GLOBAL PULLBACK ATTRACTORS OF $\mathbb{C}$–ANALYTIC NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. This is a systematic study of global pullback attractors of $\mathbb{C}$–analytic cocycles. For the large class of $\mathbb{C}$–analytic cocycles we give the description of structure of their pullback attractors. Particularly we prove that it is trivial, i.e. the fibers of these attractors contain only one point. Several applications of these results are given (ODEs, Caratheodory's equations with almost periodic coefficients, almost periodic ODEs with impulse).

1. Introduction

$\lim^{-}$ $\lim_{-}$ One of the most studied classes of nonlinear ODEs is the class of $\mathbb{C}$ - analytic differential equations, i.e. the equations

$$\frac{dz}{dt} = f(t,z), \tag{1}$$

where the right hand side f is a holomorphic function with respect to complex variable $z \in \mathbb{C}^d$. Let $\phi(t,f,z)$ be a unique solution of equation (??) with initial condition $\phi(0,f,z) = z$ and be defined on $\mathbb{R}^+$. In virtue of fundamental theory of ODEs with holomorphic right hand side (see, for example [?] and [?]) the mapping ϕ possesses the following properties:

1. $\phi(0,f,z) = z$.

2. $\phi(t+\tau,f,z) = \phi(t,f_\tau,\phi(\tau,f,z))$ for every $t,\tau \in \mathbb{R}^+$ and $z \in \mathbb{C}^d$, where f_τ is a τ- translation of function f.

3. ϕ is continuous.

4. $\phi(t,f,\cdot) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for every t and f.

The properties 1.-4. will be the basis of our research of abstract $\mathbb{C}$–analytic nonautonomous dynamical system.

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The dissipative periodic equation (??)was studied by I.L.Zinchenko [?] and he proved that in this case the equation (??) admits a unique periodic globally uniformly asymptotically stable solution. This result was generalized for almost periodic equations (??) by D.N.Cheban [?] and [?]. He studied this problem within the framework of general $\mathbb{C}$- analytic nonautonomous dynamical systems.

In this paper we study the structure of global pullback attractors of general $\mathbb{C}$-analytic cocycles with noncompact base (in terminology of equation (??): the right hand side f is unbounded with respect to time $t \in \mathbb{R}$).

Our paper is organized as follows. In section 2 we recall some notions of dynamical systems and introduce the class of $\mathbb{C}-$ analytic cocycle, which is studied detailed in this paper.

In section 3 we establish some general facts about nonautonomous dynamical systems. We introduce the semigroup E^+_ω, E^-_ω and E_ω acting on the fiber X_ω of stratification (X, h, Ω). These semigroups are subsemigroups of Ellis semigroup (in the case of compact base Ω) and play an important role in the study of nonautonomous dynamical system.

Section 4 is devoted to positively uniformly stable cocycles. For this class of cocycles we prove that on every compact invariant set the corresponding cocycle can be prolonged uniquely in the negative direction.

In section 5 we study the structure of compact global pullback attractor of $\mathbb{C}-$ analytic cocycles with compact base. The main result in this section is Theorem ?? which states that for considered class of cocycles the pullback attractor $\{I_\omega | \omega \in \Omega\}$ is trivial, i.e. the section I_ω contains a single point.

Section 6 is devoted to study of the uniform dissipative cocycles with noncompact base. For this class of cocycles we prove the triviality of its global pullback attractor (see Theorem 6.5).

In section 7 we introduce the class of cocycles possessing the property of dissipativity (nonuniform) with noncompact base. The main result in this section is Theorem 7.7 which describes the structure of compact pullback attractor of mentioned class of cocycles. In particular its triviality is proved.

Section 8 is devoted to application of our general results, obtained in sections 3-7 to study of differential equations (ODEs, Caratheodory equations with almost periodic coefficients, almost periodic ODEs with impulse).

2. $\mathbb{C}$- ANALYTIC COCYCLES.

Let Ω be a complete metric space, let $\mathbb{T}$, the time set, be either $\mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}^+ = \{t \in \mathbb{T} | \quad t \geq 0\}$ $\quad (\mathbb{T}^- = \{t \in \mathbb{T} | \quad t \leq 0\})$, let $(\Omega, \mathbb{T}, \theta)$ be an autonomous two-sided dynamical system on Ω and E^d be a d-dimensional real $(\mathbb{R}^d)$ or complex $(\mathbb{C}^d)$ Euclidean space with the norm $|\cdot|$.

Definition 2.1. *(Cocycle on the state space E^d with the base $(\Omega, \mathbb{T}, \theta)$). The triplet $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$(or briefly ϕ) is said to be a cocycle (see, for example,* [?] *and* [?]*)*

on the state space E^d with the base $(\Omega, \mathbb{T}, \theta)$ if the mapping $\phi : \mathbb{T}^+ \times \Omega \times E^d \to E^d$ satisfies the following conditions:

i) $\phi(0, \omega, u) = u$ for all $u \in E^d$ and $\omega \in \Omega$.

ii) $\phi(t + \tau, \omega, u) = \phi(t, \theta_\tau \omega, \phi(\tau, \omega, u))$ for all $t, \tau \in \mathbb{T}^+, u \in E^d$ and $\omega \in \Omega$.

iii) the mapping ϕ is continuous.

Definition 2.2. *(Skew-product dynamical system). Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a cocycle on $E^d, X := E^d \times \Omega$ and π is a mapping from $\mathbb{T}^+ \times X$ to X defined by equality $\pi = (\phi, \theta)$, i.e. $\pi(t, (u, \omega)) = (\phi(t, \omega, u), \theta_t\omega)$ for all $t \in \mathbb{T}^+$ and $(u, \omega) \in E^d \times \Omega$, the triplet $(X, \mathbb{T}^+, \pi)$ is an autonomous dynamical system and it is called [?] a skew-product dynamical system.*

Definition 2.3. *(Nonautonomous dynamical system). Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two subsemigroup of group $\mathbb{T}, (X, \mathbb{T}_1, \pi)$ and $(\Omega, \mathbb{T}_2, \theta)$ are two autonomous dynamical systems and $h : X \to \Omega$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(\Omega, \mathbb{T}_2, \theta)$ (i.e. $h(\pi(t, x)) = \theta_t(h(x))$ for all $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle(X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \theta), h\rangle$ is called (see [?] and [?]) nonautonomous dynamical system. In this connection $(\Omega, \mathbb{T}_2, \theta)$ is called the factor of dynamical system $(X, \mathbb{T}_1, \pi)$ and $(X, \mathbb{T}_1, \pi)$ is called the extension of dynamical system $(\Omega, \mathbb{T}_2, \theta)$ (see, for example, [?]).*

Example 2.4. *(The nonautonomous dynamical system generated by cocycle ϕ). Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a cocycle, $(X, \mathbb{T}^+, \pi)$ be a skew-product dynamical system ($X = E^d \times \Omega, \pi = (\phi, \theta)$) and $h = pr_2 : X \to \Omega$, then the triplet $\langle(X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \theta), h\rangle$ is a nonautonomous dynamical system.*

Denote by $HC(\mathbb{C}^d \times \Omega, \mathbb{C}^d)$ the space of all the continuous functions $f : \mathbb{C}^d \times \Omega \to \mathbb{C}^d$ holomorphic in $z \in \mathbb{C}^d$ and equipped by compact-open topology. Consider the differential equation

$$\frac{dz}{dt} = f(z, \theta_t\omega), \quad (\omega \in \Omega) \tag{2}$$

where $f \in HC(\mathbb{C}^d \times \Omega, \mathbb{C}^d)$. Let $\phi(t, \omega, z)$ be the solution of equation (??) passing through the point z for $t = 0$ and defined on $\mathbb{R}^+$. The mapping $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{C}^d \to \mathbb{C}^d$ has the following properties (see, for example, [?] and [?]):

a) $\phi(0, \omega, z) = z$ for all $z \in \mathbb{C}^d$.

b) $\phi(t + \tau, \omega, z) = \phi(t, \theta_\tau\omega, \phi(\tau, \omega, z))$ for all $t, \tau \in \mathbb{R}^+, \omega \in \Omega$ and $z \in \mathbb{C}^d$.

c) the mapping ϕ is continuous.

d) the mapping $\phi(t, \omega) := \phi(t, \omega, \cdot) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for any $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Definition 2.5. *($\mathbb{C}$-analytic cocycle). The cocycle $\langle\mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is called [?],[?] $\mathbb{C}$-analytic if the mapping $\phi(t, \omega) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$.*

Example 2.6. *Let $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ be a dynamical system of translations on $HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ (Bebutov's dynamical system (see, for example, [?])). Denote by F the mapping from $\mathbb{C}^d \times HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ to $\mathbb{C}^d$ defined by equality $F(z, f) := f(0, z)$ for all $z \in \mathbb{C}^d$ and $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$. Let Ω be the hull $H(f)$ of a given function*

$f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$, that is $\Omega = H(f) := \overline{\{f_\tau | \tau \in \mathbb{R}\}}$, where $f_\tau(t,z) := f(t+\tau, z)$ for all $t, \tau \in \mathbb{R}$ and $z \in \mathbb{C}^d$. Denote the restriction of $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ on Ω by $(\Omega, \mathbb{R}, \sigma)$. Then, under appropriate restriction on the given function $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ defining Ω, the differential equation $\frac{dz}{dt} = f(t,z) = F(z, \sigma_t f)$ generates a $\mathbb{C}-$analytic cocycle.

Remark 2.7. *Analogously as above every difference equation with holomorphic right hand side generates a C-analytic cocycle with discrete time $\mathbb{Z}^+$.*

3. Some general facts about nonautonomous dynamical systems.

Definition 3.1. *(Poisson stability). The point $\omega \in \Omega$ is called (see, for example, [?] and [?]) positively (negatively) stable in the sense of Poisson if there exists a sequence $t_n \to +\infty (t_n \to -\infty$ respectively) such that $\theta_{t_n}\omega \to \omega$. If the point ω is Poisson stable in both directions, in this case it is called Poisson stable.*

Denote by $\mathfrak{N}_\omega = \{\{t_n\} | \theta_{t_n}\omega \to \omega\}$, $\mathfrak{N}_\omega^+ := \{\{t_n\} \in \mathfrak{N}_\omega | t_n \to +\infty\}$ and $\mathfrak{N}_\omega^- := \{\{t_n\} \in \mathfrak{N}_\omega | t_n \to -\infty\}$.

