Asymptotics of Solutions of Infinite-Dimensional Homogeneous Dynamical Systems

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ABSTRACT. In this paper we study the connection between the uniform asymptotic stability and the power-law or exponential asymptotics of the solutions of infinite-dimensional systems (differential equations in Banach spaces, functional differential equations, and completely solvable multidimensional differential equations).

KEY WORDS: infinite-dimensional dynamical system, uniform asymptotic stability, power-law asymptotics, exponential asymptotics.

Krasovskii [1, 2], Zubov [3], and Coleman [4] showed that for homogeneous autonomous systems in a finite-dimensional space the existence of a power-law asymptotics is equivalent to asymptotic stability.

Filippov [5, 6] generalized this result to homogeneous generalized differential equations.

Ladis [7] showed that in the general case this result does not apply to periodic systems. For nonautonomous homogeneous systems (of order $k=1$), uniform asymptotic stability is equivalent to exponential stability (see, for example, [8]). Morozov [9] obtained a similar result for periodic generalized differential equations.

The goal of the present paper is to study the connection between the uniform asymptotic stability and the power-law (exponential) asymptotics of the solutions of infinite-dimensional systems. This problem is studied and solved within the framework of general dynamical systems with infinite-dimensional phase space. The general results obtained are applied to various differential equations with infinite-dimensional phase spaces (such as ordinary differential equations in Banach spaces, functional differential equations, some types of evolution partial differential equations, and completely solvable multidimensional differential equations).

§1. Abstract dynamical systems

Throughout the following, we shall use the notation and terminology from [10, 11]. Recall some of the terms. Suppose that (X, ρ) is a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) is the group of real numbers (integers), $S=\mathbb{R}$ or $\mathbb{Z}$, and $T=S_{+}=\{s : s \in S,\ s \geq 0\}$. Let $\rho(x, A)$ be the distance from the point x to the set A, let $C(X)$ be the set of all nonempty compact sets from X, and let 2^X be the family of all bounded closed subsets of X equipped with the Hausdorff metric.

By a *dispersive dynamical system* on X we mean a triple (X, T, f), where f is a mapping of $T \times X$ into $C(X)$ satisfying the following conditions:

1) $f(x, 0)=x$ $(x \in X)$;
2) $f(f(x, t_1), t_2)=f(x, t_1+t_2)$ $(x \in X,\ t_1, t_2 \in T)$;
3) $\beta(f(x, t), f(x_0, t_0)) \to 0$ as $x \to x_0$ and $t \to t_0$, where $\beta(A, B)=\sup\{\rho(a, B) : a \in A\}$.

A continuous one-to-one mapping $\varphi_x : T \to X$ is said to be a *motion of a dispersive dynamical system* (X, T, f) *issuing from a point* $x \in X$ if

a) $\varphi_x(0)=x$;
b) $\varphi_x(t_2) \in f(\varphi_x(t_1), t_2-t_1)$ for any $t_1, t_2 \in T$ $(t_2>t_1)$.

The set of all motions issuing from a point $x \in X$ is denoted by Φ_x, and we write $\Phi(f)=\bigcup\{\Phi_x : x \in X\}$.

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A dispersive dynamical system (X, T, f) is said to be a *system with uniqueness* (or an *ordinary dynamical system*) if for all $x \in X$ and $t \in T$ the set $f(x,t)$ is a singleton, i.e., a unique trajectory of the dynamical system (X, T, f) passes through each point $x \in X$.

By a *nonautonomous dispersive dynamical system* we mean a triple $\langle (X,T,f), (Y,T,\sigma), h\rangle$, where (X,T,f) is a dispersive dynamical system on X, (Y,T,σ) is an ordinary dynamical system on Y, and h is a homomorphism of (X,T,f) onto (Y,T,σ), i.e., a continuous mapping of X onto Y satisfying the condition $h\big(f(x,t)\big) = \sigma(h(x),t)$ for all $x \in X$ and $t \in T$.

Let (X,h,Y) be a locally trivial vector bundle with fiber E. An autonomous dynamical system $(X, \mathbb{R}_+, f)$ is said to be *homogeneous of order* $k \in \mathbb{R}$ if for any $x \in X$, $\lambda \geq 0$, and any $\varphi_x \in \Phi_x$, the function $\psi\colon T \to X$ defined by the relation $\psi(t) = \lambda \varphi_x(\lambda^{k-1}t)$ is a motion of (X,T,f) issuing from the point $\lambda x \in X$, i.e., $\psi \in \Phi_{\lambda x}$.

