_{x^1} \cap E_{y_0} \neq \emptyset$. That is why for any $x \in \omega_{x_1} \cap E_{y_0}$ the inequality (3.3) takes place , but this contradicts (3.1) . Thus the assertion we need is proved. Now for ending the proof of the theorem it is sufficiently to cite theorem 2.2 .

Remark 3.6. *Let us notice , that theorem 3.5 (like theorem 3.1) without demanding the minimality of Y does not take place even in class of linear systems. The last assertion is proved by the example 3.3 .*

Theorem 3.7. *Let (E, h, Y) be a finite-dimensional vector stratification, Y is a compact minimal set and $y_0 \in Y$, then the next conditions are equivalent:*

1. *the nonautonomous dynamical system $< (E, T_1, \pi), (Y, T_2, \sigma), h >$ is dissipative;*
2. *there is $R > 0$ such that*

$$\overline{\lim_{t \to +\infty}} |x \cdot t| < R \tag{3.11}$$

   for all $x \in E_{y_0}$ and all motions (E, T_1, π) are bounded on T_+ ;

3. *there is a positive number r such that for any $x \in E_{y_0}$ and $l > 0$ there will be $\tau = \tau(x) \geq l$ for which*

$$|x \cdot \tau| < r \tag{3.12}$$

   and all the motions (E, T_1, π) are bounded on T_+;

4. *there is a nonempty compact $K_1 \subset E$ such that $\omega_x \cap K_1 \neq \emptyset$ for all $x \in E_{y_0}$ and all the motions (E, T_1, π) are bounded on T_+;*
5. *there is a nonempty compact $K_2 \subseteq E$ such that $\omega_x \neq \emptyset$ and $\omega_x \subseteq K_2$ for all $x \in E_{y_0}$ and all the motions (E, T_1, π) are bounded on T_+ ;*
6. *there is a positive number R_0 such that for any $R_1 > 0$ there will be $l(R_1) > 0$, that*

$$|x \cdot t| < R_0 \tag{3.13}$$

   for all $t \geq L(R_1), |x| \leq R_1 (x \in E_{y_0})$ and all the motions (E, T_1, π) are bounded on T_+.

Proof. Implications 1. $\Longrightarrow$ 6. $\Longrightarrow$ 2. $\Longrightarrow$ 5. $\Longrightarrow$ 4. $\Longrightarrow$ 3. are evident. According to theorem 3.1 from 3. it follows 1..Theorem is proved.

§ 4. Global attractors of skew products of dynamical systems.

Let W and Y be full metrical spaces, (Y, T, σ) is a group dynamical system on Y and $< W, \varphi, (Y, T, \sigma) >$ is a skew product over (Y, T, σ) with the layer W (that is φ is a continuous mapping $W \times Y \times T_+$ in W,satisfying conditions: $\varphi(0, w, y) = w$ and $\varphi(t + \tau, w, y) = \varphi(t, \varphi(\tau, w, y), y_\tau)$ for all $t \in T_+, \tau \in T, w \in W$ and $y \in Y$), $X = W \times Y, (X, T_+, \pi)$ is a semi-group dynamical system on X defined by the equality $\pi = (\varphi, \sigma)$ and $< (X, T_+, \pi), (Y, T, \sigma), h >$ $(h = pr_2)$ is the corresponding nonautonomous dynamical system.

If $M \subseteq W$, then suppose

$$\Omega_y(M) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M, y^{-\tau})} \tag{4.1}$$

for every $y \in Y$, where $y^{-\tau} = \sigma(y, -\tau)$.

Lemma 4.1. *The next assertions take place:*

1. *the point* $p \in \Omega_y(M)$ *then and only then , when there are* $t_n \to +\infty$ *and* $\{x_n\} \subseteq M$ *such that* $p = \lim_{n\to+\infty} \varphi(t_n, x_n, y^{-t_n})$*;*
2. $U(t,y)\Omega_y(M) \subseteq \Omega_{y\cdot t}(M)$ *for all* $y \in Y$ *and* $t \in T_+$*, where* $U(t,y) = \varphi(t,\cdot,y)$ *;*
3. *if it were any point* $w \in \Omega_y(M)$ *the motion* $\varphi(t,w,y)$ *is defined on* T*;*
4. *if there is a nonempty compact* $K \subset W$ *such that*
$$\lim_{t\to+\infty} \beta(\varphi(t,M,y^{-t}),K) = 0, \tag{4.2}$$
then $\Omega_y(M) \neq \emptyset$*,is compact,*
$$\lim_{t\to+\infty} \beta(\varphi(t,M,y^{-t}),\Omega_y(M)) = 0 \tag{4.3}$$
and
$$U(t,y)\Omega_y(M) = \Omega_{y\cdot t}(M) \tag{4.4}$$
for all $y \in Y$ *and* $t \in T_+$ *.*

Proof. The first assertion of the lemma directly follows from the equality (4.1).

