Pullback attractors in dissipative nonautonomous differential equations under discretization*

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1 Introduction

A defining characteristic of an autonomous dynamical system is its dependence on the time that has elapsed only and not on the absolute time itself. Consequently limiting objects, such as attractors, actually exist for all time as invariant sets under the evolution of the autonomous system. Although such concepts can also be used for general nonautonomous systems, where the absolute starting time is as important as the time elapsed since starting, they are often too restrictive and exclude many interesting types of dynamical behaviour. A simple example is that of an asymptotically stable solution that is neither constant nor periodic. What are the limiting or attracting points here? What are the corresponding invariant sets, and, important for numerical considerations, how can one assure convergence to a particular point in such an invariant set? The forwards running convergence of an asymptotically stable solution is of little direct use in constructing the limiting solution since this solution may itself be changing with increasing time. An alternative is to use *pullback convergence,* that is to hold fixed an absolute time instant and to consider the limiting values at this time instant of other solutions

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that start progressively early in absolute time. The limiting sets now depend on absolute time and are invariant under the evolution of the system, that is, are carried forward onto each other as time increases. This idea was introduced several years ago in the context of random dynamical systems [7, 8, 9, 16], which are intrinsically nonautonomous, but had been used already in the 1960s by M. Krasnoselskii [14] to establish the existence of solutions of deterministic systems that are bounded over the entire time axis. It has also been applied recently [12, 13] to investigate variable timestep (hence nonautonomous) numerical approximations of global attractors of autonomous systems governed by dissipative ordinary differential equations.

Another key idea in [7, 8, 9, 16] is to formulate the nonautonomous dynamics on $\mathbb{R}^d$ in terms of a cocycle mapping ϕ that is driven by an underlying autonomous sytem θ on some parameter set P. At its simplest, P is just the absolute time set $\mathbb{R}$ and θ is the shift operator that essentially resets the starting time to the current absolute time value. More useful is to consider for P a function space of admissible vector fields as proposed by Sell [17] or as a probability sample space as in [7, 8, 9, 16], where the current parameter value takes the role of absolute time and is adjusted by θ with the passage of time. The advantage here is, in the first case at least, that the parameter space can be topologized (often as a compact space) and the product system (θ, ϕ) is an *autonomous* semi–dynamical system known as as *skew product flow* on the new product state space $P \times \mathbb{R}^d$. The extensive theory of autonomous dynamical systems can then be applied to such skew product flows, in particular concepts such as invariant sets, limit sets and attractors, but just how these manifest themselves in terms of the original dynamics on the original state space $\mathbb{R}^d$ and what relationship, if any, they have with pullback convergence need to be clarified.

In this paper we investigate the effect of time discretization on the pullback attractor of a nonautonomous ordinary differential equation for which the vector fields depend on a parameter that varies in time rather than depending directly on time itself. The parameter space is assumed to be compact so the skew product flow formalism as well as cocycle formalism also applies and the vector fields have a strong dissipative structure that implies the existence of a compact set that absorbs all compact sets under the resulting nonautonomous dynamics. The numerical scheme considered is a general 1–step scheme such as the Euler scheme with variable timesteps. Our main result is to show that the numerical scheme interpreted as a discrete time nonautonomous dynamical system, hence discrete time cocycle mapping and skew product flow on an extended parameter space, also possesses a cocycle attractor and that its component subsets converge upper semi–continuously to those of the cocycle attractor of the original system governed by the differential equation. This is a nonautonomous analogue of a result of Kloeden and Lorenz

[11] on the discretization of an autonomous attractor; see also [10, 21]. We will also see that the corresponding skew product flow systems have global attractors with the cocycle attractor component sets as their cross-sectional sets in the original state space $\mathbb{R}^d$. Finally, we investigate the periodicity and almost periodicity of the discretized pullback attractor when the parameter dynamics in the ordinary differential equation is periodic or almost periodic and the pullback attractor consists of singleton valued component sets, i.e. the pullback attractor is a single trajectory.

The paper is organised as follows. Pullback attractors, cocycles and skew product flows are defined in Section 2 and a theorem is stated, summarising results from the literature on the relationship between pullback attractors and global attractors of skew product flows. The class of nonautonomous differential equations and the corresponding variable timestep 1–step schemes to be considered are introduced in Section 3 and their cocycle formalism is then established in Section 4. The main result, Theorem 6, is formulated in Section 5 and then proved in Section 6. Section 7 is devoted to periodic and almost periodic behaviour when the pullback attractor is a single trajectory. Finally, the Appendix contains the proof of a lemma used earlier, which compares the cocycle mappings of the original continuous time system and of the discrete time numerical systems.

