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UNIFORM EXPONENTIAL STABILITY OF LINEAR ALMOST PERIODIC SYSTEMS IN BANACH SPACES

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Abstract. This article is devoted to the study linear non-autonomous dynamical systems possessing the property of uniform exponential stability. We prove that if the Cauchy operator of these systems possesses a certain compactness property, then the uniform asymptotic stability implies the uniform exponential stability. For recurrent (almost periodic) systems this result is precised. We also show application for different classes of linear evolution equations: ordinary linear differential equations in a Banach space, retarded and neutral functional differential equations, and some classes of evolution partial differential equations.

Introduction

Let $A(t)$ be a continuous $n \times n$ matrix-function and $H(A)$ be the family of all matrix-functions $B = \lim_{n\to+\infty} A_{t_n}$, where $\{t_n\} \subset \mathbb{R}$, $A_{t_n}(t) = A(t_n + t)$ and the convergence $A_{t_n} \to B$ is uniform on every compact subset of $\mathbb{R}$. The following result is well known.

Theorem [25,2,6]. *Let A be a bounded and uniformly continuous matrix-function on $\mathbb{R}$, then the following conditions are equivalent:*
1. The trivial solution of equation

$$x' = A(t)x \tag{0.1}$$

is uniformly exponentially stable.
2. The trivial solution of equation (0.1) is uniformly asymptotically stable.
3. The trivial solution of equation (0.1) and every equation

$$y' = B(t)y \quad (B \in H(A)) \tag{0.2}$$

is asymptotically stable.

For equations in infinite-dimensional spaces conditions 1, 2, and 3 are not equivalent; see examples in [10, 24, 15]. However, in the general infinite-dimensional case condition 1 implies condition 2, and condition 2 implies condition 3.

Linear non-autonomous dynamical systems satisfying one of the conditions 1, or 2, or 3 are studied in [10]; see also Theorem 1.1 below. In this article we show that

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if the operator corresponding to the the Cauchy problem for (0.1) satisfies some compactness condition, then condition 3 implies condition 1 (see Theorems 2.3 and 2.4).

For recurrent (almost periodic) systems this result is made precise in Theorems 3.2, 3.3 and 3.4. Applications of this result to different classes of linear evolution equations (ordinary linear differential equations in a Banach space, retarded and neutral functional differential equations, some classes of evolution partial differential equations) are given.

1. Linear non-autonomous dynamical systems

Assume that X and Y are complete metric spaces, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integer numbers, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}_+ = \{t \in \mathbb{T} : t \geq 0\}$ and $\mathbb{T}_- = \{t \in \mathbb{T} | t \leq 0\}$. Denote by $(X, \mathbb{T}_+, \pi)$ a semigroup dynamical system on X and $(Y, \mathbb{T}, \sigma)$ a group dynamical system on Y. A triple $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$, where h is a homomorphism of $(X, \mathbb{T}_+, \pi)$ onto $(Y, \mathbb{T}, \sigma)$, is called a non-autonomous dynamical system.

Systems $(X, \mathbb{T}_+, \pi)$ have be classified as follows: (see [7,9])

- Point dissipative, if there is $K \subseteq X$ such that for all $x \in X$
$$\lim_{t\to+\infty} \rho(xt, K) = 0, \tag{1.1}$$
where $xt = \pi^t x = \pi(t, x)$.
- Compactly dissipative, if (1.1) holds uniformly with respect to x on compact subsets of X.
- Locally dissipative, if for any point $p \in X$ there is $\delta_p > 0$ such that (1.1) takes place uniformly with respect to $x \in B(p, \delta_p) = \{x \in X : \rho(x, p) < \delta_p\}$.

Let (X, T, π) be compactly dissipative and K be a compact set that is attractor of all compact subsets of X, and let
$$J = \Omega(K) = \cap_{t\geq 0} \overline{\cup_{\tau\geq t} \pi^\tau K}\,.$$
Then the set J does not depend on selection of the attractor K, and is characterized by the properties of the dynamical system (X, T, π) only; see, for example, [7,9,19,20]). The set J is called the Levinson centre of the compactly dissipative system (X, T, π).

A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be point (compactly, locally) dissipative, if the autonomous dynamical system $(X, \mathbb{T}_+, \pi)$ is so.

Let (X, h, Y) be a locally trivial Banach fibre bundle over Y [3]. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be linear if the mapping $\pi^t : X_y \to X_{yt}$ is linear for every $t \in \mathbb{T}_+$ and $y \in Y$, where $X_y = \{x \in X | h(x) = y\}$ and $yt = \sigma(t, y)$. Let $|\cdot|$ be some norm on (X, h, Y) such that $|\cdot|$ is co-ordinated with the metric ρ on X (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$). Point, compactly, and locally dissipativity criteria for linear systems are obtained in [10].

The entire trajectory of the semigroup dynamical system $(X, \mathbb{T}_+, \pi)$ passing through the point $x \in X$ at $t = 0$ is defined as the continuous map $\gamma : \mathbb{T} \to X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t\gamma(s) = \gamma(s + t)$ for all $t \in \mathbb{T}_+$ and $s \in \mathbb{T}$. Let Φ_x be the set of all entire trajectories of $(X, \mathbb{T}_+, \pi)$ passing through x at $t = 0$ and $\Phi = \cup\{\Phi_x : x \in X\}$.

Theorem 1.1 [10]. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system and Y be a compact set. Then the following assertions hold:*

1. *$\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is point dissipative if and only if $\lim_{t\to+\infty}|xt|=0$ for all $x\in X$.*
2. *The non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is compactly dissipative if and only if $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is point dissipative and there exists a number $M\geq 0$ such that the inequality*
$$|xt|\leq M|x| \tag{1.2}$$
takes place for all $x\in X$ and $t\in\mathbb{T}_+$.
3. *The non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is locally dissipative if and only if there exist positive numbers N and ν such that the inequality $|xt|\leq Ne^{-\nu t}|x|$ takes place for all $x\in X$ and $t\in\mathbb{T}_+$.*

From the Banach-Steinhauss theorem it follows that point dissipativity and compact dissipativity are equivalent for autonomous linear systems. An example of linear autonomous dynamical system which is compactly dissipative, but is not locally dissipative is constructed in [10].

Theorem 1.2 [7,8]. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be a compact set. Then the following assertions take place:*

1. *If $(X,\mathbb{T}_+,\pi)$ is completely continuous (i.e. for all bounded subset $A\subset X$ there exists a positive number $l=l(A)$ such that $\pi^l A$ is precompact), then from point dissipativity of $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ follows its local dissipativity;*
2. *If $(X,\mathbb{T}_+,\pi)$ is asymptotically compact (i.e. for all bounded sequences $\{x_n\}\subset X$ and $\{t_n\}\to+\infty$ the sequence $\{x_nt_n\}$ is precompact if it is bounded), then from compact dissipativity of $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ results its local dissipativity.*

Recall that a measure of noncompactness [20, 27] on a complete metric space X is a function β from the bounded sets of X to the nonnegative real numbers satisfying:

(i) $\beta(A)=0$ for $A\subset X$ if and only if A is precompact

(ii) $\beta(A\cup B)=\max[\beta(A),\beta(B)]$

(iii) $\beta(A+B)\leq\beta(A)+\beta(B)$ for all $A,B\subset X$ if the space X is linear.

The Kuratowski measure of non-compactness α is defined by
$$\alpha(A)=\inf\{d : A \text{ has a finite cover of diameter } < d\}.$$

The dynamical system $(X,\mathbb{T}_+,\pi)$ is said to be conditionally β-condensing [20] if there exists $t_0>0$ such that $\beta(\pi^{t_0}B)<\beta(B)$ for all bounded sets B in X with $\beta(B)>0$. The dynamical system $(X,\mathbb{T}_+,\pi)$ is said to be β-condensing if it is conditionally β-condensing and the set $\pi^{t_0}B$ is bounded for all bounded sets $B\subseteq X$.

