An Analog of the Cameron–Johnson Theorem for Linear $\mathbb{C}$-Analytic Equations in Hilbert Space

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UDC 517.9

ABSTRACT. The well-known Cameron–Johnson theorem asserts that the equation $\dot{x} = \mathcal{A}(t)x$ with a recurrent (Bohr almost periodic) matrix $\mathcal{A}(t)$ can be reduced by a Lyapunov transformation to the equation $\dot{y} = \mathcal{B}(t)y$ with a skew-symmetric matrix $\mathcal{B}(t)$, provided that all solutions of the equation $\dot{x} = \mathcal{A}(t)x$ and of all its limit equations are bounded on the whole line. In the note, a generalization of this result to linear $\mathbb{C}$-analytic equations in a Hilbert space is presented.

KEY WORDS: Cameron–Johnson theorem, Hilbert space, linear $\mathbb{C}$-analytic differential equation, dynamical system, Lyapunov transformation, Bohr almost periodic matrix.

The well-known Cameron–Johnson Theorem [1, 2] asserts that the equation

$$\dot{x} = \mathcal{A}(t)x \tag{1}$$

with a recurrent (Bohr almost periodic) matrix $\mathcal{A}(t)$ can be reduced by a Lyapunov transformation to the equation $\dot{y} = \mathcal{B}(t)y$ with a skew-symmetric matrix $\mathcal{B}(t)$, provided that all solutions of (1) and of all its limit equations are bounded on the whole line. The goal of the present note is to generalize this result to linear $\mathbb{C}$-analytic equations in a Hilbert space.

We shall use the following notation: $\mathbb{R}$ ($\mathbb{C}$) is the set of all real (complex) numbers; $\mathbb{C}^m$ is the m-dimensional complex Euclidean space; G is a domain in $\mathbb{C}^m$; H is a real or complex Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|^2 = \langle \cdot, \cdot \rangle$; H_w is the space H endowed with the weak topology; $[H]$ (respectively, $[H_w]$) is the set of all continuous linear operators acting in H (respectively, in H_w), endowed with the operator norm (respectively, weak topology); $\mathcal{H}(G, [H])$ (respectively, $\mathcal{H}(G, \mathbb{C}^m)$, $\mathcal{H}(G, [H_w])$ is the set of all holomorphic functions $h\colon G \to [H]$ (respectively, $h\colon G \to \mathbb{C}^m$, $h\colon G \to [H_w]$), endowed with the open-compact topology.

Consider the system

$$\begin{cases} \dot{x} = \mathcal{A}(z)x\,; \\ \dot{z} = \Phi(z), \end{cases} \tag{2}$$

where $\Phi \in \mathcal{H}(G, \mathbb{C}^m)$ and $\mathcal{A} \in \mathcal{H}(G, [H])$. We shall assume that the second equation in (2) generates a dynamical system $(G, \mathbb{R}, \sigma)$ on G. By $U(t, z)$ denote the Cauchy operator of the equation

$$\dot{x} = \mathcal{A}(z \cdot t)x \qquad (z \in G), \tag{3}$$

where $z \cdot t = \sigma(z, t)$. General solutions of the differential equations (see, for example, [3–5]) imply that the operators $\{U(t, z) \mid t \in \mathbb{R}, z \in G\}$ satisfy the following conditions:

1) $U(0, z) = I$ for all $z \in G$, where I is the identity operator in H;
2) $U(t + \tau, z) = U(t, z \cdot \tau)U(\tau, z)$ for all t, $\tau \in \mathbb{R}$, and $z \in G$;
3) the map $U\colon \mathbb{R} \times G \to [H]$ $(U\colon (t, z) \to U(t, z))$ is continuous, and for each $t \in \mathbb{R}$, the map $U(t, \cdot)\colon G \to [H]$ is holomorphic.

The following theorem holds.

Translated from *Matematicheskie Zametki*, Vol. 68, No. 6, pp. 935–938, December, 2000.
Original article submitted May 5, 1997.

 0001–4346/2000/6856–0790$25.00

Theorem 1. *Suppose that there exists a constant $C > 0$ for which*

$$\|U(t, z)\| \le C \tag{4}$$

for all $t \in \mathbb{R}$ and $z \in G$. Then there exists a function $P \in \mathcal{H}(G, [H])$ such that

a) *the map P is biholomorphic, i.e., for each $z \in G$, the operator $P(z)$ is invertible and the map $P^{-1}\colon G \to [H]$ ($P^{-1}\colon z \to P^{-1}(z)$) is holomorphic;*
b) *$P^*(z) = P(z)$ for all $z \in G$, where $P^*(z)$ is the operator adjoint to $P(z)$;*
c) *for each $z \in G$, the operator $P(z)$ is positive definite;*
d) *the estimate $C^{-1}|x| \le |P(z)x| \le C|x|$ holds for all $z \in G$ and $x \in H$;*
e) *by the change of variables $x = P(z \cdot t)y$, Eq. (3) is reduced to the equation*

$$\dot{y} = \mathcal{B}(z \cdot t)y \tag{5}$$

with a skew-Hermitian operator $\mathcal{B} \in \mathcal{H}(G, [H])$, i.e., $\mathcal{B}^*(z) = -\mathcal{B}(z)$ for all $z \in G$.

