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Bounded solutions of linear almost periodic differential equations

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Abstract. The paper deals with bounded (on $\mathbb{R}_+$ or $\mathbb{R}$) solutions of the equation $\dot{x} = \mathcal{A}(t)x$ with recurrent (almost periodic) coefficients. We show that the zero solution of this equation is uniformly stable (bistable) if and only if all its solutions and the solutions of its limit equations are bounded on $\mathbb{R}_+$ ($\mathbb{R}$). These results are generalizations of the well-known theorem of Cameron–Johnson.

Introduction

This paper deals with bounded (on $\mathbb{R}_+$ or $\mathbb{R}$) solutions of the equation

$$\dot{x} = \mathcal{A}(t)x \tag{0.1}$$

with recurrent (in particular, almost Bohr periodic) coefficients.

The well-known theorem of Cameron–Johnson theorem [1], [2] for equation (0.1) in a finite-dimensional space states that this equation can be reduced by a Lyapunov transformation to an equation

$$\dot{y} = \mathcal{B}(t)y, \tag{0.2}$$

where $\mathcal{B}(t)$ is a skew-symmetric matrix, if all the solutions of equation (0.1) and the solutions of all its limit equations are bounded on $\mathbb{R}$. It is obvious that the converse statement is also valid. This theorem implies that the solutions of equation (0.1) and of all its limit equations are bounded on $\mathbb{R}$ if and only if the zero solution of equation (0.1) is Lyapunov bistable. We show that the last statement is valid for equations (0.1) in an arbitrary Banach space. Moreover, we prove that the solutions of equation (0.1) and all its limit equations are bounded on $\mathbb{R}_+$ if and only if the zero solution of equation (0.1) is uniformly stable. We establish that equation (0.1) has at most finitely many solutions that are linearly independent and bounded on $\mathbb{R}$ if its zero solution is uniformly stable and the shift operator along the trajectories of equation (0.1) is asymptotically compact.

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§ 1. Bounded motions of linear non-autonomous dynamical systems

Assume that X and Y are complete metric spaces, $\mathbb{R}$ ($\mathbb{Z}$) is the group of real numbers (integers), $T=\mathbb{R}$ or $T=\mathbb{Z}$, $T_+=\{t\in T \mid t\geqslant 0\}$, $T_-=\{t\in T\mid t\leqslant 0\}$, and $S=T_+,\ T_-,\ T$. Let (X,S,π) $((Y,T,\sigma))$ be a semigroup (group) dynamical system on X (Y). A triple $\langle (X,S,\pi),(Y,T,\sigma),h\rangle$, where h is a homomorphism of (X,S,π) onto (Y,T,σ), is called [3] a *non-autonomous dynamical system.*

A set $A\subseteq X$ is said [4] to be *positively invariant* (*quasi-invariant*) if $\pi^t A\subseteq A$ ($\pi^t A\supseteq A$) for all $t\in T_+$, where $\pi^t x=\pi(x,t)=xt$ for all $x\in X$. A set A is said to be *invariant* if it is both positively invariant and quasi-invariant.

A closed positively invariant set A in the space of the system (X,S,π) is said [4] to be *minimal* if it contains no proper closed positively invariant subset. A point $x\in X$ is said to be *recurrent* in (X,S,π) if $H(x)=\overline{\{xt\mid t\in T\}}$ is a compact minimal set of (X,S,π).

A non-autonomous dynamical system $\langle (X,T_+,\pi),(Y,T,\sigma),h\rangle$ is said [5]–[8] to be *distal on* T_+ *in the fibre* $X_y=\{x\in X\mid h(x)=y\}$ if $\inf\{\rho(x_1t,x_2t)\mid t\in T_+\}>0$ for all $x_1,x_2\in X_y,\ \ x_1\neq x_2$. For group non-autonomous dynamical systems the distalness on T_- and T in the fibre X_y can be defined likewise. Finally, a non-autonomous system is said to be *distal on* T_+ (T_-,T) if it is distal in every fibre $X_y,\ y\in Y$.

Assume that (X_i,T_+,π_i) is a dynamical system on X_i, $i=1,\dots,k$; let $X=X_1\times\dots\times X_k$, and let $\pi=(\pi_1,\dots,\pi_k)\colon X\times T_+\to X$ be defined by the formula

$$\pi(x,t)=(\pi_1(x_1,t),\dots,\pi_k(x_k,t))$$

for all $t\in T_+$ and $x=(x_1,\dots,x_k)\in X$. The dynamical system (X,T_+,π), where $X=X_1\times\dots\times X_k$ and $\pi=(\pi_1,\dots,\pi_k)$, is called the *direct product* of the dynamical systems (X_i,T_+,π_i), $i=1,\dots,k$ and denoted by $(X_1,T_+,\pi_1)\times\dots\dots\times(X_k,T_+,\pi_k)$. If $X_i=X$, $i=1,\dots,k$, and $\pi_i=\pi$, $i=1,\dots,k$, then

$$(X,T_+,\pi)\times(X,T_+,\pi)\times\dots\times(X,T_+,\pi)=(X^k,T_+,\pi).$$

The direct product of group dynamical systems is defined likewise.

The points $x_1,\dots,x_k\in X$ are said to be *jointly recurrent* if the point $(x_1,\dots,x_k)\in X^k$ is recurrent in the dynamical system (X^k,T_+,π).

Lemma 1.1 [5]–[8]**.** *The following assertions hold.*

1) *Assume that X is compact and (Y,T,σ) is minimal. If the group non-autonomous dynamical system $\langle (X,T,\pi),(Y,T,\sigma),h\rangle$ is distal on T_+ (T_-), then it is distal on T.*

2) *Assume that X is compact, (Y,T,σ) is minimal, and $y\in Y$. Then the following conditions are equivalent*:

(i) *the group non-autonomous system $\langle (X,T,\pi),(Y,T,\sigma),h\rangle$ is distal on T in the fibre X_y;*

(ii) *for any points $x_1,\dots,x_k\in X$, where k is any positive integer ≥ 2, the point $(x_1,\dots,x_k)\in X^k$ is recurrent in (X^k,T,π);*

(iii) *for any two points $x_1,x_2\in X_y$ the point $(x_1,x_2)\in X\times X$ is recurrent in $(X,T,\pi)\times(X,T,\pi)$.*

The *entire* (*full*) trajectory of the semigroup system (X, T_+, π) passing through the point $x \in X$ at $t = 0$ is defined to be the continuous map $\gamma\colon T \to X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t\gamma(s) = \gamma(s+t)$ for all $t \in T_+$ and $s \in T$. Let Φ_x be the set of all entire trajectories of (X, T_+, π) passing through x at $t = 0$.

Let $C(T, X)$ be the space of all continuous maps $\varphi\colon T \to X$ equipped with the compact-open topology and let $\bigl(C(T, X), T, \sigma\bigr)$ be the dynamical system of shifts on $C(T, X)$. Let d be a metric on $C(T, X)$ consistent with its topology (for example, the Bebutov metric).

Lemma 1.2 [9]**.** *Let (X, T_+, π) be a semigroup dynamical system and assume that for any $t \in T_+$ the map $\pi^t\colon X \to X$ is a homeomorphism and $\hat{\pi}$ is the map of $X \times T$ to X defined by the equality*

$$\hat{\pi}(x,t) = \begin{cases} \pi(x,t), & (x,t) \in X \times T_+, \\ (\pi^{-t})^{-1}(x), & (x,t) \in X \times T_-. \end{cases}$$

Then the triple $(X, T, \hat{\pi})$ is a group dynamical system.

Lemma 1.3 [5]**.** *Assume that $\langle (X, T_+, \pi), (Y, T, \sigma), h \rangle$ is a non-autonomous dynamical system, Y is compact, (Y, T, σ) is minimal, and $x_0 \in X_y$ has a relatively compact semitrajectory $\{x_0 t \mid t \in T\}$. Then one can find a recurrent point $x \in \omega_{x_0} = \bigcap_{t\geqslant 0,\, \tau \geqslant t} \overline{U x_0 \tau}$ $(x \in X_y)$ and a sequence $t_n \to +\infty$ such that $\rho(x_0 t_n, x t_n) \to 0$ as $n \to +\infty$.*

A dynamical system (X, S, π) is said [10]–[13] to be *asymptotically compact* if for any bounded positively invariant set $B \subseteq X$ there is a non-empty compact set $K \subseteq X$ such that

$$\lim_{t\to+\infty} \sup\{\rho(xt, K)\colon x \in B\} = 0.$$

Remark 1.4. (i) Assume that $x \in X$ is such that $\{xt \mid t \in T_+\}$ is bounded and (X, T_+, π) is asymptotically compact. Then $\{xt \mid t \in T_+\}$ is relatively compact.

(ii) Let $\mathbb{M} \subset X$ be bounded and invariant. Then $\mathbb{M}$ is relatively compact if the dynamical system (X, T_+, π) is asymptotically compact. In particular, if $x \in X$ and $\gamma \in \Phi_x$ is such that $\gamma(T)$ is bounded, then $\gamma(T)$ is relatively compact.