According to Lemma 2.3.5 in [20, p.15] and Lemma 3.3 in [7] the conditional condensing dynamical system $(X,\mathbb{T}_+,\pi)$ is asymptotically compact.

Let $X=E\times Y$, $A\subset X$, and $A_y=\{x\in A : pr_2x=y\}$. Then $A=\cup\{A_y : y\in Y\}$. Let $\tilde{A}_y=pr_1A_y$ and $\tilde{A}=\cup\{\tilde{A}_y : y\in Y\}$. Note that if the space Y is compact, then a set $A\subset X$ is bounded in X if and only if the set $\tilde{A}$ is bounded in E.

Lemma 1.3. *The equality $\alpha(A) = \alpha(\tilde{A})$ takes place for all bounded sets $A \subset X$, where $\alpha(A)$ and $\alpha(\tilde{A})$ are the Kuratowski measure of non-compactness for the sets $A \subset X$ and $\tilde{A} \subset E$.*

Proof. Let $\varepsilon > 0$ and A be a bounded subset in X, then there are sets $A_1, A_2, \dots, A_n$ such that $A = \cup\{A_i : i = 1, 2, \dots, n\}$ and $\operatorname{diam} A_i < \alpha(A) + \varepsilon$. Note that $\tilde{A} = \cup\{\tilde{A}_i : i = 1, 2, \dots, n\}$ and $\operatorname{diam} \tilde{A}_i \le \operatorname{diam} A_i < \alpha(A) + \varepsilon$, and consequently, $\alpha(\tilde{A}) \le \alpha(A)$.

Let ε be a positive constant, A be a bounded set in X, $\tilde{A} = \cup\{\tilde{A}_k : k = 1, 2, \dots, m\}$ and $\operatorname{diam} \tilde{A}_k < \alpha(\tilde{A}) + \varepsilon$. Since Y is compact, there are sets $Y_1, Y_2, \dots, Y_\ell$ such that $Y_1 \cup Y_2 \cup \dots \cup Y_\ell = Y$ and $\operatorname{diam} Y_j < \varepsilon$ $(j = 1, 2, \dots \ell)$. Let $A_i = pr_1^{-1}(\tilde{A}_i) \cap A$, and

$$A_{ij} = pr_2^{-1}(pr_2(pr_1^{-1}(\tilde{A}_i) \cap A) \cap Y_j) \cap A_i \,.$$

Note that $A_{ij} \subseteq \tilde{A}_i \times Y_j$, and that

$$\operatorname{diam} A_{ij} \le \operatorname{diam} \tilde{A}_i + \operatorname{diam} Y_j < \alpha(\tilde{A}) + \varepsilon + \varepsilon = \alpha(A) + 2\varepsilon \,.$$

Since $A = \cup\{A_{ij} : i = 1, 2, \dots, n, j = 1, 2, \dots, \ell\}$, it follows that $\alpha(A) \le \alpha(\tilde{A})$ and $\alpha(A) = \alpha(\tilde{A})$. wich concludes the present proof.

Let E be a Banach space and $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E$ be a continuous mapping with $\varphi(0, u, y) = u$ and $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $u \in E$, $y \in Y$ and $t, \tau \in \mathbb{T}_+$. The triplet $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is called a continuous cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre E.

The dynamical system $(X, \mathbb{T}_+, \pi)$ is called a skew-product system [25] if $X = E \times Y$ and $\pi = (\varphi, \sigma)$ (i.e. $\pi(t, (u, y)) = (\varphi(t, u, y), \sigma(t, y))$ for all $u \in E$, $y \in Y$ and $t, \tau \in \mathbb{T}_+$).

A cocycle φ is called conditionally α-condensing if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ the inequality $\alpha(\varphi(t_0, B, Y)) < \alpha(B)$ holds if $\alpha(B) > 0$. The cocycle φ is called α-condensing if it is a conditional α-condensing cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded set $B \subseteq E$.

A cocycle φ is called conditional α-contraction of order $k \in [0, 1)$, if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ for which $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded the inequality $\alpha(\varphi(t_0, B, Y)) \le k\alpha(B)$ holds. The cocycle φ is called α-contraction if it is a conditional α-contraction cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded sets $B \subseteq E$.

Lemma 1.4. *Let Y be compact and the cocycle φ be α-condensing. Then the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$, generated by the cocycle φ, is α-condensing.*

Proof. Let $A \subset X$ be a bounded subset, $t_0 > 0$ and $\alpha(A) > 0$, then

$$\begin{aligned} \pi(t_0, A) &= \cup\, \{\pi(t_0, A_y | y \in Y\} \\ &= \cup\, \{(\varphi(t_0, A_y, y), yt) | y \in Y\} \subseteq \varphi(t_0, \tilde{A}, Y) \times Y. \end{aligned} \tag{1.3}$$

Since A is bounded, $\tilde{A}$ is also bounded in E and according to the condition of the lemma the set $\varphi(t_0, \tilde{A}, Y)$ is bounded and, consequently, $\pi(t_0, A)$ is bounded. According to Lemma 1.3 and (1.3) we have

$$\alpha(\pi(t_0, A)) = \alpha(\cup\{(\varphi(t_0, A_y, y), yt_0) | y \in Y\}) \leq \alpha(\varphi(t_0, \tilde{A}, Y)) < \alpha(\tilde{A}) = \alpha(A).$$

The lemma is proved.

Theorem 1.5. *Let E be a Banach space, φ be a cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre E and the following conditions be fulfilled:*
1. *$\varphi(t, u, y) = \psi(t, u, y) + \gamma(t, u, y)$ for all $t \in \mathbb{T}_+, u \in E$ and $y \in Y$.*
2. *There exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the condition $m(t) \to 0$ as $t \to +\infty$ such that $|\psi(t, u_1, y) - \psi(t, u_2, y)| \leq m(t)|u_1 - u_2|$ for all $t \in \mathbb{T}_+, u_1, u_2 \in E$ and $y \in Y$.*
3. *$\gamma(t, A, Y)$ is compact for all bounded $A \subset X$ and $t > 0$.*

Then the cocycle φ is an α-contraction.

Proof. Let $\varepsilon > 0$ and A be a bounded set in E, then there are sets $A_1, A_2, \dots, A_n$ such that $A = \cup\{A_i : i = 1, 2, \dots, n\}$ and $\operatorname{diam} A_i < \alpha(A) + \varepsilon$ for $i = 1, 2, \dots, n$. Since Y is compact, then there are a sets $Y_1, Y_2, \dots, Y_m$ such that $Y_1 \cup Y_2 \cup \dots \cup Y_m = Y$ with condition $\operatorname{diam} Y_j < \varepsilon$ for all $j = 1, 2, \dots, m$.