Proof. For each $t \in \mathbb{R}$, we define the operator S^t by

$$(S^t f)(z) = U^*(t, z) f(z \cdot t) U(t, z)$$

for all $z \in G$ and $f \in \mathcal{H}(G, [H])$ (or $\mathcal{H}(G, [H_w])$). It is easy to check that the operators $\{S^t : t \in \mathbb{R}\}$ satisfy the following conditions:

4) the operator S^t maps $\mathcal{H}(G, [H])$ (or $\mathcal{H}(G, [H_w])$) into itself for each $t \in \mathbb{R}$;
5) $S^0 = I$, where I is the identity operator in $\mathcal{H}(G, [H])$ (or in $\mathcal{H}(G, [H_w])$);
6) $S^t S^\tau = S^{t+\tau}$ for all t, $\tau \in \mathbb{R}$;
7) the map S^t ($t \in \mathbb{R}$) is linear and continuous in the topology of $\mathcal{H}(G, [H])$ (or $\mathcal{H}(G, [H_w])$).

Conditions 4)–7) imply that $\{S^t \mid t \in \mathbb{R}\}$ is a commutative group of continuous linear operators acting in $\mathcal{H}(G, [H])$ (or $\mathcal{H}(G, [H_w])$).

We set $K = \{\mathcal{A} \in [H] \mid \|\mathcal{A}\| \le C^2\}$ and note that the set K is weakly closed. Since any bounded and closed set in H is weakly compact, it follows by the Tikhonov theorem that the set K is weakly compact. Let

$$\Gamma = \{\gamma \in \mathcal{H}(G, [H]) \mid \|\gamma(z)\| \le C^2\}.$$

The Cauchy integral formula [6, p. 339 of the Russian translation] and the Ascoli Theorem [6] imply that Γ is compact in $\mathcal{H}(G, [H_w])$. Let

$$V = \overline{\mathrm{conv}}\{S^t Q \mid t \in \mathbb{R}\},$$

where $Q(z) = I$ for all $z \in G$, I is the identity operator in H, and $\overline{\mathrm{conv}}$ is the weak closure of the convex hull of $\{S^t Q \mid t \in \mathbb{R}\}$. By the linearity of the operators $\{S^t \mid t \in \mathbb{R}\}$ and their continuity in the topology of $\mathcal{H}(G, [H_w])$, the space V is invariant, i.e., $S^t V \subseteq V$ for all $t \in \mathbb{R}$. Note that

$$\big(S^t Q(z)x, x\big) = \big(U^*(t, z)U(t, z)x, x\big) = |U(t, z)x|^2 \tag{6}$$

for all $z \in G$, $t \in \mathbb{R}$, and $x \in H$. It follows from (4) and (6) that

$$C^{-2}|x|^2 \le \big(S^t Q(z)x, x\big) \le C^2|x|^2, \tag{7}$$

and hence

$$C^{-2}|x|^2 \le \big(\mathcal{R}(z)x, x\big) \le C^2|x|^2 \tag{8}$$

for all $\mathcal{R} \in V$, $z \in G$, and $x \in H$. Besides, $\mathcal{R}^*(z) = \mathcal{R}(z)$ for all $z \in G$ and $\mathcal{R} \in V$. Inequality (8) implies that $\|\mathcal{R}(z)\| \le C^2$ for all $z \in G$, and hence $V \subseteq \Gamma$. By the Markov–Kakutani Theorem [7], the group $\{S^t\}$ has a common fixed point $\mathcal{R}$ in V, i.e., $U^*(t, z)\mathcal{R}(z \cdot t)U(t, z) = \mathcal{R}(z)$ for all $z \in G$ and $t \in \mathbb{R}$. By (8), the operator $\mathcal{R}(z)$ is positive definite, and since $\mathcal{R}^*(z) = \mathcal{R}(z)$, it follows by [8, Theorem 12.33] (see also [9, p. 65 of the Russian translation]) that there exists a unique invertible, self-adjoint, and positive definite operator $\mathcal{M}(z)$ such that $\mathcal{M}^2(z) = \mathcal{R}(z)$ for all $z \in G$. It follows from (8) that $C^{-1}|x| \le |\mathcal{M}(z)x| \le C|x|$ for all $z \in G$ and $x \in H$. Finally, the relation

$$\mathcal{R}^{\alpha}(z) = -\frac{1}{2\pi i}\oint_L \lambda^{\alpha}\big(\mathcal{R}(z) - \lambda \cdot I\big)^{-1}\, d\lambda$$