Lemma 1.5 [10]**.** *Let B be a bounded subset of X. Then the following conditions are equivalent*:

(i) *for any sequences $\{x_k\} \subset B$ and $t_k \to +\infty$ the sequence $\{x_k t_k\}$ is relatively compact*;

(ii) $\Omega(B) = \bigcap_{t\geqslant 0,\, \tau\geqslant t} \overline{U\pi^\tau(B)} \neq \varnothing$ *is compact and invariant, and*

$$\lim_{t\to+\infty} \sup\{\rho\bigl(xt, \Omega(B)\bigr)\colon x \in B\} = 0;$$

(iii) *there is a non-empty compact set $K \subseteq X$ such that*

$$\lim_{t\to+\infty} \sup\{\rho(xt, K)\colon x \in B\} = 0.$$

If $X = E\times Y$, $\pi = (\varphi,\sigma)$, that is, $\pi\big((u,y),t\big) = \big(\varphi(t,x,y),\sigma(y,t)\big)$ for all $(u,y)\in E\times Y$ and $t\in S$, then the non-autonomous dynamical system $\langle (X,S,\pi),(Y,T,\sigma),h\rangle$, where $h = \mathrm{pr}_2\colon X\to Y$, is called [14] a *skew product over* (Y,T,σ) *with the fibre* E.

Let (X,h,Y) be a locally trivial Banach fibre bundle [15]. A non-autonomous dynamical system $\langle (X,S,\pi),(Y,T,\sigma),h\rangle$ is said [14], [16] to be *linear* if the map $\pi^t\colon X_y\to X_{yt}$ is linear for every $t\in S$ and $y\in Y$.

If $\langle (X,S,\pi),(Y,T,\sigma),h\rangle$ is a skew product over (Y,T,σ) with the fibre E (that is, $X = E\times Y$, $\pi = (\varphi,\sigma)$, and $h = \mathrm{pr}_2$), then it is linear if and only if E is a Banach space and the map $\varphi(t,\cdot,y)\colon E\to E$ is linear for every $y\in Y$ and $t\in S$.

Throughout the rest of this paper we assume that Y is compact, the dynamical system (Y,T,σ) is minimal, $X = E\times Y$, E is a Banach space with the norm $|\cdot|$, the non-autonomous dynamical system $\langle (X,S,\pi),(Y,T,\sigma),h\rangle$ is linear, $\pi = (\varphi,\sigma)$, and $h = \mathrm{pr}_2$.

Let $F\subset E\times Y$ be a closed vectorial subset of the trivial fibre bundle $(E\times Y,\mathrm{pr}_2,Y)$ that is positively invariant relative to (X,S,π). We put

$$\mathbb{B}^+ = \big\{(x,y)\in F \mid \sup\{|\varphi(t,x,y)|\colon t\in T_+\} < +\infty\big\}.$$

The set $\mathbb{B}^-$ is defined likewise. If $\langle (X,T_+,\pi),(Y,T,\sigma),h\rangle$ is a semigroup non-autonomous dynamical system, then $\mathbb{B}$ is the set of all points of F with the following property: there is an entire trajectory of the dynamical system (F,T_+,π) bounded on T that passes through this point. We put $\mathbb{B}^+_y = \mathbb{B}^+\cap X_y$ and $\mathbb{B}_y = \mathbb{B}\cap X_y$, $y\in Y$.

Theorem 1.6. *The following conditions are equivalent*:

(i) *there is an* $M>0$ *such that*

$$|\varphi(t,x,y)| \leqslant M|x| \tag{1.1}$$

for all $(x,y)\in\mathbb{B}^+$ $(\mathbb{B}^-,\mathbb{B})$ *and* $t\in T_+$ (T_-,T);

(ii) $\mathbb{B}^+$ $(\mathbb{B}^-,\mathbb{B})$ *is closed in* F.

Proof. We prove this theorem in the case when $S = T_+$. In the case when $S = T_-$ or $S = T$ it can be proved in a similar way. We claim that (i) implies (ii). Assume that $(x,y)\in\overline{\mathbb{B}^+}$. Then there is an $\{x_n,y_n\}\subseteq\mathbb{B}^+$ such that $\{x_n,y_n\}\to(x,y)$ as $n\to+\infty$. By condition (i), the inequality

$$|\varphi(t,x_n,y_n)| \leqslant M|x_n| \tag{1.2}$$

is valid for all $n = 1,2,\dots$ and $t\in T_+$. Passing to the limit in (1.2) as $n\to+\infty$, we obtain that $|\varphi(t,x,y)|\leqslant M|x|$ for all $t\in T_+$, that is, $(x,y)\in\mathbb{B}^+$.

Now we claim that (ii) implies (i). Let $\mathbb{B}^+$ be closed and let $y\in Y$. We put

$$d(y) = \sup\big\{|\varphi(t,x,y)|\colon t\in T_+,\ (x,y)\in\mathbb{B}^+,\ |x|\leqslant 1\big\}. \tag{1.3}$$

We claim that the function $d\colon Y\to\mathbb{R}_+$ defined by formula (1.3) is lower semi-continuous. Assume the contrary. Then there are $\epsilon>0$, $y\in Y$, and $y_n\to y$ such that

$$\lim_{n\to+\infty} d(y_n) = d(y)-\epsilon. \tag{1.4}$$

Formula (1.3) implies that there are $|x_n| \leqslant 1$ and $\{t_n\} \subseteq T_+$ $\big((x_n, y) \in \mathbb{B}^+\big)$ such that

$$d(y) = \lim_{n\to+\infty} |\varphi(t_n, x_n, y)|.$$

Therefore, for $\epsilon > 0$ one can find a k such that the inequality

$$\big||\varphi(t_n, x_n, y)| - d(y)\big| < \epsilon/4 \tag{1.5}$$

is valid for all $n \geqslant k$. Since the map $\varphi(t_k, x_k, \cdot)\colon Y \to X$ is continuous, there is an $n = n(k)$ such that

$$|\varphi(t_k, x_k, y_n) - \varphi(t_k, x_k, y)| < \epsilon/4 \tag{1.6}$$

for all $n \geqslant n(k)$. Inequalities (1.5) and (1.6) imply that

$$\big|d(y) - |\varphi(t_k, x_k, y_n)|\big| < \epsilon/4 \tag{1.7}$$

for all $n \geqslant n(k)$. Hence,

$$d(y) - d(y_n) \leqslant \epsilon/2 \tag{1.8}$$

for all $n \geqslant n(k)$. On the other hand, formula (1.4) implies that

$$d(y) - d(y_n) \geqslant 3\epsilon/4 \tag{1.9}$$

if n is sufficiently large.

Inequality (1.9) contradicts (1.8). This contradiction proves that $d\colon Y \to \mathbb{R}_+$ is lower semicontinuous. Hence, this function has a set of points of continuity $D \subset Y$ of the type G_δ. Since Y is a complete metric space, D has an interior point $p \in D$, that is, there is a $\delta_p > 0$ such that

$$S[p, \delta_p] = \{y \in Y \mid \rho(y, p) \leqslant \delta_p\} \subset D.$$

Since Y is minimal, there are negative numbers $t_1, \dots, t_m$ such that $Y = \bigcup_{i=1}^m \sigma(S[p, \delta_p], t_i)$ (see [6], Russian p. 134).

We put $L = \max\{|t_i|\colon i = 1, \dots, m\}$. Since d is continuous on $S[p, \delta_p]$ and Y is compact, there is an $M_p > 0$ such that $d(y) \leqslant M_p$ for all $y \in S[p, \delta_p]$.

We claim that the family of operators $\{\pi^t \mid t \in [0, L]\}$ is uniformly continuous, that is, for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $|x| \leqslant \delta$ implies that $|xt| \leqslant \epsilon$ for all $t \in [0, L]$. Assume the contrary. Then there are $\epsilon_0 > 0$, $\delta_n \downarrow 0$, $|x_n| < \delta_n$, $y_n \in Y$, and $t_n \in [0, L]$ such that

$$|\varphi(t_n, x_n, y_n)| \geqslant \epsilon_0. \tag{1.10}$$

Since Y and $[0, L]$ are compact, we can assume that the sequences $\{y_n\}$ and $\{t_n\}$ are convergent. Put $y_0 = \lim_{n\to+\infty} y_n$ and $t_0 = \lim_{n\to+\infty} t_n$. Passing to the limit in (1.10) as $n \to +\infty$, we obtain $0 \geqslant \epsilon_0$. The last inequality contradicts the choice of ϵ_0. This contradiction proves the above assertion.

If $\alpha > 0$ is such that $|\varphi(t, x, y)| \leqslant 1$ for all $t \in [0, L]$, $|x| \leqslant \alpha$, and $y \in Y$, then

$$|\varphi(t, x, y)| \leqslant \alpha^{-1}|x| \tag{1.11}$$

for all $t \in [0, L]$, $x \in E$, and $y \in Y$. Assume that $q \in Y$, $y \in S[p, \delta_p]$, and t_i are such that $q = yt_i$. Then

$$\begin{aligned} |\varphi(t, x, q)| &= |\varphi(t, x, yt_i)| = \big|\varphi\big(t + t_i, \varphi(-t_i, x, yt_i), y\big)\big| \\ &= |\varphi(-t_i, x, yt_i)| \left|\varphi\left(t + t_i, \frac{\varphi(-t_i, x, yt_i)}{|\varphi(-t_i, x, yt_i)|}, y\right)\right|. \end{aligned} \tag{1.12}$$

The set $\mathbb{B}^+$ is positively invariant and contains (x, q). Therefore, $\mathbb{B}^+$ contains $\pi^{-t_i}(x, q)$, where $\pi^{-t_i}(x, q) = \big(\varphi(-t_i, x, q), \sigma(q, -t_i)\big)$. Hence, $\pi^{-t_i}(x, q) = \big(\varphi(-t_i, x, yt_i), y\big) \in \mathbb{B}^+_y$ and

$$\left|\varphi\left(t + t_i, \frac{\varphi(-t_i, x, yt_i)}{|\varphi(-t_i, x, yt_i)|}, y\right)\right| \leqslant M_p \tag{1.13}$$

for all $t \geqslant L$. On the other hand, inequality (1.11) implies that

$$|\varphi(-t_i, x, yt_i)| \leqslant \alpha^{-1}|x| \tag{1.14}$$

for all $x \in E$. Formulae (1.12)–(1.14) imply that $|\varphi(t, x, q)| \leqslant M|x|$ for all $t \in T_+$ and $x \in E$, where $M = \alpha^{-1} \max\{1, M_p\}$. This completes the proof of the theorem.