Definition 3.2. *(Conditional compactness). Let (X, h, Ω) be a fiber space, i.e. X and Ω be two metric spaces and $h : X \to \Omega$ be a homomorphism from X into Ω. The subset $M \subseteq X$ is said to be conditionally precompact, if the preimage $h^{-1}(\Omega') \bigcap M$ of every precompact subset $\Omega' \subseteq \Omega$ is a precompact subset of X, in particularly $M_\omega = h^{-1}(\omega) \bigcap M$ is precompact for every ω. The set M is called conditionally compact if it is closed and conditionally precompact.*

Example 3.3. *Let K be a compact space, $X := K \times \Omega$, $h = pr_2 : X \to \Omega$, then the triplet (X, h, Ω) be a fiber space, the space X is conditionally compact, but not compact.*

Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \theta), h \rangle$ be a nonautonomous dynamical system and $\omega \in \Omega$ be a positively Poisson stable point. Denote by

$$E_\omega^+ := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}_\omega^+ \quad \text{such that} \quad \pi^{t_n}|_{X_\omega} \to \xi\},$$

where $X_\omega := \{x \in X | \quad h(x) = \omega\}$ and $\to$ means the pointwise convergence.

Lemma 3.4. *Let $\omega \in \Omega$ be a positively Poisson stable point, $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \theta), h \rangle$ be a nonautonomous dynamical system and X be a conditionally compact space, then E_ω^+ is a nonempty compact subsemigroup of the semigroup $X_\omega^{X_\omega}$ (w.r.t. composition of mappings).*

Proof. Let $\{t_n\} \in \mathfrak{N}_\omega^+$, then $\theta_{t_n}\omega \to \omega$ and, consequently, the set

$$Q := \overline{\bigcup \{\pi^{t_n}(X_\omega) | n \in \mathbb{N}\}}$$

is compact, because X is conditionally compact. Thus $\{\pi^{t_n}|_{X_\omega}\} \subseteq Q^{X_\omega}$ and according to Tyhonov's Theorem this sequence is precompact. Let ξ be a limit point of $\{\pi^{t_n}|_{X_\omega}\}$, then $\xi \in E_\omega^+$ and, consequently, $E_\omega^+ \neq \emptyset$.

We note that $E_\omega^+ \subseteq X_\omega^{X_\omega}$ and, consequently, E_ω^+ is precompact. Let now $\xi_1, \xi_2 \in E_\omega^+$, we will prove that $\xi_1 \cdot \xi_2 \in E_\omega^+$. Since $\xi_1, \xi_2 \in E_\omega^+$, then there are two sequences

$\{t_n^i\} \in \mathfrak{N}_\omega^+ \quad (i=1,2)$ such that

$$\pi^{t_n^i}|_{X_\omega} \to \xi_i \quad (i=1,2).$$

Denote by $\xi := \xi_1 \cdot \xi_2 \in X_\omega^{X_\omega} \subseteq Q^{X_\omega}$, then we have $\pi^{t_n} \cdot \xi_2 \to \xi_1 \cdot \xi_2 = \xi$ as $n \to +\infty$. Let $U_\xi \subset X_\omega^{X_\omega}$ be an arbitrary open neighborhood of point ξ in $X_\omega^{X_\omega}$, then from relation (??) results that there exists a number $n_1(\xi) \in \mathbb{N}$ such that $\pi^{t_n} \cdot \xi_2 \in U_\xi$ for all $n \geq n_1(\xi)$. Now we fix $n \geq n_1(\xi)$, then there exist an open neighborhood $U_{\pi^{t_n^1} \cdot \xi_2} \subset U_\xi$ of point $\pi^{t_n^1} \cdot \xi_2 \in Q^{X_\omega}$ and a number $m_n \in \mathbb{N}$ such that

$$\pi^{t_n^1} \cdot \pi^{t_m^2}|_{X_\omega} \in U_{\pi^{t_n^1}} \cdot \xi_2$$

for any $n \geq n_1(\xi)$ and $m \geq m_n(\xi)$ and, consequently,

$$\pi^{t_n^1} \cdot \pi^{t_m^2}|_{X_\omega} \in U_\xi$$

for any $n \geq n_1(\xi)$ and $m \geq m_n(\xi)$. Thus from sequence $\{\pi^{t_n^1+t_m^2}|_{X_\omega}\}$ it is possible to extract a subsequence $\{\pi^{t_{n_k}^1+t_{m_k}^2}|_{X_\omega}\} \quad (t_{n_k}^1+t_{m_k}^2 \to +\infty)$ such that $\pi^{t_{n_k}^1+t_{m_k}^2}|_{X_\omega} \to \xi$ and, consequently, $\xi = \xi_1 \cdot \xi_2 \in E_\omega^+$. The Lemma is proved. □

Corollary 3.5. *Let $\omega \in \Omega$ be a negatively Poisson stable point, $\langle (X,\mathbb{T},\pi), (\Omega,\mathbb{T},\theta), h\rangle$ be a two-sided nonautonomous dynamical system and X be a conditionally compact space, then $E_\omega^- = \{\xi| \quad \exists \{t_n\} \in \mathfrak{N}_\omega^- \quad$ such that $\quad \pi^{t_n}|_{X_\omega} \to \xi\}$ is a nonempty compact subsemigroup of semigroup $X_\omega^{X_\omega}$.*

This assertion follows from Lemma ?? by change of time $t \to -t$.

Lemma 3.6. *Let $\omega \in \Omega$ be a two-sided Poisson stable point, $\langle (X,\mathbb{T},\pi), (\Omega,\mathbb{T},\theta), h\rangle$ be a two-sided nonautonomous dynamical system and X be a conditionally compact space, then $E_\omega = \{\xi| \quad \exists \{t_n\} \in \mathfrak{N}_\omega \quad$ such that $\quad \pi^{t_n}|_{X_\omega} \to \xi\}$ is a nonempty compact subsemigroup of the semigroup $X_\omega^{X_\omega}$.*

Proof. This assertion can be proved using the same type of arguments as well as in the proof of Lemma ?? and therefore we omit the details. □

Corollary 3.7. *Under the conditions of Lemma ?? E_ω^+ and E_ω^- are two nonempty subsemigroups of the semigroup E_ω.*

Lemma 3.8. *Under the conditions of Lemma ?? the following assertions hold:*

1. *if $\xi_1 \in E_\omega^-$ and $\xi_2 \in E_\omega^+$, then $\xi_1 \cdot \xi_2 \in E_\omega^- \bigcap E_\omega^+$.*
2. *$E_\omega^- \bigcap E_\omega^+$ is a subsemigroup of the semigroup E_ω^-, E_ω^+ and E_ω.*
3. *$E_\omega^- \cdot E_\omega \subseteq E_\omega^-$ and $E_\omega^+ \cdot E_\omega \subseteq E_\omega^+$, where $A_1 \cdot A_2 := \{\xi_1 \cdot \xi_2 | \xi_i \in A_i \quad (i=1,2)\}$ and $A_i \subseteq E_\omega$.*
4. *if at least one of the subsemigroups E_ω^- or E_ω^+ is a group, then $E_\omega^- = E_\omega^+ = E_\omega$.*

Proof. Let $\xi_1 \in E_\omega^-$ and $\xi_2 \in E_\omega^+$, then there are $t_n^1 \to -\infty$ and $t_n^2 \to +\infty$ such that $\theta_{t_n^i}\omega \to \omega$ and $\pi^{t_n^i}|_{X_\omega} \to \xi_i (i=1,2)$. Using the same type arguments as well as in the proof of Lemma ?? we may choose the subsequence $\{t_{n_k}^1+t_{m_k}^2\} \subset \{t_n^1+t_m^2\}$ with the following properties: a) $t_{n_k}^1+t_{m_k}^2 \geq k$ or $t_{n_k}^1+t_{m_k}^2 \leq -k$ and b) $\pi^{t_{n_k}^1+t_{m_k}^2} \to \xi_1 \cdot \xi_2$, i.e. $\xi_1 \cdot \xi_2 \in E_\omega^+ \bigcap E_\omega^-$.

The second statement follows from the first one.

Let $\xi_1 \in E^+_\omega \quad (E^-_\omega, \quad$ respectively) and $\xi_2 \in E_\omega$, then there exist two sequences $t^1_n \to +\infty$ (or $-\infty$, respectively) and t^2_n such that $\pi^{t^i_n} \to \xi_i \quad (i = 1, 2)$. Then we may choose a subsequence $\{t^1_{n_k} + t^2_{m_k}\}$ with the following properties:

$$a) t^1_{n_k} + t^2_{m_k} \geq k \quad (\leq -k, \quad \text{respectively}) \quad \text{and} \quad b) \pi^{t^1_{n_k} + t^2_{m_k}} \to \xi_1 \cdot \xi_2,$$

and consequently, $\xi_1 \cdot \xi_2 \in E^+_\omega \quad (E^-_\omega$, respectively).

Finally, let E^-_ω be a subgroup of the semigroup E_ω. According to the third statement of Lemma ?? $E^-_\omega \cdot E_\omega \subseteq E^-_\omega$. Since E^-_ω is a nonempty compact invariant set w.r.t. E_ω, then in E^-_ω exists a compact minimal subset $I \subset E^-_\omega$, i.e. $I \neq \emptyset$, compact and $u \cdot E_\omega = I$ for every $u \in I$. Let now $u \in I$ be an idempotent element of right ideal I of semigroup E_ω, then u is an unit element $(u(x) = x \quad \forall x \in X_\omega)$ of I because $I \subseteq E^-_\omega$ and E^-_ω according to conditions of Lemma ?? is a subgroup of the semigroup E_ω. Thus we have $E_\omega = u \cdot E_\omega = I \subseteq E^-_\omega$ and, consequently, $E^-_\omega = E_\omega$. Analogously $E^+_\omega = E_\omega$. The theorem is proved. □

Lemma 3.9. *Let $\omega \in \Omega$ be a two-sided Poisson stable point, $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \theta), h \rangle$ be a two-sided nonautonomous dynamical system and X be a conditionally compact space and*

$$\inf_{n \in \mathbb{N}} \rho(x_1 t_n, x_2 t_n) > 0 \tag{3}$$

for all $\{t_n\} \in \mathfrak{N}^-_\omega$ and $x_1, x_2 \in X_\omega \quad (x_1 \neq x_2)$, then E^-_ω is a subgroup of the semigroup E_ω.