2 Nonautonomous dynamical systems and pullback attractors

Consider an autonomous dynamical system on a metric space P described by a group $\theta = \{\theta_t\}_{t\in\mathbb{T}}$ of mappings of P into itself, where the time set $\mathbb{T}$ is either $\mathbb{Z}$ (discrete time) or $\mathbb{R}$ (continuous time).

Let X be a complete metric space and consider a continuous mapping $\phi : \mathbb{T}^+ \times P \times X \to X$ satisfying the properties

$$\phi(0, p, \cdot) = \mathrm{id}_X, \qquad \phi(\tau + t, p, x) = \phi(\tau, \theta_t p, \phi(t, p, x))$$

for all t, $\tau \in \mathbb{T}^+$, $p \in P$ and $x \in X$. The mapping ϕ is called a (continuous) cocycle on X with respect to θ. Then the mapping $\pi : \mathbb{T}^+ \times P \times X \to P \times X$ defined by

$$\pi(t, p, x) := (\theta_t p, \phi(t, p, x))$$

for all $t \in \mathbb{T}^+$, $(p, x) \in P \times X$ forms an autonomous semi–dynamical system on the state space $P \times X$, i.e. the set of mappings $\{\pi(t, \cdot, \cdot)\}_{t\in\mathbb{T}^+}$ of $P \times X$ into itself is a semi–group, which is called a continuous *skew product flow* [17].

The usual concept of a global attractor for the autonomous semi–dynamical system π on the state space $P \times X$ can be used here. It is the maximal nonempty compact subset $\mathcal{A}$ of $P \times X$ which is π–invariant, that is

$$\pi(t, \mathcal{A}) = \mathcal{A} \quad \text{for all} \quad t \in \mathbb{T}^+,$$

and attracts all compact subsets of $P \times X$, that is

$$\lim_{t\to\infty} H^*_{P\times X}\left(\pi(t,\mathcal{D}), \mathcal{A}\right) = 0 \qquad \text{for all} \quad \mathcal{D} \in K(P\times X),$$

where $K(P \times X)$ is the space of all nonempty compact subsets of $P \times X$ and $H^*_{P\times X}$ is the Hausdorff pseudo–metric on $K(P \times X)$.

Another type of attractor, called a *pullback attractor*, consists of subsets of the original state space X, which is advantageous for discretizations of the nonautonomous system.

Definition 1 *A family $\widehat{A} = \{A_p\}_{p\in P}$ of nonempty compact sets of X is called a* pullback attractor *of the cocycle ϕ if it is ϕ–invariant, that is*

$$\phi(t,p,A_p) = A_{\theta_t p} \qquad \textit{for all} \quad t \in \mathbb{T}^+, p \in P,$$

and pullback attracting*, that is*

$$\lim_{t\to\infty} H^*_X\left(\phi(t,\theta_{-t}p,D), A_p\right) = 0 \qquad \textit{for all} \quad D \in K(X),\ p \in P.$$

In the sequel we will require the following theorem which combines several known results.

Theorem 2 *Let ϕ be a continuous cocycle on X with respect to a group θ of continuous mappings on P and let $\pi = (\phi,\theta)$ be the corresponding skew product flow on $P \times X$. In addition, suppose that there is a nonempty compact subset B of X and for every $D \in K(X)$ there exists a $T(D) \in \mathbb{T}^+$, which is indepedent of $p \in P$, such that*

$$\phi(t,p,D) \subset B \quad \textit{for all} \quad t > T(D). \tag{1}$$

Then

1. *there exists a unique pullback attractor $\widehat{A} = \{A_p\}_{p\in P}$ of the cocycle ϕ on X, where*

$$A_p = \bigcap_{\tau\in\mathbb{T}^+} \overline{\bigcup_{\substack{t>\tau\\ t\in\mathbb{T}^+}} \phi\left(t,\theta_{-t}p,B\right)}. \tag{2}$$

2. *there exists a global compact attractor $\mathcal{A}$ of the autonomous semi–dynamical system π on $P \times X$, where*

$$\mathcal{A} = \bigcap_{\tau\in\mathbb{T}^+} \overline{\bigcup_{\substack{t>\tau\\ t\in\mathbb{T}^+}} \pi\left(t, P\times B\right)}.$$

3. *assertions* **1** *and* **2** *above are equivalent and*

$$\mathcal{A} = \bigcup_{p \in P} \{p\} \times A_p.$$

See Crauel and Flandoli [7] and Schmalfuß [16] for the proof of Assertion **1** and Cheban and Fakeeh [5] and Hale [10] for the proof of Assertion **2**. Assertion **3** has been proved by Cheban [4].