Lemma 1.7. *Assume that $\langle (X, T_+, \pi), (Y, T, \sigma), h\rangle$ is a linear non-autonomous dynamical system, (X, T_+, π) is asymptotically compact, and there is an $M > 0$ such that*

$$|\gamma(t)| \leqslant M|x| \tag{1.15}$$

for all $\gamma \in \Phi_{(x,y)}$, $(x, y) \in \mathbb{B}$, and $t \in T$. Then the set $\Phi_0 = \cup\{\Phi_{(x,y)} \mid (x, y) \in \mathbb{B},\ |x| \leqslant 1\}$ is relatively compact in $C(T, X)$.

Proof. Consider the set $K = \{\gamma(t) \mid t \in T,\ \gamma \in \Phi_0\} \subseteq \mathbb{B}$. It is obvious that K is invariant. By (1.15), it is bounded. Since (X, T_+, π) is asymptotically compact, K is relatively compact (see Remark 1.4(ii)). We claim that the family of functions $\Phi_0 \subset C(T, X)$ is equicontinuous. Assume the contrary. Then there are $\epsilon_0 > 0$, $\delta_n \downarrow 0$, $\{t_n^i\}(i = 1, 2)$, and $\{\gamma_n\} \subset \Phi_0$ such that $|t_n^1 - t_n^2| < \delta_n$ and

$$|\gamma_n(t_n^1) - \gamma_n(t_n^2)| \geqslant \epsilon_0. \tag{1.16}$$

Without loss of generality we can assume that $t_n^2 > t_n^1$. Then inequality (1.16) implies that

$$|\pi^{\tau_n} x_n - x_n| \geqslant \epsilon_0 \tag{1.17}$$

for all $n = 1, 2, \ldots$, where $\tau_n = t_n^2 - t_n^1$ and $x_n = \gamma_n(t_n^1)$. Since $\{x_n\} \subseteq K$, we can assume that the sequence $\{x_n\}$ converges. Let $x_0 = \lim_{n\to+\infty} x_n$. Passing to the limit in inequality (1.17) as $n \to +\infty$, we obtain the inequality $0 \geqslant \epsilon_0$, which contradicts the choice of ϵ_0. To complete the proof of the lemma, it is sufficient to apply the Ascoli–Arzela theorem.

Theorem 1.8. *Assume that $\langle (X,T_+,\pi),(Y,T,\sigma),h\rangle$ is a linear non-autonomous dynamical system, (X,T_+,π) is asymptotically compact, and there is an $M>0$ such that inequality* (1.15) *is valid for all $\gamma\in\Phi_{(x,y)}$, $(x,y)\in\mathbb{B}$, and $t\in T$. Then the following assertions hold.*

(i) *Any two entire trajectories $\gamma_1\in\Phi_{(x_1,y)}$ and $\gamma_2\in\Phi_{(x_2,y)}$ $\big((x_1,y),(x_2,y)\in\mathbb{B}\big)$ are jointly recurrent.*

(ii) *For any $(x,y)\in\mathbb{B}$ the set $\Phi_{(x,y)}$ consists of a single entire recurrent trajectory.*

(iii) *$\mathbb{B}$ is closed in F.*

(iv) *(X,T_+,π) induces a group dynamical system $(\mathbb{B},T,\pi)$ on $\mathbb{B}$.*

(v) *For any $y\in Y$ the set $\mathbb{B}_y$ is finite-dimensional and $\dim\mathbb{B}_y$ does not depend on $y\in Y$.*

Proof. Assume that $(x,y)\in\mathbb{B}$, and let $\gamma\in\Phi_{(x,y)}$ be bounded on T. By Lemma 1.7, the set $H(\gamma)=\overline{\{\gamma_\tau\mid\tau\in T\}}$ is compact in $C(T,X)$, since $\gamma_\tau\in\Phi_{\gamma(\tau)}$, where γ_τ is the shift of γ by τ and the bar denotes closure in $C(T,X)$. Consider the group non-autonomous dynamical system $\langle (H(\gamma),T,\lambda),(Y,T,\sigma),h\rangle$, where $(H(\gamma),T,\lambda)$ is the dynamical system of shifts on $H(\gamma)$ induced by the Bebutov system $\big(C(T,X),T,\sigma\big)$ and $\mu\colon H(\gamma)\to Y$ is the map defined by the equality $\mu(\psi)=h(\psi(0))$. Under the conditions of Theorem 1.8 (see also inequality (1.15)) the non-autonomous dynamical system $\langle (H(\gamma),T,\lambda),(Y,T,\sigma),\mu\rangle$ is negatively distal, that is, $\inf\{d\big(\lambda(\gamma_1,t),\lambda(\gamma_2,t)\big)\colon t\in T_-\}>0$ for any $\gamma_1,\gamma_2\in H(\gamma)$ such that $\gamma_1\ne\gamma_2$ and $\mu(\gamma_1)=\mu(\gamma_2)$. By Lemma 1 in [6], Russian p. 104, $\langle (H(\gamma),T,\lambda),(Y,T,\sigma),h\rangle$ is distal on T. Therefore, γ_1 and γ_2 are jointly recurrent. In particular, γ is recurrent. Moreover, by Lemma 1.1, any two entire trajectories $\gamma_1\in\Phi_{(x,y)}$ and $\gamma_2\in\Phi_{(x,y)}$ are jointly recurrent.

We claim that for any $(x,y)\in\mathbb{B}$ the set $\Phi_{(x,y)}$ consists of a single entire recurrent trajectory. Assume the contrary. Then there are $\gamma_1,\gamma_2\in\Phi_{(x,y)}$ such that $\gamma_1\ne\gamma_2$. Putting $\gamma(t)=\gamma_1(t)-\gamma_2(t)$, we obtain a recurrent function $\gamma\ne0$ such that $\gamma(t)=0$ for all $t\in T_+$, which is impossible.

Now let $(x,y)\in\overline{\mathbb{B}}$. Then there is an $\{x_n,y_n\}\subset\mathbb{B}$ such that $\{x_n,y_n\}\to\{x,y\}$. Let γ_n be the (unique) entire trajectory of (F,T_+,π) bounded on T and satisfying the condition $\gamma_n(0)=(x_n,y_n)$. By Lemma 1.7, we can assume that the sequence $\{\gamma_n\}$ converges in $C(T,X)$. Inequality (1.15) implies that $\gamma=\lim_{n\to+\infty}\gamma_n$ is an entire trajectory of (F,T_+,π) bounded on T. Moreover, $\gamma(0)=(x,y)$. Thus, $\gamma\in\Phi_{(x,y)}$, whence $(x,y)\in\mathbb{B}$.

Since $\mathbb{B}$ is closed and invariant, (X,T_+,π) induces a dynamical system $(\mathbb{B},T_+,\pi)$ on $\mathbb{B}$, and $\pi^t\mathbb{B}=\mathbb{B}$ for all $t\in T_+$. By assertion (ii) of the theorem, $\pi^t\colon\mathbb{B}\to\mathbb{B}$ ($t\in T_+$) is a one-to-one map and $(\pi^t)^{-1}(b)=\gamma_b(-t)$ for all $t\in T_+$ and $b\in\mathbb{B}$, where $\{\gamma_b\}=\Phi_b$. Lemma 1.7 implies that $\pi^t\colon\mathbb{B}\to\mathbb{B}$ is a homeomorphism. To prove assertion (iv), it is sufficient to apply Lemma 1.2.

We now prove the last assertion of the theorem. Since (X,T_+,π) is asymptotically compact, $K=\{(x,y)\mid(x,y)\in\mathbb{B},|x|\leqslant1,y\in Y\}$ is compact. Therefore, every linear subspace $\mathbb{B}_y$ of $X_y=E\times\{y\}$ is finite-dimensional. Let $y\in Y$ and let $x_1,\dots,x_k\in\mathbb{B}_y$ be a basis in $\mathbb{B}_y$. Then Lemma 1.1 implies that the points $x_1,\dots,x_k$ are jointly recurrent. Let q be an arbitrary point of Y. Since Y is minimal, there is a sequence $\{t_n\} \subset T$ such that $yt_n \to q$ and $\pi(x_i, t_n) \to \xi_i \quad (i = 1, \ldots, k)$ as $n \to +\infty$, where $\xi_1, \ldots, \xi_k \in \mathbb{B}_q$ and the points $\xi_1, \ldots, \xi_k$ are jointly recurrent.