A nonautonomous dynamical system $\langle (X,T,f), (Y,T,\sigma), h\rangle$ is said to be *homogeneous of order* $k=1$ if the autonomous dynamical system (X,T,f) is homogeneous of order $k=1$.

Everywhere below we assume that the fiber bundle (X,h,Y) is normed. Set $|A| = \sup\{|a| : a \in A\}$ if $A \subseteq X$ is bounded. Let X^s be the stable manifold of the homogeneous system $\langle (X,T,f), (Y,T,\sigma), h\rangle$, i.e.,

$$X^s = \Big\{x : x \in X,\ \lim_{t\to+\infty} |f(x,t)| = 0\Big\},$$

and $\Theta = \{\theta_y : y \in Y\}$, where θ_y is the zero element of X_y, is the zero section of the fiber bundle (X,h,Y).

Lemma 1.1. *Let a nonautonomous system* $\langle (X,T,f), (Y,T,\sigma), h\rangle$ *be homogeneous of order* $k=1$. *Then the following assertions are equivalent*:

a) *the zero section* Θ *of the fiber bundle* (X,h,Y) *is stable, i.e., for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that* $|x| < \delta$ *implies* $|f(x,t)| < \varepsilon$ *for all* $t \geq 0$;

b) *there exists a positive number* N *such that*

$$|f(x,t)| \leq N|x| \tag{1}$$

for all $x \in X$ *and* $t \geq 0$.

Proof. Let us show that a) implies b). Let $\varepsilon_0 = 1$ and $\delta_0 > 0$ be such that $|x| \leq \delta_0$ implies $|f(x,t)| \leq 1$ for all $|x| \leq \delta_0$ and $t \geq 0$. Now let $x \in X$ and $\varphi_x \in \Phi_x$. Then, in view of the homogeneity of the system (X,T,f), we have $\delta_0|x|^{-1}\varphi_x \in \Phi_{\delta_0|x|^{-1}x}$, and since $f(x,t) = \{\varphi_x(t) : \varphi_x \in \Phi_x\}$ (see, for example, [10]), from (1) we obtain $|f(x,t)| \leq \delta_0^{-1}|x|$ for all $t \geq 0$ and $x \in X$. The converse implication is obvious. The lemma is proved. □

Lemma 1.2. *Let the assumptions of Lemma* 1.1 *be valid. Then the following assertions are equivalent*:

a) *the zero section* $\langle X,h,Y\rangle$ *is uniformly asymptotically stable*; *i.e.,* Θ *is stable, and there exists a* $\gamma > 0$ *such that*

$$\lim_{t\to+\infty} \sup_{|x|\leq\gamma} |f(x,t)| = 0; \tag{2}$$

b) *the following relation is valid*:

$$\lim_{t\to+\infty} \sup_{|x|\leq 1} |f(x,t)| = 0, \tag{3}$$

and there exists an $N > 0$ *such that*

$$\sup_{|x|\leq 1} |f(x,t)| \leq N$$

for all $t \geq 0$.

Proof. Let us show that a) implies b). By Lemma 1.1, it suffices to show that (2) implies (3). Let $x \in X$ ($|x| \le 1$), and let $\varphi_x \in \Phi_x$; then, in view of the homogeneity of (X, T, f), we have $\gamma |x|^{-1} \varphi_x \in \Phi_{\gamma |x|^{-1} x}$, and since $f(x, t) = \{\varphi_x(t) : \varphi_x \in \Phi_x\}$, from (2) we obtain

$$\sup_{|x|\le 1} |f(x, t)| \le \gamma^{-1} \sup_{|y| \le \gamma} |f(y, t)| \to 0$$

as $t \to +\infty$. In a similar way, we can prove the converse implication. The lemma is thereby proved. □

Remark 1.1. Lemma 1.1 and 1.2 are also valid for autonomous homogeneous (of order $k > 1$) systems.

Lemma 1.3. *Let $\mathfrak{M}$ be a family of functions $m \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

a) *there exists an $M > 0$ such that $0 < m(t) \le M$ for all $t \ge 0$ and $m \in \mathfrak{M}$;*

b) *$m(t) \to 0$ as $t \to +\infty$ uniformly in $m \in \mathfrak{M}$, i.e., for any $\varepsilon > 0$ and $m \in \mathfrak{M}$ there exists an $L(\varepsilon, m) > 0$ such that $m(t) < \varepsilon$ for all $t \ge L(\varepsilon, m)$.*

Then we have the following assertion:

1) *if $m(t + \tau) \le m(t) m(\tau)$ for all $t, \tau \ge 0$ and $m \in \mathfrak{M}$, then there exist positive numbers N and ν such that $m(t) \le N e^{-\nu t}$ for all $t \ge 0$ and $m \in \mathfrak{M}$;*

2) *if $m(t + \tau) \le m(t) m\big(\tau m^{\alpha - 1}(t)\big)$ $(\alpha > 1)$ for all $t, \tau \ge 0$ and $m \in \mathfrak{M}$, then there exist positive numbers a and b such that*

$$m(t) \le M (a + bt)^{-1/(\alpha - 1)} \tag{4}$$

for all $t \ge 0$ and $m \in \mathfrak{M}$.