Remark: Assertion **1** true remains under weaker conditions. For instance, the sets D in the absorbing condition (1) could be parameter dependent, the parameter space P need not be compact nor the mappings θ_t continuous (which is the situation in random dynamical systems, see Arnold [1]). Note that the validity of Assertion **3** for nonuniform absorbing times, a situation which occurs in important applications, remains open. See [6] for a systematic investigation of the relationship between the pullback and forwards attractors of the cocycle system and the global attractor of the associated skew product flow.

3 Nonautonomous quasilinear differential equation

We consider a nonautonomous quasilinear differential equation

$$\dot{x} = A(p)x + f(p, x) \tag{3}$$

on $\mathbb{R}^d$ where $p \in P$, on which there exists a group of mappings $\theta_t : P \to P$ for all $t \in \mathbb{R}$. A solution $x(t) = \phi(t, p, x_0)$ of (3) with initial value $x(t) = x_0$ satisfies the equation

$$\frac{d\phi}{dt}(t, p, x_0) = A(\theta_t p)\phi(t, p, x_0) + f(\theta_t p, \phi(t, p, x_0)).$$

Our assumptions are

D1. P is a compact metric space and $(t, p) \mapsto \theta_t p$ is continuous.

D2. $p \mapsto A(p)$ is continuous and satisfies

$$(A(p)x, x) \leq -\alpha(p) |x|^2$$

for all $(p, x) \in P \times \mathbb{R}^d$.

D3. $(p,x) \mapsto f(p,x)$ is continuous, is locally Lipschitz in x uniformly in $p \in P$ and satisfies

$$(f(p,x),x) \leq a(p)\,|x|^2 + c(p)$$

for all $(p,x) \in P \times I\!R^d$.

D4. $\alpha(p) > 0$, $\alpha(p) - a(p) \geq \alpha_0 > 0$ and $c(p) \leq c_0 < \infty$ for all $p \in P$.

Remark. The mapping f in **D3** denotes the remaining part of the differential equation. It may also contain linear terms that has not been included in linear part with the matrix $A(p)$. Though seemingly superfluous, it is sometimes convenient to distinguish the matrix operator $A(p)$ in this way (especially in infinite dimensional generalizations, which are not considered here).

Remark. By the above continuity and the compactness of P we have the finite uniform upper bounds

$$A_0 := \sup_{p \in P} \|A(p)\|, \qquad F_R := \sup_{p \in P, |x| \leq R} |F(p,x)|.$$

We also consider a variable timestep one-step explicit numerical scheme corresponding to the differential equation (3), such as the Euler scheme, which we write as

$$x_{n+1} = x_n + h_n F\left(h_n, \theta_{t_n} p, x_n\right) \tag{4}$$

where $t_0 = 0$ and $t_n = \sum_{j=0}^{n-1} h_j$, $t_{-n} = -\sum_{j=1}^{n} h_{-j}$ for $n \geq 1$ for $\{h_j\}_{n \in Z\!\!\!Z}$ a given two sided sequence of positive terms and $F : [0,1] \times P \times I\!R^d \to I\!R^d$ is the increment function. We make the following assumptions:

N1. F is continuous in all of its variables and locally Lipschitz in x uniformly in $(h,p) \in [0,1] \times P$;

N2. The numerical scheme (4) satisfies a local discretization error estimate of the form

$$|\phi(h_0,p,x_0) - x_1| \leq h_0 \mu_R(h_0), \qquad |x_0| \leq R,$$

for each $R > 0$, where $\mu_R(h) > 0$ for $h > 0$ and $\mu_R(h) \to 0$ as $h \to 0+$.