We claim that $\xi_1, \ldots, \xi_k$ are linearly independent. Assume the contrary. Then there are constants $c_1, \ldots, c_k$ such that $c_1\xi_1 + \cdots + c_k\xi_k = 0$ and $\sum_{i=1}^k |c_i| \neq 0$. Since $(\xi_1, \ldots, \xi_k) \in H(x_1, \ldots, x_k)$ and $(x_1, \ldots, x_k)$ is recurrent, there is a $\{\tau_n\} \subset T$ such that $q\tau_n \to y$ and $\pi(\xi_i, \tau_n) \to x_i \quad (i = 1, \ldots, k)$ as $n \to +\infty$. Therefore,

$$c_1x_1 + \cdots + c_kx_k = \lim_{n\to+\infty} \pi(c_1\xi_1 + \cdots + c_k\xi_k, \tau_n) = 0.$$

The last relation contradicts the choice of $x_1, \ldots, x_k$. Thus, $\dim \mathbb{B}_q \geqslant \dim \mathbb{B}_y$ for all $q \in Y$. Since Y is minimal, the reverse inequality also holds. Hence, $\dim \mathbb{B}_q = \dim \mathbb{B}_y$ for all $q \in Y$, which completes the proof of the theorem.

Theorem 1.9. *Let $\langle (X, T_+, \pi), (Y, T, \sigma), h\rangle$ be a linear non-autonomous dynamical system and assume that (X, T_+, π) is asymptotically compact. Then the following conditions are equivalent*:

(i) *there is an $M > 0$ such that* (1.15) *is valid for all $\gamma \in \Phi_{(x,y)}$, $(x, y) \in \mathbb{B}$, and $t \in T$;*

(ii) *$\mathbb{B}$ is closed in F.*

Proof. By Theorem 1.8, the theorem will be proved if we prove that (ii) implies (i). By Theorem 1.6, there is an $M > 0$ such that (1.1) is valid for all $(x, y) \in \mathbb{B}$ and $t \in T_+$. Since $\mathbb{B}$ is invariant, for any $(x, y) \in \mathbb{B}$ and $\gamma \in \Phi_{(x,y)}$ the inequality

$$|\gamma(t)| \geqslant M^{-1}|x| \tag{1.18}$$

is valid for all $t \in T_-$. Repeating the arguments used in Lemma 1.7, we can show that $H(\gamma) = \overline{\{\gamma_\tau \mid \tau \in T\}}$ is compact in $C(T, X)$. Consider the group dynamical system $\langle (H(\gamma), T, \lambda), (Y, T, \sigma), \mu\rangle$ (see the proof of Theorem 1.8). Inequality (1.18) implies that the non-autonomous system $\langle (H(\gamma), T, \lambda), (Y, T, \sigma), \mu\rangle$, is distal on T_-. By Lemma 1.1, γ is recurrent. Therefore,

$$\sup\{|\gamma(t)| \colon t \in T\} = \sup\{|\gamma(t)| \colon t \in T_+\} \leqslant M|x|$$

for all $\gamma \in \Phi_{(x,y)}$ and $(x, y) \in \mathbb{B}$. Hence, condition (i) holds and the theorem is proved.

Theorem 1.10. *Let $\langle (X, T_+, \pi), (Y, T, \sigma), h\rangle$ be a linear non-autonomous dynamical system. Assume that (X, T_+, π) is asymptotically compact and $\mathbb{B}^+$ is closed. Then*

(i) *$\mathbb{B}$ is closed, and*

(ii) *for any $(x, y) \in \mathbb{B}^+$ there is a recurrent point $(x, y) \in \mathbb{B}$ such that* $\lim_{t\to+\infty} |\varphi(t, x_0, y) - \varphi(t, x, y)| = 0$.

Proof. Assume that $\mathbb{B}^+$ is closed. By Theorem 1.6, there is an $M > 0$ such that (1.1) is valid for all $(x, y) \in \mathbb{B}^+$ and $t \in T_+$. Assume that $(x, y) \in \mathbb{B} \subseteq \mathbb{B}^+$. Then inequality (1.1) implies that (1.18) is valid for all $t \in T_-$ and $\gamma \in \Phi_{(x,y)}$.

Repeating the arguments used in the proof of Theorem 1.9, we obtain (1.15) for all $\gamma \in \Phi_{(x,y)}$, $(x,y) \in \mathbb{B}$ and $t \in T$. To complete the proof of the first assertion of the theorem, it is sufficient to apply Theorem 1.8.

Now we prove the second assertion of the theorem. Let $(x_0, y) \in \mathbb{B}^+$. Since (X, T_+, π) is asymptotically compact, the semitrajectory $\{\pi^t(x_0, y) \mid t \in T_+\}$ of the point (x_0, y) is relatively compact. Therefore, $\omega_{(x_0,y)} \neq \varnothing$ is compact and invariant. By Lemma 1.3, one can find a recurrent point $(x, y) \in \omega_{(x_0,y)}$ and a sequence $t_n \to +\infty$ such that

$$\lim_{n\to+\infty} |\varphi(t_n, x_0, y) - \varphi(t_n, x, y)| = 0. \tag{1.19}$$

Inequality (1.15) implies that

$$|\varphi(t, x_0, y) - \varphi(t, x, y)| \leqslant M|\varphi(t_n, x_0, y) - \varphi(t_n, x, y)| \tag{1.20}$$

for all $t \geqslant t_n$. Formulae (1.19) and (1.20) imply the desired assertion, which completes the proof of the theorem.

Remark 1.11. The second assertion of Theorem 1.9 remains true even if we do not assume that $\mathbb{B}^+$ is closed.

We conclude this section with a condition under which a linear non-autonomous system is asymptotically compact.

Let $P\colon X \to X$ be a projection of the vector bundle, that is, $P_y = P\,|_{X_y}$ is a projection in X_y for every $y \in Y$. Then P is said to be *completely continuous* if $P(\mathbb{M})$ is relatively compact for any bounded set $\mathbb{M} \subset X$.

Lemma 1.12. *Let $\langle (X, T_+, \pi), (Y, T, \sigma), h\rangle$ be a linear non-autonomous dynamical system. Assume that the maps $\pi^t = \pi(\,\cdot\,, t)\colon X \to X$ can be represented as sums $\pi(x,t) = \pi_1(x,t) + \pi_2(x,t)$ for all $t \in T_+$ and $x \in X$, and that the following conditions hold.*

(i) *$|\pi_1(x,t)| \leqslant m(t,r)$ for all $t \in T_+$ and $x \in X$, where $m\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ and for every $r \geqslant 0$ the function $m(t,r)$ tends to zero as $t \to +\infty$.*

(ii) *The maps $\pi_2(\,\cdot\,, t)\colon X \to X$ $(t > 0)$ are conditionally completely continuous, that is, $\pi_2(A,t)$ is relatively compact for any $t > 0$ and any bounded positively invariant set $A \subseteq X$.*

Then the dynamical system (X, T_+, π) is asymptotically compact.

Proof. Let $A \subseteq X$ be a bounded positively invariant set. Then $A = \Sigma^+(A) = \{\pi^t A \mid t \in T_+\}$. Since Y is compact and (X, h, Y) is locally trivial, there is an $r > 0$ such that $A \subseteq \{x \in X \colon |x| \leqslant r\}$. We claim that for any $\{x_k\} \subset A$ and $t_n \to +\infty$ the sequence $\{x_k t_k\}$ is relatively compact. We can cover $\mathbb{M} = \{x_k t_k\}_{k=1}^{\infty}$ with a finite ϵ-net for any $\epsilon > 0$. Assume that $\epsilon > 0$ and $l > 0$ are such that $m(l, r) < \epsilon/2$. We represent $\mathbb{M}$ as the union $\mathbb{M}_1 \cup \mathbb{M}_2$, where $\mathbb{M}_1 = \{x_k t_k\}_{k=1}^{k_1}$, $\mathbb{M}_2 = \{x_k t_k\}_{k=k_1+1}^{\infty}$ and $k_1 = \max\{k \mid t_k \leqslant l\}$. The set $\mathbb{M}_2$ is a subset of $\pi^l(\Sigma^+(A))$ whose elements can be represented in the form $\pi_1(x,t) + \pi_2(x,t)$ $(x \in \Sigma^+(A))$. Since $\pi_2(\Sigma^+(A), l)$ is relatively compact, we can cover it with a finite $(\epsilon/2)$-net. For any $y \in \pi_1(\Sigma^+(A), l)$ there is an $x \in \Sigma^+(A)$ such that $y = \pi_1(x, l)$ and $|y| = |\pi_1(x,l)| \leqslant m(l,r) < \epsilon/2$. Therefore, the zero section Θ of the vector bundle (X, h, Y) is an $(\epsilon/2)$-net of $\pi_1(\Sigma^+(A), l)$. Since Y is compact and (X, h, Y) is locally trivial, the zero section

Θ is compact. Hence, $\mathbb{M}_2$ and $\mathbb{M}$ are covered by the ϵ-net $\Theta \cup \mathbb{M}_1$. Since Θ is compact and the space Y is complete, $\mathbb{M} = \{x_k t_k\}_{k=1}^{\infty}$ is relatively compact. We complete the proof by applying Lemma 1.5.

Corollary 1.13. *Let $\langle (X, T_+, \pi), (Y, T, \sigma), h \rangle$ be a linear non-autonomous dynamical system and let $P\colon X \to X$ be a completely continuous projection. Assume that there are positive numbers N and ν such that $|\pi^t Q(x)| \leqslant N e^{-\nu t}|x|$ for all $x \in X$ and $t \in T_+$, where $Q\colon X \to X$ and $Q_y = Q|_{X_y} = I_y - P_y$ for all $y \in Y$ $(I_y = \mathrm{id}_{X_y})$.*

Then (X, T_+, π) is asymptotically compact.