Let t_0 be a positive number such that $m(t_0) < 1$. We note that

$$\begin{aligned} \varphi(t_0, A, Y) &\subseteq \psi(t_0, A, Y) + \gamma(t_0, A, Y) \\ &= \cup\{\psi(t_0, A_i, Y_j) | i = 1, 2, \dots, n; j = 1, 2, .., m\} + \gamma(t_0, A, Y). \end{aligned} \tag{1.4}$$

According to the conditions of Theorem 1.5, $\alpha(\gamma(t_0, A, Y)) = 0$ and

$$\operatorname{diam} \psi(t_0, A_i, y) \leq m(t_0) \operatorname{diam} A_i$$

for all $y \in Y$. Thus we have

$$\begin{aligned} |\psi(t_0, u_1, y_1) - \psi(t_0, u_2, y_2)| \leq & |\psi(t_0, u_1, y_1) - \psi(t_0, u_2, y_1)| \\ & + |\psi(t_0, u_2, y_1) - \psi(t_0, u_2, y_2)| \end{aligned} \tag{1.5}$$

and, consequently,

$$\operatorname{diam} \psi(t_0, A_i, y) \leq m(t_0) \operatorname{diam} A_i + \operatorname{diam} \psi(t_0, u_2, Y_j) \quad \text{for all} \quad y \in Y_j\,. \tag{1.6}$$

Since Y is compact, from (1.5)-(1.6) follows the inequality

$$\operatorname{diam} \psi(t_0, A_i, Y_j) \leq m(t_0) \operatorname{diam} A_i \leq m(t_0)(\alpha(A) + \varepsilon)$$

and, consequently, $\alpha(\varphi(t_0, A, Y)) \leq m(t_0)\alpha(A)$. The theorem is proved.

2. Exponential stable linear non-autonomous dynamical systems

Lemma 2.1 [14]. *Let $m : \mathbb{T}_+ \to \mathbb{T}_+$ satisfy the following conditions:*
1. *There exists a positive constant M such that $m(t) \leq M$ for all $t \in \mathbb{T}_+$.*
2. *$m(t) \to 0$ as $t \to +\infty$.*
3. *$m(t + \tau) \leq m(t)m(\tau)$ for all $t, \tau \in \mathbb{T}_+$.*

Then there exist two positive constants N and ν such that $m(t) \leq Ne^{-\nu t}$ for all $t \in \mathbb{T}_+$.

Theorem 2.2. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be a compact set. Then the following conditions are equivalent:*

1. *The non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable, i.e. there exist two positive constants N and ν such that $|\pi(t,x)| \le Ne^{-\nu t}|x|$ for all $t\in\mathbb{T}_+$ and $x\in X$.*
2. *$\|\pi^t\|\to 0$ as $t\to+\infty$, where $\|\pi^t\| = \sup\{|\pi^t x| : x\in X, |x|\le 1\}$.*
3. *The non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is locally dissipative.*

Proof. According to Theorem 1.1, conditions 1 and 3 are equivalent. Now we will prove that the conditions 1 and 2 are equivalent. It is clear that from 1 follows 2. According to condition 2 there exists $L>0$ such that

$$\|\pi^t\| \le 1 \tag{1.7}$$

for all $t\ge L$. We claim that the family of operators $\{\pi^t : t\in[0,L]\}$ is uniformly continuous, that is, for any $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that $|x|\le\delta$ implies $|xt|\le\varepsilon$ for all $t\in[0,L]$. On the contrary, assume that there are $\varepsilon_0>0$, $\delta_n>0$ with $\delta_n\to 0$, $|x_n|<\delta_n$ and $t_n\in[0,L]$ such that

$$|x_n t_n| \ge \varepsilon_0. \tag{1.8}$$

Since (X,h,Y) is a locally trivial Banach fibre bundle and Y is compact, the zero section $\Theta=\{\theta_y : y\in Y, \theta_y\in X_y, |\theta_y|=0\}$ of (X,h,Y) is compact and, consequently, we can assume that the sequences $\{x_n\}$ and $\{t_n\}$ are convergent. Put $x_0=\lim\limits_{n\to+\infty} x_n$ and $t_0=\lim\limits_{n\to+\infty} t_n$, then $x_0=\theta_{y_0}$ $\quad (y_0=h(x_0))$. Passing to the limit in (1.8) as $n\to+\infty$, we obtain $0=|x_0t_0|\ge\varepsilon_0$. This last inequality contradicts the choice of ε_0, and hence proves the above assertion. If $\gamma>0$ is such that $|\pi^t x|\le 1$ for all $|x|\le\gamma$ and $t\in[0,L]$, then

$$|xt| \le \frac{1}{\gamma}|x| \tag{1.9}$$

for all $t\in[0,L]$ and $x\in X$. We put $M=\max\{\gamma^{-1},1\}$, then from (1.7) and (1.9) follows

$$\|\pi^t\| \le M \tag{1.10}$$

for all $t\ge 0$ and $x\in X$. Consider the function $m(t)=\|\pi^t\|$. We note that $m(t+\tau)\le m(t)m(\tau)$ for all $t,\tau\in\mathbb{T}_+$ and $m(t)\le M$ for all $t\in\mathbb{T}_+$ and $m(t)\to 0$ as $t\to+\infty$. According to Lemma 2.1 there exist positive numbers N and ν such that $m(t)\le Ne^{-\nu t}$ for all $t\in\mathbb{T}_+$. Thus $|\pi(t,x)|\le\|\pi^t\||x|\le Ne^{-\nu t}|x|$ for all $t\in\mathbb{T}_+$ and $x\in X$. The theorem is proved.

Let $\mathbb{B}=\{x\in X : \exists\gamma\in\Phi_x \text{ such that } \sup\limits_{t\in\mathbb{T}}|\gamma(t)|<+\infty\}$.

Theorem 2.3. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be compact and $(X,\mathbb{T}_+,\pi)$ be conditionally α-condensing. Then the following assertions are equivalent:*

1. *The non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is point dissipative and this system doesn't admit non-trivial bounded trajectories on $\mathbb{T}$, i.e. $\mathbb{B}\subseteq\Theta=\{\theta_y : y\in Y, \theta_y\in X_y, |\theta_y|=0\}$.*

2. *The non-autonomous dynamical system* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *is uniformly exponentially stable.*

Proof. Denote by $\Theta = \{\theta_y : y \in Y, \theta_y \in X_y, |\theta_y| = 0\}$ the zero section of the vector fibering (X, h, Y). Since (X, h, Y) is locally trivial and Y is compact, the zero section Θ is compact and an invariant set of the dynamical system $(X, \mathbb{T}_+, \pi)$. Taking into account that the dynamical system $(X, \mathbb{T}_+, \pi)$ is conditionally α-condensing, according to Theorem 2.4.8 [20] the set Θ is orbitally stable and in particular there exists a positive constant N such that $|xt| \le N|x|$ for all $t \in \mathbb{T}_+$ and $x \in X$. By virtue of Theorem 1.1 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative and according to Theorem 1.2 $(X, \mathbb{T}_+, \pi)$ is locally dissipative. It follows from Theorem 2.2 that $(X, \mathbb{T}_+, \pi)$ is uniformly exponentially stable.

Let now the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be uniformly exponentially stable, then according to Theorem 2.2 it is locally dissipative. Let J be its Levinson's centre (i.e. maximal compact invariant set of dynamical system $(X, \mathbb{T}_+, \pi)$) . We note that according to the linearity of non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ we have $J = \Theta$. Let φ be a entire bounded trajectory of dynamical system $(X, \mathbb{T}_+, \pi)$. Since the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is conditionally α-condensing, in particulary, it is asymptotically compact and the set $M = \varphi(\mathbb{T})$ is precompact. In fact, the set M is invariant $\Omega(M) = \overline{M}$ and in view of Lemma 3.3 [7] the set M is precompact. We note that $\varphi(\mathbb{T}) \subseteq J = \Theta$ because J is the maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$. The theorem is proved.

Remark 2.4. *Theorem A in [26] implies a version of Theorem 2.3 under slightly stronger assumptions.*

Theorem 2.5. *Let* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *be a linear non-autonomous dynamical system,* Y *be compact and* $(X, \mathbb{T}_+, \pi)$ *be completely continuous, i.e. for any bounded set* $A \subseteq X$ *there exists a positive number* ℓ *such that* $\pi^\ell(A)$ *is precompact. Then the following assertions are equivalent:*

1. *The non-autonomous dynamical system* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *is uniformly exponentially stable.*
2. $\lim_{t\to+\infty} |\pi^t x| = 0$ *for all* $x \in X$.

