Global attractors of nonautonomous dynamical systems and almost periodic limit regimes of some class of evolutionary equations.

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Abstract

One special class of the nonautonomous dynamical systems with the global attractor in the paper is studies. These systems model the properties of differential equations with convergence, i.e. the equations having the global limit regime.

Introduction

In this paper we study the limit regimes of almost periodic equations

$$x' = f(t, x), \qquad (0.1)$$

where $x \in E((E, |\cdot|)$ is a Banach space), $f : \mathbb{R} \times E \to E$ is a closed mapping and for any $t_0 \in \mathbb{R}$ and $x_0 \in E$ the equation (0.1) admits a unique solution $x(t; t_0, x_0)$ defined for all $t \geq t_0$ and satisfying the initial condition $x(t; t_0, x_0) = x_0$.

A bounded (compact) solution $p : \mathbb{R} \to E$ is said to be limit regime if it is globally asymptotic stable (see, for example [1]). There are many works [1-4], where one studies systems with convergence, i.e. systems which admit the limit regime. The majority of these works are devoted to the study of periodic equations and only in the last 15-20 years one starts to study systematically the nonperiodical systems with convergence (see, for exemple [5-13]). It is necessary to underline that the notion of system with convergence is not satisfactory in the nonperiodical case.

In this paper we propose a more general point of view on the notion of system with convergence (0.1). We study the systems with convergence in the liame of general nonautonomous dynamical systems admitting the global compact attractor with special property.

1 Nonautonomous dynamical systems with convergence.

Let (X,ρ) and (Y,d) be complete metric spaces, $\mathbb{R}(\mathbb{Z})$ be a group of real (integer) numbers, $\mathbb{S}=\mathbb{R}$ or $\mathbb{Z}, \mathbb{S}_{+}=\{t\in\mathbb{S}|t\geq 0\}$ and $\mathbb{T}(\mathbb{S}_{+}\subseteq\mathbb{T})$ be a subgroup of group $\mathbb{S}$.

By $(X,\mathbb{T},\pi)$ we denote a dynamical system on X and $xt=\pi(t,x)=\pi^{t}x$.

Dynamical system $(X,\mathbb{T},\pi)$ is called [14-17] compact dissipative, if there exists a nonempty compact $K\subseteq X$ such that

$$\lim_{t\to+\infty}\rho(xt,K)=0 \tag{1.1}$$

for all $x\in X$, moreover equality (1.1) holds uniformly with respect to $x\in X$ on each compact from X. In this case the set K is called attractor of family of all compacts $C(X)$ from space X .

We denote

$$J=\Omega(K)=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi^{\tau}K},$$

then [14-17] the set J does not depend of the choice of attractor K and is characterized by the properties of dynamical system $(X,\mathbb{T},\pi)$. The set J is called [18] Levinson's center of dynamical system $(X,\mathbb{T},\pi)$.

Let us mention some facts, which we will use below.

Will say that a space X has property (S), if for any compact $K \subseteq X$ there exists a connected set $M \subseteq X$ such that $K \subseteq M$.

Theorem 1.1 *[14-17] If $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J is its Levinson's center, then:*

1.J is invariant, i.e. $\pi^t J = J$ for all $t \in T$;

2. J is orbitally stable, i.e. for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\rho(xt, J) < \varepsilon$ for all $t \geq 0$;

3. J is attractor for the family of all compact subsets of X;

4. $J$is maximal compact invariant set of $(X, \mathbb{T}, \pi)$;

5. J is connected if the space X possesses the (S)-property.

Let Y be a compact metric space and $(X, \mathbb{T}_1, \pi)((Y, \mathbb{T}_2, \sigma))$ be a dynamical system on $X(Y)$, $(\mathbb{T}_1 \subseteq \mathbb{T}_2)$ and $h : X \to Y$ be a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, then the triple $< (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h >$ is called [13,19-20] a nonautonomous dynamical system.

Let W and Y be complete metric spaces, $(Y, \mathbb{S}, \sigma)$ be a group dynamical system on Y and $< W, \varphi, (Y, \mathbb{S}, \sigma) >$ be a skew product [21] (cocycle [22-23]) over $(Y, \mathbb{S}, \sigma)$ with fibre W, i.e. φ is a continuous mapping of $W \times Y \times \mathbb{T}$ into W, satisfying the following conditions: $\varphi(0, w, y) = w$ and $\varphi(t + \tau, w, y) = \varphi(t, \varphi(\tau, w, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}, w \in W$ and $y \in Y$.

We denote $X = W \times Y$ and define on X a dynamical system $(X, \mathbb{T}, \pi)$ by the equality $\pi = (\varphi, \sigma)$ i.e. $\pi(t, (w, y)) = (\varphi(t, w, y), \sigma(t, y))$ for all $t \in \mathbb{T}$ and $(w, y) \in W \times Y$,then the triple $< (X, \mathbb{T}, \pi), ((Y, \mathbb{S}, \sigma), h >$, where $h = pr_2$, is a nonautonomous dynamical system.

For any two bounded subsets A and B from X we denote by $\beta(A, B)$ the semi-deviation of A to B , i.e. $\beta(A, B) = \sup\{\rho(a, B) | a \in A\}$ and $\rho(a, B) = \inf\{\rho(a, b) | b \in B\}$.

The skew product over $(Y,\mathbb{S},\sigma)$ with fibre W is called [16] compact dissipative, if there exists a nonempty compact $K\subseteq W$ such that

$$\lim_{t\to+\infty}\sup\{\beta(U(t,y)M,K): y\in Y\}=0 \qquad (1.2)$$

for all $M\in C(W)$, where $U(t,y)=\varphi(t,\cdot,y)$.

Lemma 1.2 *In order for the skew product $<W,\varphi,(Y,\mathbb{S},\sigma)>$ over $(Y,\mathbb{T},\sigma)$ with fibre W to be compact dissipative, it is necessary and sufficiently that the autonomous dynamical system $(X,\mathbb{T},\pi)$ $\quad (X=W\times Y$ and $\pi=(\varphi,\sigma))$ should be compact dissipative.*

By an entire trajectory of semi-group dynamical system $(X,\mathbb{T},\pi)$ (of skew product $<W,\varphi,(Y,\mathbb{S},\sigma)>$ over $(Y,\mathbb{T},\sigma)$ with fibre W), passing through point $x\in X$ ($(u,y)\in W\times Y$) we mean a continuous mapping $\gamma:\mathbb{S}\to X(\nu:\mathbb{S}\to W)$ satisfying conditions : $\gamma(0)=x(\nu(0)=w)$ and $\gamma(t+\tau)=\pi^t\gamma(\tau)$ $(\gamma(t+\tau)=\varphi(t,\nu(\tau),y\tau))$ for all $t\in\mathbb{T}$ and $\tau\in\mathbb{S}$.

Theorem 1.3 *[16] Let Y be a compact, $<W,\varphi,(Y,\mathbb{S},\sigma)>$ be compact dissipative and K be a non-empty compact, appearing in the equality (1.2), then:*

1. $I_y=\Omega_y(K)\neq\emptyset$, is compact, $I_y\subseteq K$ and

$$\lim_{t\to+\infty}\beta(U(t,y^{-t})K,I_y)=0$$

for every $y\in Y$, where

$$\Omega_y(M)=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}U(\tau,y^{-\tau})M}$$

and $y^{-\tau}=\sigma(-\tau,y)$;

2. $U(t,y)I_y=I_{yt}$ for all $y\in Y$ and $t\in\mathbb{T}$;

3.

$$\lim_{t\to+\infty}\beta(U(t,y^{-t})M,I_y)=0 \qquad (1.3)$$

for all $M \in C(W)$ *and* $y \in Y$;

4.

$$\lim_{t\to+\infty} \sup\{\beta(U(t,y^{-t})M, I) : y \in Y\} = 0$$

whatever is $M \in C(W)$, *where* $I = \bigcup\{I_y : y \in Y\}$;

5. $I = pr_1 J$ *and* $I_y = pr_1 J_y$, *where* J *is a center of Levinson of* (X, T, π) *and* $J_y = J \bigcap X_y$;

6. *the set* I *is compact* ;

7. *the set* I *is connected if the space* $W \times Y$ *has property* (S).

A nonautonomous dynamical system $< (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h >$ is said to be convergent if the following conditions are valid:

a. the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative;

b. the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$, where $X_y = h^{-1}(y) = \{x | x \in X, h(x) = y\}$ and $J_X(J_Y)$ is a Levinson's centre of dynamical system $(X, \mathbb{T}_1, \pi)((Y, \mathbb{T}_2, \sigma))$.

Let $M \subseteq X$ and $M \dot{\times} M = \{(x_1, x_2) | x_1, x_2 \in M, h(x_1) = h(x_2)\}$.

Lemma 1.4 *Let* $< (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h >$ *be a nonautonomous dynamical system,* $K \subseteq X$ *be a compact invariant set and* $M = h(K)$. *If the equality*

$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in K \dot{\times} K} \rho(x_1 t, x_2 t) = 0 \qquad (1.4)$$

takes place, then the set $K_y = K \bigcap X_y$ *contains a single point for all* $y \in M$.