To deduce this corollary from Lemma 1.12, it is sufficient to observe that $\pi(x,t) = \pi_1(x,t) + \pi_2(x,t)$ for all $x \in X$ and $t \in T_+$, where $\pi_1(x,t) = \pi^t Q(x)$ and $\pi_2(x,t) = \pi^t P(x)$. Under the hypotheses of Corollary 1.13, for every $t > 0$ the map $\pi_2(\,\cdot\,, t) = \pi^t P$ is completely continuous and $|\pi_1(x,t)| \leqslant N e^{-\nu t}|x|$. Hence, Lemma 1.12 is applicable.

Remark 1.14. In the proofs of Lemma 1.12 and Corollary 1.13 we used only the compactness of Y. Hence, these statements are also valid in the case when the dynamical system (Y, T, σ) is not minimal.

§ 2. Bounded solutions of linear differential equations in a Banach space with almost periodic coefficients

Let $[E]$ be the Banach space of all bounded linear operators that act on a Banach space E equipped with the operator norm. Let Λ be a complete metric space of closed linear operators that act on E (for example, $\Lambda = [E]$ or $\Lambda = \{A_0 + B \mid B \in [E]\}$, where A_0 is a closed operator that acts on E). Let $C(\mathbb{R}, \Lambda)$ be the space of all continuous operator-valued functions $\mathcal{A}\colon \mathbb{R} \to \Lambda$ equipped with the compact-open topology and let $\bigl(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma\bigr)$ be the dynamical system of shifts on $C(\mathbb{R}, \Lambda)$.

2.1. Ordinary linear differential equations. Let $\Lambda = [E]$ and consider the differential equation

$$\dot{u} = \mathcal{A}(t)u, \tag{2.1}$$

where $\mathcal{A} \in C(\mathbb{R}, [E])$. Consider the H-class of equation (2.1), that is, the family of equations

$$\dot{v} = \mathcal{B}(t)v, \tag{2.2}$$

with $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau \mid \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = \mathcal{A}(t+\tau)$, and $t \in \mathbb{R}$, where the bar denotes closure in $C(\mathbb{R}, [E])$. Let $\varphi(t, v, \mathcal{B})$ be the solution of equation (2.2) that satisfies the condition $\varphi(0, v, \mathcal{B}) = v$.

We put $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y, \mathbb{R}, \sigma)$. We put $X = E \times Y$ and define a dynamical system on X by setting

$$\pi\bigl((v, \mathcal{B}), t\bigr) = \bigl(\varphi(t, v, \mathcal{B}), \mathcal{B}_t\bigr)$$

for all $(v, \mathcal{B}) \in E \times Y$ and $t \in \mathbb{R}$. Then $\langle (X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a linear group non-autonomous dynamical system, where $h = \mathrm{pr}_2\colon X \to Y$. Applying the results of § 1 to this system, we obtain the following assertions.

Lemma 2.1 [17], [18]. (i) *The map* $(t, u, \mathcal{A}) \mapsto \varphi(t, u, \mathcal{A})$ *of* $\mathbb{R} \times E \times C(\mathbb{R}, [E])$ *to* E *is continuous, and*

(ii) *the map* $\mathcal{A} \mapsto U(\,\cdot\,, \mathcal{A})$ *of* $C(\mathbb{R}, [E])$ *to* $C(\mathbb{R}, [E])$ *is continuous, where* $U(t, \mathcal{A})$ *is the Cauchy operator* [19] *of equation* (2.1).

Theorem 2.2. *Assume that* $\mathcal{A} \in C(\mathbb{R}, [E])$ *is recurrent* (*that is,* $H(\mathcal{A})$ *is a compact minimal set of* $(C(\mathbb{R}, [E]), \mathbb{R}, \sigma)$. *Then the following conditions are equivalent*:

(a) *the set*

$$\mathbb{B}^+ \quad (\mathbb{B}^-, \mathbb{B}) = \left\{ (v, \mathbb{B}) \in E \times H(\mathcal{A}) \;\middle|\; \sup_{t \in \mathbb{R}_+ \ (\mathbb{R}_-, \mathbb{R})} |\varphi(t, v, \mathcal{B})| < +\infty \right\} \tag{2.3}$$

is closed in $E \times H(\mathcal{A})$, *and*

(b) *there is a positive number* M *such that*

$$|\varphi(t, v, \mathcal{B})| \leqslant M|v| \tag{2.4}$$

for all $t \in \mathbb{R}_+ \;\; (\mathbb{R}_-, \mathbb{R})$ *and* $(v, \mathcal{B}) \in \mathbb{B}^+ \;\; (\mathbb{B}^-, \mathbb{B})$.

Corollary 2.3. *Let* $\mathcal{A} \in C(\mathbb{R}, [E])$ *be recurrent. Then the following assertions are equivalent*:

(i) *all solutions of all equations* (2.2) *are bounded on* $\mathbb{R}_+ \;\; (\mathbb{R}_-, \mathbb{R})$, *and*

(ii) *there is an* $M > 0$ *such that* (2.4) *is valid for all* $v \in E$, $\mathcal{B} \in H(\mathcal{A})$, *and* $t \in \mathbb{R}_+ \;\; (\mathbb{R}_-, \mathbb{R})$.

Theorem 2.4. *Assume that* $\mathcal{A} \in C(\mathbb{R}, [E])$ *is recurrent, the linear non-autonomous dynamical system generated by equation* (2.1) *is asymptotically compact, and all solutions of all equations* (2.2) *are bounded on* $\mathbb{R}_+$. *Then*

(i) *there is an* $M > 0$ *such that* (2.4) *is valid for all* $t \in \mathbb{R}_+$, $v \in E$ *and* $\mathcal{B} \in H(\mathcal{A})$,

(ii) *the set* $\mathbb{B}$ *defined by formula* (2.3) *is closed in* $E \times H(\mathcal{A})$,

(iii) *all solutions of all equations* (2.2) *bounded on* $\mathbb{R}$ *are recurrent*,

(iv) *for any* $\mathcal{B} \in H(\mathcal{A})$ *equation* (2.2) *has only a finite number* $n_{\mathcal{B}}$ *of solutions that are linearly independent and bounded on* $\mathbb{R}$, *and* $n_{\mathcal{B}} = n_{\mathcal{A}}$ *for all* $\mathcal{B} \in H(\mathcal{A})$,

(v) *for any* $v_0 \in E$ *and* $\mathcal{B} \in H(\mathcal{A})$ *there is a* $(v, \mathcal{B}) \in \mathbb{B}$ *such that*

$$\lim_{t \to +\infty} |\varphi(t, v_0, \mathcal{B}) - \varphi(t, v, \mathcal{B})| = 0,$$

that is, any solution of any equation (2.2) *is asymptotically recurrent*.

We now formulate some sufficient conditions for the asymptotical compactness of the linear non-autonomous dynamical system generated by equation (2.1).

Theorem 2.5. *Let* $\mathcal{A} \in C(\mathbb{R}, [E])$, $\mathcal{A}(t) = \mathcal{A}_1(t) + \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$, *and assume that* $H(\mathcal{A}_i)$, $i = 1, 2$, *are compact and the following conditions hold.*

(i) *The zero solution of the equation*

$$\dot{u} = \mathcal{A}_1(t)u \tag{2.5}$$

is uniformly asymptotically stable, that is, there are positive numbers N and ν such that

$$\|U(t,\mathcal{A}_1)U^{-1}(\tau,\mathcal{A}_1)\| \leqslant Ne^{-\nu(t-\tau)} \tag{2.6}$$

for all $t \geqslant \tau \quad (t,\tau \in \mathbb{R})$, where $U(t,\mathcal{A}_1)$ is the Cauchy operator of the equation $\dot{u} = \mathcal{A}_1(t)u$.

(ii) *The family of operators $\{\mathcal{A}_2(t) \mid t \in \mathbb{R}_+\}$ is uniformly completely continuous, that is, for any bounded set $A \subset E$ the set $\{\mathcal{A}_2(t)A \mid t \in \mathbb{R}_+\}$ is relatively compact.*

Then the linear non-autonomous dynamical system generated by equation (2.1) *is asymptotically compact.*

Proof. Let $\mathcal{B} \in H(\mathcal{A})$. Then there are $\{t_n\} \subset T$ and $\mathcal{B}_i \in H(\mathcal{A}_i)$, $i = 1,2$, such that $\mathcal{B}(t) = \mathcal{B}_1(t) + \mathcal{B}_2(t)$ and $\mathcal{B}_i(t) = \lim_{n\to+\infty} \mathcal{A}_i(t+t_n)$. Note that

$$\varphi(t,v,\mathcal{B}) = U(t,\mathcal{B}_1)v + \int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})\,d\tau. \tag{2.7}$$

By Lemma 2.1,

$$U(t,\mathcal{B}_i) = \lim_{n\to+\infty} U(t,\mathcal{A}_{it_n}), \qquad \mathcal{A}_{it_n}(t) = \mathcal{A}_i(t+t_n),$$

and the equality

$$U(t,\mathcal{A}_{1t_n})U^{-1}(\tau,\mathcal{A}_{1t_n}) = U(t+t_n,\mathcal{A}_1)U^{-1}(\tau+t_n,\mathcal{A}_1)$$

and inequality (2.6) imply that

$$\|U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\| \leqslant Ne^{-\nu(t-\tau)} \tag{2.8}$$

for all $t \geqslant \tau$ and $\mathcal{B}_1 \in H(\mathcal{A}_1)$. By Lemma 1.12, Theorem 2.5 will be proved if we can prove that the set

$$\left\{\int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})\,d\tau \mid (v,\mathcal{B}) \in A\right\}$$

is relatively compact for every $t > 0$ and every bounded positively invariant set $A \subseteq E \times Y$. We put

$$K_A = \overline{\{\mathcal{B}_2(t)\varphi(t,v,\mathcal{B}) \mid t \in \mathbb{R}_+, (v,\mathcal{B}) \in A\}}.$$

Then

$$\begin{aligned} &\int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})\,d\tau \\ &\in t\cdot \overline{\operatorname{conv}}\big\{U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)w \bigm| 0 \leqslant \tau \leqslant t,\ \mathcal{B}_1 \in H(\mathcal{A}_1),\ w \in K_A\big\}. \end{aligned} \tag{2.9}$$

Since $H(\mathcal{A}_1)$, $H(\mathcal{A})$, and K_A are compact sets, formula (2.9), condition (ii) of Theorem 2.5, and Lemma 2.1 imply that $U\{U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)w \mid 0 \leqslant \tau \leqslant t,$ $\mathcal{B}_1 \in H(\mathcal{A}_1), w \in K_A\}$ is compact, which completes the proof of the theorem.