Proof. It is clear that condition 1 implies 2. Now we will show that condition 1 follows from 2. According to Theorem 1.1 the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is point dissipative. Since the dynamical system $(X, \mathbb{T}_+, \pi)$ is completely continuous, by virtue of Theorem 1.2 the dynamical system $(X, \mathbb{T}_+, \pi)$ is locally dissipative. To prove the theorem it is sufficient to refer to Theorem 2.2.

3. Linear non-autonomous dynamical system with a minimal base

In this section we study a linear non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ with compact minimal base $(Y, \mathbb{T}, \sigma)$.

Theorem 3.1 [4,5]. *Let* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *be a linear non-autonomous dynamical system and the following conditions hold:*

1. Y *is compact and minimal (i.e.* $Y = H(y) = \overline{\{yt : t \in \mathbb{T}\}}$ *for all* $y \in Y$ *);*
2. *for any* $x \in X$ *there exists* $C_x \ge 0$ *such that* $|xt| \le C_x$ *for all* $t \in \mathbb{T}_+$*;*

3. the mapping $y \mapsto \|\pi^t_y\|$ is continuous, where $\|\pi^t_y\|$ is the norm of the linear operator $\pi^t_y = \pi^t|_{X_y}$, for every $t \in \mathbb{T}_+$ or $(X, \mathbb{T}_+, \pi)$ is a skew-product dynamical system.

Then there exists $M \geq 0$ such that

$$|\pi(t, x)| \leq M|x| \tag{3.1}$$

holds for all $t \in \mathbb{T}_+$ and $x \in X$.

Theorem 3.2. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system and the following conditions hold:*

1. *Y is compact and minimal.*
2. *The dynamical system $(X, \mathbb{T}_+, \pi)$ is asymptotically compact.*
3. *The mapping $y \mapsto \|\pi^t_y\|$ is continuous, where $\|\pi^t_y\|$ is the norm of the linear operator $\pi^t_y = \pi^t|_{X_y}$, for every $t \in \mathbb{T}_+$ or $(X, \mathbb{T}_+, \pi)$ is a skew-product dynamical system.*

Then the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is uniformly exponentially stable if and only if

$$\lim_{t \to +\infty} |\pi^t x| = 0 \tag{3.2}$$

for all $x \in X$.

Proof. It is clear that from uniform exponential stability of the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ follows the equality (3.2).

From condition (3.2) and minimality of $(Y, \mathbb{T}, \sigma)$ by virtue of Theorem 3.1 it follows that for the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ the inequality (3.1) holds and according to Theorem 1.1 this system is compactly dissipative. Since the dynamical system $(X, \mathbb{T}_+, \pi)$ is asymptotically compact, to finish the proof of the Theorem it is sufficient to remark that according to Theorem 2.13 [8] every compactly dissipative and asymptotically compact dynamical system is locally dissipative. Thus a linear non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is locally dissipative and, consequently, it is uniform exponentially stable. The theorem is proved.

Theorem 3.3 [5]. *Assume that $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is a linear non-autonomous dynamical system, generated by a cocycle φ, $(X, \mathbb{T}_+, \pi)$ is asymptotically compact, and there is an $M > 0$ such that $|\varphi(t, u, y)| \leq M|u|$ for all $(u, y) \in X = E \times Y$ and $t \in \mathbb{T}_+$. Then the following assertions hold.*

- *(i) For any $(u, y) \in \mathbb{B} = \{x \in X : \exists \gamma \in \Phi_x$ such that $\sup_{t \in \mathbb{T}} |\gamma(t)| < +\infty\}$ the set $\Phi_{(u,y)}$ consists of a single entire recurrent trajectory.*
- *(ii) $\mathbb{B}$ is closed in X.*
- *(iii) $(X, \mathbb{T}_+, \pi)$ induces a group dynamical system $(\mathbb{B}, \mathbb{T}, \pi)$ on $\mathbb{B}$.*
- *(iv) $(\mathbb{B}, h, Y)$ is a finite dimensional vector subfibering of (X, h, Y), i.e. $\dim \mathbb{B}_y$ does not depend on $y \in Y$.*

Theorem 3.4. *Suppose that the following conditions are satisfied:*

1. *A dynamical system $(Y, \mathbb{T}, \sigma)$is compact and minimal.*
2. *A linear non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is generated by cocycle φ (i.e. $X = E \times Y$, $\pi = (\varphi, \sigma)$ and $h = pr_2 : X \mapsto Y$).*

3. The dynamical system $(X, \mathbb{T}_+, \pi)$ is conditionally α-condensing.
4. There exists a positive number M such that $|\varphi(t, u, y)| \le M|u|$ for all $t \in Y$ and $t \in \mathbb{T}_+$.

Then there are two vectorial positively invariant subfiberings (X^0, h, Y) and (X^s, h, Y) of (X, h, Y) such that:

a. $X_y = X^0_y + X^s_y$ and $X^0_y \cap X^s_y = 0_y$ for all $y \in Y$, where $0_y = (0, y) \in X = E \times Y$ and 0 is the zero in the Banach space E.
b. The vector subfibering (X^0, h, Y) is finite dimensional, invariant (i.e. $\pi^t X^0 = X^0$ for all $t \in \mathbb{T}_+$) and every trajectory of the dynamical system $(X, \mathbb{T}_+, \pi)$ belonging to X^0 is recurrent.
c. There exist two positive numbers N and ν such that $|\varphi(t, u, y)| \le Ne^{-\nu t}|u|$ for all $(u, y) \in X^s$ and $t \in \mathbb{T}_+$.

Proof. Let $X^0 = \mathbb{B}$, then according to Theorem 3.3, statement b holds. Denote by P_y the projection of $X_y = h^{-1}(y)$ to $\mathbb{B}_y = \mathbb{B} \cap h^{-1}(y)$, then $P_y(u, y) = (\mathcal{P}(y)u, y)$ for all $u \in E, \mathcal{P}^2(y) = \mathcal{P}(y)$ and the mapping $\mathcal{P} : Y \to [E]$ ($y \mapsto \mathcal{P}(y)$) is continuous, where by $[E]$ denotes the set of all linear continuous operators acting on E. Now we set $X^s_y = \mathcal{Q}(y)X_y$ and $X^s = \cup\{X^s_y : y \in Y\}$, where $\mathcal{Q}(y) = Id_E - \mathcal{P}(y)$. We will show that X^s is closed in X. In fact, let $\{x_n\} = \{(u_n, y_n)\} \subseteq X^s$ and $x_0 = (u_0, y_0) = \lim_{n\to\infty} x_n$. Note that $P_{y_0}(x_0) = (\mathcal{P}(y_0)u_0, y_0) = (\lim_{n\to\infty} \mathcal{P}(y_n)u_n, y_0) = (0, y_0) = 0_{y_0}$ and, consequently, $x_0 \in X^s_{y_0} \subseteq X^s$.

Let $(X^s, \mathbb{T}_+, \pi)$ be the dynamical system induced by $(X, \mathbb{T}_+, \pi)$. It is clear that under the conditions of Theorem 3.4 the dynamical system $(X^s, \mathbb{T}_+, \pi)$ is asymptotically compact and every positive semi-trajectory is precompact and, consequently, $\lim_{t\to\infty} |\pi(t, x)| = 0$ for all $x \in X^s$ because the dynamical system $(X^s, \mathbb{T}_+, \pi)$ doesn't have a non-trivial entire trajectory bounded on $\mathbb{T}$. In fact, if we suppose that it is not true, then there exist $x_0 = (u_0, y_0)$ and $t_n \to +\infty$ such that: $|u_0| \ne 0$, $\lim_{n\to+\infty} \pi(t_n, x) = x_0$ and through point x_0 pass a non-trivial entire trajectory bounded on $\mathbb{T}$. This contradiction proves the necessary assertion. Thus we can apply Theorem 2.3 according which there exist two positive constants N and ν such that $|\varphi(t, u, y)| \le Ne^{-\nu t}|u|$ for all $(u, y) \in X^s$ and $t \in \mathbb{T}_+$. The theorem is proved.