Proof. Indeed, if $u \in E^-_\omega$ is an arbitrary idempotent element of E^-_ω, then $u^2 = u$ and there exists a sequence $\{t_n\} \in \mathfrak{N}^-_\omega$ such that $\pi^{t_n} \to u$. According to (??) we have $u(x_1) \neq u(x_2)$ for all $x_1 \neq x_2 \quad (x_1, x_2 \in X_\omega)$. On the other hand $u^2(x) = u(x)$ for all $x \in X_\omega$ and, consequently, $u(x) = x$ for all $x \in X_\omega$. Thus every idempotent of semigroup E^-_ω is an unit element of E_ω (in particular E^-_ω) and, consequently, E^-_ω is a group (see, for example [?]). □

Lemma 3.10. *Let $\omega \in \Omega$ be a two-sided Poisson stable point, $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \theta), h \rangle$ be a two-sided nonautonomous dynamical system and X be a conditionally compact space and the condition (??) holds for all $\{t_n\} \in \mathfrak{N}^-_\omega$ and $x_1, x_2 \in X_\omega (x_1 \neq x_2)$, then inequality (??) is fulfilled for any $\{t_n\} \in \mathfrak{N}^+_\omega$ and $x_1, x_2 \in X_\omega (x_1 \neq x_2)$.*

Proof. According to Lemma ?? under the conditions of Lemma ?? we have $E^-_\omega = E^+_\omega = E_\omega$ and E_ω is a group. Suppose that for some sequence $\{t_n\} \in \mathfrak{N}^+_\omega$

$$\inf_{n \in \mathbb{N}} \rho(x_1 t_n, x_2 t_n) = 0. \tag{4}$$

Since $\theta_{t_n} \omega \to \omega$ and the space X is conditionally compact, the sequence $\{\pi^{t_n}|_{X_\omega}\}$ is precompact in Q^{X_ω}, where $Q := \overline{\bigcup \{\pi^{t_n}(X_\omega) | n \in \mathbb{N}\}}$ and, consequently, we may assume that it is convergent. Let $\xi = \lim\limits_{n \to +\infty} \pi^{t_n}|_{X_\omega}$, then from equality (??) results that $\xi(x_1) = \xi(x_2) \quad (x_1 \neq x_2)$, but $\xi \in E_\omega$ and E_ω is a group and, consequently, ξ is a one-to-one mapping. The obtained contradiction proves our assertion. □

Definition 3.11. *(Entire trajectory) Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ $((X, \mathbb{T}^+, \pi))$ be a cocycle (onesided dynamical system, respectively). The continuous mapping $\nu : \mathbb{T} \to E^d$ $(\gamma : \mathbb{T} \to X$, respectively) is called an entire trajectory of cocycle ϕ (of dynamical system $(X, \mathbb{T}^+, \pi)$) passing through point $(\omega, u) \in \Omega \times E^d$ $(x \in X)$ for $t = 0$ if $\phi(t, \theta_s\omega, \nu(s)) = \nu(t+s)$ and $\nu(0) = u$ $(\pi^t\gamma(s) = \gamma(t+s)$ and $\gamma(0) = x$, respectively) for all $t \in \mathbb{T}^+$and $s \in \mathbb{T}$.*

Lemma 3.12. *Let $\omega \in \Omega$ be a positively Poisson stable point, $\langle(X, \mathbb{T}, (\Omega, \mathbb{T}, \theta), h\rangle$ be a nonautonomous dynamical system, generated by cocycle ϕ (see example ??), $pr_1(\bigcup\limits_{t\ge 0} \pi^t X_\omega)$ be precompact and*

$$A_\omega(X_\omega) := (\bigcap_{t\ge 0}\overline{\bigcup \pi^\tau X_\omega})\bigcap X_\omega,$$

then for any $x \in A_\omega(X_\omega)$ there exists an entire trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$ passing through point x for $t = 0$ and $pr_1(\gamma(\mathbb{T}))$ $(\gamma(\mathbb{T}) := \{\gamma(t) | t \in \mathbb{T}\})$ is precompact.

Proof. Let $A_\omega(X_\omega)$, then there are $\{t_n\} \in \mathfrak{N}_\omega$ and $x_n \in X_\omega$ such that $x = \lim\limits_{n\to\infty} \pi^{t_n}x_n, \theta_{t_n}\omega \to \omega$ and $t_n \to +\infty$. We consider the sequence $\{\gamma_n\} \subset C(\mathbb{T}, M)$, where $M := \overline{\bigcup\limits_{t\ge 0} \pi^t X_\omega}$, defined by equality

$$\gamma_n(t) = \pi^{t+t_n}x_n, \quad \text{if} \quad t \ge -t_n \quad \text{and} \quad \gamma_n(t) = x_n \quad \text{for} \quad t \le t_n.$$

Now we will prove that the sequence $\{\gamma_n\}$ is equicontinuous on every segment $[-l, l] \subset \mathbb{T}$. If we suppose that it is not true, then there exist $\varepsilon_0, l_0 > 0, t^i_n \in [-l_0, l_0]$ and $\delta_n \to 0$ $(\delta_n > 0)$ such that

$$|t^1_n - t^2_n| \le \delta_n \quad \text{and} \quad \rho(\gamma_n(t^1_n), \gamma_n(t^2_n)) \ge \varepsilon_0. \tag{5}$$

We may suppose that $t^i_n \to t_0$ $(i = 1, 2)$. From (??) we obtain

$$\varepsilon_0 \le \rho(\gamma_n(t^1_n), \gamma_n(t^2_n)) = \rho(\pi^{t^1_n+l_0}(\pi^{t_n-l_0}x_n), \pi^{t^2_n+l_0}(\pi^{t_n-l_0}x_n)) \tag{6}$$

for sufficiently large n $(t_n \ge l_0)$. Note that the sequence $\{\pi^{t_n-l_0}x_n\}$ is precompact and $h(\pi^{t_n-l_0}x_n) = \theta_{t_n-l_0}h(x_n) = \theta_{t_n-l_0}\omega \to \theta_{-l_0}\omega$. Let $\bar{x} = \lim\limits_{n\to\infty} \pi^{t_n-l_0}x_n$, then passing to limit in the inequality (??) we obtain $\varepsilon_0 \le 0$. The obtained contradiction proves our assertion.

Now taking into account the conditional compactness of set K we can affirm that $\{\gamma_n\}$ is a precompact sequence of $C(\mathbb{T}, M)$. Let γ be a limit point of sequence $\{\gamma_n\}$, then there exists a subsequence $\{\gamma_{k_n}\}$ such that $\gamma(t) = \lim\limits_{n\to\infty} \gamma_{k_n}(t)$ uniformly on every segment $[-l, l] \subset \mathbb{T}$. In particular $\gamma \in C(\mathbb{T}, M)$. We note that $\pi^t\gamma(s) = \lim\limits_{n\to\infty} \pi^t\gamma_{k_n}(s) = \lim\limits_{n\to\infty} \gamma_{k_n}(s + t) = \gamma(s + t)$ for all $t \in \mathbb{T}^+$ and $s \in \mathbb{T}$. Finally, we see that $\gamma(0) = \lim\limits_{n\to\infty} \gamma_{k_n}(0) = \lim\limits_{n\to\infty} \pi^{t_{k_n}}x_{k_n} = x$, i.e. γ is an entire trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$ passing through point x. The Lemma is completely proved. □

4. Positively uniformly stable cocycles

Let E^d be a $d-$dimensional real ($\mathbb{R}^d$) or complex ($\mathbb{C}^d$) Euclidean space with the norm $|\cdot|, \rho$ be the distance generated by this norm, Ω be a metric space and the triplet $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a cocycle on the state space E^d.

Definition 4.1. *(Compact global pullback attractor). The family of compact sets $\{I_\omega | \omega \in \Omega\}$ $(I_\omega \subset E^d$ is nonempty compact for every $\omega \in \Omega)$ is called (see, for example [?]) the compact global pullback attractor of cocycle ϕ if the following conditions are fulfilled:*

1. The set $I := \bigcup\{I_\omega | \quad \omega \in \Omega\}$ is precompact.

2. $\{I_\omega | \quad \omega \in \Omega\}$ is invariant w.r.t. the cocycle ϕ, i.e. $\phi(t,\omega,I_\omega) = I_{\theta_t\omega}$ for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$.

3. The equality $\lim\limits_{t\to+\infty} \beta(\phi(t,\theta_{-t}\omega)K, I_\omega) = 0$ holds for every nonempty compact $K \subset E^d$ and $\omega \in \Omega$, where $\beta(A,B) := \sup\limits_{a\in A} \rho(a,B)$ is the semi distance of Hausdorff.

Remark 4.2. *If $\{I_\omega | \quad \omega \in \Omega\}$ is a compact global pullback attractor, then the set $J := \bigcup\{J_\omega | \quad \omega \in \Omega\}$, where $J_\omega := I_\omega \times \omega$, is the maximal conditionally compact invariant set of skew-product system $(X, \mathbb{T}^+, \pi)$ and, consequently the for the given cocycle ϕ there exists at most one compact global pullback attractor.*

Remark 4.3. *It is clear that ν is an entire trajectory of a cocycle ϕ passing through point (ω, u) if and only if $\gamma = (\nu, Id_\Omega)$ is an entire trajectory of the skew-product dynamical system, passing through the point $x = (\omega, u)$.*

Definition 4.4. *(Positively uniformly stable cocycles) The cocycle ϕ is called positively uniformly stable on the family of compact sets $K := \{K_\omega | \omega \in \Omega\}$ $(K_\omega \subset E^d)$ if for arbitrary $\varepsilon > 0$ there exists a $\delta(\varepsilon, K) > 0$ such that $|u_1 - u_2| < \delta$ implies $|\phi(t,\omega,u_1) - \phi(t,\omega,u_2)| < \varepsilon$ for all $t \geq 0, \omega \in \Omega$ and $u_1, u_2 \in K_\omega$. The cocycle ϕ is called positively uniformly stable if it is positively uniformly stable on every family of compact sets from E^d.*

Theorem 4.5. *Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a cocycle with the following properties:*

1) It admits a conditionally precompact invariant set $\{I_\omega | \quad \omega \in \Omega\}$ (i.e. $\bigcup\{I_\omega | \omega \in \Omega'\}$ is precompact subset of E^d for any precompact subset Ω' of Ω).

2) The cocycle ϕ is positively uniformly stable on $\{I_\omega | \quad \omega \in \Omega\}$.

Then all motions on $J := \bigcup\{J_\omega | \quad \omega \in \Omega\}$ $(J_\omega := I_\omega \times \{\omega\})$ may be continued uniquely to the left and define on J a two-sided dynamical system $(J, \mathbb{T}, \pi)$, i.e. the skew-product system $(X, \mathbb{T}^+, \pi)$ generates on J a two-sided dynamical system $(J, \mathbb{T}, \pi)$.

Proof. First step: we will prove that the set $J \subset X$ is distal in the negative direction w.r.t. the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \theta), h\rangle$ (see example ??), i.e. for all $\omega \in \Omega$ and $u_1, u_2 \in I_\omega (u_1 \neq u_2)$ the following inequality holds

$$\inf_{t\leq 0} \rho(\gamma_1(t), \gamma_2(t)) > 0 \tag{7}$$

for all $\gamma_i \in \Phi_{(\omega,u_i)} (i = 1,2)$, where by $\Phi_{(\omega,u)}$ it is denoted the family of all the entire trajectories of $(X, \mathbb{T}^+, \pi)$ passing through point (ω, u) and belonging to J. If it is not true, then there exist $\omega_0 \in \Omega, u_i^0 \in I_{\omega_0} \quad (u_1^0 \neq u_2^0), \gamma_i^0 \in \Phi_{(\omega_0,u_i^0)} (i = 1,2)$ and $-t_n \to -\infty$ such that

$$\rho(\gamma_1^0(-t_n), \gamma_2^0(-t_n)) \to 0 \tag{8}$$

as $n \to \infty$. Let $\varepsilon := \rho(u_1^0, u_2^0) > 0$ and $\delta = \delta(\varepsilon) > 0$ be chosen from positively uniformly stability of cocycle ϕ on family of compact subsets $\{I_\omega | \omega \in \Omega\}$, then for sufficiently large n from (??) we have $\rho(\gamma_1^0(-t_n), \gamma_2^0(-t_n)) < \delta$ and, consequently, $\varepsilon = \rho(u_1^0, u_2^0) = \rho(\pi^{t_n} \gamma_1^0(-t_n), \pi^{t_n} \gamma_2^0(-t_n)) < \varepsilon$. The obtained contradiction proves our assertion.