Theorem 2.6. *Let $H(\mathcal{A})$ be compact and assume that there is a finite-dimensional projection $P \in [E]$ such that*

(i) *the family of projections $\{P(t) \mid t \in \mathbb{R}\}$, where $P(t) = U(t,\mathcal{A})PU^{-1}(t,\mathcal{A})$, is relatively compact in $[E]$, and*

(ii) *there are positive numbers N and ν such that*

$$\|U(t,\mathcal{A})QU^{-1}(\tau,\mathcal{A})\| \leqslant Ne^{-\nu(t-\tau)}$$

for all $t \geqslant \tau$, where $Q = I - P$.

Then the linear non-autonomous dynamical system generated by equation (2.1) *is asymptotically compact.*

Proof. Since the family of projections $\{P(t) \mid t \in \mathbb{R}\}$ is relatively compact, we can assume that the sequence $\{P(t_n)\}$ converges. Let $P(\mathcal{B}) = \lim_{n\to+\infty} P(t_n)$. We claim that the family $\mathbb{H} = \overline{\{P(t) \mid t \in \mathbb{R}\}}$ is uniformly completely continuous, where the bar denotes closure in $[E]$. Indeed, let A be a bounded subset of E, $\{x_n\} \subseteq \{QA \mid Q \in H\}$, and $\epsilon_n \downarrow 0$. Then there are $t_n \in \mathbb{R}$ and $v_n \in A$ such that $\rho\bigl(x_n, P(t_n)v_n\bigr) \leqslant \epsilon_n$. Since the sequence $\{P(t_n)\}$ is relatively compact, we can assume that it converges. Let $L = \lim_{n\to+\infty} P(t_n)$. Then L is completely continuous, which implies that the sequence $\{x'_n\} = \{Lv_n\}$ is relatively compact. Note that

$$\rho(x_n, x'_n) \leqslant \rho\bigl(x_n, P(t_n)v_n\bigr) + \rho\bigl(P(t_n)v_n, Lv_n\bigr) \leqslant \epsilon_n + \|P(t_n) - L\| \cdot |v_n|,$$

which implies that $\rho(x_n, x'_n) \to 0$ as $n \to +\infty$. Hence, $\{x_n\}$ is relatively compact.

Assume that $\mathcal{B} \in H(\mathcal{A})$ and $\{t_n\} \subset \mathbb{R}$ are such that

$$\mathcal{B} = \lim_{n\to+\infty} \mathcal{A}_{t_n}, \qquad P(\mathcal{B}) = \lim_{n\to+\infty} P(\mathcal{A}_{t_n}),$$

where $P(\mathcal{A}_{t_n}) = U(t_n,\mathcal{A})PU^{-1}(t_n,\mathcal{A})$. The assertions proved above imply that the family $\{P(\mathcal{B}) \mid \mathcal{B} \in H(\mathcal{A})\}$ is uniformly completely continuous. Note that $Q(\mathcal{B}) = \lim_{n\to+\infty} Q(\mathcal{A}_{t_n})$, where $Q(\mathcal{B}) = I - P(\mathcal{B})$ and $Q(\mathcal{A}_{t_n}) = I - P(\mathcal{A}_{t_n})$. Moreover, condition (ii) of Theorem 2.6 implies that

$$\|U(t,\mathcal{A}_{t_n})Q(\mathcal{A}_{t_n})U^{-1}(\tau,\mathcal{A}_{t_n})\| \leqslant Ne^{-\nu(t-\tau)} \tag{2.10}$$

for all $t \geqslant \tau$. Passing to the limit in (2.10) as $n \to +\infty$ and taking Lemma 2.1 into account, we obtain that

$$\|U(t,\mathcal{B})Q(\mathcal{B})U^{-1}(\tau,\mathcal{B})\| \leqslant Ne^{-\nu(t-\tau)}$$

for all $t \geqslant \tau$ and $\mathcal{B} \in H(\mathcal{A})$. We complete the proof of the theorem by observing that $U(t,\mathcal{B})Q(\mathcal{B}) + U(t,\mathcal{B})P(\mathcal{B}) = U(t,\mathcal{B})$ and applying Lemma 1.12.

2.2. Linear functional-differential equations. Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi\colon [a,b] \to \mathbb{R}^n$ with the norm sup. For $[a,b] = [-r,0]$ we put $\mathcal{C} = C([-r,0],\mathbb{R}^n)$. Let $c \in \mathbb{R}$, $a \geqslant 0$, and $u \in C([c-r, c+a], \mathbb{R}^n)$. We define $u_t \in \mathcal{C}$ for any $t \in [c, c+a]$ by the relation $u_t(\theta) = u(t+\theta)$, $-r \leqslant \theta \leqslant 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C}, \mathbb{R}^n)$ be the Banach space of all linear operators that act from $\mathcal{C}$ to $\mathbb{R}^n$ equipped with the operator norm, let $C(\mathbb{R}, \mathfrak{A})$ be the space of all operator-valued functions $\mathcal{A}\colon \mathbb{R} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{R}, \mathfrak{A})$. Let $H(\mathcal{A}) = \{\overline{\mathcal{A}_\tau \mid \tau \in \mathbb{R}}\}$, where $\mathcal{A}_\tau$ is the shift of the operator-valued function $\mathcal{A}$ by τ and the bar denotes closure in $C(\mathbb{R}, \mathfrak{A})$.

Consider the linear functional-differential equation with delay

$$\dot{u} = \mathcal{A}(t) u_t \tag{2.11}$$

along with the family of equations

$$\dot{v} = \mathcal{B}(t) v_t, \tag{2.12}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi(t, v, \mathcal{B})$ be the solution of equation (2.12) satisfying the condition $\varphi(0, v, \mathcal{B}) = v$ and defined for all $t \geqslant 0$. Let $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y, \mathbb{R}, \sigma)$. Let $X = C \times Y$ and let $\pi = (\varphi, \sigma)$ be the dynamical system on X defined by the equality $\pi((v, \mathcal{B}), \tau) = (\varphi(\tau, v, \mathcal{B}), \mathcal{B}_\tau)$. The non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ $(h = \mathrm{pr}_2\colon X \to Y)$ is linear.

Lemma 2.7. *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R}, \mathfrak{A})$. Then the linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ generated by equation* (2.11) *is completely continuous, that is, for any bounded set $A \subset X$ there is an $l = l(A) > 0$ such that $\pi^l A$ is relatively compact.*

This follows from general properties of solutions of linear functional-differential equations with delay (see, for example, [12], Lemmas 2.2.3 and 3.6.1) since $Y = H(\mathcal{A})$ is compact.

Applying the results obtained in § 1 to the linear non-autonomous dynamical system generated by equation (2.11), we obtain the following assertions.

Theorem 2.8. *Let $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ be recurrent. Then the following conditions are equivalent*:

(i) *all solutions of all equations* (2.12) *are bounded on $\mathbb{R}_+$,*

(ii) *there is a positive number M such that $|\varphi(t, v, \mathcal{B})| \leqslant M|v|$ for all $t \geqslant 0$, $v \in \mathcal{C}$, and $\mathcal{B} \in H(\mathcal{A})$.*

Theorem 2.9. *Let $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ be recurrent, and assume that all solutions of all equations* (2.12) *are bounded on $\mathbb{R}_+$. Then*

(i) *the set of all the solutions of all equations* (2.12) *that are bounded on $\mathbb{R}$ is closed in $C(\mathbb{R}, \mathcal{C}) \times H(\mathcal{A})$,*

(ii) *all the solutions of all equations* (2.12) *that are bounded on $\mathbb{R}$ are recurrent,*

(iii) *for any $\mathcal{B} \in H(\mathcal{A})$ equation* (2.12) *has only a finite number $n_{\mathcal{B}}$ of solutions that are linearly independent and bounded on $\mathbb{R}$, and $n_{\mathcal{B}} = n_{\mathcal{A}}$ for all $\mathcal{B} \in H(\mathcal{A})$,*

(iv) *all solutions of all equations* (2.12) *are asymptotically recurrent.*

Now consider the neutral functional-differential equation

$$\frac{d}{dt} D u_t = \mathcal{A}(t) u_t, \tag{2.13}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ and the operator $D \in \mathfrak{A}$ is atomic at zero [12], Russian p. 67. Like (2.11), equation (2.13) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = \mathcal{C} \times Y$, $Y = H(\mathcal{A})$, and $\pi = (\varphi, \sigma)$.