4. Some classes of linear uniformly exponentially stable non-autonomous differential equations

Let Λ be a complete metric space of linear operators that act on a Banach space E and $C(\mathbb{R}, \Lambda)$ be the space of all continuous operator-functions $A : \mathbb{R} \to \Lambda$ equipped with the open-compact topology and $(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{R}, \Lambda)$.

Ordinary linear differential equations. Let $\Lambda = [E]$ and consider the linear differential equation

$$u' = \mathcal{A}(t)u\,, \tag{4.1}$$

where $\mathcal{A} \in C(\mathbb{R}, \Lambda)$. Along with equation (4.1), we shall also consider its H-class, that is, the family of equations

$$v' = \mathcal{B}(t)v\,, \tag{4.2}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = \mathcal{A}(t + \tau)$ $(t \in \mathbb{R})$, and the bar denotes closure in $C(\mathbb{R}, \Lambda)$. Let $\varphi(t, u, \mathcal{B})$ be the solution of equation (4.2) that satisfies the condition $\varphi(0,u,\mathcal{B})=u$. We put $Y=H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y,\mathbb{R},\sigma)$. Then the triple $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a linear non-autonomous dynamical system, where $X=E\times Y$, $\pi=(\varphi,\sigma)$; i.e., $\pi((v,\mathcal{B}),\tau)=(\varphi(\tau,v,\mathcal{B}),\mathcal{B}_\tau)$ and $h=pr_2:X\to Y$).

Lemma 4.1 [11,13].

(i) The mapping $(t,u,\mathcal{A})\mapsto\varphi(t,u,\mathcal{A})$ of $\mathbb{R}\times E\times C(\mathbb{R},[E])$ to E is continuous, and

(ii) the mapping $\mathcal{A}\mapsto U(\cdot,\mathcal{A})$ of $C(\mathbb{R},[E])$ to $C(\mathbb{R},[E])$ is continuous, where $U(\cdot,\mathcal{A})$ is the Cauchy operator [17] of equation (4.1).

Theorem 4.2. *Let $\mathcal{A}\in C(\mathbb{R},\Lambda)$ be compact (i.e. $H(\mathcal{A})$ is a compact set of $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$), then the following conditions are equivalent:*

1. *The trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t,\mathcal{A})U(\tau,\mathcal{A})^{-1}\|\leq Ne^{-(t-\tau)}$ for all $t\geq\tau$.*
2. *There exist positive numbers N and ν such that $\|U(t,\mathcal{B})U(\tau,\mathcal{B})^{-1}\|\leq Ne^{-(t-\tau)}$ for all $t\geq\tau$ and $\mathcal{B}\in H(\mathcal{A})$.*
3. $\lim\limits_{t\to+\infty}\sup\{\|U(t,\mathcal{B})\|:\mathcal{B}\in H(\mathcal{A})\}=0$.

Proof. Applying Theorem 2.2 to the non-autonomous system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, generated by equation (4.1), we obtain the equivalence of conditions 2 and 3. According to Lemma 3 [11] conditions 1 and 2 are equivalent. The theorem is proved.

Theorem 4.3. *Let $\mathcal{A}\in C(\mathbb{R},\Lambda)$ be recurrent with respect to $t\in\mathbb{T}$ (i.e. $H(\mathcal{A})$ is a compact and minimal set of $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$), the non-autonomous system $\langle(X,\mathbb{R}_+,\pi)$,
$(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4.1) is asymptotically compact. Then the following conditions are equivalent:*

1. *The trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t,\mathcal{A})U(\tau,\mathcal{A})^{-1}\|\leq Ne^{-(t-\tau)}$ for all $t\geq\tau$.*
2. $\lim\limits_{t\to+\infty}\sup|\varphi(t,u,\mathcal{B})|=0$ *for every $u\in E$ and $\mathcal{B}\in H(\mathcal{A})$.*

Proof. According to Lemma 4.1 the mapping $U(t,\cdot):[E]\to[E]$ is continuous and, consequently, the mapping $\mathcal{B}\mapsto\|U(t,\mathcal{B})\|$ is also continuous for every $t\in\mathbb{T}$. Now applying Theorem 3.2 to non-autonomous system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4.1), we obtain the equivalence of conditions 1. and 2.. The theorem is proved.

We now formulate some sufficient conditions for the $\alpha-$ condensedness (in particular, asymptotical compactness) of the linear non-autonomous dynamical system generated by equation (4.1).

Theorem 4.4. *Let $\mathcal{A}\in C(\mathbb{R},[E]),\mathcal{A}(t)=\mathcal{A}_1(t)+\mathcal{A}_2(t)$ for all $t\in\mathbb{R}$, and assume that $H(\mathcal{A}_i)$ $(i=1,2)$ are compact and the following conditions hold:*

(i) The zero solution of the equation

$$u'=\mathcal{A}_1(t)u \tag{4.3}$$

is uniformly asymptotically stable, that is, there are positive numbers N and ν such that

$$\|U(t,\mathcal{A}_1)U^{-1}(\tau,\mathcal{A}_1)\|\leq Ne^{-\nu(t-\tau)} \tag{4.4}$$

for all $t \geq \tau$ $(t, \tau \in \mathbb{R})$, where $U(t, \mathcal{A}_1)$ is the Cauchy operator of equation (4.3).

(ii) The family of operators $\{\mathcal{A}_2(t) : t \in \mathbb{R}_+\}$ is uniformly completely continuous, that is, for any bounded set $A \subset E$ the set $\{\mathcal{A}_2(t)A : t \in \mathbb{R}_+\}$ is precompact.

Then the linear non-autonomous dynamical system generated by equation (4.1) is an α-contraction.

Proof. First of all we note that the set $\varphi(t, A, Y)$ is bounded for every $t > 0$ and bounded set $A \subseteq E$. Let $\mathcal{B} \in H(\mathcal{A})$. Then there are $\{t_n\} \subset \mathbb{T}$ such that $\mathcal{B}(t) = \mathcal{B}_1(t) + \mathcal{B}_2(t)$ and $\mathcal{B}_i(t) = \lim_{t\to+\infty} \mathcal{A}_i(t + t_n)$. Note that

$$\varphi(t, v, \mathcal{B}) = U(t, \mathcal{B}_1)v + \int_0^t U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau, v, \mathcal{B})d\tau. \tag{4.5}$$