Proof. Suppose that there exists $y_0 \in M$ such that K_{y_0} contains at least two points $\bar{x}_1$ and $\bar{x}_2 (\bar{x}_1 \neq \bar{x}_2)$. Since set K is invariant, then there exists a trajectory φ_i, passing trough the point $\bar{x}_i (i = 1, 2)$ such that $\varphi_i(\mathbb{S}) \subseteq K$. Let $0 < \varepsilon < \frac{\rho(\bar{x}_1, \bar{x}_2)}{2}$ and $L(\varepsilon) > 0$, so that $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \geq L(\varepsilon)$ and $(x_1, x_2) \in K \dot{\times} K$. Thus, we have

$$\rho(\bar{x}_1, \bar{x}_2) = \rho(\pi^t \varphi_1(-t), \pi^t \varphi_2(-t)) < \varepsilon \tag{1.5}$$

for all $t \geq L(\varepsilon)$. The obtained contradiction shows that K_y contains a single point for all $y \in M$. The lemma is proved.

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be satisfying condition (A) if the set $\bigcup\{\pi^t K | t \geq 0\}$ is relatively compact for every $K \in C(X) = \{K | K \subseteq X$ and K is compact $\}$.

We denote by $L_Y = \{x | x \in X$, so that at least one entire trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passes through $x\}$.

Remark 1.5 *For a compact dissipative system $(X, \mathbb{T}, \pi)$ we have $L_X = J_X$, where J_X is a Levinson's centre of $(X, \mathbb{T}, \pi)$.*

Theorem 1.6 *Let $(X, \mathbb{T}, \pi)$ be a dynamical system satisfying the condition (A) and $(Y, \mathbb{T}, \sigma)$ be compact dissipative, then the following conditions are equivalent:*

1. the set $L_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$;

2. every semi-trajectory $\Sigma_x^+ = \{xt | \quad t \geq 0\}$ is asymptotically stable, i.e.

2.a. for all $\varepsilon > 0$ and $p \in X$ there exists $\delta(\varepsilon, p) > 0$ such that $\rho(x,p) < \delta (h(x) = h(p))$ implies $\rho(xt, pt) < \varepsilon$ for any $t \geq 0$.

2.b. there exists $\gamma(p) > 0$ such that $\rho(x,p) < \gamma(p)(h(x) = h(p))$ implies $\lim\limits_{t \to +\infty} \rho(xt, pt) = 0$.

3. a. for all ε and $K \in C(X)$ there exists $\delta(\varepsilon, K) > 0$ such that $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2), x_1, x_2 \in K)$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \geq 0$.

b. $\lim\limits_{t \to +\infty} \rho(x_1 t, x_2 t) = 0$ for all $(x_1, x_2) \in X \dot{\times} X$

4. the equality (1.4) takes place for all $K \in C(X)$.

Proof. We will prove that 1. implies 2.. Really, if we suppose that it is not correct, then there are $p_0 \in X, \varepsilon_0 > 0, p_n \to p_0 (h(p_n) = h(p_0))$ and $t_n \to +\infty$ so that

$$\rho(p_n t_n, p_0 t_n) \geq \varepsilon_0. \tag{1.6}$$

Since $(X,\mathbb{T},\pi)$ satisfies the condition (A), then we may suppose that the sequences $\{p_n t_n\}$ and $\{p_0 t_n\}$ are convergent. Letting $\bar{p}=\lim\limits_{n\to+\infty} p_n t_n,\quad \bar{p}_0=\lim\limits_{n\to+\infty} p_0 t_n$ and taking into consideration (1.6) we will have $\bar{p}\neq\bar{p}_0$. On the other hand $h(\bar{p})=\lim\limits_{n\to+\infty} h(p_n)t_n=\lim\limits_{n\to+\infty} h(p_0)t_n=h(\bar{p}_0)=\bar{y}\in J_Y$ and according to the lemma 4 [10] $\bar{p},\bar{p}_0\in L_X\bigcap X_{\bar{y}}$, but in virtue of condition 1. we have $\bar{p}=\bar{p}_0$. The obtained contradiction proves the necessary affirmation.

Now we will note that 1. implies 2.b.. To prove this implication it is sufficient to show that

$$\lim_{t\to+\infty}\rho(x_1t,x_2t)=0 \tag{1.7}$$

for all $(x_1,x_2)\in X\dot{\times}X$. Assuming the contrary we obtain

$$\rho(x_1^0t_n,x_2^0t_n)\geq\varepsilon_0. \tag{1.8}$$

Dynamical system $(X,\mathbb{T},\pi)$ satisfies the condition (A) and, consequently, we may assume that sequences $\{x_i^0t_n\}(i=1,2)$ and $\{y_0t_n\}(y_0=h(x_1^0)=h(x_2^0))$ are convergent. We denote by $\bar{x}_i^0=\lim\limits_{n\to+\infty}x_i^0t_n$ and $\bar{y}_0=\lim\limits_{n\to+\infty}y_0t_n$, then $\bar{x}_1^0,\bar{x}_2^0\in L_X\bigcap X_{\bar{y}_0}$ and according to condition 1. $\bar{x}_1^0=\bar{x}_2^0$. The last equality and inequality (1.8) are contradictory. This contradiction proves the necessary affirmation.

We will show that 2. implies 3. . Note that

$$\lim_{t\to+\infty}\rho(xt,pt)=0 \tag{1.9}$$

for all $p\in X$ and $x\in X_q(q=h(p))$. In fact, we denote by $G_q=\{x|x\in X$ such that the equality (1.9) takes place $\}$ and suppose that $G_q\neq X_q$. In virtue of condition 2. G_q is open in the X_q. Let $\Gamma_q=\partial G_q(\partial G_q$ is the boundary of $G_q)$ and $\bar{p}\in\Gamma_q$, then $B(\bar{p},\gamma(\bar{b}))\bigcap(X_q\setminus G_q)\neq\emptyset(B(\bar{p},\gamma(\bar{b}))=\{x|h(x)=h(\bar{p}),\rho(x,\bar{p})<\gamma(\bar{p})\}$. It is easy to see that the last relations are not satisfied simultaneously and, consequently, $\Gamma_q=\emptyset$ for all $q\in Y$, i.e. $X_q=G_q$. Let $K\in C(X)$ and $\varepsilon>0$, then there exists $\delta(\varepsilon,K)>0$ such that $\rho(x_1,x_2)<\delta(h(x_1)=h(x_2),x_1,x_2\in K)$ implies $\rho(x_1t,x_2t)<\varepsilon$ for any $t\geq0$. Assuming the contrary, we obtain $K_0 \in C(X), \varepsilon_0 > 0, \delta_n \to 0 (\delta_n > 0), \{x_n^i\} \subseteq K_0 (i = 1,2)$ and $t_n \to +\infty$ such that $\rho(x_n^1, x_n^2) < \delta_n$ and

$$\rho(x_n^1 t_n, x_n^2 t_n) \geq \varepsilon_0 \tag{1.10}$$

Since K_0 is a compact we may suppose that sequences $\{x_n^i\}(i = 1,2)$ are convergent and we denote by $\bar{x} = \lim_{n\to+\infty} x_n^1 = \lim_{n\to+\infty} x_n^2 (\bar{x} \in K_0)$. According to condition 2. for $\varepsilon_0 > 0$ and $\bar{x} \in K_0$ there exists $\delta(\frac{\varepsilon_0}{3}, \bar{x}) > 0$ so that $\rho(x, \bar{x}) < \delta(\frac{\varepsilon_0}{3}, \bar{x})(h(x) = h(\bar{x}))$ implies $\rho(xt, \bar{x}t) < \frac{\varepsilon_0}{3}$ for all $t \geq 0$. Since $x_n^i \to \bar{x}(i = 1,2)$, then there exists $\bar{n}$ such that $\rho(x_n^i, \bar{x}) < \delta(\frac{\varepsilon_0}{3}, \bar{x})(n \geq \bar{n})$ and, consequently,

$$\rho(x_n^1 t, x_n^2 t) \leq \frac{2\varepsilon_0}{3} \tag{1.11}$$

for all $t \geq 0$ and $n \geq \bar{n}$. But the inequalities (1.10) and (1.11) are contradictory. Thus we showed that 2. implies 3. .

We will prove that 3. implies 4.. If we suppose the contrary, then there exist $\varepsilon_0 > 0, K_0 \in C(X), t_n \to +\infty$ and $\{x_n^i\} \subseteq K_0 (i = 1,2; h(x_n^1) = h(x_n^2))$ such that the inequality (1.10) takes place. We may assume without loss of generality that sequences $\{x_n^i\}(i = 1,2)$ are convergent, because K_0 is compact. Let $x^i = \lim_{n\to+\infty} x_n^i, 0 < \varepsilon < \varepsilon_0$ and $\delta(\frac{\varepsilon}{3}, K_0) > 0$ be chosen according to condition 3.a.. Since $h(x^1) = h(x^2)$ and $x^1, x^2 \in K_0$, then for $\frac{\varepsilon}{3}$ there exists $L(\frac{\varepsilon}{3}, x^1, x^2) > 0$ so that $\rho(x^1 t, x^2 t) < \frac{\varepsilon}{3}$ for all $t \geq L(\frac{\varepsilon}{3}, x^1, x^2)$ and, consequently,

$$\rho(x_n^1 t_n, x_n^2 t_n) \leq \rho(x_n^1 t_n, x^1 t_n) + \rho(x^1 t_n, x^2 t_n) + \rho(x^2 t_n, x_n^2 t_n) < \varepsilon \tag{1.12}$$

for sufficiently large n. The inequalities (1.12) and (1.10) are contradictory. Thus the necessary affirmation is proved.

Finally, we note that 4. implies 1. . In fact, if we suppose that there exists $y_0 \in J_Y$ such that $L_X \bigcap X_{y_0}$ contains at least two points x_1 and $x_2 (x_1 \neq x_2)$ and denoting by K a compact invariant set such that $x_1, x_2 \in K$, we will have $x_1, x_2 \in K_{y_0} = K \bigcap X_{y_0}$. On the other hand, according to lemma 1.4

K_{y_0} contains no more than one point. The obtained contradiction proves the theorem 1.2 .