Second step: we will prove that for any $\omega \in \Omega$ and $u \in I_\omega$ the set $\Phi_{(\omega,u)}$ contains only one entire trajectory of $(X, \mathbb{T}^+, \pi)$ belonging to J. Let $\Phi := \bigcup \{\Phi_{(\omega,u)} | (\omega, u) \in J\} \subset C(\mathbb{T}, X)$, where $C(\mathbb{T}, X)$ is a space of all the continuous functions $f : \mathbb{T} \to X$ equipped with compact-open topology and $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$ is Bebutov's dynamical system (dynamical system of translations (see, for example, [?, ?])). It is easy to verify that Φ is a closed and invariant subset of dynamical system $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$ and, consequently, induces on the set Φ the dynamical system $(\Phi, \mathbb{T}, \sigma)$. Let H be a mapping from Φ into Ω, defined by equality $H(\gamma) := h(\gamma(0))$, then it is possible to verify (see [?]) that the triplet $\langle (\Phi, \mathbb{T}, \sigma), (\Omega, \mathbb{T}, \theta), H \rangle$ is a nonautonomous dynamical system. Now we will show that this nonautonomous dynamical system is distal on the negative direction, i.e.

$$\inf_{t \leq 0} \rho(\gamma_1^t, \gamma_2^t) > 0$$

for all $\gamma_1, \gamma_2 \in \Phi_\omega \quad (\gamma_1 \neq \gamma_2)$ and $\omega \in \Omega$. Indeed, otherwise there exist $\omega_0, \gamma_1, \gamma_2 \in \Phi_{\omega_0} (\gamma_1 \neq \gamma_2)$ and $t_n \to +\infty$ such that $\rho(\gamma_1^{-t_n}, \gamma_2^{-t_n}) \to 0$ (where $\gamma^\tau := \sigma(\gamma, \tau)$, i.e. $\gamma^\tau(s) := \gamma(\tau + t)$ for all $s \in \mathbb{T}$) as $n \to \infty$ and, consequently,

$$|\gamma_1(-t_n) - \gamma_2(-t_n)| \leq \rho(\gamma_1^{-t_n}, \gamma_2^{-t_n}) \to 0. \tag{9}$$

Since $\gamma_1 \neq \gamma_2$, then there exists $t_0 \in \mathbb{T}$ such that $\gamma_1(t_0) \neq \gamma_2(t_0)$. Let $\tilde{\gamma}_i(t) := \gamma_i(t + t_0)$ for all $t \in \mathbb{T}$, then $\tilde{\gamma}_i \in \Phi_{\omega_0}$ and from inequality (??) we have

$$|\tilde{\gamma}_1(-t_n) - \tilde{\gamma}_2(-t_n)| \to 0. \tag{10}$$

as $n \to \infty, -t_n - t_0 \to -\infty$. Thus we found $\omega_0 := h(\gamma_i(t_0))$ and $u_i := pr_1 \gamma_i(t_0) \quad (i = 1,2), u_1, u_2 \in I_{\omega_0} \quad (u_1 \neq u_2)$ and the entire trajectories $\tilde{\gamma}_i \in \Phi_{(\omega,u_i)} (i = 1,2)$ such that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are proximal (see (??)). But (??) and (??) are contradictory. Thus the negative distality of the nonautonomous dynamical system $\langle (\Phi, \mathbb{T}, \sigma), (\Omega, \mathbb{T}, \theta), H \rangle$ is proved.

Now we can prove that for any $\omega \in \Omega$ and $u \in I_\omega$ the set $\Phi_{(\omega,u)}$ contains a unique entire trajectory. In fact, if it is not true, then there exists $(\omega_0, u_0) \in \Omega \times E^d$ and two different trajectories $\gamma_1, \gamma_2 \in \Phi_{(\omega_0,u_0)} (\gamma_1 \neq \gamma_2)$. In virtue of above γ_1 and γ_2 are negatively distal with respect to $\langle (\Phi, \mathbb{T}, \sigma), (\Omega, \mathbb{T}, \theta), H \rangle$, i.e.

$$\alpha(\gamma_1, \gamma_2) := \inf_{t \leq 0} \rho(\gamma_1^t, \gamma_2^t) > 0$$

and, consequently, $\rho(\gamma_1(t), \gamma_2(t)) \geq \alpha(\gamma_1, \gamma_2) > 0$ for all $t \geq 0$. In particular $\gamma_1(0) \neq \gamma_2(0)$. The obtained contradiction proves our statement.

Third step: let now $\tilde{\pi}$ be a mapping from $\mathbb{T} \times J$ into J defined by equality

$$\tilde{\pi}(t,x) = \pi(t,x) \quad \text{if} \quad t \le 0 \quad \text{and} \quad \gamma_x(t) \quad \text{if} \quad t < 0$$

for all $x \in J$, where γ_x is a unique entire trajectory of the dynamical system $(X, \mathbb{T}^+, \pi)$ passing through point x and belonging to J. To prove that $(J, \mathbb{T}, \tilde{\pi})$ is a two-sided dynamical system on J it is sufficient to verify the continuity of the mapping $\tilde{\pi}$. Let $x \in J, \quad t \in \mathbb{T}^-, x_n \to x$ and $t_n \to t$, then there is a $l_0 > 0$ such that $t_n \in [-l_0, l_0]$ and, consequently,

$$\rho(\tilde{\pi}(t_n, x_n), \tilde{\pi}(t,x)) = \rho(\pi^{t_n+l_0}\gamma_{x_n}(-l_0), \pi^{t+l_0}\gamma_x(-l_0)) \le \tag{11}$$
$$\rho(\pi^{t_n+l_0}\gamma_{x_n}(-l_0), \pi^{t_n+l_0}\gamma_x(-l_0)) + \rho(\pi^{t_n+l_0}\gamma_x(-l_0), \pi^{t+l_0}\gamma_x(-l_0)).$$

Reasoning as in the proof of Lemma **??** it is possible to establish that the sequence $\{\gamma_{x_n}\}$ is precompact in $C(\mathbb{T}, J)$ and that every limit point of this sequence $\gamma \in \Phi$ and $\gamma(0) = x$. Taking into account the result of the second step we claim that $\gamma_{x_n} \to \gamma_x$ uniformly on every segment $[-l,l] \subset \mathbb{T}(l > 0)$. In particular, $\gamma_{x_n}(-l_0) \to \gamma_x(-l_0)$. Passing now to limit in inequality (**??**) when $n \to \infty$ we obtain the continuity of mapping $\tilde{\pi}$ in the point (t,x). The theorem is completely proved. □

Remark 4.6. *Theorem* **??** *is true and in the case if we replace the condition 2) by the following: 2.1) for arbitrary $\varepsilon > 0$ there exist two positive numbers $\delta(\varepsilon)$ and $L(\varepsilon)$ such that*

$$\rho(\phi(t,\omega,u_1), \phi(t,\omega,u_2)) < \varepsilon \tag{12}$$

for all $\omega \in \Omega, t \ge L(\varepsilon)$ and $u_1, u_2 \in I_\omega$ with condition $\rho(u_1,u_2) < \delta$.

5. The compact global pullback attractors of $\mathbb{C}-$analytic cocycles with compact base

In this section we suppose that $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is a $\mathbb{C}-$analytic cocycle and Ω is a compact space.

Theorem 5.1. *Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a $\mathbb{C}-$analytic cocycle admitting a compact global pullback attractor $\{I_\omega | \omega \in \Omega\}$, then:*

1. The compact invariant set $J = \bigcup\{J_\omega | \omega \in \Omega\}$ of the skew-product dynamical system $(X, \mathbb{T}^+, \pi) \quad (X := \mathbb{C}^d \times \Omega, \ \pi := (\phi, \theta))$ is asymptotically stable.

2. There exists a positive number δ_0 such that the cocycle ϕ is positively uniformly stable on the compact set $\mathfrak{B}[I,\delta] := \bigcup\{B[I_\omega,\delta] | \quad \omega \in \Omega\}$, where $B[I_\omega, \delta] := \{z \in \mathbb{C}^d | \quad \rho(z, I_\omega) \le \delta\}$, for all $0 < \delta < \delta_0$.

3. The skew-product dynamical system $(X, \mathbb{T}^+, \pi)$ generates on J a group dynamical system $(J, \mathbb{T}, \pi)$.

Proof. Denote by $X = \mathbb{C}^d \times \Omega$ and by $(X, \mathbb{T}^+, \pi)$ the skew-product dynamical system. Then under the conditions of the theorem the set $J = \bigcup\{J_\omega | \omega \in \Omega\}$ is a nonempty compact invariant set and according to Theorem 4.1 [?] is asymptotically stable with respect to $(X, \mathbb{T}^+, \pi)$. In particular there exists a $\delta_0 > 0$ such that the set $B[J, \delta_0] := \{x \in X | \rho(x, J) \le \delta_0\}$ is positively invariant. Since Ω is compact and $\pi^t(u,\omega) = (\phi(t,\omega,u), \theta_t\omega)$, then there exists a positive number $C = C(\delta_0)$ such that $|\phi(t,\omega,u)| \le C$ for all $\omega \in \Omega$ and $u \in B[I_\omega, \delta_0]$. Taking into account the connectedness of set I_ω (see, for example [?]) according to Cauchy's Theorem for all $\delta < \delta_0$ there exists a positive number $L(\delta)$ such that

$$|\phi(t,\omega,u_1) - \phi(t,\omega,u_2)| \le L(\delta)|u_1 - u_2| \tag{13}$$

for all $\omega \in \Omega, t \in \mathbb{R}^+$ and $u_1, u_2 \in B[I_\omega, \delta]$. It is easy to see that from inequality (??) results the positively uniformly stability of set $B[I,\delta]$ for every $0 < \delta < \delta_0$. Particularly the set $I := \bigcup\{I_\omega | \omega \in \Omega\}$ will be positively uniformly stable and to finish the proof of Theorem it is sufficiently to apply Theorem ?? to our situation for the skew-product system $(X, \mathbb{T}^+, \pi)$. The Theorem is completely proved. □

Definition 5.2. *(Linear cocycle) The cocycle $\langle \mathbb{E}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is called linear (see, for example, [?],[?] and [?]) if the mapping $\phi(t,\omega) : E^d \to E^d$ is linear for every $t \in \mathbb{T}^+$ and $\omega \in \Omega$.*

Theorem 5.3. *([?],[?] and [?]) Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a linear cocycle, then the following conditions are equivalent:*

1. $\lim\limits_{t\to+\infty} |\phi(t,\omega,u)| = 0$ *for all $u \in E^d$ and $\omega \in \Omega$.*
2. *There exist positive numbers N, ν such that $|\phi(t,\omega,u)| \le N \exp(-\nu t)|u|$ for all $t \in \mathbb{T}^+, \omega \in \Omega$ and $u \in E^d$.*

Theorem 5.4. *Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a $\mathbb{C}-$analytic cocycle admitting a compact pullback attractor $\{I_\omega | \quad \omega \in \Omega\}$, and let every point $\omega \in \Omega$ be positively Poisson stable. Then the following assertions hold:*

1. *For every $\omega \in \Omega$ the set I_ω consists of a unique point $\nu(\omega)$.*
2. *$\nu(\theta_t\omega) = \phi(t,\omega,\nu(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.*
3. *The mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma := (\nu, Id_\Omega)$.*
4. *Every point $\gamma(\omega)$ is positively Poisson stable.*
5. *The continuous invariant section ν is uniformly asymptotically stable, i.e.*

5.a) for arbitrary $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(z,\nu(\omega)) < \delta$ implies $\rho(\phi(t,\omega,z),\nu(\theta_t\omega)) < \varepsilon$ for all $t \ge 0$ and $\omega \in \Omega$.