N3. F satisfies the consistency condition

$$F(0,p,x) = A(p)x + f(p,x)$$

for all $(p,x) \in P \times I\!R^d$;

N4. F satisfies the Lipschitz consistency condition

$$|(F(h,p,x) - A(p)x - f(p,x)) - (F(h,p,y) - A(p)y - f(p,y))| \leq \bar{\mu}_R(h)|x-y|$$

for all $|x|$, $|y| \leq R$ uniformly in $p \in P$, where $\bar{\mu}_R(h) > 0$ for $h > 0$ and $\bar{\mu}_R(h) \to 0$ as $h \to 0+$.

For example, $F(h,p,x) \equiv A(p)x + f(p,x)$ for the Euler scheme applied to the differential equation (3). Note that for one-step order schemes such as the Euler and Runge–Kutta schemes, $\mu(h)$ is typically of the form $K_R h^p$ for some integer $p \geq 1$. In our case here this would require the differentiability of F in p as well as x and of θ_t in t, which is too restrictive for certain applications.

Our main result (see Theorem 6 below) is that nonautonomous dynamical systems generated by the differential equation (3) and the numerical scheme (4) both have pullback and global attractors, and that the numerical attractors converge upper semi–continuously to the corresponding attractors of the differential equation as the step size goes to zero. We will apply Theorem 2 to establish the existence of such attractors, but first we need to show in what sense the numerical scheme (4) generates a discrete time cocycle mapping and skew product flow.

3.1 An example

We consider the 3–dimensional Lorenz system with time dependent coefficients

$$\begin{aligned} \dot{x_1} &= -\sigma(t)x_1 + \sigma(t)x_2 \\ \dot{x_2} &= r(t)x_1 - x_2 - x_1x_3 \\ \dot{x_3} &= -b(t)x_3 + x_1x_2 \end{aligned}$$

(see Temam [22], Chapter I.2.3). Assuming that σ and r are differentiable functions of $\mathbb{R}$ into itself, we can rewrite this system after the transformation $x_3 := x_3 - r(t) - \sigma(t)$ in the form (3) with the matrix

$$A(p) = \begin{pmatrix} -\sigma(0) & \sigma(0) & 0 \\ -\sigma(0) & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the mapping

$$f(p,x) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 - \dot{\sigma}(0) - \dot{r}(0) - b(0)(r(0)+\sigma(0)) - (b(0)-1)x_3 \end{pmatrix},$$

where $x = (x_1, x_2, x_3)$ and $p := p(\cdot) = (b(\cdot), r(\cdot), \sigma(\cdot)) \in C(I\!R, I\!R) \times C^1(I\!R, I\!R) \times C^1(I\!R, I\!R)$.

Suppose that b is almost periodic in $C(I\!R, I\!R)$ and that σ, r are almost periodic in $C^1(I\!R, I\!R)$ with

$$0 < \sigma_{\min} := \inf_{t\in R} \sigma(t), \qquad 1 < \inf_{t\in R} b(t).$$

Then define θ_t for each $t \in I\!R$ by $\theta_t p(\cdot) := p(\cdot + t)$. Finally, define the parameter set P to be the closed hull [17]

$$P := \overline{\bigcup_{t\in I\!R} \theta_t(b(\cdot), r(\cdot), \sigma(\cdot))},$$

which is a compact subset of $C(I\!R, I\!R) \times C^1(I\!R, I\!R) \times C^1(I\!R, I\!R)$.

We need to check that the assumptions **D1**–**D4** are satisfied by the differential equation with this matrix A and function f. The first assumption **D1** follows straighforwardly and the second **D2** holds with $\alpha(p) \equiv \alpha_0 := \min(\sigma_{\min}, 1) > 0$ with the assumptions on σ and r ensuing the continuity of A. These assumptions on σ and r also ensure the continuity of f needed in **D3** and the estimate is given by

$$\begin{aligned} (f(p,x),x) &= (-\dot{\sigma}(0) - \dot{r}(0))x_3 - b(0)(r(0)+\sigma(0))x_3 - (b(0)-1)x_3^2 \\ &\le \frac{(\dot{\sigma}(0)+\dot{r}(0))^2}{2(b(0)-1)} + \frac{b^2(0)(r(0)+\sigma(0))^2}{2(b(0)-1)} =: c(p) \end{aligned}$$

with $a(p) \equiv 0$. Finally, it is obvious from these definitions of the constants that **D4** is also satisfied, in particular with $c_0 := \sup_{p\in P} c(p) < \infty$.