Proof. The first assertion of the lemma is an insignificant modification of a lemma due to Massera and Sheffer [12, p. 167]. Let us prove the second assertion of the lemma. Let $\tau > 0$ be such that $m(t) \le 1/2$ for all $t \ge \tau$ and $m \in \mathfrak{M}$. Since

$$0 < m(t) \le M \text{ and } m(t + \tau) \le m(t) m\big(\tau m^{\alpha-1}(t)\big) \text{ for all } t, \tau \ge 0 \text{ and } m \in \mathfrak{M},$$

we have

$$m(t) \le M, \qquad 0 \le t \le q\tau, \quad q = 2^{\alpha - 1}, \tag{5}$$

and

$$m(t) \le \frac{1}{2}, \qquad q\tau \le t < +\infty, \tag{6}$$

for all $m \in \mathfrak{M}$. Set $t_0 = 0$ and $t_{i+1} = t_i + \tau q_i$ ($q_i = q^i$), and note that

$$m(t_i) \le \frac{1}{2^i}, \qquad i \ge 1, \tag{7}$$

for all $m \in \mathfrak{M}$. Indeed, according to (6), we have $m(t_1) \le 2^{-1}$ Moreover,

$$m(t_{i+1}) = m(t_i + \tau q_i) \le m(t_i) m\big(\tau q_i\, m^{\alpha-1}(t_i)\big) \tag{8}$$

for all $m \in \mathfrak{M}$. Suppose that (7) is valid for all $i \le n$; then it follows from (8) that $m(t_{n+1}) \le 1/2^{n+1}$, since $\tau q_n\, m^{\alpha - 1}(t_n) \ge \tau$ (for any $m \in \mathfrak{M}$) in view of the choice of q_n and the inductive assumption (7). Thus it follows from (5) and (7) that $m(t) \le 1/2^n$ for all $t \ge t_n$ and $n \ge 1$. Note that $t_{n+1} = \tau(q^{n+1} - 1)(q - 1)^{-1}$, and therefore

$$2^{-n} = 2\left(\frac{2^{\alpha-1} - 1}{\tau} t_{n+1} + 1\right)^{-1/(\alpha-1)}.$$

Now let $t \in [t_n, t_{n+1})$; then we have

$$2^{-n} < 2\left(1 + \frac{2^{\alpha-1} - 1}{\tau} t\right)^{-1/(\alpha-1)}. \tag{9}$$

It follows from (5) and (9) that

$$m(t) \le M\left(2^{1-\alpha} + \frac{1 - 2^{1-\alpha}}{\tau} t\right)^{-1/(\alpha-1)}$$

for all $t \ge 0$ and $m \in \mathfrak{M}$. By setting $a = 2^{1-\alpha}$ and $b = \tau^{-1}(1 - 2^{1-\alpha})$, we obtain the required assertion. The lemma is proved. □

Theorem 1.1. *Let the dispersive nonautonomous dynamical system* $\langle (X,T,f),(Y,T,\sigma),h\rangle$ *be homogeneous of order* $k=1$. *Then the following assertions are equivalent*:

a) *the zero section of the fiber bundle* (X,h,Y) *is uniformly asymptotically stable*;
b) *there exist positive numbers* N *and* ν *such that* $|f(x,t)| \le Ne^{-\nu t}|x|$ *for all* $x\in X$ *and* $t\ge 0$.