5.b) There exists $\delta_0 > 0$ such that

$$\lim_{t\to+\infty} \rho(\phi(t,\omega,z),\nu(\theta_t\omega)) = 0$$

for all $\omega \in \Omega$ and z with the condition $\rho(z,\nu(\omega)) \le \delta_0$.

Proof. Under the conditions of Theorem ?? there exists a positive number δ_0 such that the set $M := \mathfrak{B}[J,\delta_0]$ is a compact and positively invariant set of skew-product system $(X, \mathbb{T}^+, \pi)$ (see the proof of Theorem ??), where $\mathfrak{B}[I,\delta_0] := \bigcup\{B[I_\omega,\delta_0] \times \{\omega\} \mid \omega \in \Omega\}$. Denote by $E = E(M, \mathbb{T}^+, \pi)$ the Ellis semigroup of the dynamical system $(M, \mathbb{T}^+, \pi)$, $E_\omega := \{\xi \in E | \xi M_\omega \subseteq M_\omega\}$, where $M_\omega := \{(u,\omega) | \quad (u,\omega) \in M\}$ and $E^+_\omega := \{\xi \in E_\omega | \quad \exists \{t_n\} \in \mathfrak{N}^+_\omega$ such that $\pi^{t_n}|_{M_\omega} \to \xi\}$. According to Theorem ?? and Lemma ?? E^+_ω is a nonempty compact subsemigroup of the Ellis semigroup E. Note that every mapping $\xi \in E^+_\omega$, which maps $B[I_\omega,\delta_0]$ into I_ω, is holomorphic because, according to Theorem ??, the convergence $\pi^{t_n}|_{M_\omega} \to \xi$ is uniform on

M_ω. Consider an idempotent $v \in E^+_\omega$, then $v(v(u)) = v(u)$ for all $u \in M_\omega$ and, consequently, $v(p) = p$ for every $p \in v(M_\omega) = v(I_\omega)$. Since v is holomorphic and $v(I_\omega)$ is a compact connected set, then [?] $v(I_\omega)$ contains only one point $\nu(\omega)$. On the other hand we have $v(v(u)) = v(u)$ for all $u \in M_\omega$, i.e. $v(\nu(\omega)) = v(u)$. Thus there exists a sequence $t_n \to +\infty$ such that

$$|\phi(t_n,\omega,u) - \phi(t_n,\omega,\nu(u))| = 0 \tag{14}$$

for all $u \in M_\omega$. Taking into account the positively uniformly stability of cocycle ϕ from (??) we obtain the equality

$$|\phi(t,\omega,u) - \phi(t,\omega,\nu(u))| = 0 \tag{15}$$

for all $u \in B(I_\omega,\delta_0) := \{u \in \mathbb{C}^d \,|\rho(u,I_\omega) < \delta_0\}$. Now we will prove that $I_\omega = \{\nu(\omega)\}$ for every $\omega \in \Omega$. Let $0 < \delta < \delta_0, u \in I_\omega$ and $h \in \mathbb{C}^d$ with condition $|h| < \delta$, then according to equality (??) we have

$$\lim_{t\to+\infty} \sup_{|h|\le\delta} |\phi(t,\omega,u+h) - \phi(t,\omega,u)| = 0 \tag{16}$$

for all $\omega \in \Omega$ and $u \in I_\omega$. In virtue of Cauchy's formula (see [?] and also [?])

$$U(t,(u,\omega))w = \tag{17}$$

$$\frac{1}{(2\pi i)^d} \int\limits_{|v_1|=\frac{\delta}{2}} \cdot \ldots \cdot \int\limits_{|v_d|=\frac{\delta}{2}} \frac{\phi(t,\omega,u+v) - \phi(t,\omega,u)}{v_1 \cdot \ldots \cdot v_d} \sum_{k=1}^{d} \frac{w_k}{v_k} dv_1 \cdot \ldots \cdot dv_d,$$

where $U(t,(u,\omega)) := \frac{\partial\phi(t,\omega,u)}{\partial u}$ for all $(u,\omega) \in M$ and $t \in \mathbb{T}^+$. From (??) and (??) it follows that $\lim\limits_{t\to+\infty} ||U(t,(u,\omega))|| = 0$ for all $(u,\omega) \in M$. According to Theorem ?? there exist positive numbers N and α such that

$$||U(t,(u,\omega))|| \le N \exp(-\alpha t) \tag{18}$$

for any $(u,\omega) \in M$. Let now $u_1, u_2 \in I_\omega$ and $\psi : [0,1] \to B(I_\omega,\delta)$ be a continuously differentiable function with properties: $\psi(0) = u_1$ and $\psi(1) = u_2$. Consider the function $\Delta(s) := \phi(t,\omega,\psi(s))$, then according to Lagrange's formula we have

$$\Delta(1) - \Delta(0) = \Delta'(\tau), \tag{19}$$

where $0 < \tau < 1$. Hence from (??) and (??) we have

$$|\phi(t,\omega,u_1) - \phi(t,\omega,u_2)| \le N_1 \exp(-\alpha t)|u_1 - u_2| \tag{20}$$

for all $t \in \mathbb{T}^+, \omega \in \Omega$ and $u_1, u_2 \in M_\omega$, where $N_1 = N \cdot m$ and $m = \max\limits_{0\le s\le 1} |\psi'(s)|$.

To finish the proof of theorem it is sufficient to remark that according to Theorem ?? on set J there is defined a two-sided dynamical system $(J,\mathbb{T},\pi)$ and, in particular, through every point $u \in I_\omega$ passes a unique entire trajectory of cocycle ϕ, i.e. the function $\phi(t,\omega,u)$ ($u \in I_\omega$ and $\omega \in \Omega$) is defined on $\mathbb{T}$. If $u_1 \neq u_2 \quad (u_1,u_2 \in I_\omega)$, then from (??) follows that

$$|\phi(-t,\omega,u_1) - \phi(-t,\omega,u_2)| \ge N_1 \exp(\alpha t)|u_1 - u_2|$$

for all $t \in \mathbb{T}^+$. But the trajectories $\phi(t,\omega,u_i) \in I_{\theta_t\omega} (i = 1,2; t \in \mathbb{T})$ are bounded on $\mathbb{T}$. The obtained contradiction proves our assertion. Thus we have $I_\omega = \{\nu(\omega)\}$.

Now it is easy to see that the mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma = (\nu, Id_\Omega)$, and $\pi^t\gamma(\omega) = \gamma(\theta_t\omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$ and, consequently, $\nu(\theta_t\omega) = \phi(t,\omega,\nu(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.

Next we will note that every point $\gamma(\omega)$ is positively Poisson stable. Indeed, let $\{t_n\} \in \mathfrak{N}_\omega$, then $\pi^{t_n}\gamma(\omega) = \gamma(\theta_{t_n}\omega) \to \gamma(\omega)$ and, consequently, $\{t_n\} \in \mathfrak{N}_{\gamma(\omega)}$. Finally, the uniformly asymptotically stability of continuous and invariant section ν results from Theorem ??. The theorem is completely proved. □

6. The uniform dissipative cocycles with noncompact base

Let Ω be a complete metric space (generally speaking noncompact), $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a cocycle on the state space E^d and $(X, \mathbb{T}^+, \pi)$ be the corresponding skew-product dynamical system, where $X = E^d \times \Omega$ and $\pi = (\phi, \theta)$.

Definition 6.1. *(Dissipative cocycle) The cocycle $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is said to be dissipative if for any $\omega \in \Omega$ there is a positive number r_ω such that*

$$\lim_{t\to+\infty} \sup |\phi(t,\omega,u)| < r_\omega$$

for all $\omega \in \Omega$ and $u \in E^d$, i.e. for all $u \in E^d$ and $\omega \in \Omega$ there exists a positive number $L(\omega,u)$ such that $|\phi(t,\omega,u)| < r_\omega$ for all $t \geq L(\omega,u)$.

Definition 6.2. *(Uniformly dissipative cocycle) The cocycle $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is said to be uniformly dissipative if there exists a positive number r (r is not dependant upon $\omega \in \Omega$) such that for any $R > 0$ there is a positive number $L(R)$ such that $|\phi(t,\omega,u)| < r$ for all $\omega \in \Omega$ and $|u| \leq R$ and $t \geq L(R)$.*

Theorem 6.3. *([?, ?, ?]) Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be an uniformly dissipative cocycle, then it admits a compact global pullback attractor $\{I_\omega | \quad \omega \in \Omega\}$ with $|u| \leq r$ for all $u \in I_\omega$ and $\omega \in \Omega$, where r is the positive number in definition ??.*

Theorem 6.4. *Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a $\mathbb{C}-$analytic uniformly dissipative cocycle, then the following statements hold:*

i). The cocycle ϕ admits a compact global pullback attractor $\{I_\omega | \quad \omega \in \Omega\}$ with $|u| \leq r$ for all $u \in I_\omega$ and $\omega \in \Omega$, where r is the positive number in condition (??).

ii). For any $R > 0$ there exist positive constants $C = C(R)$ and $L(R)$ such that

$$\rho(\phi(t,\omega,u_1), \phi(t,\omega,u_2)) \leq C\rho(u_1,u_2) \tag{21}$$

for all $t \geq L(R), \omega \in \Omega$ and $u_1, u_2 \in \mathbb{C}^d$ with the condition $|u_i| \leq R \quad (i = 1,2)$.

iii). For arbitrary $\varepsilon > 0$ there exist $L(\varepsilon) > 0$ and $\delta(\varepsilon) > 0$ such that

$$|\phi(t,\omega,u+h) - \phi(t,\omega,u))| < \varepsilon$$

for all $t \geq L(\varepsilon)$, $u \in I_\omega, \omega \in \Omega$ and $|h| < \delta$.

iv). The set J of $(X, \mathbb{T}^+, \pi)$ is negatively distal, i.e.

$$\inf_{t\leq 0} \rho(\gamma_1(t), \gamma_2(t)) > 0,$$

where γ_i (i=1,2) is a entire trajectory passing through point $(u_i, \omega) \in J, u_1 \neq u_2$ and $\gamma_i(\mathbb{S}) \subseteq J$.

v). On the set J there is defined a two-sided dynamical system $(J, \mathbb{T}, \pi)$ generated by skew-product system $(X, \mathbb{T}^+, \pi)$.