Corollary 1.7 *Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two compact dissipative dynamical systems, then the following conditions are equivalent:*

1. a nonautonomous dynamical system $<(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h>$ is convergent;

2. every semi-trajectory $\sum_x^+ (x \in X)$ is asymptotically stable;

3. 3.a and 3.b from theorem 1.6 are fulfilled;

4. the equality (1.4) takes place for all $K \in C(X)$.

Theorem 1.8 *Let $<(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h>$ be a nonautonomous dynamical system, $(Y, \mathbb{T}, \sigma)$ be compact dissipative and its Levinson's centre J_Y be minimal (i.e. every semi-trajectory $\sum_y^+ (y \in J_Y)$ is dense in J_Y), then the following conditions are equivalent:*

1. nonautonomous dynamical system $<(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h>$ is convergent;

2. dynamical system $(X, \mathbb{T}, \pi)$ satisfies condition (A) and for every $K \in C(X)$ the equality (1.4) takes place.

Proof. In virtue of the corollary 1.7, 1. implies 2. . We will show the converse assertion. Let $K \in C(X)$, then $\sum_K^+ = \bigcup \{ \sum_x^+ | x \in K\}$ is relatively compact and according to lemma 4 [10] the set

$$\Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K}$$

is nonempty, compact, invariant and, consequently, $h(\Omega(K)) \subseteq \Omega(h(K)) \subseteq J_Y$, because J_Y is a maximal compact invariant set in Y. Thus J_Y is minimal, then the equality

$$h(\Omega(K)) = J_Y \tag{1.13}$$

takes place. We note that $\Omega(K_1) = \Omega(K_2)$ for all K_1 and K_2 from $C(X)$. In fact, since $M = \Omega(K_1)\bigcup\Omega(K_2)$ is compact and invariant and in virtue of minimality of J_Y, we have $h(M) = J_Y$. On the other hand, according to lemma 1.4 the set $M_y = M\bigcap X_y$ contains a single point for every $y \in J_Y$. We have $\Omega(K_i)\bigcap X_y \subseteq M\bigcap X_y (i = 1, 2)$ and $\Omega(K_1)\bigcap X_y = \Omega(K_2)\bigcap X_y = M\bigcap X_y$ for any $y \in J_Y$ and, consequently, $\Omega(K_1) = \Omega(K_2)$ for all K_1 and K_2 from $C(X)$. From this it follows that $(X, \mathbb{T}, \pi)$ is compact dissipative and according to theorem 1.6 $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ is convergent.

Corollary 1.9 *Let $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ be a nonautonomous dynamical system, $(Y, \mathbb{T}, \sigma)$ be compact dissipative, J_Y be minimal and $(X, \mathbb{T}, \pi)$ satisfies the condition (A), then the conditions 1.-4. from corollary 1.7 are equivalent.*

Theorem 1.10 *Suppose that the following conditions are fulfilled:*

1. let $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ be a nonautonomous dynamical system ;

2. $(Y, \mathbb{T}, \sigma)$ is compact dissipative;

3. $(X, \mathbb{T}, \pi)$is locally compact, i.e. for all $x \in X$ there exist $\delta = \delta_x > 0$ and $l = l_x > 0$ such that $\pi^l B(x, \delta_x)$ is relatively compact.

In order for nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ to be convergent it is necessary and sufficient that every semi-trajectory $\sum_x^+$ of dynamical system $(X, \mathbb{T}, \pi)$ should be relatively compact and that system $< (h^-(J_Y), \mathbb{T}, \pi), (J_Y, \mathbb{T}, \sigma), h >$ be convergent, where J_Y is Levinson's centre of system $(Y, \mathbb{T}, \sigma)$.

Proof. The necessity of theorem is evident. Now we will show that under the conditions of theorem system $(X, \mathbb{T}, \pi)$ is point dissipative. To this end it is sufficient to show that the set $\Omega_X = \overline{\bigcup\{\omega_x | x \in X\}}$ is compact. We note that $h(\omega_x) \subseteq \omega_{h(x)} \subseteq J_Y$ and, consequently, $\omega_x \subseteq h^{-1}(J_Y)$. Since ω_x is compact and invariant, then $\omega_x \subseteq \bar{J}$, where $\bar{J}$ is the Levinson's centre of dynamical system $(h^{-1}(J_Y), \mathbb{T}, \pi)$. Thus $\Omega_X \subseteq \bar{J}$ and, consequently, Ω_X is compact. In virtue of theorem 1.3.1 [13], the point dissipativeness and compact dissipativeness for the locally compact dynamical systems are equivalent and, consequently,$(X, \mathbb{T}, \pi)$ is compact dissipative. Let J_X be a Levinson's center of $(X, \mathbb{T}, \pi)$, then $h(J_X) \subseteq J_Y$ and, consequently, $J_X \subseteq h^{-1}(J_Y)$. Since $\bar{J}$ is a maximal compact invariant set in $h^{-1}(J_Y)$, then $J_X \subseteq \bar{J}$. From this it results that $J_X \bigcap X_y \subseteq \bar{J} \bigcap X_y$ for all $y \in J_Y$ and, consequently, $J_X \bigcap X_y$ contains no more than one point for any $y \in J_Y$. The theorem is proved.

Theorem 1.11 *Suppose that the following conditions are fulfilled:*

1. *let* $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *be a nonautonomous dynamical system ;*
2. $(Y, \mathbb{T}, \sigma)$ *is compact dissipative;*
3. *there exists a point* $y_0 \in Y$ *such that* $Y = H^+(y_0)$. *A nonautonomous dynamical system* $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *will be convergent if and only if the following conditions are fulfilled:*

a. *dynamical system* $(X, \mathbb{T}, \pi)$ *satisfies the condition (A);*

b. *set* $L_X \bigcap X_y$ *contains no more than one point for any* $y \in J_Y = \omega_{y_0}$.

Proof. The necessity of theorem is evident. Let $x_0 \in X_{y_0}$, then $h(H^+(x_0)) = H^+(y_0)$ and $h(\omega_{x_0}) = \omega_{y_0}$. We denote that $h(\Omega_X) \subseteq \Omega_Y \subseteq J_Y = \omega_{y_0}$ and since $\omega_{x_0} \subseteq \Omega_X$, then $h(\Omega_X) = \omega_{y_0}$. Since $\Omega_X \subseteq L_X$ and $L_X \bigcap X_y$ contains no more than one point for every $y \in J_Y = \omega_{y_0}$, we have $\omega_{x_0} \bigcap X_y = \Omega_X \bigcap X_y$ for all $y \in J_Y$ and, consequently, $\Omega_X = \omega_{x_0}$. Thus $\Omega_X = \omega_{x_0}$ is compact and, consequently, the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative. We have that $(X, \mathbb{T}, \pi)$ is point dissipative and satisfies the condition (A) and according to the theorem 1.5 [24] $(X, \mathbb{T}, \pi)$ is compact dissipative. We denote by J_X a Levinson's centre of dynamical system $(X, \mathbb{T}, \pi)$, then $J_X \subseteq L_X$ and, consequently, $J_X \bigcap X_y$ contains no more than one point for any $y \in J_Y$. The theorem is proved.

A point $y_0 \in Y$ is called [24-26] asymptotically stationary (asymptotically ω- periodic, asymptotically almost periodic, asymptotically recurrent) if there exists a stationary ($\omega-$ periodic , almost periodic, recurrent) point $q \in Y$ such that

$$\lim_{t \to +\infty} d(y_0 t, qt) = 0. \tag{1.14}$$

Remark 1.12 *a. Let* $Y = H^+(y_0) = \overline{\{y_0 t | t \geq 0\}}$ *be compact, then the dynamical system* $(Y, \mathbb{T}, \sigma)$ *is compact dissipative and* $J_Y = \omega_{y_0} (\omega_{y_0} = \bigcap \overline{\bigcup_{\tau \geq t} \sigma^\tau y_0})$.

b. Let y_0 *be asymptotically stationary (asymptotically* ω*- periodic, asymptotically almost periodic, asymptotically recurrent) and* $Y = H^+(y_0)$*, then* $(Y, \mathbb{T}, \sigma)$ *is compact dissipative and the set* $J_Y = \omega_{y_0}$ *is minimal.*

A point $x \in X$ is called [25,26] comparable with regard to the recurrence property in the limit with a point $y \in Y$ if the inclusion $\mathbb{L}_y \subseteq \mathbb{L}_x$ takes place, where $\mathbb{L}_y = \{\{t_n\} | t_n \to +\infty$ and $\{yt_n\}$ is convergent$\}$.

It is known [25,26] that if $\mathbb{L}_y \subseteq \mathbb{L}_x$, then the point x possesses the same character of recurrence property in the limit as point $y \in Y$. In particular, if the point $y \in Y$ is asymptotically stationary (asymptotically ω- periodic, asymptotically almost periodic, asymptotically recurrent) and $\mathbb{L}_y \subseteq \mathbb{L}_x$, then the point x will be asymptotically stationary (asymptotically ω- periodic, asymptotically almost periodic, asymptotically recurrent) .