By Lemma 4.1,

$$U(t, \mathcal{B}_i) = \lim_{t\to+\infty} U(t, \mathcal{A}_{it_n}), \quad \mathcal{A}_{it_n}(t) = \mathcal{A}_i(t + t_n),$$

and the equality

$$U(t, \mathcal{A}_{1t_n})U^{-1}(\tau, \mathcal{A}_{1t_n}) = U(t + t_n, \mathcal{A}_1)U^{-1}(\tau + t_n, \mathcal{A}_1)$$

and inequality (4.4) imply that

$$\|U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)\| \leq Ne^{-\nu(t-\tau)} \tag{4.6}$$

for all $t \geq \tau$ and $\mathcal{B}_1 \in H(\mathcal{A}_1)$. By Theorem 1.5, Theorem 4.4 will be proved if we can prove that the set

$$\{\int_0^t U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau, v, \mathcal{B})d\tau : (v, \mathcal{B}) \in A\}$$

is precompact for every $t > 0$ and every bounded positively invariant set $A \subseteq E \times Y$. We put

$$K_A^t = \overline{\{\mathcal{B}_2(\tau)\varphi(\tau, v, \mathcal{B}) : \tau \in [0, t], (v, \mathcal{B}) \in A\}}$$

and we note that the set K_A^t is compact. Really, the set $\varphi([0, t], A) = \cup\{\varphi(\tau, v, \mathcal{B}) : \tau \in [0, t], (v, \mathcal{B}) \in A\}$ is bounded because $|\varphi(\tau, v, \mathcal{B})| \leq e^{Mt}r$ for all $\tau \in [0, t]$ and $(v, \mathcal{B}) \in A$, where $r = \sup\{|v| : \exists \mathcal{B} \in H(\mathcal{A}),$ such that $(v, \mathcal{B}) \in A\}$ and $M = \sup\{\|\mathcal{A}(t)\| : t \in \mathbb{T}\}$. Then

$$\int_0^t U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau, v, \mathcal{B})d\tau$$

$$\in t\ \overline{\text{conv}}\{U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)w : 0 \leq \tau \leq t, \mathcal{B}_1 \in H(\mathcal{A}_1), w \in K_A^t\}. \tag{4.7}$$

Since $H(\mathcal{A}_1), H(\mathcal{A})$, and K_A^t are compact sets, then formula (4.7), condition (ii) of Theorem 4.4 , and Lemma 4.1 imply that $\{U(t, \mathcal{B}_1)U^{-1}(\tau, \mathcal{B}_1)w : 0 \leq \tau \leq t, \mathcal{B}_1 \in H(\mathcal{A}_1), w \in K_A^t\}$ is compact, which completes the proof of the theorem.

Theorem 4.5. *Let $H(\mathcal{A})$ be compact and assume that there is a finite-dimensional projection $P \in [E]$ such that*

(i) the family of projections $\{P(t) : t \in \mathbb{R}\}$, where $P(t) = U(t,\mathcal{A})PU^{-1}(t,\mathcal{A})$, is precompact in $[E]$, and

(ii) there are positive numbers N and ν such that

$$\|U(t,\mathcal{A})QU^{-1}(\tau,\mathcal{A})\| \leq Ne^{-\nu(t-\tau)}$$

for all $t \geq \tau$, where $Q = I - P$.

Then the linear non-autonomous dynamical system generated by equation (4.1) is an α-contraction.

Proof. Since the family of projections $P(t) = U(t,\mathcal{A})PU^{-1}(t,\mathcal{A})$ is precompact in $[E]$, the family $\mathbb{H} = \overline{\{P(t) : t \in \mathbb{R}\}}$ is uniformly completely continuous, where the bar denotes closure in $[E]$. Indeed, let A be a bounded subset of E, $\{x_n\} \subseteq \{QA : Q \in \mathbb{H}\}$, and $\varepsilon_n \downarrow 0$. Then there are $t_n \in \mathbb{R}$ and $v_n \in A$ such that $|x_n - P(t_n)v_n| \leq \varepsilon_n$. Since the sequence $\{P(t_n)\}$ is precompact, we can assume that it converges. Let $L = \lim_{n\to+\infty} P(t_n)$. Then L is completely continuous, which implies that the sequence $\{x'_n\} = \{Lv_n\}$ is precompact. Note that

$$|x_n - x'_n| \leq |x_n - P(t_n)v_n| + |P(t_n)v_n - Lv_n| \leq \varepsilon_n + \|P(t_n) - L\||v_n|,$$

which implies that $|x_n - x'_n| \to 0$ as $n \to +\infty$. Hence, $\{x_n\}$ is precompact.

Assume that $\mathcal{B} \in H(\mathcal{A})$ and $\{t_n\} \subset \mathbb{R}$ are such that

$$\mathcal{B} = \lim_{n\to+\infty} \mathcal{A}_{t_n}, P(\mathcal{B}) = \lim_{n\to+\infty} P(\mathcal{A}_{t_n}),$$

where $P(\mathcal{A}_{t_n}) = U(t_n,\mathcal{A})PU^{-1}(t_n,\mathcal{A})$. The assertions proved above imply that the family $\{P(\mathcal{B}) : \mathcal{B} \in H(\mathcal{A})\}$ is uniformly completely continuous. Note that $Q(\mathcal{B}) = \lim_{n\to+\infty} Q(\mathcal{A}_{t_n})$, where $Q(\mathcal{B}) = I - P(\mathcal{B})$ and $Q(\mathcal{A}_{t_n}) = I - P(\mathcal{A}_{t_n})$. Moreover, condition (ii) of Theorem 4.5 implies that

$$\|U(t,\mathcal{A}_{t_n})QU^{-1}(\tau,\mathcal{A}_{t_n})\| \leq Ne^{-\nu(t-\tau)} \tag{4.7}$$

for all $t \geq \tau$. Passing to the limit in (4.7) as $n \to +\infty$ and taking into account Lemma 4.1, we obtain that

$$\|U(t,\mathcal{B})QU^{-1}(\tau,\mathcal{B})\| \leq Ne^{-\nu(t-\tau)}$$

for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A})$. We complete the proof of the theorem by observing that $U(t,\mathcal{B})Q(\mathcal{B}) + U(t,\mathcal{B})P(\mathcal{B}) = U(t,\mathcal{B})$ and applying Theorem 1.5.

Theorem 4.6. *Suppose that the following conditions are satisfied:*

1. *The operator-function $\mathcal{A} \in C(\mathbb{R},[E])$ is recurrent with respect to $t \in \mathbb{R}$.*
2. *The linear non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ generated by equation (4.1) is conditionally α-condensing.*
3. *the trivial solution of equation (4.1) is uniformly stable in the positive direction, i.e. there exist a positive number M such that*

$$\|U(t,\mathcal{A})U^{-1}(\tau,\mathcal{A})\| \leq M \tag{4.8}$$

for all $t \geq \tau$.

Then there are two vectorial positively invariant subfibers (X^0, h, Y) *and* (X^s, h, Y) *of* (X, h, Y) *such that:*

a. $X_y = X_y^0 + X_y^s$ *and* $X_y^0 \cap X_y^s = 0_y$ *for all* $y \in Y$, *where* $0_y = (0, y) \in X = E \times Y$ *and* 0 *is the zero in the Banach space* E.

b. *The vectorial subfiber* (X^0, h, Y) *is finite dimensional, invariant (i.e.* $\pi^t X^0 = X^0$ *for all* $t \in \mathbb{T}_+$*) and every trajectory of a dynamical system* $(X, \mathbb{T}_+, \pi)$ *belonging to* X^0 *is recurrent.*

c. *There exist two positive numbers* N *and* ν *such that* $|\varphi(t, u, \mathcal{B})| \leq N e^{-\nu t}|u|$ *for all* $(u, \mathcal{B}) \in X^s$ *and* $t \in \mathbb{T}_+$, *where* $\varphi(t, u, \mathcal{B}) = U(t, \mathcal{B})u$.

Proof. Assume that $\mathcal{B} \in H(\mathcal{A})$ and $\{t_n\} \subset \mathbb{R}$ are such that $\mathcal{B} = \lim_{n \to +\infty} \mathcal{A}_{t_n}$, then condition (3) of Theorem 4.6 implies that

$$\|U(t, \mathcal{A}_{t_n})U^{-1}(\tau, \mathcal{A}_{t_n})\| \leq N \tag{4.8}$$

for all $t \geq \tau$. Passing to the limit in (4.8) as $n \to +\infty$ and taking into account Lemma 4.1, we obtain that

$$\|U(t, \mathcal{B})U^{-1}(\tau, \mathcal{B})\| \leq N$$

for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A})$ and, consequently,

$$\|U(t, \mathcal{B})\| \leq N$$

for all $t \geq 0$ and $\mathcal{B} \in H(\mathcal{A})$. Now to finish the proof of Theorem 4.6 it is sufficiently to refer on Theorem 3.4.