($\alpha = \pm 1/2$), where L is a simple contour encircling the spectrum of the operator $\mathcal{R}(z)$, implies the biholomorphicity of the operator $\mathcal{M}\colon G \to [H]$. Since $U^*(t, z)\mathcal{M}^2(z \cdot t)U(t, z) = \mathcal{M}^2(z)$, it follows that

$$\mathcal{M}(z \cdot t)U(t, z)\mathcal{M}^{-1}(z) = \mathcal{M}^{-1}(z \cdot t)U^{*^{-1}}(t, z)\mathcal{M}(z)$$

for all $t \in \mathbb{R}$ and $z \in G$. We set

$$\mathcal{V}(t, z) = \mathcal{M}(z \cdot t)U(t, z)\mathcal{M}^{-1}(z)$$

and note that the operators $\{\mathcal{V}(t, z) \mid t \in \mathbb{R}, z \in G\}$ satisfy conditions 1)–3) and, besides,

8) $\mathcal{V}^*(t, z) = \mathcal{V}^{-1}(t, z)$ for all $t \in \mathbb{R}$ and $z \in G$;

9) $U(t, z)P(z) = P(z \cdot t)\mathcal{V}(t, z)$ for all $t \in \mathbb{R}$ and $z \in G$, where $P(z) = \mathcal{M}^{-1}(z)$.

Let

$$\mathcal{B}(z) = \frac{d}{dt}\mathcal{V}(t, z)|_{t=0} = [\dot{\mathcal{M}}(z) + \mathcal{M}(z)\mathcal{A}(z)]P(z), \tag{9}$$

where

$$\dot{\mathcal{M}}(z) = \frac{d}{dt}\mathcal{M}(z \cdot t)|_{t=0} = \frac{d}{dt}\mathcal{M}(z)\Phi(z);$$

then $\mathcal{B} \in \mathcal{H}(G, [H])$ and

$$\frac{d}{dt}\mathcal{V}(t, z) = \mathcal{B}(z \cdot t)\mathcal{V}(t, z) \tag{10}$$

for all $t \in \mathbb{R}$ and $z \in G$. Thus $\mathcal{V}(t, z)$ is a Cauchy operator of (5). On the other hand, by condition 8), we have

$$\begin{cases} \dot{\mathcal{V}}^{*^{-1}}(t, z) = -\mathcal{B}^*(z \cdot t)\mathcal{V}^{*^{-1}}(t, z), \\ \mathcal{V}^{*^{-1}}(0, z) = I \end{cases} \tag{11}$$

for all $t \in \mathbb{R}$ and $z \in G$. Relations (10), (11) and condition 8) imply $\mathcal{B}^*(z) = -\mathcal{B}(z)$ for all $z \in G$. Finally, to conclude the proof of the theorem, it suffices to note that by condition 9), the change of variables $x = P(z \cdot t)y$ reduces Eq. (3) to (5). The theorem is completely proved. $\square$

Remark 1. If the point $z \in G$ is stationary (ω-periodic, quasiperiodic, etc.), then by [10] the operator-function $P(z \cdot t)$ is also stationary (ω-periodic, quasiperiodic, etc.).

Remark 2. Theorem 1 also holds for systems of difference equations of the form

$$\begin{cases} x(k+1) = \mathcal{A}(z \cdot k)x(k) \\ z(k+1) = \Phi\big(z(k)\big) \end{cases}, \qquad k \in \mathbb{Z},$$

where $\mathcal{A} \in \mathcal{H}(G, [H])$ and $\Phi \in \mathcal{H}(G, \mathbb{C}^m)$, and for systems of differential and difference equations with multidimensional time.

References

1. R. H. Cameron, "Almost periodic properties of bounded solutions of linear differential equations with almost periodic coefficients," *J. Math. Phys.*, **15**, 73–81 (1936).
2. R. A. Johnson, "On a Floquet theory for almost periodic, two-dimensional linear systems," *J. Differential Equations*, **37**, 184–205 (1980).
3. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York–Toronto–London (1955).
4. B. P. Demidovich, *Lectures on the Mathematical Theory of Stability* [in Russian], Nauka, Moscow (1967).
5. Yu. L. Daletskii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space* [in Russian], Nauka, Moscow (1970); English transl.: American Mathematical Society, Providence, R.I. (1974).
6. L. Schwartz, *Analyse mathématique. II*, Hermann, Paris (1967).
7. N. Bourbaki, *Éspaces vectoriels topologiques*, Ch. I, II, Hermann et Cie, Paris (1953); Ch. III–V, Hermann et Cie, Paris (1955).
8. W. Rudin, *Functional Analysis*, McGraw–Hill, New York, (1973).
9. P. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N.J. (1967).
10. B. A. Shcherbakov, *Poisson Stability of Motions of Dynamical Systems and of Solutions of Differential Equations* [in Russian], Shtiintsa, Kishinev (1985).

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