Theorem 1.13 *Let* $y \in Y$ *be asymptotically stationary (asymptotically* ω*-periodic, asymptotically almost periodic, asymptotically recurrent) and* $Y = H^+(y_0)$*, then the nonautonomous dynamical system* $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *will be convergent if and only if the following conditions are fulfilled:*

a. the dynamical system $(X, \mathbb{T}, \pi)$ *satisfies condition (A);*

b. every point $x \in X$ *is comparable with regard to the recurrence property in the limit with a point* $y = h(x)$ *and, in particular,* x *is asymptotically stationary (asymptotically* ω*- periodic, asymptotically almost periodic, asymptotically recurrent);*

c. for any $\varepsilon > 0$ *and* $K \in C(X)$ *there exists* $\delta = \delta(\varepsilon, K) > 0$ *such that* $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2); x_1, x_2 \in K)$ *implies* $\rho(x_1 t, x_2 t) < \varepsilon$ *for all* $t \geq 0$.

d. the equality $\lim_{t \to +\infty} \rho(x_1 t, x_2 t) = 0$ *takes place for all* $(x_1, x_2) \in X \dot{\times} X$;

Proof. The necessity of conditions a.,c. and d. is assured by corollary 1.7 . Now we will show that under the conditions of theorem the condition b. takes place. Let $x \in X$ and $y = h(x)$, then according to the convergence of nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ the set $H^+(x) = \overline{\{xt | t \geq 0\}}$ is compact. We note that $\omega_x \bigcap X_q \subseteq J_X \bigcap X_q$ for all $q \in \omega_y$ and since
$< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ is convergent, then $\omega_x \bigcap X_q$ contains a single point. According to theorem 1 [27] the point x is comparable with regard to the recurrence property in the limit with point y. If $y \in H^+(y_0)$, then it is evident that point y will be asymptotically stationary (asymptotically ω- periodic, asymptotically almost periodic, asymptotically recurrent) and, consequently, the point x possesses the same character of recurrence property in the limit as point y does.

We will show that the conditions a., b., c. and d. imply the convergence of nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$. First of all, according to condition b. we have that $\omega_x \neq \emptyset$ is compact, minimal and $h(\omega_x) = \omega_{y_0}$ for all $x \in X$. We note that $\omega_x \bigcap X_q$ contains a single point for every $q \in \omega_{y_0}$. In the opposite case there exist $q_0 \in \omega_{y_0}, p_1, p_2 \in \omega_x \bigcap X_{q_0} (p_1 \neq p_2)$ and $t_n^i \to +\infty (i = 1, 2)$ such that $xt_n^i \to p_i (i = 1, 2)$ as $n \to +\infty$. We note that $yt_n^i \to q_0 (i = 1, 2)$ as $n \to +\infty$, where $y = h(x)$. Let $\bar{t}_{2n-1} = t_n^1$ and $\bar{t}_{2n} = t_n^2$ for every $n \in \mathbb{N}$, then $\{\bar{t}_n\} \in \mathbb{L}_y$ and, consequently, $\{\bar{t}_n\} \in \mathbb{L}_x$, i.e. $\{xt_n\}$ is convergent, therefore $p_1 = p_2$. The last equality contradicts to the choice of points p_1 and p_2. The obtained contradiction proves the necessary assertion. Now we will prove that $\omega_{x_1} \bigcap X_q = \omega_{x_2} \bigcap X_q$ for all $x_1, x_2 \in X$ and $q \in \omega_{y_0}$. Let $q \in \omega_{y_0}, \{p_i\} = \omega_{x_i} \bigcap X_q (i = 1, 2)$ and $\{t_n\} \in \mathbb{L}_q$ such that $qt_n \to q$. In virtue of condition c. and minimality of $\omega_{x_i} (i = 1, 2)$ we have

$\rho(p_1 t_n, p_2 t_n) \to 0$ as $n \to +\infty$ and, consequently, $p_1 = p_2$. Thus, $\omega_{x_1} = \omega_{x_2}$ for all $x_1, x_2 \in X$ and, consequently, $(X, \mathbb{T}, \pi)$ is point dissipative and since $(X, \mathbb{T}, \pi)$ satisfies the condition (A), then according to the theorem 1.5 [24] $(X, \mathbb{T}, \pi)$ is compact dissipative. To finish the proof of the theorem it is sufficient to apply the theorem 1.6 and remark 1.5 .

Corollary 1.14 *Under the conditions of theorem 1.10 if the space X is locally compact, then the condition a. results from conditions b., c. and d.* .

Theorem 1.15 *Let $(Y, \mathbb{T}, \sigma)$ be compact dissipative and $h(L_X) = J_Y$. In order that nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ be convergent it is necessary and if $J_Y = Y$, then it is also sufficient that the following conditions be fulfilled:*

1. $\sum_x^+$ is relatively compact for all $x \in X$;

2. L_X is relatively compact;

3. $L_X \bigcap X_y$ contains a single point for every $y \in Y$;

4. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $\rho(x, x_y) < \delta(\{x_y\} = L_X \bigcap X_y$ and $h(x) = y \in J_Y)$ implies $\rho(xt, x_{yt}) < \varepsilon$ for all $t \geq 0$ and $x \in X$.

Proof. Necessity. Let $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ be convergent, then $(X, \mathbb{T}, \pi)$ is compact dissipative and $J_X = L_X$. It is evident that the conditions 1.,2. and 3. are fulfilled. We will prove that the condition 4. takes place. Suppose that it is not true, then there are $\varepsilon_0 > 0, \delta_n \to 0 (\delta_n > 0), \{x_n\}, \{y_n\} \subseteq J_Y$ and $t_n \to +\infty$ such that $\rho(x_n, x_{y_n}) < \delta_n (y_n = h(x_n))$ and

$$\rho(x_n t_n, x_{y_n t_n}) \geq \varepsilon_0. \tag{1.15}$$

Thus J_Y and J_X are compacts, then we may assume without loss of generality that sequences $\{y_n\}$ and $\{x_{y_n}\}$ are convergent. Let $y_0 = \lim\limits_{n \to +\infty} y_n$, then $x_{y_0} = \lim\limits_{n \to +\infty} x_{y_n} = \lim\limits_{n \to +\infty} x_n$.In virtue of compact dissipativeness of dynamical system $(X,\mathbb{T},\pi)$ we may suppose that the sequence $\{x_nt_n\}$ is convergent. We denote by $\bar{x}=\lim\limits_{n\to+\infty} x_nt_n$. Since J_Y is compact, then we may suppose that the sequence $\{y_nt_n\}\subseteq J_Y$ is convergent and then we denote by $\bar{y}=\lim\limits_{n\to+\infty} y_nt_n$. We note that $h(\bar{x})=\lim\limits_{n\to+\infty} h(x_n)t_n=\lim\limits_{n\to+\infty} y_nt_n=\bar{y},\bar{x}\in J_X$ and, consequently, $\bar{x}\in J_X\bigcap X_{\bar{y}}=\{x_{\bar{y}}\}$, i.e. $\bar{x}=x_{\bar{y}}$. On the other hand, passing to the limit in (1.15) as $n\to+\infty$ we have $\rho(\bar{x},x_{\bar{y}})\geq\varepsilon_0$. This contradiction proves the necessary assertion.

Sufficiency. Suppose that conditions 1.-4. are fulfilled. In order to prove that nonautonomous dynamical system $<(X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h>$ is convergent under the conditions of theorem 1.15 it is sufficient to show that the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative. In virtue of conditions of theorem 1.15 the dynamical system $(X,\mathbb{T},\pi)$ is point dissipative and $\Omega_X\subseteq L_X$. We note that set L_X is closed. We will show that L_X is orbitally stable. Suppose that this assertion is not true, then there are $\varepsilon_0>0, x_n\to x_0\in L_X$ and $t_n\to+\infty$ such that

$$\rho(x_nt_n,L_X)\geq\varepsilon_0. \tag{1.16}$$

Since $y_n\to y_0=h(x_0)(y_n=h(x_n))$ then under the conditions of theorem 1.15 $x_{y_n}\to x_{y_0}=x_0$ and, consequently, $\rho(x_n,x_{y_n})\to 0$. From the last relation and the condition 4. of theorem 1.15 it results that $\rho(x_nt_n,x_{y_nt_n})\to 0$ as $n\to+\infty$. But the last relation contradicts the inequality (1.16). Thus, L_X is compact, invariant and orbitally stable. Taking into account that $\Omega_X\subseteq L_X$, we have $J^+(\Omega_X)\subseteq L_X$. We will show that $J^+(\Omega_X)=L_X$. Really, let $\bar{x}\in L_X$ and $\varphi:\mathbb{S}\to L_X$ be the whole trajectory of dynamical system $(X,\mathbb{T},\pi)$ passing through the point $\bar{x}$. We denote by $\alpha^{\varphi}_{\bar{x}}=\bigcap\limits_{t\leq 0}\overline{\bigcup\limits_{\tau\leq t}\varphi(\tau)}$, then $\Omega_X\bigcap\alpha^{\varphi}_{\bar{x}}\neq\emptyset$. Let $p\in\Omega_X\bigcap\alpha^{\varphi}_{\bar{x}}$ and $t_n\to-\infty$ such that $\varphi(t_n)\to p$, then $\pi^{-t_n}\varphi(t_n)=\varphi(0)=\bar{x}$, i.e. $\bar{x}\in J^+_p\subseteq J^+(\Omega_X)$ and, consequently, $L_X=J^+(\Omega_X)$ is compact and orbitally stable and in virtue of theorem 2.5 [28] the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative. The theorem is proved.