Let $w \in \Omega_y(M)$, then there are $t_n \to +\infty$ and $x_n \subseteq M$ such that
$$w = \lim_{n\to+\infty} \varphi(t_n, x_n, y^{-t_n})$$
and ,hence,
$$\varphi(t,w,y) = \lim_{n\to+\infty} \varphi(t, \varphi(t_n, x_n, y^{-t_n}), y) = \lim_{n\to+\infty} \varphi(t + t_n, x_n, y^{-t_n}). \tag{4.5}$$
Thus $\varphi(t,w,y) \in \Omega_{y\cdot t}(M)$, that is $U(t,y)\Omega_y(M) \subseteq \Omega_{y\cdot}(M)$ for all $y \in Y$ and $t \in T_+$.

From the equality (4.5) it follows , that the motion $\varphi(t,w,y)$ is defined on T like $\varphi(t+t_n, x_n, y^{-t_n})$ is defined on $[-t_n, +\infty)$ and $t_n \to +\infty$.

The fourth assertion of the lemma is proved like theorem 1.1.1 and lemma 1.1.3 from [8].

The skew product over (Y,T,σ) with the layer W we will define by a compactly dissipative one , if there is a nonempty compact $K \subseteq W$ such that
$$\lim_{t\to+\infty} \sup\{\beta(U(t,y)M,K) | y \in Y\} = 0 \tag{4.6}$$
for any $M \in C(W)$.

Lemma 4.2. *Let* Y *is compact and* $< W, \varphi, (Y,T,\sigma) >$ *is a skew product over* (Y,T,σ) *with the layer* W*. In order to* $< W, \varphi, (Y,T,\sigma) >$ *, were a compact dissipative one , it is necessary and sufficiently that the semi-group autonomous system* (X, T_+, π) *should be a compactly dissipative one .*

This assertion directly follows from the corresponding definitions .

We will say , that the space X possesses the (S)-property, if for any compact $K \subseteq X$ there is a coherent set $M \subseteq X$ such that $K \subseteq M$.

By the whole trajectory of the semi-group dynamical system (X, T_+, π) (of the skew product $< W, \varphi, (Y,T,\sigma) >$ over (Y,T,σ) with the layer W), which goes through the point $x \in X((u,y) \in W \times Y)$ we will call the continuous mapping $\gamma : T \to X(\nu : T \to W)$ which satisfies conditions : $\gamma(0) = x(\nu(0) = u)$ and $\pi^t\gamma(\tau) = \gamma(t+\tau)(\varphi(t+\tau,u,y) = \varphi(t,\gamma(\tau), y\cdot t))$ for all $t \in T_+$ and $\tau \in T$.

Theorem 4.3. *Let Y be compact, $< W, \varphi, (Y, T, \sigma) >$ is compactly dissipative and K is the nonempty compact, figuring in the equality (4.6), then :*

1. $I_y = \Omega_y(K) \neq \emptyset$, *is compact,* $I_y \subseteq K$ *and*
$$\lim_{t\to+\infty} \beta(U(t, y^{-t})K, I_y) = 0 \tag{4.7}$$
for every $y \in Y$*;*
2. $U(t, y)I_y = I_{y\cdot t}$ *for all* $y \in Y$ *and* $t \in T_+$*;*
3. $$\lim_{t\to+\infty} \beta(U(t, y^{-t})M, I_y) = 0 \tag{4.8}$$
for all $M \in C(W)$ *and* $y \in Y$ *;*
4. $$\lim_{t\to+\infty} \sup\{\beta(U(t, y^{-t})M, I)|y \in Y\} = 0 \tag{4.9}$$
for any $M \in C(W)$, *where* $I = \cup\{I_y|y \in Y\}$*;*
5. $I_y = pr_1 I_y$ *for all* $y \in Y$, *where J is a Levinson centre of* (X, T_+, π), *and ,hence ,* $I = pr_1 J$*;*
6. *the set I is compact;*
7. *the set I is coherent if one of the next two conditions is fulfilled :*
   a. $T_+ = R_+$ *and the spaces W and Y are coherent;*
   b. $T_+ = \mathbb{Z}_+$ *and the space* $W \times Y$ *possesses the* (S)*-property or it is coherent and locally coherent.*

Proof. The first two assertions of the theorem follows from lemma 4.1 .