Proof. For the proof of Theorem 1.1, it suffices to establish the implication a)$\Longrightarrow$b), since the converse assertion is obvious. Set

$$m(t) = \sup_{|x|\le 1} |f(x,t)|. \tag{10}$$

By Lemma 1.2, the mapping $m\colon \mathbb{R}_+ \to \mathbb{R}_+$ is well defined by (10); moreover, $0 < m(t) \le M$ ($M \ge m(0) = 1$) and $m(t)\to 0$ as $t\to +\infty$. Further, note that, in view of the first-order homogeneity of the system $\langle (X,T,f),(Y,T,\sigma),h\rangle$, we have

$$\begin{aligned}\frac{m(t+\tau)}{m(t)} &= \frac{1}{m(t)}\sup_{|x|\le 1}|f(x,t+\tau)| = \frac{1}{m(t)}\sup_{|x|\le 1}|f(f(x,t),\tau)| \\ &\le \sup_{|x|\le 1}\left|f\left(\frac{f(x,t)}{m(t)},\tau\right)\right| \le \sup_{|x|\le 1}|f(x,\tau)| = m(\tau)\end{aligned}$$

for all $t,\tau\ge 0$. Now let $x\in X$ ($|x|\ne 0$). Then, since the system $\langle (X,T,\pi),(Y,T,\sigma),h\rangle$ is homogeneous, we can write

$$\left|\frac{1}{|x|}f(x,t)\right| \le \sup_{|x|\le 1}|f(x,t)| = m(t),$$

i.e., $|f(x,t)|\le m(t)|x|$ for all $t\ge 0$ and $x\in X$. Next, to complete the proof of the theorem, it suffices to refer to Lemma 1.3. The theorem is proved. ☐

Theorem 1.2. *For an autonomous homogeneous* (*of order* $k>1$) *dispersive dynamical system* $(X,\mathbb{R}_+,f)$ *the following assertions are equivalent*:

1) *the zero motion* $(X,\mathbb{R}_+,f)$ *is uniformly asymptotically stable*;
2) *there exist positive numbers* α *and* β *such that*

$$|f(x,t)| \le (\alpha|x|^{1-k}+\beta t)^{-1/(k-1)} \tag{11}$$

for all $t\ge 0$ *and* $x\in X$.

Proof. Let us show that under the conditions of Theorem 1.2 assumption 1) implies 2). Let $x\in X$ ($x\ne 0$). Then, in view of the homogeneity of order $k>1$ of the system $(X,\mathbb{R}_+,f)$, we have

$$\frac{1}{|x|}|f(x,t)| \le \sup_{|y|\le 1}|f(y,t|x|^{k-1})| = m(t|x|^{k-1}),$$

and therefore

$$|f(x,t)| \le |x|\, m(t|x|^{k-1}) \tag{12}$$

for all $x\in X$ and $t\ge 0$, where $m\colon\mathbb{R}_+\to\mathbb{R}_+$ is defined by relation (10). According to Lemmas 1.1 and 1.2, and also Remark 1.1, the function m satisfies the assumptions of Lemma 1.3. Moreover, note that

$$\begin{aligned}\frac{m(t+\tau)}{m(t)} &= \frac{1}{m(t)}\sup_{|x|\le 1}|f(x,t+\tau)| = \frac{1}{m(t)}\sup_{|x|\le 1}|f(f(x,t),\tau)| = \sup_{|x|\le 1}\frac{1}{m(t)}|f(f(x,t),\tau)| \\ &\le \sup_{|x|\le 1}\left|f\left(\frac{f(x,t)}{m(t)},\tau\, m^{k-1}(t)\right)\right| \le m\big(\tau\, m^{k-1}(t)\big),\end{aligned}$$

i.e.,

$$m(t+\tau) \le m(t)m\big(\tau\, m^{k-1}(t)\big)$$

for all $t, \tau \ge 0$. Now, to complete the proof of the required assertion, note that, by Lemma 1.3, there exist numbers $a, b > 0$ for which inequality (4) holds, and (12) and (4) implies (11) if we set $\alpha = M^{1-k}a$ and $\beta = M^{1-k}b$.

Let us now show that 2) implies 1). To this end, note that the function $\omega\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by the relation

$$\omega(r,t) = (\alpha r^{1-k} + \beta t)^{-1/(k-1)}$$

is monotone increasing with respect to r for each $t \ge 0$, and for a given $r \ge 0$ we have the relation

$$\lim_{t\to+\infty} \omega(r,t) = 0.$$

Since $|f(x,t)| \le \omega(r,t)$ for all $|x| \le r$ and $t \ge 0$, it follows that

$$m(t) = \sup\{|f(x,t)| : |x| \le 1\} \le \omega(1,t),$$

and therefore m is bounded and tends to 0 as $t \to +\infty$. By Lemma 1.2 and Remark 1.1, the zero motion $(X, \mathbb{R}_+, f)$ is uniformly asymptotically stable. The proof of the theorem is complete. □

§2. Dynamical systems with multidimensional time

Recall that a set $G \subseteq E$ is called a *cone* in a Banach space E if $-t \notin G$ whenever $t \in G \setminus \{0\}$ and the inclusion $t \in G$ implies $\lambda t \in G$ for all $\lambda \ge 0$.