Theorem 1.16 *Let* $< (X, \mathbb{T}, \pi), (Y, ,\sigma), h >$ *be a nonautonomous dynamical system and* Y *is a compact minimal set, then the following conditions are equivalent:*

1. $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *is convergent;*

2. every semi-trajectory $\sum_x^+ (x \in X)$ *is relatively compact and asymptotically stable;*

3.a. every semi-trajectory $\sum_x^+ (x \in X)$ *is relatively compact .*

3.b. $\lim_{t \to +\infty} \rho(x_1 t, x_2 t) = 0$ *for all* $(x_1, x_2) \in X \dot{\times} X$ *.*

3.c. for any $\varepsilon > 0$ *and* $K \in C(X)$ *there exists* $\delta(\varepsilon, K) > 0$ *such that* $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2); x_1, x_2 \in K)$ *implies* $\rho(x_1 t, x_2 t) < \varepsilon$ *for all* $t \geq 0$.

4. every semi-trajectory $\sum_x^+ (x \in X)$ *is relatively compact and the equality (1.4) takes place for all* $K \in C(X)$.

Proof. In [27,29] the equivalence of conditions 1., 2. and 3. is proved . According to the theorem 1.2 1. implies 4. . To finish the proof of theorem is sufficient to establish that the condition 4. implies, for example, 3. . We note that from the condition 4. follow 3.a and 3.b . We will show that from the condition 4. results condition 3.c . In fact, if we suppose that it is not true, then there are $\varepsilon_0 > 0, K_0 \in C(X), \delta_n \to 0, \{x_n^i\} \subseteq K_0 (i = 1, 2; h(x_n^1) = h(x_n^2))$ and $t_n \to +\infty$ such that $\rho(x_n^1, x_n^2) < \delta_n$ and

$$\rho(x_n^1 t_n, x_n^2 t_n) \geq \varepsilon_0. \tag{1.17}$$

According to the equality (1.4) for compact $K_0 \in C(X)$ there exists $L(\frac{\varepsilon_0}{2}, K_0) > 0$ such that

$$\rho(x_n^1 t, x_n^2 t) < \frac{\varepsilon_0}{2}. \tag{1.18}$$

for all $t \geq L(\frac{\varepsilon_0}{2}, K_0)$. But the inequalities (1.17) and (1.18) are contradictory. The obtained contradiction proves the theorem.

Theorem 1.17 *Let* $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *be a nonautonomous dynamical system,* $M \neq \emptyset$ *be a compact and positive invariant. Suppose that the following conditions are fulfilled:*

1. $h(M) = Y$*;*

2. $M \bigcap X_y$ *contains a single point for all* $y \in Y$*;*

3. M *is globally asymptotically stable, i.e. for any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $\rho(x, p) < \delta (x \in X_y, p \in M_y = M \bigcap X_y)$ *implies* $\rho(xt, pt) < \varepsilon$ *for all* $t \geq 0$ *and* $\lim_{t \to +\infty} \rho(xt, M_{h(x)t}) = 0$ *for all* $x \in X$*.*

Then the nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *is convergent.*

Proof. We note that under the conditions of theorem the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative and $\Omega_X \subseteq M$. We will show that set M is orbitally stable in $(X, \mathbb{T}, \pi)$. Suppose that it is not true, then there are $\varepsilon_0 > 0, \delta_n \to 0, x_n \in B(M, \delta_n)$ and $t_n \to +\infty$ such that

$$\rho(x_n t_n, M) \geq \varepsilon_0. \tag{1.19}$$

Since M is compact, then we may suppose that the sequence $\{x_n\}$ is convergent. Let $x_0 = \lim_{n \to +\infty} x_n, x_{y_n} \in M_{y_n}, \rho(x_n, M) = \rho(x_n, x_{y_n})$ and $y_0 = h(x_0)$, then $x_0 = \lim_{n \to +\infty} x_{y_n}$ and $x_0 \in M_{y_0}$. Let $q_n = h(x_n)$ and we note that

$$\rho(x_n, x_{q_n}) \leq \rho(x_n, x_{y_n}) + \rho(x_{y_n}, x_{q_n}) \to 0 \tag{1.20}$$

as $n \to +\infty$, because $q_n \to y_0$ and $x_{q_n} \to x_0$. Taking into account (1.20) and the asymptotical stability of set M we have

$$\rho(x_n t_n, x_{q_n t_n}) = \rho(x_n t_n, x_{q_n} t_n) \to 0. \tag{1.21}$$

But the equality (1.21) and inequality (1.19) are contradictory. Hence, the set M is orbitally stable in $(X, \mathbb{T}, \pi)$ and in virtue of lemma 7 [29] the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and $J_X \subseteq M$. To finish the proof of theorem is sufficient to note that $h(J_X) = J_Y$ and for all $y \in J_Y$ we have $J_X \bigcap X_y \subseteq M \bigcap X_y$ and, consequently, $J_X \bigcap X_y$ contains a single point for any $y \in J_Y$. The theorem is proved.

Remark 1.18 *If there exists $y_0 \in Y$ such that $Y = H^+(y_0)$, then it is evident that the theorem 1.17 is invertible. To this end we may take set $H^+(x_0)$, where $x_0 \in X_{y_0}$, in the quality of set M, which figures in the theorem.*

Theorem 1.19 *Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two compact dissipative dynamical systems, then the following conditions are equivalent:*

1. nonautonomous dynamical system $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ is convergent ;

2. there exists a continuous function $V : X\dot{\times}X \to \mathbb{R}_+$ which satisfies the following conditions:

2.a. V is positive definite, i.e. $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$;

2.b. $V(x_1t, x_2t) \leq V(x_1, x_2)$ for all $t \geq 0$ and $(x_1, x_2) \in X\dot{\times}X$;

2.c. $V(x_1t, x_2t) = V(x_1, x_2)$ for any $t \geq 0$ if and only if $x_1 = x_2$;

3. there exists a continuous function $V : X\dot{\times}X \to \mathbb{R}_+$ which satisfies the following conditions:

3.a. V is positive definite;

3.b. $V(x_1t, x_2t) < V(x_1, x_2)$ for all $t > 0$ and $(x_1, x_2) \in X\dot{\times}X \setminus \Delta_X$, where $\Delta_X = \{(x, x) | x \in X\}$.

In the case when J_Y is minimal the theorem 1.19 is proved in [12]. In general case the same type argument that in [12] proves the theorem 1.19 utilizing the results given above.

Theorem 1.20 *Suppose that the following conditions are fulfilled:*

1. $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ are two compact dissipative dynamical systems;

2. there exists a continuous function $V : X\dot{\times}X \to \mathbb{R}_+$ which satisfies the following conditions:

2.a. V is positive definite;

2.b. $V(x_1t, x_2t) \leq \omega(V(x_1, x_2), t)$ for all $t \geq 0$ and $(x_1, x_2) \in X\dot{\times}X$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function with respect to first variable and $\omega(x, t) \to 0$ as $t \to +\infty$ for every $x \in \mathbb{R}_+$.

Then $<(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h>$ is convergent.

Proof. According to corollary 1.7 in order for the nonautonomous dynamical system $<(X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h>$ to be convergent it is necessary and sufficient that the equality (1.4) takes place for all $K \in C(X)$. First of all we will show that

$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in K\dot{\times}K} V(x_1t, x_2t) = 0 \tag{1.22}$$

for any $K \in C(X)$. In fact, in virtue of compactness K there exists $\alpha > 0$ so that $V(x_1, x_2) \leq \alpha$ for all $(x_1, x_2) \in K\dot{\times}K$ and $V(\bar{x}_1, \bar{x}_2) = \alpha$ for certain $(\bar{x}_1, \bar{x}_2) \in K\dot{\times}K$ and, consequently,

$$V(x_1t, x_2t) \leq \omega(V(x_1, x_2), t) \leq \omega(\alpha, t). \tag{1.23}$$

Taking into account that $\omega(\alpha, t) \to 0$ as $t \to +\infty$ we note that (1.23) implies (1.22). We will show that (1.22) implies (1.4). If we suppose that it is not true, then there are $K \in C(X), \varepsilon_0 > 0, \{x_n^i\} \subseteq K (i = 1, 2)$ and $t_n \to +\infty$ such that

$$\rho(x_n^1t_n, x_n^2t_n) \geq \varepsilon_0. \tag{1.24}$$

Without loss of generality we may assume that the sequences $\{x_n^it_n\} (i = 1, 2)$ are convergent because $(X, \mathbb{T}, \pi)$ is compact dissipative. Let $\bar{x}_i = \lim_{n\to+\infty} x_n^it_n$, then according to (1.24) we have $\bar{x}_1 \neq \bar{x}_2$. On the other hand, according to (1.23) we have $0 \leq V(x_n^1t_n, x_n^2t_n) \leq \omega(\alpha, t_n) \to 0$ as $n \to +\infty$ and, consequently, $V(\bar{x}_1, \bar{x}_2) = \lim_{n\to+\infty} V(x_n^1t_n, x_n^2t_n) = 0$. Therefore the equality $\bar{x}_1 = \bar{x}_2$ takes place. The contradiction obtained proves the theorem.