If we suppose that the equality (4.8) does not take place , then there will be $\epsilon_0 > 0, y_0 \in Y, M_0 \in C(W), \{x_n\} \subseteq M_0$ and $t_n \to +\infty$ such that
$$\rho(U(t_n, y_0^{-t_n})x_n, I_{y_0}) \geq \epsilon_0. \tag{4.10}$$
According to the equality (4.7) for ϵ_0 and $y_0 \in Y$ there will be $t_0 = t_0(\epsilon_0, y_0) > 0$ such that
$$\beta(U(t, y_0^{-t})K, I_{y_0}) < \frac{\epsilon_0}{2} \tag{4.11}$$
for all $t \geq t_0$. Let us notice , that
$$U(t_n, y_0^{-t_n})x_n = U(t_0, y_0^{-t_0})U(t_n - t_0, y_0^{-t_n})x_n. \tag{4.12}$$

As $< W, \varphi, (Y, T, \sigma) >$ is compactly dissipative,then the sequence $\{U(t_n - t_0, y_0^{-t_n})x_n\}$ we may consider to be a convergent one. Suppose $\overline{x} = \lim_{n\to+\infty} \varphi(t_n - t_0, x_n, y_0^{-t_n})$, then according to lemma 4.1 $\overline{x} \in \Omega_{y_0^{-t_0}}(M_0)$ and $U(t_0, y_0^{-t_0})\overline{x} \in \Omega_{y_0}(M_0)$. From the equality (4.6) it follows that $\overline{x} \in K$. Passing to the limit in (4.10), when $n \to +\infty$ and taking into consideration (4.12) we will get
$$U(t_0, y_0^{-t_0})\overline{x} \notin B(I_{y_0}, \varepsilon_0). \tag{4.13}$$
On the other hand as $\overline{x} \in K$, then from (4.11) we have
$$U(t_0, y_0^{-t_0})\overline{x} \in B(I_{y_0}, \frac{\varepsilon}{2}), \tag{4.14}$$

,and this contradicts (4.13). This contradiction proves the assertion we need .

Let us prove now the equality (4.9). If we suppose that it does not take place , then there will be $\varepsilon_0 > 0, M_0 \in C(W), y_n \in Y, \{x_n\} \subseteq M_0$ and $t_n \to +\infty$ such that

$$\rho(U(t_n, y_n^{-t_n})x_n, I) \geq \varepsilon_0. \tag{4.15}$$

As Y is compact, then the sequences $\{y_n\}$ and $\{y_n \cdot t_n\}$ we may consider to be convergent. Suppose $y_0 = \lim_{n\to+\infty} y_n$ and $\overline{y} = \lim_{n\to+\infty} y_n \cdot t_n$. According to (4.8) for the number $\varepsilon_0 > 0$ and $y_0 \in Y$ there will be $t_0 = t_0(\varepsilon_0, y_0)$ such that

$$\beta(U(t_0, {y_0}^{-t})M_0, I_{y_0}) < \frac{\varepsilon_0}{2} \tag{4.16}$$

for all $t \geq t_0(\varepsilon_0, y_0)$. Let us notice , that

$$U(t_n, y_n^{-t_n})x_n = U(t_0, y_n^{-t_0})U(t_n - t_0, y_n^{-t_n})x_n. \tag{4.17}$$

As $< W, \varphi, (Y, T, \sigma) >$ is compactly dissipative,then the sequence $\{U(t_n - t_0, y_n^{-t_n})x_n\}$ we may consider to be a convergent one.Suppose $x' = \lim_{n\to+\infty} \varphi(t_n - t_0, x_n, y_n^{t_n})$ and let us notice, that according to (4.6) $x' \in K$. From the equality (4.17) it follows,that $U(t_n, y_n^{-t_n})x_n \to U(t_0, y^{-t_0})x'$ and ,hence, from (4.15) we have

$$U(t_0, {y_0}^{-t_0})x' \in B(I_{y_0}, \frac{\varepsilon_0}{2}). \tag{4.18}$$

The last inclusion contradicts (4.17), and this finishes the proving of the fourth assertion of the theorem.

Let us prove the fifth assertion of the theorem.In order to do this ,let us notice, that $w \in I_y$, if $\varphi(t, w, y)$ is defined on T and $\varphi(T, w, y)$ is relatively compact. Really, as $w = \varphi(t, \varphi(-t, w, y), y^{-t})$ for all $t \in T$, then from the equality (4.8) follows the inclusion we need. Thus we get the following description of the set I_y : $I_y = \{w \in W|$ at least one whole trajectory of $< W, \varphi, (Y, T, \sigma) >\}$ goes through the point (x, y). Now it remains to notice ,that Levinson centre J is compact and consists of the whole trajectories of (X, T_+, π) , and ,hence, $pr_1 J_y \subseteq I_y$ for all $y \in Y$.

The compactness of the set I follows from the equality $I = pr_1 J$, from the compactness of J and from the continuity of $pr_1 : X \to W$.

The last assertion follows from the next: in conditions of theorem 4.3 Levinson centre J of the dynamical system (X, T_+, π), according to corollary 1.8.7 and theorem 1.8.15 from [8] , is coherent, and ,hence, I as a continuous image of a coherent set, also is coherent . Theorem is proved in full.

Remark 4.4 Theorem 4.3 intensifies and defines more precisely the main results of [11-12,23].

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