Let G be a closed convex cone in E. A semigroup dynamical system (X, G, π) is said to be a *dynamical system with multidimensional time.*

A dynamical system (X, G, π) with multidimensional time $G \subseteq E$ is called *homogeneous of order* k ($k \ge 1$) if $\pi(\lambda x, t) = \lambda\, \pi(x, \lambda^{k-1} t)$ for all $\lambda \ge 0$, $x \in E$ and $t \in G$.

Lemma 2.1. *Let $m\colon G \to \mathbb{R}_+$, $0 < m(t) \le M$ $(M \ge 1)$ and $m(t) \to 0$ as $\|t\| \to +\infty$. Then the following assertions are valid*:

1) *if $m(t+\tau) \le m(t)m(\tau)$ for all $t, \tau \in G$, then there exist positive numbers N and ν such that*

$$m(t) \le N e^{-\nu\|t\|} \tag{13}$$

for all $t \in G$;

2) *if $m(t+\tau) \le m(\tau)m\big(\tau\, m^{\alpha-1}(t)\big)$ $(\alpha > 1)$ for all $t, \tau \in G$, then there exist positive numbers a and b such that*

$$m(t) \le M(a + b\|t\|)^{-1/(\alpha-1)} \tag{14}$$

for all $t \in G$.

Proof. Set $H = \{h : h \in G,\ \|h\| = 1\}$ and $m_h(\lambda) = m(\lambda h)$ for all $h \in H$ and $\lambda \ge 0$. Then the family of functions $\mathfrak{M} = \{m_h : h \in H\}$ satisfies the assumptions of Lemma 1.3, and therefore there exist positive numbers a and b such that

$$m_h(\lambda) \le M(a + b\lambda)^{-1/(\alpha-1)}$$

for all $h \in H$ and $\lambda \ge 0$. Now let $t \in G \setminus \{0\}$ and $h = t\|t\|^{-1}$. Then

$$m(t) = m(\|t\|h) = m_h(\|t\|) \le M(a + b\|t\|)^{-1/(\alpha-1)}$$

for all $t \in G$. The lemma is proved. □

Theorem 2.1. *Let X be a Banach space, and let (X, G, π) be a homogeneous (of order $k \geq 1$) dynamical system with multidimensional time. Then the following assertions are equivalent:*

1) *the zero motion (X, G, π) is uniformly asymptotically stable, i.e., for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|x\| \leq \delta$ implies $\|\pi(x, t)\| < \varepsilon$ for all $t \in G$, and there exists a $\gamma > 0$ such that*

$$\lim_{\|t\| \to +\infty} \|\pi(x, t)\| = 0$$

uniformly with respect to $\|x\| < \gamma$;

2a) *if $k = 1$, then there exist positive numbers N and ν such that*

$$\|\pi(x, t)\| \leq N e^{-\nu \|t\|} \|x\| \tag{15}$$

for all $x \in X$ and $t \in G$;

2b) *if $k > 1$, then there exist positive numbers a and b such that*

$$\|\pi(x, t)\| \leq (\alpha \|x\|^{1-k} + \beta \|t\|)^{-1/(k-1)} \tag{16}$$

for all $x \in X$ and $t \in G$.

Proof. Let us show that under the assumptions of Theorem 1.3 1) implies 2). Let $x \neq 0$. Then, in view of the homogeneity (X, G, π) of order k ($k \geq 1$), we have

$$\|\pi(x, t)\| = \|x\| \cdot \left\| \pi\left(\frac{x}{\|x\|}, t\|x\|^{k-1} \right) \right\| \leq \|x\| \, m(t\|x\|^{k-1}) \tag{17}$$

for all $x \in X$ and $t \in G$, where

$$m(t) = \sup_{\|x\| \leq 1} \|\pi(x, t)\|.$$

Note that the uniform asymptotic stability of the zero motion of the dynamical system (X, G, π) implies that the function $m\colon G \to \mathbb{R}_+$ defined by relation (17) satisfies the assumptions of Lemma 2.1, and therefore it satisfies inequality (13) for $k = 1$ and (14) for $k > 1$. Inequalities (13), (14), and (17) imply inequalities (15) and (16).