Remark 1.21 *a. Let* $V(x_1t, x_2t) \leq Ne^{-\nu t}V(x_1, x_2)$ *for all* $t \geq 0$ *and* $(x_1, x_2) \in X\dot{\times}X$*, then the condition 2. of theorem 1.20 is fulfilled and in this case* $\omega(x, t) = Nxe^{-\nu t}$*.*

b. If the inequality $V(x_1t, x_2t) \leq (V^{2-\alpha}(x_1, x_2)+(\alpha-2)t)^{\frac{1}{2-\alpha}} (\alpha > 2)$ *takes place for all* $(x_1, x_2) \in X\dot{\times}X$ *and* $t \geq 0$*, then the condition 2. of theorem 1.20 is satisfied with function* $\omega(x, t) = (x^{2-\alpha} + (\alpha - 2)t)^{\frac{1}{2-\alpha}}$*.*

c. All the results from § 1 are true also in the case when spaces X *and* Y *are not metric, but pseudometric.*

d. The nonautonomous dynamical systems with convergence are the simplest among nonautonomous dissipative dynamical systems. If $< (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h >$ *is a nonautonomous dynamical system with convergence and* $J_X(J_Y)$ *is a Levinson's centre of dynamical system* $(X, \mathbb{T}, \pi)$ $((Y, \mathbb{T}, \sigma))$*, then* J_X *and* J_Y *are homeomorphic. In spite of the fact that Levinson's centre* J_X *of nonautonomous dynamical system with convergence admits a complete description, it is necessary to note that the structure of* J_X *may be very complicated (for example,* J_X *may be a strange attractor [12]).*

2 The periodic, almost periodic and recurrent limit regimes of some class of nonautonomous differential equations.

2.1 Let $(E, |\cdot|)$ be a Banach space, $C(\mathbb{R} \times E, E)$ is a space of all continuous mappings from $\mathbb{R} \times E$ into E equipped with the topology of convergence on every compact (open-compact topology). For $f \in C(\mathbb{R} \times E, E)$ and $\tau \in \mathbb{R}$ we denote by f_τ the τ translation of f with respect to t, i.e. $f_\tau(t, x) = f(t + \tau, x)$, $H^+(f) = \overline{\{f_\tau | \tau \in \mathbb{R}_+\}}$ and $\omega_f = \{g | \exists \tau_n \to +\infty, g = \lim_{n\to+\infty} f_{\tau_n}\}$.

Consider the differential equation

$$x' = f(t, x), \tag{2.1}$$

where $f \in C(\mathbb{R} \times E, E)$, and a family of equations

$$y' = g(t, y) \tag{2.2}$$

with $g \in H^+(f)$ or ω_f. Throughout this section we suppose that $f \in C(\mathbb{R} \times E, E)$ is regular, i.e. for all $g \in H^+(f)$ and $x \in E$ the equation (2.2) admits a unique solution $\varphi(t, x, g)$ defined on $\mathbb{R}_+$ with condition that $\varphi(0, x, g) = x$ and mapping $\varphi : \mathbb{R}_+ \times E \times H^+(f) \to E$ is continuous.

The solution $\varphi(t, x_0, f)$ is called uniformly stable, if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $|\varphi(t_0, x, f) - \varphi(t_0, x_0, f)| < \delta$ implies $|\varphi(t, x, f) - \varphi(t, x_0, f)| < \varepsilon$ for all $t \geq t_0 \geq 0$.

The solution $\varphi(t, x_0, f)$ is called globally asymptotically stable, if $\varphi(t, x_0, f)$ is uniformly stable and for all $\varepsilon > 0$ and $K \in C(X)$ there is $T(\varepsilon, K) > 0$ so that $|\varphi(t, x, f) - \varphi(t, x_0, f)| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon, K)(\varphi(t_0, x, f) \in K)$.

We will call the equation (2.1) convergent if it admits at least one compact solution on $\mathbb{R}_+$ which is globally asymptotically stable.

Remark 2.1 *We note that the notion of convergence generalized above is the well known concept of convergence (see, for example [1]). We note that the equation $x' = -x + e^{-t}$ is convergent according to our definition, but is not convergent by usual definition [1].*

Lemma 2.2 *If the equation (2.1) is convergent, then for all $g \in \omega_f$ the equation (2.2) admits a single compact solution defined on $\mathbb{R}$, which is globally asymptotically stable.*

The proof of lemma 2.2 is similar to the proof theorem A (see [30, p.176-177]).

Example 2.3 *Let $Y = H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be a dynamical system of translations, i.e. $\sigma(g, \tau) = g_\tau$. Let $X = E \times Y$ and we will define on X the dynamical system $(X, \mathbb{R}_+, \pi)$ in the following way: $\pi = (\varphi, \sigma)$, i.e.*

$\pi(< x, g >, \tau) =< \varphi(\tau, x, g), g_\tau >$ for all $\tau \in \mathbb{R}_+, x \in X$ and $g \in H^+(f)$, then the triple $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h >$, where $h = pr_2 : X \to Y$ is a nonautonomous dynamical system, generated by equation (2.1).

Applying the theorems 1.6, 1.17 and the lemma 2.2 to the so-constructed nonautonomous dynamical system we will obtain the following assertion.

Theorem 2.4 *Let $f \in C(\mathbb{R} \times E, E)$ be a regular function and $H^+(f)$ be compact. In order for equation (2.1) to be convergent it is necessary and sufficient that the nonautonomous dynamical system $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h >$, generated by equation (2.1) (see example 2.1) is convergent.*

A function $f \in C(\mathbb{R} \times E, E)$ is called stationary ($\omega-$ periodic, almost periodic, recurrent, asymptotically stationary, asymptotically $\omega-$ periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to x on every compact from E, if the motion $\sigma(f, \tau)$ possesses the same property in the dynamical system $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ of translations, i.e. $\sigma(f, \tau) = f_\tau$.

Applying the results from § 1 to the constructed in example 2.3 nonautonomous dynamical system constructed in example 2.3 we obtain the following results.

Theorem 2.5 *Let E be a finite dimensional space, $f \in C(\mathbb{R} \times E, E)$ be a regular function and asymptotically stationary (asymptotically $\omega-$ periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{T}$ uniformly with respect to x on every compact from E, then the following conditions are equivalent:*

1. the equation (2.1) is convergent;

2. all the solutions of equation (2.1) are bounded on $\mathbb{R}_+$ and for any $g \in \omega_f$ the equation (2.2) admits a single bounded solution defined on $\mathbb{R}$ which is globally asymptotically stable;

3. for all $g \in H^+(f)$ every solution of equation (2.2) is bounded on $\mathbb{R}_+$ and is asymptotically stable.

Theorem 2.6 *If the function $f \in C(\mathbb{R}\times E, E)$ is regular and almost periodic (recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to x on every compact from E, then the following conditions are equivalent:*

1. the equation (2.1) is convergent;

2. for any $g \in H^+(f) = \omega_f$ the equation (2.2) admits a single compact solution , defined on $\mathbb{R}$ which is asymptotically stable;

3. for all $g \in H^+(f) = \omega_f$ every solution of equation (2.2) is defined on $\mathbb{R}_+$, compact and is asymptotically stable.

2.2. Let $(H, < \cdot, \cdot >)$ be a Hilbert space and $f \in C(\mathbb{R} \times H, H)$ be a function satisfying the condition

$$Re < x_1 - x_2, f(t, x_1) - f(t, x_2) > \leq -\kappa |x_1 - x_2|^{\alpha} \tag{2.3}$$

for all $t \in \mathbb{R}_+$ and $x \in H$ $(\kappa > 0$ and $\alpha > 2)$. We note that every function $g \in H^+(f)$ satisfies the condition (2.3) with the same constants κ and α. According to the results of [31] if the function f satisfies condition (2.3), then it is regular.

Lemma 2.7 *If the function $f \in C(\mathbb{R} \times H, H)$ satisfies the condition (2.3) and $H^+(f)$ is compact, then :*

1. for every $u \in H$ the solution $\varphi(t, u, f)$ of equation (2.1) is compact on $\mathbb{R}_+$, i.e. the set $\varphi(\mathbb{R}_+, u, f)$ is relatively compact in H;

2. for any $t \geq 0$ and $x_1, x_2 \in H$ we have

$$|\varphi(t, x_1, f) - \varphi(t, x_2, f)| \leq (|x_1 - x_2|^{2-\alpha} + (\alpha - 2)t)^{\frac{1}{2-\alpha}}. \tag{2.4}$$

Proof. Suppose that the conditions of lemma 2.7 are fulfilled and $f \in C(\mathbb{R} \times H, H)$, then we will define the function $F \in C(\mathbb{R} \times H, H)$ in the following way

$$F(t,x) = \begin{cases} f(t,x) & \text{if} \quad (t,x) \in \mathbb{R}_+ \times H \\ f(0,x) & \text{if} \quad (t,x) \in \mathbb{R}_+ \times H \end{cases}$$

It is easy to see that function F possesses the following properties:

a. $\{F_\tau | \tau \in \mathbb{R}\}$ is relatively compact in $C(\mathbb{R} \times H, H)$;

b. $Re < x_1 - x_2, F(t,x_1) - F(t,x_2) > \quad \leq -\kappa |x_1 - x_2|^\alpha$ for all $t \in \mathbb{R}$ and $x_1, x_2 \in H$.

In virtue of theorem 2.2.3.1 [31] the equation $x' = F(t,x)$ admits a single solution $\varphi(t, x_0, F)$ compact on $\mathbb{R}$ and for all $x_1, x_2 \in H$ and $t \geq 0$ we have

$$|\varphi(t,x_1,F) - \varphi(t,x_2,F)| \leq (|x_1 - x_2|^{2-\alpha} + (\alpha - 2)t)^{\frac{1}{2-\alpha}}. \tag{2.5}$$

and, consequently, $\lim_{t \to +\infty} |\varphi(t,x,F) - \varphi(t,x_0,F)| = 0$ for all $x \in H$. From the last relation it follows that every solution of equation (2.1) is compact on $\mathbb{R}_+$, because $\varphi(t,x,f) = \varphi(t,x,F)$ for all $t \geq 0$. The lemma is proved.