The Euler scheme for this differential equation also satisfies assumptions **N1**–**N4**. Since $F(h,p,x) \equiv A(p)x + f(p,x)$ here, assumptions **N3** and **N4** hold trivially, while assumption **N1** follows from **D1**–**D3** above. Assumption **N2** also follows from **D1**–**D3** with the proof being almost the same as in the first part of the proof in the Appendix. Note that if b is also assumed to be continuously differentiable, then we have the usual second order local discretization error here. One can show that assumptions **N1**–**N4** are also satisfied by higher order schemes such as Runge–Kutta schemes, but the details are not as straightforward or clean as for the Euler scheme.

4 Cocycle property

The solution $\phi(t, p, x_0)$ of the differential equation (3) satisfies assumptions **N1–DN4** the initial condition

$$\phi(0, p, x_0) = x_0 \quad \text{for all} \quad (p, x_0) \in P \times I\!R^d$$

and the cocycle property

$$\phi(s + t, p, x_0) = \phi(s, \theta_t p, \phi(t, p, x_0)) \quad \text{for all} \quad s, t \in I\!R^+, \ \ (p, x_0) \in P \times I\!R^d$$

with respect to the autonomous dynamical system generated by the group $\{\theta_t\}_{t \in I\!R}$ on P. (Existence of such solutions for all $t \in I\!R^+$ is assured by that of an absorbing set to be established in the proof of Theorem 6 below). By our assumptions the mapping $(t, p, x) \mapsto \phi(t, \theta_t p)$ is continuous. Morever, the mapping $\Phi := (\phi, \theta)$ defined on $I\!R^+ \times P \times I\!R^d$

$$\Phi(t, p, x) := (\phi(t, p, x), \theta_t p) \quad \text{for all} \quad (t, p, x_0) \in I\!R^+ \times P \times I\!R^d$$

generates an autonomous semi-dynamical system, that is a skew product flow, on the state space $\mathcal{X} := P \times I\!R^d$.

The situation is somewhat more complicated for the discrete time system generated by the numerical scheme with variable time steps. For this we will restrict the choice of admissible stepsize sequences. For each $\delta > 0$, we define $\mathcal{H}^\delta$ to be the set of all two sided sequences $\{h_n\}_{n \in Z\!\!\!Z}$ satisfying

$$\frac{1}{2}\delta \leq h_n \leq \delta \tag{5}$$

for each $n \in Z\!\!\!Z$ (the particular factor 1/2 here is chosen just for convenience). The set $\mathcal{H}^\delta$ is compact metric space with the metric

$$\rho_{\mathcal{H}^\delta}\left(\mathbf{h}^{(1)}, \mathbf{h}^{(2)}\right) = \sum_{n=-\infty}^{\infty} 2^{-|n|} \left| h_n^{(1)} - h_n^{(2)} \right|.$$

We then consider the shift operator $\tilde{\theta} : \mathcal{H}^\delta \to \mathcal{H}^\delta$ defined by $\tilde{\theta}\mathbf{h} = \tilde{\theta}\{h_n\}_{n \in Z\!\!\!Z} := \{h_{n+1}\}_{n \in Z\!\!\!Z}$. The operator $\tilde{\theta}$ is a homeomorphism with respect to the above metric on $\mathcal{H}^\delta$ and its group of iterates $\{\tilde{\theta}_n\}_{n \in Z\!\!\!Z}$ forms a discrete time autonomous dynamical system on the compact metric space $\left(\mathcal{H}^\delta, \rho_{\mathcal{H}^\delta}\right)$. Finally, for a given sequence $\{h_n\}_{n \in Z\!\!\!Z}$ we set $t_0 = 0$ and define $t_n = t_n(\mathbf{h}) := \sum_{j=0}^{n-1} h_j$ and $t_{-n} = t_{-n}(\mathbf{h}) := -\sum_{j=1}^{n} h_{-j}$ for $n \geq 1$.

Now we introduce the parameter space $\mathcal{Q}^\delta := \mathcal{H}^\delta \times P$ for a fixed $\delta > 0$ and we use the following lemma to introduce a discrete time autonomous dynamical system $\Theta = \{\Theta_n\}_{n \in Z\!\!\!Z}$ on $\mathcal{Q}^\delta$.