Lemma 2.10. *Let $H(\mathcal{A})$ be compact, and assume that the operator D is stable, that is, the zero solution of the homogeneous difference equation $Dy_t = 0$ is uniformly asymptotically stable* [12]*. Then the linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation* (2.13) *is asymptotically compact.*

This follows from Theorem 12.3.2 and Lemma 12.4.1 in [12], since $Y = H(\mathcal{A})$ is compact. Therefore, Theorems 2.8 and 2.9 hold for equations (2.13).

2.3. Linear partial differential equations. Consider the differential equation (2.1) with unbounded coefficients. Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$, where Λ is a complete metric space of linear closed operators that act on E (for example, $\Lambda = \{\mathcal{A}_0 + \mathcal{B} \mid \mathcal{B} \in [E]\}$, where $\mathcal{A}_0$ is a closed operator that acts on E). Consider the H-class (2.2) of equation (2.1), where $\mathcal{B} \in H(\mathcal{A})$. We assume that the following conditions are fulfilled for equation (2.1) and its H-class:

(i) for any $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (2.2) has precisely one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, v, \mathcal{B}) = v$;

(ii) the map $\varphi\colon (t, v, \mathcal{B}) \to \varphi(t, v, \mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}, \Lambda)$;

(iii) for every $t \in \mathbb{R}_+$ the map $U(t, \cdot)\colon H(\mathcal{A}) \to [E]$ is continuous, where $U(t, \mathcal{B})$ is the Cauchy operator of equation (2.2), that is, $U(t, \mathcal{B})v = \varphi(t, v, \mathcal{B})$ for all $t \in \mathbb{R}_+$ and $v \in E$.

Under the above assumptions equation (2.1) generates a linear non-autonomous dynamical system to which the results obtained in § 1 can be applied. Therefore, Theorems 2.2 and 2.4 hold in this case.

In conclusion we consider a partial differential equation that satisfies conditions (i)–(iii).

Example 2.11. A closed linear operator $\mathcal{A}\colon D(\mathcal{A}) \to E$ with dense domain $D(\mathcal{A})$ is said [20] to be *sectorial* if one can find a $\varphi \in (0, \pi/2)$, an $M \geqslant 1$, and a real number a such that the sector

$$S_{a,\varphi} = \{\lambda \mid \varphi \leqslant |\arg(\lambda - a)| \leqslant \pi, \quad \lambda \neq a\}$$

lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ and $\|(\lambda \cdot I - \mathcal{A})^{-1}\| \leqslant M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. An important class of sectorial operators is formed by elliptic operators [20], [21].

Consider the differential equation

$$\dot{u} = (\mathcal{A}_1 + \mathcal{A}_2(t))u, \tag{2.14}$$

where $\mathcal{A}_1$ is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $\mathcal{A}_2 \in C(\mathbb{R}, [E])$. The results of [6], [20] imply that equation (2.14) satisfies conditions (i)–(iii). Therefore, analogues of Theorems 2.2 and 2.4 hold for equation (2.14).

Remark 2.12. Statements similar to Theorems 2.2 and 2.4 hold for difference equations and can be deduced from the results of § 1 by applying these results to linear non-autonomous dynamical systems with discrete time generated by the corresponding difference equations.

§ 3. Finite-dimensional systems

Throughout this section we assume that the Banach space E is finite-dimensional and its norm $|\cdot|$ is induced by the scalar product $\langle\cdot,\cdot\rangle$, that is, $|\cdot|^2=\langle\cdot,\cdot\rangle$. We propose several conditions for equations (0.1) in a finite-dimensional space that are equivalent to the uniform bistability of the zero solution of equation (0.1), and prove that the uniform Lyapunov stability of the zero solution of equation (0.1) implies that there is a frame of solutions of equation (0.1) bounded on $\mathbb{R}$ whose Gram determinant is separated from zero.

Let $x_1,\dots,x_k$ be a set of vectors in E. Let us recall [22] that the *Gram determinant* $\Gamma(x_1,\dots,x_k)$ of the vectors $x_1,\dots,x_k$ is defined to be the determinant $|\langle x_i,x_j\rangle|_{i,j=1}^n$. The Gram determinant of the vectors $x_1,\dots,x_k$ is non-negative, and equals zero only if the vectors $x_1,\dots,x_k$ are linearly dependent.

Theorem 3.1. *The following assertions are equivalent*:

(i) *there is an* $M>0$ *such that* (1.1) *is valid for all* $t\in T$ *and* $(x,y)\in\mathbb{B}$;

(ii) $\mathbb{B}$ *is closed*;

(iii) $\mathbb{B}$ *is a subbundle of* F, *that is,* $\mathbb{B}$ *is closed and there is a* k *such that* $\dim\mathbb{B}_y=k$ *for all* $y\in Y$;

(iv) *all the motions in* F *that are non-trivial and bounded on* T *are separated from zero, that is,* $\inf\{|\varphi(t,x,y)|\colon t\in T\}>0$ *for any* $(x,y)\in\mathbb{B}$ *such that* $x\neq 0$;

(v) *one can find a* $y_0\in Y$ *and a basis* $\xi_1,\dots,\xi_k\in\mathbb{B}_{y_0}$ *such that*

$$\inf_{t\in\mathbb{R}}\Gamma(\xi_1,\dots,\xi_k;t)=\alpha>0, \tag{3.1}$$

where $\xi_i=(x_i,y_0)$, $i=1,\dots,k$, *and* $\Gamma(\xi_1,\dots,\xi_k;t)$ *is the Gram determinant of the vectors* $\pi(\xi_i,t)\in E\times Y$, $i=1,\dots,k$.

Proof. Assertions (i) and (ii) are equivalent by Theorem 1.6. By Theorems 1.6 and 1.8, (ii) implies (iii) (the reverse implication is obvious). The equivalence of conditions (iii) and (iv) follows from [16], Theorem 8.22.

Assume that (iv) is fulfilled, let $y_0\in Y$, and let $\xi_1,\dots,\xi_k\in\mathbb{B}_{y_0}$ be a basis in $\mathbb{B}_{y_0}$. We claim that (3.1) holds. Assume the contrary. Then one can find a $\{t_n\}\subset T$ such that $|t_n|\to+\infty$ and $\Gamma(\xi_1,\dots,\xi_k;t_n)\to 0$ as $n\to+\infty$. Since all non-zero motions in F are separated from zero, Lemma 1.1 implies that $\xi_1,\dots,\xi_k$ and y are jointly recurrent. Without loss of generality, we can assume that the sequences $\{\pi(\xi_i,t_n)\}$, $i=1,\dots,k$, and $\{\sigma(y,t_n)\}$ are convergent.

Let

$$\eta_i=\lim_{n\to+\infty}\pi(\xi_i,t_n),\qquad i=1,\dots,k,\qquad q=\lim_{n\to+\infty}\sigma(y_0,t_n).$$

Then

$$\Gamma(\eta_1,\dots,\eta_k)=\lim_{n\to+\infty}\Gamma\bigl(\pi(\xi_1,t_n),\dots,\pi(\xi_k,t_n)\bigr)=0.$$

Hence, $\eta_1,\dots,\eta_k$ are linearly dependent. Repeating the above argument, we deduce from the last fact that $\xi_1,\dots,\xi_k$ are linearly dependent, which contradicts their choice. Hence, (iv) implies (v).

We claim that (v) implies (iv). First, we prove that for any $q \in Y$ one can find a basis $\eta_1, \dots, \eta_k$ such that $\Gamma(\eta_1, \dots, \eta_k; t) \geqslant \alpha > 0$. Indeed, since Y is minimal, there is a sequence $\{t_n\} \subset T$ such that $\sigma(y_0, t_n) \to q$. Since $\xi_1, \dots, \xi_k \in \mathbb{B}_{y_0}$, we can assume that the sequences $\{\pi(\xi_i, t_n)\}$, $i = 1, \dots, k$, are convergent. Let $\eta_i = \lim_{n\to+\infty} \pi(\xi_i, t_n)$. Then $\eta_1, \dots, \eta_k \in \mathbb{B}_q$ and

$$\Gamma(\eta_1, \dots, \eta_k; t) = \lim_{n\to+\infty} \Gamma(\xi_1, \dots, \xi_k; t + t_n) \tag{3.2}$$

for all $t \in T$. Therefore, $\eta_1, \dots, \eta_k$ are linearly independent. Hence, $n_q = \dim \mathbb{B}_q \geqslant n_{y_0} = \dim \mathbb{B}_{y_0}$. Since Y is minimal, the reverse inequality also holds. Therefore, $n_q = n_{y_0}$ for all $q \in Y$. This implies that $\eta_1, \dots, \eta_k$ is a basis in $\mathbb{B}_q$. Thus, we have shown that for any $q \in Y$ there is a basis $\eta_1, \dots, \eta_k$ in $\mathbb{B}_q$ that satisfies condition (3.2). For any $q \in Y$ and $(x, q) \in \mathbb{B}_q$ we have $\inf\{|\varphi(t, x, q)| \colon t \in T\} > 0$. Indeed, if we assume the contrary, then there are $(x_0, q) \in \mathbb{B}_q$, $x_0 \neq 0$, and $|t_n| \to +\infty$ such that

$$|\varphi(t_n, x_0, y)| \to 0 \tag{3.3}$$

as $n \to +\infty$. Let $\eta'_1, \dots, \eta'_k$ be a basis in $\mathbb{B}_q$, and let $\eta'_1 = (x_0, q)$. Then (3.3) implies that

$$\Gamma(\eta'_1 t_n, \dots, \eta'_k t_n) \to 0. \tag{3.4}$$

Since $\eta_1, \dots, \eta_k$ and $\eta'_1, \dots, \eta'_k$ are two bases in $\mathbb{B}_q$, there is a non-degenerate linear transformation that transforms the first basis into the second, and vice versa. Formula (3.4) implies that a similar relation holds for the basis $\eta_1, \dots, \eta_k$, which contradicts inequality (3.2). This contradiction completes the proof of the implication (v) $\Rightarrow$ (iv). The theorem is proved.