Proof. The first assertion of theorem results from Theorem ??. Let now $R > 0$ and $R' > R$, according to uniformly dissipativity of cocycle ϕ there exists $L(R') > 0$ such that $|\phi(t, \omega, u)| < r$ for all $t \geq L(R'), \omega \in \Omega$ and $|u| \leq R'$. In virtue of Cauchy's formula for $R < R'$ there is a constant $C(R) > 0$ such that $|\frac{\partial \phi}{\partial u}(t, \omega, u)| \leq C(R)$ for all $t \geq L(R), \omega \in \Omega$ and $|u| \leq R$ and, consequently the inequality (??) holds.

The third assertion we will prove by method of contradiction. If it is not true, then there exist $\varepsilon_0 > 0, \delta_n \to 0 \quad (\delta_n > 0), \ |h_n| < \delta_n \quad (h_n \in \mathbb{C}^d), t_n \geq n, \quad \omega_n \in \Omega$ and $u_n \in I_{\omega_n}$ such that

$$|\phi(t_n, \omega_n, u_n + h_n) - \phi(t_n, \omega_n, u_n))| \geq \varepsilon_0$$

Let now $R > r$ and $C(R), L(R)$ be positive constants figuring in the inequality (??), then we have the following inequality

$$\varepsilon_0 \leq |\phi(t_n, \omega_n, u_n + h_n) - \phi(t_n, \omega_n, u_n))| \leq C(R)|h_n| \leq C(R)\delta_n. \tag{22}$$

Passing to limit in the inequality (??) as $n \to \infty$ we obtain $\varepsilon_0 \leq 0$. The obtained contradiction proves our assertion.

The fourth and fifth statements follow from Theorem ?? (see also Remark ??) because from condition iii) results that for arbitrary $\varepsilon > 0$ there exist two positive constants $\delta(\varepsilon)$ and $L(\varepsilon)$ satisfying the inequality (??). The Theorem is completely proved. □

Theorem 6.5. *Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta) \rangle$ be a $\mathbb{C}-$analytic uniformly dissipative cocycle and every point $\omega \in \Omega$ be positively Poisson stable, then:*

1. *The set I_ω consists of only one point $\nu(\omega)$ for every ω.*
2. *The mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma := (\nu, Id_\Omega)$.*
3. *$\nu(\theta_t \omega) = \phi(t, \omega, \nu(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.*
4. *$\lim_{t \to +\infty} \rho(\phi(t, \theta_{-t}\omega)z, \nu(\omega)) = 0$ for every $\omega \in \Omega$ uniformly with respect to z in compact subsets of $\mathbb{C}^d$.*
5. *Every point $\gamma(\omega)$ is positively Poisson stable.*
6. *$\lim_{t \to +\infty} \rho(\phi(t, \omega, z), \nu(\theta_t \omega)) = 0$ for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$, i.e. every positive semi trajectory $\phi(t, \omega, z)$ is asymptotically Poisson stable in positive direction.*

Proof. Let $\langle \mathbb{C}^d, \phi, (\Omega, \mathbb{T}, \theta) \rangle$ be a $\mathbb{C}-$analytic uniformly dissipative cocycle, then according to Theorem ?? this cocycle has the properties i)-v). Let $\{I_\omega | \ \omega \in \Omega\}$ be the compact global pullback attractor of cocycle ϕ and let $(X, \mathbb{T}^+, \pi)$ be the skew-product dynamical system. Denote by

$$E^+_\omega := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}^+_\omega, \ \pi^{t_n}|_{X_\omega} \to \xi\}.$$

Since the cocycle ϕ possesses the property ii), then the pointwise convergence $\pi^{t_n}|_{X_\omega} \to \xi$ coincides with uniform convergence on every compact subset of $X_\omega = \mathbb{C}^d \times \{\omega\}$ and, consequently, every mapping $\xi \in E^+_\omega$ is holomorphic. As well as in

Lemma ?? it is possible to show that E^+_ω is a nonempty compact semigroup w.r.t. composition of mappings. Consider the idempotent element v of semigroup E^+_ω. We will show that $v(X_\omega)\subseteq I_\omega$. Indeed, $v\in E^+_\omega$ and, consequently, there exists a sequence $\{t_n\}\in\mathfrak{N}^+_\omega$ such that $v=\lim\limits_{n\to\infty}\pi^{t_n}|_{X_\omega}$. Let $\bar{x}\in v(X_\omega)$, i.e. $\bar{x}=v(x)$ for some $x\in X_\omega$. This means that $\bar{x}=\lim\limits_{n\to\infty}\pi^{t_n}x$. According to Lemma ?? there exists an entire trajectory γ of the skew-product system $(X,\mathbb{T}^+,\pi)$ passing through the point $\bar{x}$ for $t=0$ and $\gamma(\mathbb{T}):=\{\gamma(t)|t\in\mathbb{T}\}$ is conditionally precompact. Taking into account that J is a maximal invariant set of $(X,\mathbb{T}^+,\pi)$ with precompact $pr_1 J$ (see remark ??) we have $\bar{x}\in J_\omega$, i.e. $v(X_\omega)\subseteq J_\omega$. Since $X_\omega=\mathbb{C}^d\times\{\omega\}$ and v is holomorphic by virtue of Liouville's Theorem the holomorphic function v is a constant, i.e. there exists $\gamma(\omega)\in J_\omega$ such that $v(X_\omega)=\{\gamma(\omega)\}$. We note that $v^2=v$ and, consequently, $v(v(x))=v(x)$ for all $x\in X_\omega$, i.e. $v(\gamma(\omega))=v(x)$. Thus, there exists a sequence $t_n\to+\infty$ such that

$$\lim_{t\to+\infty}\rho(\pi^{t_n}\gamma(\omega),\pi^{t_n}x)=0. \tag{23}$$

Taking into consideration the property ii) of cocycle ϕ we obtain from (??) the equality

$$\lim_{n\to\infty}\rho(\phi(t,\omega,z),\phi(t,\omega,\nu(\omega)))=0 \tag{24}$$

for all $z\in\mathbb{C}^d$, where $\gamma:=(\nu,Id_\Omega)$.

Now we will show that there exists $\delta_0>0$ such that for arbitrary $\varepsilon>0$ there is $L(\varepsilon)>0$ with the property

$$|\phi(t,\omega,u+h)-\phi(t,\omega,u)|<\varepsilon \tag{25}$$

for all $(u,\omega)\in J$, $t\ge L(\varepsilon)$ and uniformly w.r.t. $|h|\le\delta_0$. If it is not true, then there are $\delta\to+0,\varepsilon_0>0,|h_n|\le\delta_n,\omega_n\in\Omega,u_n\in I_{\omega_n}$ and $t_n\ge n$ such that

$$|\phi(t_n,\omega_n,u_n+h_n)-\phi(t_n,\omega_n,u_n)|\ge\varepsilon_0.$$

On the other hand, according to property ii), there exists $C(R)>0$ $(R>\sup\limits_{n\in\mathbb{N}}\delta_n)$ such that for sufficiently large n we have

$$\varepsilon_0\le|\phi(t_n,\omega_n,u_n+h_n)-\phi(t_n,\omega_n,u_n)|\le C(R)\delta_n. \tag{26}$$

Taking into account that $\delta_n\to0$ from (??) it follows that $\varepsilon_0\le0$. The obtained contradiction prove our assertion.

From equality (??) and inequality (??) it follows that

$$\lim_{t\to+\infty}||U(t,(u,\omega))||=0 \tag{27}$$

uniformly with respect to $(u,\omega)\in\mathfrak{B}[J,\delta_0]$. Denote by

$$m(t):=\sup\{||U(t,(u,\omega))|| \quad |(u,\omega)\in\mathfrak{B}[J,\delta_0]\},$$

then

a) $m(t)\to0$ as $t\to+\infty$.

b) $\exists L>0$ such that m is bounded on $[L,+\infty)$.

c) $m(t+\tau)\le m(t)m(\tau)$ for all $t,\tau\ge L$.

From a)-c) it follows (see, for example, [?]) that there exist $N, \alpha > 0$ such that $m(t) \le N \exp(-\alpha t)$ for all $t \ge L$ and, consequently, from (??) we have

$$||U(t,(u,\omega))|| \le N \exp(-\alpha t)$$

for all $(u,\omega) \in \mathfrak{B}[J,\delta_0]$ and $t \ge L$. Using the same arguments as well as as in the proof of Theorem ?? we conclude that $I_\omega = \{\nu(\omega)\}$ for all $\omega \in \Omega$.

Now we will prove that the mapping $\omega \to \gamma(\omega)$ is continuous. Let $\omega_n \to \omega$ and consider the sequence $\{\gamma(\omega_n)\} \subset J$. Since J is conditionally compact, this sequence is precompact. Let $\bar{x}$ be a limit point of $\{\gamma(\omega_n)\}$, then it is easy to see that $\bar{x} \in J_\omega = \{\gamma(\omega)\}$ and, consequently, $\gamma(\omega)$ is a unique limit point of precompact sequence $\{\gamma(\omega_n)\}$. Hence $\gamma(\omega_n) \to \gamma(\omega)$.

The equality $\nu(\theta_t\omega) = \phi(t,\omega,\nu(\omega))$ follows from invariance of J and from equality $J_\omega = \{\gamma(\omega)\}$ for all $\omega \in \Omega$, taking into account that $\nu(\omega) = pr_1\gamma(\omega)$.

The equality 4 follows from equality $J_\omega = \{\gamma(\omega)\} = \{(\nu(\omega),\omega)\}$ and from the fact that $\{\nu(\omega)|\ \omega \in \Omega\}$ is a compact global pullback attractor of cocycle ϕ.

The stability in the sense of Poisson in the positive direction of point $\gamma(\omega)$ follows from the continuity of γ and the equality $\pi^t\gamma(\omega) = \gamma(\theta_t\omega)$ for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$.

The sixth assertion follows from (??) and the equality $\phi(t,\omega,\nu(\omega)) = \nu(\theta_t\omega)$ for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$. The Theorem is completely proved. □

7. The compact and local dissipative cocycles with noncompact base

Definition 7.1. *(compact dissipative cocycle) The cocycle $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is said to be compactly dissipative if for any nonempty compact $\Omega' \subset \Omega$ there is a positive number $r_{\Omega'}$ such that for arbitrary $R > 0$ there exists a positive number $L(R,\Omega')$ with the following property*

$$|\phi(t,\omega,u)| < r_{\Omega'} \tag{28}$$

for all $\omega \in \Omega'$, $|u| \le R$ and $t \ge L(R,\Omega')$.