The proof of the fact that 2) implies 1) is carried out using the same reasoning as in Theorem 1.2. The proof of the theorem is complete. $\square$

Let (X, G, π) and (Y, G, σ) be two dynamical systems with multidimensional time, and let $h\colon X \to Y$ is a homomorphism from (X, G, π) to (Y, G, σ). Then the triple $\langle (X, G, \pi), (Y, G, \sigma), h \rangle$ is called a *nonautonomous dynamical system with multidimensional time.*

The nonautonomous dynamical system with multidimensional time $\langle (X, G, \pi), (Y, G, \sigma), h \rangle$ is said to be *homogeneous of order* $k = 1$ if the triple (X, h, Y) is a vector fiber bundle and the autonomous system (X, G, π) is homogeneous of order $k = 1$.

Theorem 2.2. *Let $\langle (X, G, \pi), (Y, G, \sigma), h \rangle$ be a nonautonomous dynamical system with multidimensional time. If $\langle (X, G, \pi), (Y, G, \sigma), h \rangle$ is homogeneous of order $k = 1$, then the following conditions are equivalent:*

a) *the zero section of the fiber bundle (X, h, Y) is uniformly asymptotically stable, i.e.,*
   a1) *for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(x, t)| < \varepsilon$ for all $t \in G$;*
   a2) *there exists a $\gamma > 0$ such that*

$$\lim_{\|t\| \to +\infty} |\pi(x, t)| = 0 \tag{18}$$

   for $|x| < \gamma$; moreover, (18) *holds uniformly in x;*

b) *there exist positive numbers N and ν such that $|\pi(x, t)| \leq N e^{-\nu \|t\|} |x|$ for all $x \in X$ and $t \in G$.*

Proof. It is obvious that b) implies a); therefore, to complete the proof of Theorem 1.4, it suffices to show that a) implies b). Set

$$m(t) = \sup_{|x|\le 1} |\pi(x,t)|. \tag{19}$$

Since the zero section of the dynamical system $\langle (X, G, \pi), (Y, G, \sigma), h\rangle$ is uniformly asymptotically stable and the system itself is homogeneous of order $k = 1$, the function $m\colon G \to \mathbb{R}_+$ satisfying the assumptions of Lemma 2.1 is well defined by relation (19). Moreover, it follows from the fact that the system $\langle (X, G, \pi), (Y, G, \sigma), h\rangle$ is homogeneous of order $k = 1$ that

$$m(t+\tau) \le m(t)m(\tau) \qquad \text{and} \qquad |\pi(x,t)| \le |x| m(t)$$

for all $t, \tau \in G$ and $x \in X$. According to the first assertion of Lemma 2.1, the function m satisfies inequality (13), and therefore

$$|\pi(x,t)| \le |x| m(t) \le N e^{-\nu\|t\|} |x| \qquad \text{for all} \quad x \in X \quad \text{and} \quad t \in G.$$

The proof of the theorem is complete. □

§3. Ordinary differential equations in a Banach space

Let E be a real or complex Banach space with norm $\|\cdot\|$. Denote by $C(\mathbb{R}\times E, E)$ the family of all continuous functions $F\colon \mathbb{R}\times E \to E$ equipped with the open-compact topology. Consider the differential equation

$$\dot{x} = F(t,x), \tag{20}$$

where $F \in C(\mathbb{R}\times E, E)$. Along with Eq. (20), we shall also consider the family of equations

$$\dot{y} = G(t,y), \tag{21}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}$, F_τ is the translation of the function F along t by τ, and the bar denotes the closure in $C(\mathbb{R}\times E, E)$.

A function $F \in C(\mathbb{R}\times E, E)$ is said to be *regular* if the following conditions are satisfied:

a) whatever $v \in E$ and $G \in H(F)$, Eq. (21) has a unique solution defined on $\mathbb{R}_+$ and issuing from the point v at $t = 0$; we denote this solution by $\varphi(t, v, G)$;

b) the mapping $\varphi\colon \mathbb{R}_+ \times E \times H(F) \to E$ is continuous.

Note that condition a) also implies the following relation:

c) $\varphi(t+\tau, v, G) = \varphi(t, \varphi(\tau, v, G), G_\tau)$ for all $t, \tau \in \mathbb{R}_+$, $v \in E$ and $G \in H(F)$.

As is well known (see, for example, [13–15]), Eq. (20) with a regular right-hand side determines the nonautonomous dynamical system $\langle (X, T, f), (Y, T, \sigma), h\rangle$, where $Y = H(F)$, and $(Y, \mathbb{R}, \sigma)$ is the dynamical system of translations on $H(F)$, $X = E \times Y$, $f\colon \mathbb{R}_+ \times E \times Y \to E$ is the mapping defined by the relation $f\big(\tau, (v, G)\big) = \langle \varphi(\tau, v, G), G_\tau\rangle$ ($\tau \ge 0$, $v \in E$, and $G \in H(F)$), and $h = \mathrm{pr}_2\colon X \to Y$. Applying Theorems 1.1 and 1.2 to the nonautonomous dynamical system thus constructed, we obtain the corresponding assertions for equations of the form (20).