Applying Theorem 3.1 to the linear non-autonomous dynamical system generated by equation (0.1), we obtain the following assertion.

Theorem 3.2. *Let $\mathcal{A} \in C(\mathbb{R}, [E])$ be recurrent. Then the following conditions are equivalent*:

(a) *the set* $\mathbb{B} = \{(u, \mathcal{B}) \in E \times H(\mathcal{A}) \mid \sup_{t\in\mathbb{R}} |\varphi(t, u, \mathcal{B})| < +\infty\}$ *is closed in* $E \times H(\mathcal{A})$;

(b) *there is a positive number M such that*

$$|\varphi(t, u, \mathcal{B})| \leqslant M|u| \tag{3.5}$$

for all $t \in \mathbb{R}$ and $(u, \mathcal{B}) \in \mathbb{B}$;

(c) $\mathbb{B}$ *is closed and all fibres $\mathbb{B}_{\mathcal{B}}$ have the same dimension, that is, all the equations* (2.2) *have the same number of solutions that are linearly independent and bounded on* $\mathbb{R}$;

(d) *all non-trivial solutions of all equations in the H-class (2.2) bounded on $\mathbb{R}$ are separated from zero, that is,*

$$\inf_{t\in\mathbb{R}} |\varphi(t, u, \mathcal{B})| > 0 \tag{3.6}$$

for all $(u, \mathcal{B}) \in \mathbb{B}$, $u \neq 0$;

(e) *there is a basis $\varphi_1, \dots, \varphi_k$ ($k = \dim \mathbb{B}_{\mathcal{A}}$ and $\mathbb{B}_{\mathcal{A}} = \{(u, \mathcal{A}) \mid (u, \mathcal{A}) \in \mathbb{B}\}$) that consists of solutions of equation* (2.1) *bounded on $\mathbb{R}$ and satisfies the condition $\Gamma(\varphi_1, \dots, \varphi_k; t) \geqslant \alpha > 0$ for all $t \in \mathbb{R}$, where $\Gamma(\varphi_1, \dots, \varphi_k; t) = |\langle \varphi_i(t), \varphi_j(t)\rangle|_{i,j=1}^n$ is the Gram determinant of $\varphi_1, \dots, \varphi_k$.*

Corollary 3.3. *Let $\mathcal{A} \in C(\mathbb{R}, [E])$ be recurrent. Then the following conditions are equivalent*:

(i) *all solutions of all equations from the H-class* (2.2) *are bounded on* $\mathbb{R}$;

(ii) *there is an $M > 0$ such that $|\varphi(t, u, \mathcal{B})| \leqslant M|u|$ for all $t \in T$ and $u \in E$*;

(iii) *all non-zero solutions of all equations from the H-class* (2.2) *are bounded on $\mathbb{R}$ and separated from zero, that is,* (3.6) *is valid for all $(u, \mathcal{B}) \in \mathbb{B}$ such that $u \neq 0$*;

(iv) *there are positive numbers C and α such that $\|U(t, \mathcal{A})\| \leqslant C$ for all $t \in \mathbb{R}$ and $\inf\{|\det U(t, \mathcal{A})| \colon t \in \mathbb{R}\} = \alpha$, where $U(t, \mathcal{A})$ is the Cauchy operator of equation* (2.1).

This follows from Theorem 3.2, since $\Gamma(\varphi_1, \dots, \varphi_n; t) = |\det U(t, \mathcal{A})|^2$, where $\varphi_1, \dots, \varphi_n$ are the column-vectors of the matrix $U(t, \mathcal{A})$.

Remark 3.4. The equivalence of conditions (i) and (ii) was established in [1], [2]. The implication (iv) $\Rightarrow$ (i) sharpens a result in [2].

Theorem 3.5. *Assume that all solutions of all equations from the H-class* (2.2) *are bounded on $\mathbb{R}_+$. Then there is an $M > 0$ such that $|\varphi(t, u, \mathcal{B})| \leqslant M|u|$ for all $t \in \mathbb{R}$ and $(u, \mathcal{B}) \in \mathbb{B}$, that is, $\mathbb{B}$ is closed.*

This follows from Theorems 1.6 and 1.10.

In conclusion we consider examples that illustrate the above results.

Example 3.6. Let $a \in C(\mathbb{R}, \mathbb{R})$ be the Bohr almost periodic function defined by the equality

$$a(t) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}} \sin \frac{t}{2k+1}, \tag{3.7}$$

and let

$$h(t) = \int_0^t a(s)\, ds = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)}.$$

Note that $a(t + t_n) \to -a(t)$ uniformly on $\mathbb{R}$, where $t_n = \pi(2n+1)!!$. Therefore, $-a \in H(a) = \overline{\{a_\tau \mid \tau \in \mathbb{R}\}}$. Using the inequality $|\sin t| \geqslant \frac{1}{2}|t|$ with $|t| \leqslant 1$, we obtain that

$$\begin{aligned} h(t) &= \sum_{k=0}^{\infty} \frac{2}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)} \geqslant \sum_{k \geqslant \frac{1}{2}\left(\frac{|t|}{2} - 1\right)} \frac{t^2}{8} \cdot \frac{1}{(2k+1)^{5/2}} \\ &= \frac{t^2}{8} \int_{\frac{1}{2}\left(\frac{|t|}{2} - 1\right)} \frac{ds}{(2s+1)^{5/2}} = \frac{t^2 2^{3/2}}{24|t|^{3/2}} = \frac{1}{6\sqrt{2}} |t|^{1/2} \to +\infty \end{aligned}$$

as $|t| \to +\infty$. This implies that the moduli of all non-zero solutions of the equation

$$\dot{x} = a(t)x \tag{3.8}$$

tend to $+\infty$ as $|t| \to +\infty$, whereas those of the equation

$$\dot{y} = b(t)y, \tag{3.9}$$

with $b = -a \in H(a)$ tend to zero.

The above example is a slight modification of the well-known example of Favard (see [23] or [24], p. 435). Our case differs from Favard's example in that the solutions of equation (3.9) are not only bounded on $\mathbb{R}$, but they tend to zero as $|t| \to +\infty$. Thus, a non-zero solution of equation (3.9) is asymptotically stable, but the zero solution of equation (3.8) is not, even though $a \in H(b)$.

Example 3.7. Let $a \in C(\mathbb{R}, \mathbb{R})$ be the Bohr almost periodic function defined by the equality

$$a(t) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}} \cos \frac{t}{2k+1}, \tag{3.10}$$

and let

$$h(t) = \int_0^t a(s)\, ds = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1/2}} \sin \frac{t}{(2k+1)}.$$

Then

(i) $h_n \to h$ uniformly on any line segment $[-l, l]$, $l > 0$, where

$$h_n(t) = \sum_{k=0}^{n} \frac{1}{(2k+1)^{1/2}} \sin \frac{t}{2k+1};$$

(ii) $h_n(t + t_n) = -h_n(t)$ for all $t \in \mathbb{R}$, where $t_n = \pi(2n+1)!!$;

(iii) if $M_n = \max\{h_n(t) \colon t \in [0, 2t_n]\}$ and $\tau_n \in [0, 2t_n]$ is such that $h_n(\tau_n) = M_n$, then $M_n \geqslant \sum_{k=0}^{n} (2k+1)^{-1/2} \to +\infty$ as $n \to +\infty$;

(iv) since $\sin x \geqslant 0$ for $x \in [0,1]$, we have $\sin \frac{t}{2k+1} \geqslant 0$ for $n \geqslant n(t) = \left[\frac{t-1}{2}\right] + 1$, and $h(t) \geqslant h_n(t)$ for all $n \geqslant n(t)$ and $t \in \mathbb{R}$, and therefore, $\sup\{h(t) \colon t \in \mathbb{R}_+\} = +\infty$.

Consider the differential equation (3.8), where a is defined by formula (3.10). Assertions (i)–(iv) imply that all non-zero solutions are unbounded on $\mathbb{R}$ and $\inf_{t \in \mathbb{R}} |\varphi(t, x, a)| = 0$, where $\varphi(t, x, a) = x \exp \int_0^t a(s)\, ds$.

Example 3.8. Assume that $a \in C(\mathbb{R}, \mathbb{R})$ is defined by the formula $a(t) = -1 + \sin \sqrt[3]{t}$, $t \in \mathbb{R}$. For equation (3.8) the sets

$$\mathbb{B}^+ = \{(x, b) \mid x \in \mathbb{R}, b \in H(a)\} = \mathbb{R} \times H(a),$$
$$\mathbb{B} = \{(0, b) \mid b \in H(a)\} \cup \{(x, \theta) \mid x \in \mathbb{R}\},$$

where $\theta \in C(\mathbb{R}, \mathbb{R})$ is the function identically equal to zero, are closed. Thus, the recurrence of $\mathcal{A}$ in Theorem 3.2 (Theorem 3.5) is a sufficient condition, but this condition is not necessary.

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