Corollary 2.8 *Under the conditions of lemma 2.7 we have*

$$|\varphi(t,x_1,g) - \varphi(t,x_2,g)| \leq (|x_1 - x_2|^{2-\alpha} + (\alpha - 2)t)^{\frac{1}{2-\alpha}} \tag{2.6}$$

for all $g \in H^+(f), t \in \mathbb{R}_+$ *and* $x_1, x_2 \in H$.

Theorem 2.9 *If a function* $f \in C(\mathbb{R} \times H, H)$ *satisfies the condition (2.3) and* $H^+(f)$ *is compact, then the equation (2.1) is convergent.*

Proof. According to lemma 2.7 all solutions of equation (2.1) are compact on $\mathbb{R}_+$ and in virtue of corollary 2.8 every solution $\varphi(t,x,f)$ of equation (2.1) is uniformly asymptotically stable.

Corollary 2.10 *If the function* $f \in C(\mathbb{R} \times H, H)$ *satisfies the condition (2.3) and is asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent) with respect to* $t \in \mathbb{R}$ *uniformly* *with respect to* x *on every compact from* H*, then any solution of equation (2.1) is asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent) and for all* $t \geq 0$ *and* $x_1, x_2 \in H$ *the inequality (2.4) takes place.*

Example 2.11 . *Consider the equation*

$$x' = -x|x| + p(t), \tag{2.7}$$

where $p \in C(\mathbb{R}, H)$*. It is easy to see that function* $f(t,x) = -x|x| + p(t)$ *satisfies the condition (2.3) with* $\kappa = \frac{1}{2}$ *and* $\alpha = 3$*. In fact,*

$$< x_1 - x_2, f(t,x_1) - f(t,x_2) > = < x_1 - x_2, -x_1|x_1| + x_2|x_2| >$$

$$= < x_1 - x_2, -x_1(|x_1| + |x_2|) + x_2(|x_1| > + |x_2|) > + < x_1 - x_2, x_1|x_2| - x_2|x_1| >$$

$$= -|x_1 - x_2|^2(|x_1| + |x_2|) + \frac{1}{2}(|x_1| + |x_2|)(2|x_1||x_2| - 2 < x_1, x_2 > . \tag{2.8}$$

In virtue of Schwart's inequality $< x_1 - x_2, x_1|x_2| - x_2|x_1| > \leq \frac{1}{2}|x_1 - x_2|^2(|x_1| + |x_2|)$*, consequently, we have*

$$< x_1 - x_2, f(t,x_1) - f(t,x_2) > \leq -\frac{1}{2}|x_1 - x_2|^2(|x_1| + |x_2|) \leq -\frac{1}{2}|x_1 - x_2|^3. \tag{2.9}$$

Thus, the theorem 2.9 and corollary 2.10 are applicable for equation (2.7) .

Remark 2.12 *We note that in the case when* $\alpha = 2$ *the convergence of equation (2.1) is proved in [8].*

2.3 Let H be a real Hilbert space. Recall [31-33] that an operator $A : D(A) \to H(D(A) \subseteq H)$ is called uniformly monotone if there exists $\alpha > 0$ so that

$$< Au - Av, u - v > \quad \geq \alpha |u - v|^2 \tag{2.10}$$

for all $u, v \in D(A)$.

Example 2.13 *Consider the differential equation*

$$x' + Ax = f(t), \tag{2.11}$$

where $f \in C(\mathbb{R}, H)$ *and* A *is a maximal monotone operator. It is known [33] that for all* $x_0 \in \overline{D(A)}$ *there exists a unique weak solution* $\varphi(t, x_0, f)$ *of equation (2.11) satisfying the condition* $\varphi(0, x_0, f) = x_0$ *and defined on* $\mathbb{R}_+$*. Let* $Y = H^+(f)$ *and* $(Y, \mathbb{R}_+, \sigma)$ *be a dynamical system of translations on* Y*. We denote by* $X = \overline{D(A)} \times Y$ *and by* $(X, \mathbb{R}_+, \pi)$ *a dynamical system on* X *where* $\pi(< v, g >, t) =< \varphi(t, v, g), g_t >$*, then the triple* $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h >$ $(h = pr_2 : X \to Y)$ *is a nonautonomous dynamical system [34], generated by equation (2.11). Applying the results from § 1 to the so-constructed nonautonomous dynamical system we will obtain the following results for equation (2.11).*

Theorem 2.14 *Let* $H^+(f)$ *be compact, then the equation (2.11) is convergent, i.e. the equation (2.11) admits at least one compact solution on* $\mathbb{R}_+$ *which is globally asymptotically stable.*

Proof. Modifying the results from [8,34], we obtain that under the conditions of theorem 2.14 all the solutions of equation (2.11) are compact on $\mathbb{R}_+$. On the other hand, in virtue of condition (2.10) we have

$$|\varphi(t, x_1, f) - \varphi(t, x_2, f)| \leq e^{-\alpha t}|x_1 - x_2| \tag{2.12}$$

for all $t \geq 0$ and $x_1, x_2 \in H$. And what is more, if $g \in H^+(f)$, than for the solutions of equation

$$y' + Ay = g(t) \tag{2.13}$$

the following estimation

$$|\varphi(t, y_1, g) - \varphi(t, y_2, g)| \leq e^{-\alpha t}|y_1 - y_2| \tag{2.14}$$

takes place for all $t \geq 0$ and $y_1, y_2 \in H$. From inequality (2.12) it follows that every solution of equation (2.11) is globally asymptotically stable. The theorem is proved.

Remark 2.15 *We note that the equation (2.1) is convergent if and only if the nonautonomous dynamical system* $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h >$ *constructed in the example 2.13 possesses the same property of convergence.*

Corollary 2.16 *Let* $f \in C(\mathbb{R}, H)$ *be asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent), then every solution of equation (2.11) is asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent) and globally asymptotically stable.*

Example 2.17 *Consider the equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - \phi(\frac{\partial u}{\partial t}) + f(t) \tag{2.15}$$

in the open set $\Omega \subset \mathbb{R}^n$ *with condition* $u|_{\partial\Omega} = 0$ *on the boundary* $\partial\Omega$ *of* Ω. *Suppose that the function* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfies the conditions :* $\phi(0) = 0$ *and* $0 < c_1 \leq \phi'(\xi) \leq c_2 (\xi \in \mathbb{R})$. *Then the equation may be rewritten in the following way*

$$\begin{cases} \frac{\partial u}{\partial t} & = v \\ \frac{\partial v}{\partial t} & = \Delta u - \phi(v) + f(t) \end{cases} . \tag{2.16}$$

We denote by $H = W_0^{1,2}(\Omega) \times L^2(\Omega)$ *and we will define on* H *the scalar product*

$$< (u, v), (u^*, v^*) >= \int\limits_{\Omega} [vv^* + \Delta u \Delta u^* + \lambda u v^* + \lambda u^* v] dx,$$

where λ *is a certain positive constant independent of* c_1 *and* c_2. *It is possible to verify (see, for example, [35]) that to the system (2.16) the theorem 2.14 and corollary 2.16 may be applied.*

2.4 Let $I \subseteq \mathbb{R}$, $\mathbb{D}(I, \mathbb{R})$ be a space of all infinitely differentiable functions $\varphi : I \to H$ with compact support and $[H]$ be the algebra of all linear operators on H.

Consider the equation

$$\int_{\mathbb{R}} [< u(t), \varphi'(t) > + < A(t)u(t), \varphi(t) > + < f(t), \varphi(t) >] dt = 0, \qquad (2.17)$$

where $A \in C(\mathbb{R}, [H])$ and $f \in C(\mathbb{R}, H)$. A function $u \in C(I, H)$ is called the solution of equation (2.17) if the equality (2.17) takes place for all $\varphi \in \mathbb{D}(I, H)$.

Let $x \in H, \varphi(t, x, A, f)$ be a solution of equation (2.17) defined on $\mathbb{R}_+$ and satisfying the condition $\varphi(0, x, A, f) = x$ and

$$\int_{\mathbb{R}} [< u(t), \varphi'(t) > + < B(t)u(t), \varphi(t) > + < g(t), \varphi(t) >] dt = 0 \qquad (2.18)$$

is a family of equations, where $(B, g) \in H^+(A, f) = \overline{\{(A_\tau, f_\tau) | \tau \in \mathbb{R}_+\}}$. We will suppose that the operator-function $A \in C(\mathbb{R}, [H])$ is self-adjoint and negative defined, i.e. $A(t) = -A_1(t) + iA_2(t)$ for all $t \in \mathbb{R}$, where $A_1(t)$ and $A_2(t)$ are self-adjoint and

$$< A_1(t)u, u > \quad \geq \alpha |u|^2 \qquad (2.19)$$

for all $t \in \mathbb{R}$ and $u \in H$, where $\alpha > 0$.

Lemma 2.18 *[36] We have*

$$\frac{1}{2}\frac{d}{dt}|\varphi(t, x, A, f)|^2 = - < A_1(t)\varphi(t, x, A, f), \varphi(t, x, A, f) >$$

$$+ Re < f(t), \varphi(t, x, A, f) > \qquad (2.20)$$

for all $t > 0$.

Lemma 2.19 *The following inequality*

$$|\varphi(t,x,A,f)| \le |x| + \int_0^t |f(t)|d\tau \tag{2.21}$$

takes place for all $t \ge 0$.