Definition 7.2. *(Local dissipative cocycle) The cocycle $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ is said to be locally dissipative if for any $\omega \in \Omega$ and $R > 0$ there are positive numbers r_ω, δ_ω and $L(R,\omega)$ such that*

$$|\phi(t,\tilde{\omega},u)| < r_{\Omega'} \tag{29}$$

for all $\tilde{\omega} \in B(\omega,\delta_\omega) := \{\tilde{\omega} \in \Omega| \quad \rho(\tilde{\omega},\omega) < \delta_\omega\}, |u| \le R$ and $t \ge L(R,\omega)$.

Lemma 7.3. *Every locally dissipative cocycle ϕ is compactly dissipative.*

Proof. Suppose that ϕ is locally dissipative and let $\Omega' \subset \Omega$ be a nonempty compact set. According to locally dissipativity of ϕ for every $\omega \in \Omega$, and $R > 0$ there exist $r_\omega, L(R,\omega) > 0$ and $\delta_\omega > 0$ such that the inequality (??) holds. Considering the open covering $\bigcup\{B(\omega,\delta_\omega)|\omega \in \Omega'\}$ of compact set Ω', we may extract the finite sub covering $\bigcup\{B(\omega_i,\delta_{\omega_i})|i = \overline{1,k}\}$. Let $L(R,\Omega') := \max\{L(R,\omega_i)|i = \overline{1,k}\}$, then it is clear that inequality (??) holds for all $\omega \in \Omega'$, $|u| \le R$ and $t \ge L(R,\Omega)$. The Lemma is proved. □

Remark 7.4. *Compact dissipativity, generally speaking, does not imply locally dissipativity.*

Lemma 7.5. *Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a compactly dissipative $\mathbb{C}-$analytic cocycle. Then for any nonempty compact $\Omega' \subset \Omega$ and $R > 0$ there exist $L(\Omega', R)$ and $C = C(\Omega', R) > 0$ such that*

$$\rho(\phi(t, \omega, u_1), \phi(t, \omega, u_2)) \le C(\Omega', R)\rho(u_1, u_2) \tag{30}$$

for any $\omega \in \Omega', |u_i| \le R$ $(i = 1, 2)$ and $t \ge L(\Omega', R)$.

Proof. Let $R > 0$ and $R' > R$, then according to the compact dissipativity of cocycle ϕ for nonempty compact $\Omega' \subset \Omega$ and $R' > 0$ there exist $r_{\Omega'} > 0$ and $L(\Omega', R') > 0$ such that $|\phi(t, \omega, u)| < r_{\Omega'}$ for all $\omega \in \Omega, |u| \le R'$ and $t \ge L(\Omega', R')$. In view of Cauchy's formula for $R < R'$ there exists a constant $C = C(R, \Omega) > 0$ such that $|\frac{\partial \phi}{\partial u}(t, \omega, u)| \le C(R, \Omega')$ for all $t \ge L(R', \Omega'), \omega \in \Omega'$ and $|u| \le R$ and, consequently, the inequality (??) holds for $|u_1|, |u_2| \le R, \omega \in \Omega'$ and $t \ge L(\Omega', R) := \inf\{L(\Omega', R')|R' > R\}$. ☐

Lemma 7.6. *Let $\langle E^d, \phi, (\Omega, \mathbb{T}, \theta)\rangle$ be a compactly dissipative $\mathbb{C}-$analytic cocycle, and γ_1, γ_2 are two entire bounded trajectories passing through point (u_1, ω) and (u_2, ω) for $t = 0$ respectively, then the following assertions hold:*

1. *If $\omega \in \Omega$ is negatively Poisson stable and $\{t_n\} \in \mathfrak{N}^-_\omega$, then*

$$\inf_{n \in \mathbb{N}} \rho(\gamma_1(t_n), \gamma_2(t_n)) > 0 \tag{31}$$

if $u_1 \ne u_2$.

2. *If $\omega \in \Omega$ is positively Poisson stable and $\{t_n\} \in \mathfrak{N}^+_\omega$, then the equality*

$$\inf_{n \in \mathbb{N}} \rho(\phi(t_n, \omega, u_1), \phi(t_n, \omega, u_2)) = 0 \tag{32}$$

implies

$$\lim_{t \to +\infty} \rho(\phi(t, \omega, u_1), \phi(t, \omega, u_2)) = 0. \tag{33}$$

Proof. Let $\omega \in \Omega$ be a negative Poisson stable point, $\{t_n\} \in \mathfrak{N}^-_\omega$ and $u_1 \ne u_2$. If the equality (??) is not true, then $\rho(\gamma_1(t_n), \gamma_2(t_n)) \to 0$ as $n \to \infty$. Denote by $R := \sup_{t \in \mathbb{T}} \max\{|\gamma_1(t)|, |\gamma_2(t)|\} > 0, \Omega' := \overline{\{\theta_{t_n}\omega | n \in \mathbb{N}\}} \subset \Omega$ and let $C(\Omega', R), L(\Omega', R)$ be the corresponding constants figuring in the inequality (??), then we obtain

$$\begin{aligned} \rho(u_1, u_2) &= \rho(\phi(-t_n, \theta_{t_n}\omega, \gamma_1(t_n)), \phi(-t_n, \theta_{t_n}\omega_2(t_n))) \\ &\le C(\Omega', R)\rho(\gamma_1(t_n), \gamma_2(t_n)) \end{aligned} \tag{34}$$

for sufficiently large n $(-t_n \ge L(\Omega', R))$. Passing to limit in the inequality (??) as $n \to \infty$ we have $\rho(u_1, u_2) \le 0$, but $u_1 \ne u_2$. The obtained contradiction prove the first statement of Lemma ?? .

Let now $\omega \in \Omega$ be a positive Poisson stable point and $\{t_n\} \in \mathfrak{N}^+_\omega$ such that the inequality (??) holds, then for arbitrary $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$\rho(\phi(t, \omega, u_1), \phi(t, \omega, u_2)) < \frac{\varepsilon}{2C(\Omega', R)} \tag{35}$$

and, consequently, according to Lemma ?? we obtain

$$\rho(\phi(t,\omega,u_1),\phi(t,\omega,u_2)) = \rho(\phi(t-t_n,\theta_{t_n}\omega,\phi_(t_n,\omega,u_1)), \tag{36}$$

$$\phi(t-t_n,\theta_{t_n}\omega,\phi(t_n,\omega,u_2)) \le C(\Omega',R)\rho(\phi(t_n,\omega,u_1),\phi(t_n,\omega,u_2))$$

for all $t \ge t_n + L(\Omega',R)$, where $R := \sup\{\max\{|\phi(t,\omega,u_1)|,|\phi(t,\omega,u_2)|\}|t \in \mathbb{T}^+\}$. Denote by $L(\varepsilon) := t_{n_0(\varepsilon)} + L(\Omega,R)$, then from inequalities (??) and (??) we obtain

$$\rho(\phi(t,\omega,u_1),\phi(t,\omega,u_2)) < \varepsilon$$

for all $t \ge L(\varepsilon)$ and, consequently, (??) holds. The lemma is proved. □

Theorem 7.7. *Let $\langle E^d,\phi,(\Omega,\mathbb{T},\theta)\rangle$ be a compactly dissipative $\mathbb{C}-$analytic cocycle admitting a compact pullback attractor $\{I_\omega| \quad \omega \in \Omega\}$ and every point $\omega \in \Omega$ be two-sided Poisson stable, then the following assertions hold:*

1. *The set I_ω consists of only one point $\nu(\omega)$, i.e. $I_\omega = \{\nu(\omega)\}$ for every $\omega \in \Omega$.*
2. *The mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma = (\nu, Id_\Omega)$.*
3. *$\nu(\theta_t\omega) = \phi(t,\omega,\nu(\omega))$ for all $\omega \in \Omega$ and $t \in T^+$.*
4. *The point $\gamma(\omega)$ is Poisson's stable for all $\omega \in \Omega$.*
5. *$\lim_{t\to+\infty}\beta(\phi(t,\theta_{-t}\omega)K,\nu(\omega)) = 0$ for all compact subsets K of $\mathbb{C}^d$, where $\beta(A,B)$ is the semi-distance of Hausdorff between A and B.*
6. *$\lim_{t\to+\infty}|\phi(t,\omega,z)-\nu(\theta_t\omega)| = 0$ for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$.*

Proof. To prove the first assertion of Theorem ?? we consider the nonautonomous dynamical system $\langle(\Phi,\mathbb{T},\sigma),(\Omega,\mathbb{T},\theta),H\rangle$ constructed in the proof of Theorem ??. Using the same type of argument as in Theorem ?? and taking into consideration the Lemma ?? we may state that the system $\langle(\Phi,\mathbb{T},\sigma),(\Omega,\mathbb{T},\theta),H\rangle$ possesses the following properties:

a) Φ is conditionally compact invariant set.

b) for every $\omega \in \Omega, \gamma_1,\gamma_2 \in \Phi_\omega(\gamma_1 \ne \gamma_2)$ and $\{t_n\} \in \mathfrak{N}_\omega^-$ holds

$$\inf_{n\in\mathbb{N}} \rho(\gamma_1^{t_n},\gamma_2^{t_n}) > 0.$$

Then according to Lemmas ??-?? we have that $E_\omega^- = E_\omega^+ = E_\omega$ is a group. Particularly there are two sequences $t_n^1 \to +\infty$ and $t_n^2 \to -\infty$ such that

$$\lim_{n\to\infty}\gamma^{t_n^i} = \gamma \quad (i = 1,2). \tag{37}$$

for all $\gamma \in \Phi_\omega$, i.e. every entire trajectory γ of global pullback attractor $\{I_\omega|\omega \in \Omega\}$ is two-sided Poisson stable. On the other hand according to Lemma ?? we have

$$\lim_{t\to+\infty}\rho(\gamma_1(t),\gamma_2(t)) = 0. \tag{38}$$

From (??) and (??) we obtain

$$\rho(\gamma_1(t),\gamma_2(t)) = \lim_{n\to\infty}\rho(\gamma_1(t+t_n^1),\gamma_2(t+t_n^1)) = 0$$

for all $t \in \mathbb{T}$, i.e. $\gamma_1 = \gamma_2$. Thus there exists a unique entire trajectory $\tilde{\gamma}_\omega \in \Phi_\omega$, i.e. $\Phi_\omega = \{\tilde{\gamma}_\omega\}$ and, consequently, $I_\omega = \{\tilde{\gamma}_\omega(0)\} := \{\gamma(\omega)\}$, where $\gamma(\omega) := \tilde{\gamma}_\omega(0)$.