Lemma 3 *The mappings* $\Theta_n : \mathcal{Q}^\delta \to \mathcal{Q}^\delta$, $n \in \mathbb{Z}$, *generated by iteration of*

$$\Theta_0 := \mathrm{id}_{\mathcal{Q}^\delta}, \qquad \Theta_1(\mathbf{h}, p) := \left(\tilde{\theta}_1 \mathbf{h}, \theta_{h_0} p\right), \qquad \Theta_{-1}(\mathbf{h}, p) := \left(\tilde{\theta}_{-1}\mathbf{h}, \theta_{-h_{-1}} p\right)$$

form a group of continuous mappings on $\mathcal{H}^\delta \times P$.

Proof: We first show that $\Theta_1\Theta_{-1} = \Theta_{-1}\Theta_1 = \Theta_0 = \mathrm{id}_{\mathcal{Q}^\delta}$. Indeed we have

$$\begin{aligned} \Theta_1\Theta_{-1}(\mathbf{h}, p) &= \Theta_1\left(\tilde{\theta}_{-1}\mathbf{h}, \theta_{h_{-1}} p\right) = \left(\tilde{\theta}_1\tilde{\theta}_{-1}\mathbf{h}, \theta_{h_{-1}}\theta_{-h_{-1}} p\right) = (\mathbf{h}, p) \\ \Theta_{-1}\Theta_1(\mathbf{h}, p) &= \Theta_{-1}\left(\tilde{\theta}_1\mathbf{h}, \theta_{h_0} p\right) = \left(\tilde{\theta}_{-1}\tilde{\theta}_1\mathbf{h}, \theta_{-h_0}\theta_{h_0} p\right) = (\mathbf{h}, p). \end{aligned}$$

The continuity of the Θ_n mappings follows from the facts that $(t, p) \mapsto \theta_t p$ is continuous, $\tilde{\theta}$ is a homeomorphism and the composition and cartesian products of continuous mappings are continuous. □

We define a mapping $\psi : \mathbb{Z}^+ \times \mathcal{Q}^\delta \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi(0, q, x_0) := x_0, \qquad \psi(n, q, x_0) = \psi(n, (\mathbf{h}, p), x_0) := x_n \qquad n \geq 1,$$

where x_n is the nth iterate of the numerical scheme (4) with initial value $x_0 \in \mathbb{R}^d$, initial parameter $p \in P$ and stepsize sequence $\mathbf{h} \in \mathcal{H}^\delta$. These mappings are continuous on $\mathcal{Q}^\delta \times \mathbb{R}^d$. They also satisfy a cocycle property with respect to Θ.

Lemma 4 ψ *is a discrete time cocycle* $\mathbb{R}^d$ *with respect to the group* Θ *on* $\mathcal{Q}^\delta$.

Proof: We write the numerical scheme (4) for a given $(\mathbf{h}, p)$ and x_0 as

$$x_{n+1} = x_n + h_n F\left(h_n, \theta_{t_n(\mathbf{h})} p, x_n\right), \qquad n \in \mathbb{Z}^+,$$

where $t_0(\mathbf{h}) = 0$ and $t_n(\mathbf{h}) = \sum_{j=0}^{n-1} h_j$ for $n \geq 1$. Hence, in terms of the ψ mapping we have

$$\psi(n+1, (\mathbf{h}, p), x_0) = \psi(n, (\mathbf{h}, p), x_0) + h_n F(h_n, \theta_{t_n(\mathbf{h})} p, \psi(n, (\mathbf{h}, p), x_0))$$

in general and

$$\psi(1, (\mathbf{h}, p), x_0) = x_0 + h_0 F\left(h_0, \theta_{t_0(\mathbf{h})} p, x_0\right))$$

for $n = 0$ since by definition $\psi(0, (\mathbf{h}, p), x_0) = x_0$, which gives the identity property. The cocycle property is established as follows. Let $n \geq 0$. Then

$$\begin{aligned} \psi(1+n, q, x_0) &= \psi(1+n, (\mathbf{h}, p), x_0) \\ &= \psi(n, (\mathbf{h}, p), x_0) + h_n F\left(h_n, \theta_{t_n(\mathbf{h})} p, \psi(n, (\mathbf{h}, p), x_0)\right) \\ &= \psi(n, (\mathbf{h}, p), x_0) + (\tilde{\theta}_n \mathbf{h})_0 F\left((\tilde{\theta}_n \mathbf{h})_0, \theta_{t_0(\tilde{\theta}_n \mathbf{h})} p, \psi(n, (\mathbf{h}, p), x_0)\right) \\ &= \psi\left(1, \Theta_n(\mathbf{h}, p), \psi(n, (\mathbf{h}, p), x_0)\right) = \psi\left(1, \Theta_n q, \psi(n, q, x_0)\right), \end{aligned}$$

that is $x_{1+n} = = \psi(1, \Theta_n q, x_n)$. Iterating this n times, we obtain

$$x_n = \psi(1, \Theta_{n-1} q, x_0) \circ \cdots \circ \psi(1, \Theta_0 q, x_0).$$