Proof. In virtue of equality (2.20) we have

$$\frac{1}{2}\frac{d}{dt}|\varphi(t,x,A,f)|^2 \le |f(t)||\varphi(t,x,A,f)|.$$

Let $v(t) = |\varphi(t,x,A,f)|^2$, then $\frac{dv}{dt} \le 2|f(t)|\sqrt{v(t)}$ and, consequently,

$$\sqrt{v(t)} - \sqrt{v(\tau)} \le \int_\tau^t |f(s)|ds$$

from which the inequality (2.21) follows.

Lemma 2.20 *Let* l, r *and* $\beta > 0, x_0 \in H, A \in C(\mathbb{R},[H])$ *and* $f \in C(\mathbb{R},[H])$, *then there exists* $M = M(f,l,r,\beta,x_0) >$ *such that*

$$|\varphi(t,x,B,g) - \varphi(t,x_0,A,f)| \le |x-x_0| +$$

$$M\int_0^t \|B(t)-A(t)\|d\tau + \int_0^t |g(\tau)-f(\tau)|d\tau \tag{2.22}$$

for all $t \in [0,l]$ *and* $x \in B(x_0,r) = \{x | x \in H, |x-x_0| \le r\}$ *if* $|g(t)-f(t)| \le \beta$ *and* $Re < B(t)x, x > \le 0$ *for any* $t \in [0,l]$ *and* $x \in H$.

Proof. We denote by $v(t) = \varphi(t,x,B,g) - \varphi(t,x_0,A,f)$, then

$$\int_{\mathbb{R}} [< v(t), \varphi'(t) > + < A(t)v(t), \varphi(t) > +$$

$$< B(t)-A(t))v(t), \varphi(t) > + < g(t)-f(t), \varphi(t) >]dt = 0$$

for any $\varphi \in \mathbb{D}(\mathbb{R},H)$. In virtue of lemma 2.18

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 = Re < A(t)v(t), v(t) >$$

$$+Re[< (B(t) - A(t))\varphi(t,x,B,f), v(t) > + < g(t) - f(t), v(t) >]$$

and according to lemma 2.19 we have

$$|v(t)| \le |v(0)| + \int_0^t |(B(\tau) - A(\tau))\varphi(\tau,x,B,g) + g(\tau) - f(\tau)| d\tau$$

$$\le |v(0)| + \int_0^t \|B(\tau) - A(\tau)\| |\varphi(\tau,x,B,g)| d\tau + \int_0^t |g(\tau) - f(\tau)| d\tau. \quad (2.23)$$

On the other hand according to lemma 2.19 for $\varphi(t,x,B,g)$ we have

$$|\varphi(t,x,B,g)| \le |x| + \int_0^t |g(\tau)| d\tau \le |x_0| + r + \beta l$$

$$+l \max_{0 \le t \le l} |f(t)| = M(f,l,r,\beta,x_0). \quad (2.24)$$

Taking into account the inequalities (2.23) and (2.24) we obtain (2.22). The lemma is proved.

Let $\bar{X} = H \times H^+(A,f)$ and we denote by X the set of all $< u,(b,g) > \in \bar{X}$ such that through the point $u \in H$ passes a solution $\varphi(t,u,B,g)$ of equation (2.18) defined on $\mathbb{R}_+$.

Lemma 2.21 *The set $X \subseteq H \times H^+(A,f)$ is closed in $H \times H^+(A,f)$.*

Proof. Let $< x,(A,f) > \in \bar{X}$, then there exists a sequence $< x_k,(B_k,g_k) > \in X$ such that $x_k \to x$ in space H, $B_k \to A$ in $C(\mathbb{R},[H])$ and $g_k \to f$ in $C(\mathbb{R},H)$. Let $l,\varepsilon > 0$ are such that

$$|x_k - x_m| < \varepsilon, \quad |f_k(t) - f_m(t)| < \varepsilon \quad \text{and} \quad \|B_k(t) - B_m(t)\| < \varepsilon \quad (2.25)$$

for all $t \in [0,l]$ and $k,l \ge k_0$. Denote $r = \sup\{|x_k| : k \in \mathbb{N}\}$, then according to lemma 2.20

$$|\varphi(t,x_k,B_k,f_k) - \varphi(t,x_m,B_m,f_m)| \le |x_k - x_m| + M \int_0^t \|B_k(\tau) - B_m(\tau)\| d\tau$$

$$+ \int_0^t |f_k(\tau) - f_m(\tau)| d\tau \le \varepsilon + M\varepsilon l + \varepsilon l \quad (2.26)$$

for all $t \in [0, l]$ and $k, m \geq k_0$, where M is a positive constant which is independent of r, l and f. Taking into account that space $C(\mathbb{R}_+, H)$ is complete and inequality (2.26), we conclude that the sequence $\{\varphi(t, x_k, B_k, f_k)\}$ is convergent in $C(\mathbb{R}_+, H)$ and according to inequality (2.26) $\varphi(t, x, A, f) = \lim_{k \to +\infty} \varphi(t, x_k, B_k, f_k)$. The lemma is proved.

Lemma 2.22 *The mapping* $\varphi : \mathbb{R}_+ \times X \to H(\varphi : (t, < u, B, g >) \to \varphi(t, u, G, g))$ *is continuous.*

Proof. Let $t_n \to t, x_k \to x, B_k \to B$ and $g_k \to g$ then

$$|\varphi(t, x_k, B_k, g_k) - \varphi(t, x, B, g)| \leq |\varphi(t, x_k, B_k, g_k) - \varphi(t_k, x, B, g)|$$

$$+|\varphi(t_k, x, B, g) - \varphi(t, x, B, g)| \leq \max_{0 \leq t \leq l} |\varphi(t, x_k, B_k, g_k) - \varphi(t, x, B, g)|$$

$$+|\varphi(t_k, x, B, g) - \varphi(t, x, B, g)|. \tag{2.27}$$

In virtue of inequality (2.27) and lemma 2.20 we obtain the necessary assertion. The lemma is proved.

Lemma 2.23 *For all* $(B, g) \in H^+(A, f)$ *and* $x_1, x_2 \in H$ *we have*

$$|\varphi(t, x_1, B, g) - \varphi(t, x_2, B, g)| \leq e^{-\alpha t}|x_1 - x_2| \tag{2.28}$$

for any $t \in \mathbb{R}_+$.

Proof. If the operator-function $A(t)$ is negative defined, then every operator-function $B \in H^+(A)$ is negative defined and $Re < B(t)u, u > \quad \geq \alpha|u|^2 \quad (t \in \mathbb{R}, u \in H)$, where $\alpha > 0$ is the same constant as that one figuring in (2.19) for operator-function $A(t)$. Let $\omega(t) = \varphi(t, x_1, B, g) - \varphi(t, x_2, B, g)$, then according to lemma 2.18 we have

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|^2 = Re < B(t)\omega(t), \omega(t) > \quad \leq -\alpha|\omega(t)|^2 \tag{2.29}$$

and, consequently, $|\omega(t)| \leq |\omega(0)|e^{-\alpha t}$ for all $t \in \mathbb{R}_+$. The lemma is proved.

Example 2.24 *We will define on* X *a dynamical system in the following way:* $\pi(x,t) = \pi(< u,(b,g) >,t) =< \varphi(t,u,B,g),(B_t,g_t) >$ *for all* $< u,(B,g) >\in X$ *and* $\mathbb{R}_+$. *Let* $Y = H^+(A,f)$ *and* $(Y,\mathbb{R}_+,\sigma)$ *be a dynamical system of translations on* Y *and* $h = pr_2 : X \to Y$, *then the triple* $< (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h >$ *is a nonautonomous dynamical system, generated by equation (2.17).*

We will call the equation (2.17) convergent if it admits a compact solution on $\mathbb{R}_+$ *which is globally asymptotically stable. According to the results of* § *1 the equation (2.17) will be convergent if and only if the nonautonomous dynamical system* $< (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h >$ *generated by equation (2.17) (see example 2.24) will be convergent.*

Theorem 2.25 *Let* $A \in C(\mathbb{R},[H]), f \in C(\mathbb{R},H)$ *and* $H^+(A,f)$ *be compact, then the equation (2.17) is convergent.*

Proof. According to the results from [8,34] all the solutions of equation (2.17) are compact on $\mathbb{R}_+$ and in virtue of lemma 2.23 every solution of equation (2.17) is globally asymptotically stable.

Corollary 2.26 *Let* $A \in C(\mathbb{R},[H])$ *and* $f \in C(\mathbb{R},H)$ *be asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent), then all the solutions of equation (2.17) are asymptotically stationary (asymptotically* $\omega-$ *periodic, asymptotically almost periodic, asymptotically recurrent) and globally asymptotically stable.*

We note that in the case when A and F are almost periodic the corollary 2.26 generalizes the results from [36].

Example 2.27 *Consider the equation*

$$\frac{\partial u}{\partial t} = \mathbb{L}u + f(t,x) \qquad (u|_{t=0} = \varphi(x), u|_{\partial\Omega} = 0), \tag{2.30}$$

where $\mathbb{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial u}{\partial x_j}) - a(t,x)u$ *is an uniformly elliptic operator, i.e. the following inequality*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \leq \mu|\xi|^2$$

$(\lambda, \mu > 0)$ *takes place for all* $\xi \in \mathbb{R}^n$*. It is known [36] that the equation (2.30) may be rewritten in form (2.17) if we denote by* $H = L^2(\Omega)$ *, where* Ω *is a bounded domain in* $\mathbb{R}^n$ *and* $\partial\Omega$ *is its boundary with operator-function* $A(t)$ *defined by equality*

$$< A(t)u, \varphi >= -\int_{\Omega} [\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a(t,x)u\varphi]dx.$$

Hence, to equality (2.30) the theorem 2.25 and corollary 2.26 may be applied.

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