The proof of item 2)-6) of Theorem ?? uses the same type of arguments as in Theorem ??. The Theorem is proved. □

8. Applications

8.1. **Ordinary differential equations.** We consider the equation

$$\frac{dz}{dt} = f(t,z), \tag{39}$$

where $f \in CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$ and the family of equations

$$\frac{dz}{dt} = g(t,z), \tag{40}$$

where $g \in H(f) := \overline{\{f_\tau | \tau\in\mathbb{R}\}}$ and f_τ is a $\tau-$translation of function f w.r.t. variable t, i.e. $f_\tau(t,z) := f(t+\tau,z)$ for all $t\in\mathbb{R}$ and $z\in\mathbb{C}^d$. Denote by $\phi(t,g,z)$ the solution of equation (??) with the initial condition $\varphi(0,g,z)=z$, then ϕ is a $\mathbb{C}-$analytic cocycle on $\mathbb{C}^d$.

Definition 8.1. *The equation (??) is called dissipative if there exists a positive number r such that $\lim_{t\to+\infty}\sup|\phi(t,g,z)| < r$ for all $z\in\mathbb{C}^d$ and $g\in H(f)$.*

Definition 8.2. *The function $f\in CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$ is called positively (negatively) Poisson stable in $t\in\mathbb{R}$ uniformly w.r.t. z on compact subsets of $\mathbb{C}^d$ [?],[?] if there exists $t_n\to+\infty$ ($t_n\to-\infty$, respectively) such that $f(t+t_n,z)\to f(t,z)$ as $n\to\infty$ uniformly on every compact subsets of $\mathbb{R}\times\mathbb{C}^d$.*

Theorem 8.3. *Suppose that the following conditions hold:*

1. *The set $H(f)\subset CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$ is compact.*
2. *Every function $g\in H(f)$ is positively Poisson stable (in this case function f is called [?],[?] quasi recurrent).*
3. *The equation (??) is dissipative.*

Then every equation (??) admits a unique bounded on $\mathbb{R}$ and positively Poisson stable solution. This solution is globally uniformly asymptotically stable.

Proof. We consider the dynamical system of translations (Bebutov's system [?],[?]) $(CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d),\mathbb{R},\sigma)$. Since $H(f)$ is invariant and closed subset of $CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$, then on set $H(f)$ it is induced a dynamical system $(H(f),\mathbb{R},\sigma)$. Let $\Omega := H(f)$, then on space $\mathbb{C}^d$ it is defined a $\mathbb{C}-$analytic cocycle $\langle\mathbb{C}^d,\phi,(\Omega,\mathbb{R},\sigma)\rangle$, generated by equation (??). According to the general properties of equation (??) with holomorphic right hand side f, the cocycle ϕ will be $\mathbb{C}-$analytic (see, for example, [?]). To finish the proof of this theorem it is sufficient to apply to cocycle ϕ the Theorem ??. □

Definition 8.4. *(Bohr's almost periodic function) The function $f\in CH(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$ is called almost periodic (in the sense of Bohr) in $t\in\mathbb{R}$ uniformly w.r.t. z*

on compact subsets of $\mathbb{C}^d$ [?, ?, ?] *if for every* $\varepsilon > 0$ *and nonempty compact subset* $K \subset \mathbb{C}^d$ *the set*

$$\mathfrak{T}(\varepsilon, f, K) := \{\tau \in \mathbb{R} | \max_{z\in K} |f(t+\tau, z) - f(t,z)| < \varepsilon\}$$

is relatively dense on $\mathbb{R}$*, i.e. there is a number* $l = l(\varepsilon, f, K) > 0$ *such that*

$$\mathfrak{T}(\varepsilon, f, K) \bigcap [a, a+l] \neq \emptyset$$

for all $a \in \mathbb{R}$*.*

Definition 8.5. *The equation (??) is called pullback dissipative if for every* $g \in H(f)$ *there exists a positive number* r_g *such that for all* $R > 0$

$$\lim_{t\to+\infty} \sup ||\phi(t, g_{-t}, z)| < r_g$$

uniformly w.r.t. $|z| \leq R$*.*

Theorem 8.6. *Let* $f \in CH(\mathbb{R}\times\mathbb{C}^d, \mathbb{C}^d)$ *be an almost periodic function in* $t \in \mathbb{R}$ *uniformly w.r.t. z on compact subsets of* $\mathbb{C}^d$ *and the equation (??) be pullback dissipative, then every equation (??) admits a unique bounded solution* $\nu_g(t)$*, which is almost periodic and satisfies the following conditions:*

1. $\nu_g(t)$ *is uniformly asymptotically stable (locally).*
2. $\lim\limits_{t\to+\infty} \sup\limits_{|z|\leq R} |\phi(t, \omega_{-t})z - \nu_g(0)| = 0.$

Proof. The proof of this theorem uses the same type of argument as the proof of Theorem ?? and is based on the Theorem ??. □

8.2. **Caratheodory differential equations.** Consider now the equation (??) with right hand side f satisfying the conditions of Caratheodory (see, for example,[?]) and holomorphic w.r.t. variable $z \in \mathbb{C}^d$. The space of all the Carateodory functions we denote by $\mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d, \mathbb{C}^d)$. The topology on this space is defined by family of semi-norm (see [?])

$$p_{n,m}(f) := \int_{-n}^{n} \max_{|z|\leq m} |f(t,z)| dt.$$

This space is metrizable and on $\mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d, \mathbb{C}^d)$ the dynamical system of translations $(\mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ can be defined.

Using the standard arguments for ODEs (see,for example, [?] and [?]) one can prove that every equation (??) admits a unique solution $\phi(t, g, z)$ with initial condition $\phi(0, g, z) = z$ and supplementary the mapping $\phi(t, g) := \phi(t, g, \cdot) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic. Thus if the solutions $\phi(t, g, z)$ are defined on $\mathbb{R}^+$, the mapping $\phi : \mathbb{R}^+ \times H(f) \times \mathbb{C}^d \to \mathbb{C}^d$ defines a $\mathbb{C}-$analytic cocycle on $\mathbb{C}^d$ with the base $H(f)$, where $H(f) := \overline{\{f_\tau | \tau \in \mathbb{R}\}}$ and the bar denotes the closure in the space $\mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d, \mathbb{C}^d)$. Hence we may apply the general results from sections 2-7 to cocycle ϕ, generated by equation (??) with Caratheodory's right hand side, and we will obtain some results for this type of equations. For example the following assertion holds.

Theorem 8.7. *Let $f \in \mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)$ be an almost periodic function in $t\in\mathbb{R}$ (in the sense of Stepanov [?]) uniformly w.r.t. z on compact subsets of $\mathbb{C}^d$, i.e. for every $\varepsilon>0$ and compact subset $K\subset\mathbb{C}^d$ the set*

$$\mathfrak{T}(\varepsilon,f,K):=\{\tau\in\mathbb{R}|\int_0^1\max_{x\in K}|f(t+\tau+s,z)-f(t+s,z)|ds<\varepsilon\}$$

is relatively dense on $\mathbb{R}$. Suppose that the equation (??) is dissipative, then every equation (??) admits a unique almost periodic (in the sense of Bohr) solution $\nu_g(t)$ which is globally uniformly asymptotically stable.

8.3. **ODEs with impulses.** Let $\{t_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers, $\inf\{t_{n+1}-t_n|\quad n\in\mathbb{Z}\}>0$, $p:\mathbb{R}\to\mathbb{C}^d$ be a continuously differentiable function on every interval (t_n,t_{n+1}), continuous to the right in every point $t=t_n$, almost periodic in the sense of Stepanov and

$$p'(t)=\sum_{n\in\mathbb{Z}}s_n\delta_{t_n},$$

where $s_n:=p(t_n+0)-p(t_n-0)$ (i.e. the function p is piecewise constant). More information about the function described above can be found in the books [?] and [?].

Consider the equation with impulses

$$\frac{dz}{dt}=f(t,z)+\sum_{n\in\mathbb{Z}}s_n\delta_{t_n} \tag{41}$$

or equivalently

$$\frac{dz}{dt}=f(t,z)+p'(t). \tag{42}$$

At the same time we consider the family of equations

$$\frac{dz}{dt}=g(t,z)+q'(t), \tag{43}$$

where $(g,q)\in H(f,p):=\overline{\{(f_\tau,p_\tau)|\tau\in\mathbb{R}\}}$ and by the bar we denote the closure in the product-space $\mathfrak{C}H(\mathbb{R}\times\mathbb{C}^d,\mathbb{C}^d)\times\mathfrak{C}(\mathbb{R},\mathbb{C}^d)$.

Denote by $\phi(t,g,q,z)$ the unique solution of equation (??) (see [?] and [?]) satisfying the initial condition $\phi(0,g,q,z)=z$. This solution is continuous on every interval (t_n,t_{n+1}) and continuous to the right in every point $t=t_n$ (see [?] and [?]).

Definition 8.8. *The equation (??) is called dissipative if there exists a number $r>0$ such that $\lim_{t\to+\infty}\sup|\phi(t,g,q,z)|<r$ for every $(g,q)\in H(f,p)$ and $z\in\mathbb{C}^d$.*

Using the transformation $w:=z+q(t)$ we can transform the equation (??) into the equation

$$\frac{dw}{dt}=g(t,w+q(t)). \tag{44}$$

Remark 8.9. *Denote by $\tilde{\phi}(t,g,q,z)$ the cocycle defined by the family of equations (??), then it is clear that the cocycle ϕ generated by (??) is dissipative if and only if the cocycle $\tilde{\phi}$ generated by (??) is dissipative.*

Theorem 8.10. *Let $f \in CH(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ be a Bohr's almost periodic function in $t \in \mathbb{R}$ uniformly with respect to z on every compacts subsets of $\mathbb{C}^d$, locally Lipshitz in z uniformly w.r.t. $t \in \mathbb{T}$ and $p \in \mathfrak{C}(\mathbb{R}, \mathbb{C}^d)$ be a Stepanov almost periodic function bounded on $\mathbb{R}$. If the equation (??) is dissipative, then for every $(g, q) \in H(f, p)$ the equation (??) admits a unique Stepanov almost periodic solution and this solution is globally uniformly asymptotically stable .*

Proof. Let $\phi(t, g, q, z)$ be the cocycle generated by equation (??) and let $\tilde{\phi}(t, g, q, w)$ be the cocycle generated by equation (??). Then we have the following equality

$$\phi(t, g, q, z) = q(t) + \tilde{\phi}(t, g, q, z - q(0)). \tag{45}$$

Under the conditions of Theorem ?? the cocycle $\tilde{\phi}$ is dissipative, $\mathbb{C}-$analytic and the right hand side of equation (??) under the conditions of Theorem ?? is Stepanov almost periodic in $t \in \mathbb{R}$ uniformly on every compact of $\mathbb{C}^d$ w.r.t. z. To finish the proof of this theorem we apply the Theorem (??) to the equation (??) and take into consideration the relation (??). The theorem is proved. ☐

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