Theorem 3.1. *Let $F \in C(\mathbb{R}\times E, E)$ and $F(t, \lambda x) = \lambda F(t,x)$ for all $t \in \mathbb{R}$, $x \in E$ and $\lambda \ge 0$. Then the following assertions are equivalent*:

a) *the zero solution of Eq.* (20) *is uniformly asymptotically stable*;

b) *there exist positive numbers N and ν such that*

$$\|\varphi(t, v, G)\| \le N e^{-\nu t} \|v\|$$

for all $t \ge 0$, $v \in E$, and $G \in H(F)$.

Theorem 3.2. *Suppose that Eq.* (20) *is autonomous, i.e., the right-hand side* F *is independent of* $t \in \mathbb{R}$, *and* $F(\lambda x) = \lambda^k F(x)$ $(k > 1)$ *for all* $x \in E$ *and* $\lambda \geq 0$. *Then the following assertions are equivalent*:

a) *the zero solution of the equation* $\dot{x} = F(x)$ *is uniformly asymptotically stable*;

b) *there exist positive numbers* α *and* β *such that*

$$\|\varphi(t, x)\| \leq (\alpha \|x\|^{1-k} + \beta t)^{-1/(k-1)}$$

for all $t \geq 0$ *and* $x \in E$.

Remark 3.1. a) Theorem 3.1 and 3.2 are also valid for differential equations with nonunique solutions as well as for generalized differential equations, since, under certain regularity conditions for the right-hand side F, the generalized differential equation $\dot{x} \in F(t, x)$ determines a nonautonomous dispersive dynamical system (for more details, see [11]).

b) Theorem 3.1 for generalized differential equations with right-hand side periodic in t in a finite-dimensional space was proved in [9].

c) Theorem 3.2 for finite-dimensional differential equations was proved in [1–4], and for generalized differential equations in a finite-dimensional space it was proved in [5, 6].

d) Theorems 2.1 and 2.2 imply the existence of analogs of Theorems 3.1 and 3.2 also for completely solvable differential equations in Banach spaces [16].

§4. Functional differential equations

Let $r > 0$, and let $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi\colon [a, b] \to \mathbb{R}^n$ with norm sup. If $[a, b] = [-r, 0]$, then we set $C = C([-r, 0], \mathbb{R}^n)$. Suppose that $\sigma \in \mathbb{R}$, $A \geq 0$, and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma + A]$, let us define $u_t \in C$ by the relation

$$u_t(\theta) = u(t + \theta), \qquad -r \leq \theta \leq 0.$$

Denote by $C(\mathbb{R} \times C, \mathbb{R}^n)$ the space of all continuous functions $F\colon \mathbb{R} \times C \to \mathbb{R}^n$, equipped with open-compact topology. Consider the differential equation

$$\dot{x} = F(t, x_t), \tag{22}$$

where $F \in C(\mathbb{R} \times C, \mathbb{R}^n)$. Along with Eq. (22), consider the family of equations

$$\dot{y} = G(t, y_t), \tag{23}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}$. It follows from the general properties of functional differential equations [17] that Eq. (22) with a regular right-hand side F naturally determines a nonautonomous dynamical system (for more details, see [13]), which is constructed as follows. Set $Y = H(F)$. By $(Y, \mathbb{R}, \sigma)$ denote the dynamical system of translations on Y, $X = C \times Y$, and let $(X, \mathbb{R}_+, f)$ be a dynamical system on X, where $f^\tau(x) = f^\tau(v, G) = \langle \varphi_\tau(\cdot\, v, G), G_\tau \rangle$ for all $\tau \geq 0$, $v \in C$, and $G \in H(F)$; here $\varphi(\cdot\, v, G)$ is the unique solution of Eq. (23) under the condition $\varphi(0, v, G) = v$. Then

$$\langle (X, \mathbb{R}_+, f), (Y, \mathbb{R}, \sigma), h \rangle$$

is the nonautonomous dynamical system determined by Eq. (22), where $h = \mathrm{pr}_2\colon X \to Y$. Applying Theorems 1.1 and 1.2 to the nonautonomous dynamical system thus constructed, we obtain the following assertions.