Partial linear differential equations. Let Λ be some complete metric space of linear closed operators acting into Banach space E (for example $\Lambda = \{A_0 + B | B \in [E]\}$, where A_0 is a closed operator that acts on E). We assume that the following conditions are fulfilled for equation (4.1) and its $H-$ class (4.2):

a. for any $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (4.2) has exactly one mild solution $\varphi(t, v, \mathcal{B})$ (i.e. $\varphi(\cdot, v, \mathcal{B})$ is continuous, differentiable and stisfies of equation (4.2)) defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, v, \mathcal{B}) = v$;

b. the mapping $\varphi : (t, v, \mathcal{B}) \to \varphi(t, v, \mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$;

c. for every $t \in \mathbb{R}_+$ the mapping $U(t, \cdot) : H(\mathcal{A}) \to [E]$ is continuous, where $U(t, \cdot)$ is the Cauchy operator of equation (4.2), i.e. $U(t, \mathcal{B})v = \varphi(t, v, \mathcal{B})$ $(t \in \mathbb{R}_+, v \in E$ and $\mathcal{B} \in H(\mathcal{A})$).

Under the above assumptions the equation (4.1) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $X = E \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$. Applying the results from § 2 and § 3 to this system, we will obtain the analogous assertions for different classes of partial differential equations.

We will consider examples of partial differential equations which satisfy the above conditions a-c.

Example 4.7. A closed linear operator $\mathcal{A} : D(\mathcal{A}) \mapsto E$ with dense domain $D(\mathcal{A})$ is said [21] to be sectorial if one can find a $\varphi \in (0, \frac{\pi}{2})$, an $M \geq 1$, and a real number a such that the sector

$$S_{a,\varphi} = \{\lambda : \varphi \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ and $\|(\lambda I-\mathcal{A})^{-1}\| \le M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. An important class of sectorial operators is formed by elliptic operators [21,22].

Consider the differential equation

$$u' = (\mathcal{A}_1 + \mathcal{A}_2(t))u, \tag{4.9}$$

where $\mathcal{A}_1$ is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $\mathcal{A}_2 \in C(\mathbb{R},[E])$. The results of [21,23] imply that equation (4.9) satisfies conditions a.-c..

Example 4.8. Let $\mathcal{H}$ be a Hilbert space with a scalar product $\langle \cdot,\cdot\rangle = |\cdot|^2$, $\mathcal{D}(\mathbb{R}_+,\mathcal{H})$ be the set of all infinite differentiable, bounded functions on $\mathbb{R}_+$ with values into $\mathcal{H}$.

Denote by $(C(\mathbb{R},[\mathcal{H}]),\mathbb{R},\sigma)$ a dynamical system of shifts on $C(\mathbb{R},[\mathcal{H}])$. Consider the equation

$$\int_{\mathbb{R}_+}[< u(t),\varphi'(t) > + < \mathcal{A}(t)u(t),\varphi(t) >]dt = 0, \tag{4.10}$$

along with the family of equations

$$\int_{\mathbb{R}_+}[< u(t),\varphi'(t) > + < \mathcal{B}(t)u(t),\varphi(t) >]dt = 0, \tag{4.11}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau | \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = (t+\tau)$ and the bar denotes closure in $C(\mathbb{R},[\mathcal{H}])$.

The function $u \in C(\mathbb{R}_+,\mathcal{H})$ is called a solution of equation (4.10), if (4.10) takes place for all $\varphi \in \mathcal{D}(\mathbb{R}_+,\mathcal{H})$.

Assume that the operator $\mathcal{A}(t)$ is self-adjoint and negative definite, i.e., $\mathcal{A}(t) = -\mathcal{A}_1(t) + i\mathcal{A}_2(t)$ for all $t \in \mathbb{R}$, where $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ are self-adjoint and

$$\langle \mathcal{A}_1(t)u,u\rangle \ge \alpha |u|^2 \tag{4.12}$$

for all $t \in \mathbb{R}$ and $u \in \mathcal{H}$, where $\alpha > 0$. Let $(H(\mathcal{A}),\mathbb{R},\sigma)$ be a dynamical system of shifts on $H(\mathcal{A})$, $\varphi(t,v,\mathcal{B})$ be a solution of (4.11) with condition $\varphi(0,v,\mathcal{B}) = v$, $\overline{X} = \mathcal{H} \times H(\mathcal{A})$, X be a set of all the points $\langle u,\mathcal{B}\rangle \in \overline{X}$ such that through point $u \in \mathcal{H}$ passes a solution $\varphi(t,u,\mathcal{A})$ of equation (4.10) defined on $\mathbb{R}_+$. According to Lemma 2.21 in [12] the set X is closed in $\overline{X}$. In virtue of Lemma 2.22 in [12] the triple $(X,\mathbb{R}_+,\pi)$ is a dynamical system on X (where $\pi = (\varphi,\sigma)$) and $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a linear non-autonomous dynamical system, where $h = pr_2 : X \to Y = H(\mathcal{A})$. Applying the results from [1] it is possible to show that for every t the mapping $\mathcal{B} \mapsto U(t,\mathcal{B})$ (where $U(t,\mathcal{B})v = \varphi(t,v,\mathcal{B})$) from $H(\mathcal{A})$ into $[\mathcal{H}]$ is continuous and, consequently, for this system Theorem 2.2 is applicable. Thus the following assertion takes place.

Theorem 4.9. *Let $\mathcal{A} \in C(\mathbb{R},[\mathcal{H}])$ be compact, then the following assertion hold:*

1. *The trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t,\mathcal{B})\| \le Ne^{-\nu t}$ for all $t \ge 0$ and $\mathcal{B} \in H(\mathcal{A})$.*
2. $\lim_{t\to+\infty} \sup\{\|U(t,\mathcal{B})\| : \mathcal{B} \in H(\mathcal{A})\} = 0.$

Proof. According to Lemma 2.23 in [12] from (4.12) follows the condition 1. In virtue of Theorem 2.2 the conditions 1 and 2 are equivalent. The theorem is proved.

We will give the example of a boundary value problem reducing to an equation of type (4.10). Let Ω be a bounded domain in $\mathbb{R}^n$, Γ be the boundary of Ω, $Q = \mathbb{R}_+ \times \Omega$ and $S = \mathbb{R}_+ \times \Gamma$. Consider the first initial boundary value problem in Ω for the equation

$$\frac{\partial u}{\partial t} = L(t)u \quad u|_{t=0} = \varphi,\ u|_S = 0\,, \tag{4.13}$$

where

$$L(t)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial u}{\partial x_j}) - a(t,x)u$$

According to the Riesz theorem, the operator $\mathcal{A}(t)$ is defined by

$$\langle \mathcal{A}(t)u, \varphi\rangle = -\int_{\Omega} [\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a(t,x)u\varphi]dx.$$

If $a_{ij}(t,x) = a_{ji}(t,x)$ and the functions $a_{ij}(t,x)$ and $a(t,x)$ are bounded and uniformly continuous with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in \Omega$, then for equation (4.13) Theorem 2.2 is applicable for $\mathcal{H} = \dot{W}_2^1(\Omega)$.

Linear functional-differential equations.

Let r be a positive number, $C([a,b],\mathbb{R}^n)$ be the Banach space of continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with the supremum norm, and $\mathcal{C} = C([-r,0],\mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $\alpha \geq 0$ and $u \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma+\alpha]$ we define $u_t \in \mathcal{C}$ by $u_t(\theta) = u(t+\theta)$, with $-r \leq \theta \leq 0$. Denote by $\mathfrak{A} = \mathfrak{A}(C,\mathbb{R}^n)$ the Banach space of all linear continuous operators acting from $\mathcal{C}$ into $\mathbb{R}^n$ equipped with the operator norm. Consider the equation

$$u' = \mathcal{A}(t)u_t\,, \tag{4.14}$$

where $\mathcal{A} \in C(\mathbb{R},\mathfrak{A})$. We put $H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = \mathcal{A}(t+\tau)$, where the bar denotes closure in the topology of uniform convergence on every compact of $\mathbb{R}$.

Along with equation (4.14) we also consider the family of equations

$$u' = \mathcal{B}(t)u_t\,, \tag{4.15}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi_t(v,\mathcal{B})$ be a solution of equation (4.15) with condition $\varphi_0(v,\mathcal{B}) = v$ defined on $\mathbb{R}_+$. We put $Y = H(\mathcal{A})$ and denote by $(Y,\mathbb{R},\sigma)$ the dynamical system of shifts on $H(\mathcal{A})$. Let $X = \mathcal{C} \times Y$ and $\pi = (\varphi,\sigma)$ the dynamical system on X defined by the equality $\pi(\tau,(v,\mathcal{B})) = (\varphi_\tau(v,\mathcal{B}),\mathcal{B}_\tau)$. The non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ ($h = pr_2 : X \to Y$) is linear. The following assertion takes place.

Lemma 4.10[4]. *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R},\mathfrak{A})$, then the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4.14) is completely continuous.*

Theorem 4.11. *Let $H(\mathcal{A})$ be compact. Then the following assertions are equivalent:*

1. *For any $\mathcal{B} \in H(\mathcal{A})$ the zero solution of equation (4.15) is asymptotically stable, i.e. $\lim_{t\to+\infty} |\varphi_t(v,\mathcal{B})| = 0$ for all $v \in \mathcal{C}$ and $\mathcal{B} \in H(\mathcal{A})$*

2. *The zero solution of equation (4.14) is uniformly exponentially stable, i.e. there are positive numbers N and ν such that $|\varphi_t(v,\mathcal{B})| \leq Ne^{-\nu t}|v|$ for all $t \geq 0, v \in \mathcal{C}$ and $\mathcal{B} \in H(\mathcal{A})$.*

Proof. Let $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be the linear non-autonomous dynamical system, generated by equation (4.14). According to Lemma 4.10 this system is completely continuous and to finish the proof it is sufficient to refer to Theorem 2.5.

Consider the neutral functional differential equation

$$\frac{d}{dt}Dx_t = \mathcal{A}(t)x_t\,, \tag{4.16}$$

where $\mathcal{A} \in C(\mathbb{R},\mathfrak{A})$ and $D \in \mathfrak{A}$ is an operator nonatomic at zero [19, p.67]. As well as in the case of equation (4.14), the equation (4.16) generates a linear non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X = \mathcal{C}\times Y$, $Y = H(\mathcal{A})$ and $\pi = (\varphi,\sigma)$. The following statement takes place.

Lemma 4.12. *Let $H(\mathcal{A})$ be compact and the operator D is stable; i.e., the zero solution of the homogeneous difference equation $Dy_t = 0$ is uniformly asymptotically stable. Then the linear non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, generated by equation (4.16), is conditionally α-condensing.*

Proof. According to [20, p.119, formula (5.18)] the mapping $\varphi_t(\cdot,\mathcal{B}) : \mathcal{C} \to \mathcal{C}$ can be written as

$$\varphi_t(\cdot,\mathcal{B}) = S_t(\cdot) + U_t(\cdot,\mathcal{B})$$

for all $\mathcal{B} \in H(\mathcal{A})$, where $U_t(\cdot,\mathcal{B})$ is conditionally completely continuous for $t \geq r$. Also there exist positive constants N,ν such that $\|S_t\| \leq Ne^{-\nu t}(t \geq 0)$. Then the proof is completed by referring to Theorem 1.5.

Theorem 4.13. *Let $\mathcal{A} \in C(\mathbb{R},\mathfrak{A})$ be recurrent (i.e. $H(\mathcal{A})$ is compact minimal set in the dynamical system of shifts $(C(\mathbb{R},\mathfrak{A}),\mathbb{R},\sigma)$) and D is stable, then the following assertions are equivalent:*

1. *The zero solution of equation (4.14) and all equations*

$$\frac{d}{dt}Dx_t = \mathcal{B}(t)x_t\,, \tag{4.17}$$

where $\mathcal{B} \in H(\mathcal{A})$, is asymptotically stable, i.e. $\lim\limits_{t\to+\infty}|\varphi_t(v,\mathcal{B})| = 0$ for all $v \in \mathcal{C}$ and $\mathcal{B} \in H(\mathcal{A})$ ($\varphi_t(v,\mathcal{B})$ is the solution of equation (4.17) with condition $\varphi_0(v,\mathcal{B}) = v$).

2. *The zero solution of equation (4.16) is uniformly exponentially stable, i.e. there are positive numbers N and ν such that $|\varphi_t(v,\mathcal{B})| \leq Ne^{-\nu t}|v|$ for all $t \geq 0, v \in \mathcal{C}$ and $\mathcal{B} \in H(\mathcal{A})$.*
3. *All solutions of all equations (4.17) are bounded on $\mathbb{R}_+$ and they don't have non-trivial solutions bounded on $\mathbb{R}$.*

Proof. Let $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be the linear non-autonomous dynamical system, generated by equation (4.16). According to Lemma 4.12 this system is conditionally α condensing. To finish the proof of Theorem 4.13 it is sufficient to refer to Theorems 2.3 and 3.4. The theorem is proved.

Theorem 4.14. *Let $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ be recurrent (i.e. $H(\mathcal{A})$ is compact minimal in the dynamical system of shifts $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$), D is stable, and all solutions of all equations (4.17) are bounded on $\mathbb{R}_+$.*

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be the linear non-autonomous dynamical system generated by equation (4.16).

Then there are two positively invariant vector subfiberings (X^0, h, Y) and (X^s, h, Y) of (X, h, Y) such that:

a. *$X_y = X^0_y + X^s_y$ and $X^0_y \cap X^s_y = 0_y$ for all $y \in Y$, where $0_y = (0, y) \in X = E \times Y$ and 0 is the zero in the Banach space E.*

b. *The vector subfibering (X^0, h, Y) is finite dimensional, invariant (i.e.$\pi^t X^0 = X^0$ for all $t \in \mathbb{T}_+$) and every trajectory of a dynamical system $(X, \mathbb{T}_+, \pi)$ belonging to X^0 is recurrent.*

c. *There exist two positive numbers N and ν such that $|\varphi_t(v, \mathcal{B})| \leq N e^{-\nu t} |v|$ for all $(v, \mathcal{B}) \in X^s$, where $\varphi_t(v, \mathcal{B}) = U(t, \mathcal{B})v$ and $U(t, \mathcal{B})$ is the Cauchy operator of equation (4.17).*

Proof. Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be the linear non-autonomous dynamical system generated by equation (4.16). In virtue of Lemma 4.12 this non-autonomous dynamical system is conditionally α-condensing. According to Theorem 1.6 [5] there exists a positive number M such that $|\varphi_t(v, \mathcal{B})| \leq M|v|$ for all $t \geq 0, v \in \mathcal{C}$ and $\mathcal{B} \in H(\mathcal{A})$. To finish the proof of Theorem 4.14 it is sufficient to refer to Theorem 3.4.

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