Theorem 4.1. *Let* $F \in C(\mathbb{R} \times C, \mathbb{R}^n)$ *be regular, and let* $F(t, \lambda x) = \lambda F(t, x)$ *for all* $t \in \mathbb{R}$, $x \in C$, *and* $\lambda \geq 0$. *Then the following assertions are equivalent*:

a) *the zero solution of Eq.* (22) *is uniformly asymptotically stable*;

b) *there exist positive numbers* N *and* ν *such that*

$$\|\varphi(t, v, G)\| \leq N e^{-\nu t} \|v\|$$

for all $t \geq 0$, $v \in C$, *and* $G \in H(F)$.

Theorem 4.2. *For the autonomous functional differential equation*

$$\dot{x} = F(x_t) \tag{24}$$

with a regular right-hand side F *satisfying the condition* $F(\lambda x) = \lambda^k F(x)$ $(k > 1)$ *for all* $x \in C$ *and* $\lambda \geq 0$, *the following assertions are equivalent*:

a) *the zero solution of Eq.* (24) *is uniformly asymptotically stable*;

b) *there exist positive numbers* α *and* β *such that*

$$\|\varphi(t, v)\| \leq (\alpha \|v\|^{1-k} + \beta t)^{-1/(k-1)}$$

for all $t \geq 0$ *and* $v \in C$.

Remark 4.1. Theorems 4.1 and 4.2 also hold for functional differential equations with nonunique solutions and for generalized functional differential equations if their the right-hand sides have certain regularity.

§5. Quasilinear parabolic equations

Let E be a Banach space, and let $A\colon D(A) \to E$ be a linear closed operator with dense domain. An operator A is called [18] *sectorial* if for some $\varphi \in (0, \pi/2)$, some $M \geq 1$, and some real a, the sector

$$S_{a,\varphi} = \{\lambda : \varphi \leq |\arg(\lambda - a)| \leq \pi, \ \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. If A is a sectorial operator, then there exists an $a_1 \geq 0$ such that $\operatorname{Re}\sigma(A + a_1 I) > 0$ $(\sigma(A) = \mathbb{C} \setminus \rho(A))$. Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [18]

$$A_1^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha} (\lambda I + A_1)^{-1} \, d\lambda,$$

which is linear, bounded, and one-to-one. Set $X^\alpha = D(A_1^\alpha)$, and let us equip the space X^α with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ $(x \in X^\alpha)$, $X^0 = E$, and $X^1 = D(A)$. Then X^α is a Banach space with norm $\|\cdot\|_\alpha$ and is densely and continuously embedded in E.

Consider the differential equation

$$\dot{x} + Ax = F(t, x), \tag{25}$$

where $F \in C(\mathbb{R} \times X^\alpha, E)$ and $C(\mathbb{R} \times X^\alpha, E)$ is the space all continuous functions equipped with open-compact topology.

Along with Eq. (25), consider the family of equations

$$\dot{y} + Ay = G(t, y), \tag{26}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}$. Regularity conditions for F are given in Theorems 3.3.3, 3.3.4, 3.3.6, and 3.4.1 in [18].

Assuming that F is regular, a nonautonomous dynamical system can be associated in a natural way with Eq. (25). Namely, we set $Y = H(F)$ and by $(Y, \mathbb{R}, \sigma)$ denote the dynamical system of translations on Y. Further, let $X = X^\alpha \times Y$, and let $(X, \mathbb{R}_+, f)$ be the dynamical system on X defined by the relation $f^\tau(v, G) = \langle \varphi(\tau, v, G), G_\tau \rangle$, where $\varphi(\tau, v, G)$ is the unique solution of (26) defined on $\mathbb{R}^+$ and satisfying the condition $\varphi(0, v, G) = v$. Finally, by setting $h = \mathrm{pr}_2 : X \to Y$, we obtain the nonautonomous system $\langle (X, \mathbb{R}_+, f), (Y, \mathbb{R}, \sigma), h \rangle$ determined by Eq. (25). Applying Theorem 1.1 to the last system, we obtain the following assertion.

Theorem 5.1. *Let* $F \in C(\mathbb{R} \times X^\alpha, E)$ *be regular, and let* $F(t, \lambda x) = \lambda F(t, x)$ *for all* $t \in \mathbb{R}$, $x \in X^\alpha$, *and* $\lambda \geq 0$. *Then the following two assertions for Eq.* (25) *are equivalent*:

a) *the zero solution of Eq.* (25) *is uniformly asymptotically stable*;

b) *there exist positive numbers* N *and* ν *such that*

$$\|\varphi(t, v, G)\|_\alpha \leq N e^{-\nu t} \|v\|_\alpha$$

for all $t \geq 0$ *and* $v \in X^\alpha$.

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