Similarly with $m \geq 2$ we have

$$\begin{aligned} \psi(m+n, q, x_0) = x_{m+n} &= \psi(1, \Theta_{m+n-1} q, \cdot) \circ \cdots \circ \psi(1, \Theta_n q, \cdot) \circ \\ &\quad \circ \psi(1, \Theta_{n-1} q, \cdot) \circ \cdots \circ \psi(1, \Theta_0 q, x_0) \\ &= \psi(1, \Theta_{m+n-1} q, \cdot) \circ \cdots \circ \psi(1, \Theta_n q, x_n) \\ &= \psi(1, \Theta_{m-1} \Theta_n q, \cdot) \circ \cdots \circ \psi(1, \Theta_0 \Theta_n q, x_n) \\ &= \psi(m, \Theta_n q, x_n) = \psi(m, \Theta_n q, \psi(n, q, x_0)), \end{aligned}$$

which is the desired cocycle property.

Finally we define $\Psi := (\psi, \Theta)$ and observe that the mappings $\Psi(n, \cdot, \cdot)$ are continuous on $\mathcal{Q}^\delta \times I\!R^d$ for $n \in Z\!\!Z^+$.

Remark: The mappings ψ, Θ and Ψ are defined in the same way for each $\delta > 0$, so we do not index them with δ.

¿From the cocycle and group properties we obtain

Lemma 5 *$\Psi = (\psi, \Theta)$ is a discrete time autonomous semi–dynamical system on the state space $\mathcal{Q}^\delta \times I\!R^d$.*

5 Main result

Our main result is the to establish the existence of pullback attractors for the nonautonomous dynamical systems (NDS) generated by the differential equation and numerical scheme and to show that the components of the numerical pullback attractor are upper semi–continuous in their parameter and converge upper semi–continuously to the corresponding components of the differential equation's pullback attractor. Global attractors also exist for the corresponding skew product flows and converge upper semi–continuously in an appropriate sense.

Theorem 6 *Let Assumptions* **D1–D4** *and* **N1–N3** *hold. Then the continuous time NDS (ϕ, θ) generated by the differential equation (3) has a pullback attractor $\widehat{A} = \{A_p\}_{p \in P}$ and the discrete time NDS (Ψ, Θ) generated by the numerical scheme (4) has a pullback attractor $\widehat{A}^\delta = \{A_q^\delta\}_{q \in \mathcal{Q}^\delta}$, provided the maximal stepsize δ is* *sufficiently small, such that the setvalued mappings* $p \mapsto A_p$ *and* $(p, \mathbf{h}) \mapsto A^\delta_{(p,\mathbf{h})}$ *are upper semi–continuous with respect to the Hausdorff metric and satisfy*

$$\lim_{\delta \to 0+} \sup_{\mathbf{h} \in \mathcal{H}^\delta} H^*_{\mathbb{R}^d}\left(A^\delta_{(p,\mathbf{h})}, A_p\right) = 0 \qquad \textit{for each} \quad p \in P.$$

Moreover, the corresponding skew product flows have global attractors $\mathcal{A}$ *and* $\mathcal{A}^\delta$*, respectively, of the form*

$$\mathcal{A} = \bigcup_{p \in P} \{p\} \times A_p, \qquad \mathcal{A}^\delta = \bigcup_{q \in \mathcal{Q}^\delta} \{q\} \times A^\delta_q,$$

which satisfy

$$\lim_{\delta \to 0+} H^*_{P \times \mathbb{R}^d}\left(\mathrm{Pr}_{P \times \mathbb{R}^d} \mathcal{A}^\delta, \mathcal{A}\right) = 0.$$

6 Proof of Theorem 6

The proof of the convergence assertions in Theorem 6 will follow immediately from an application of Theorem 2 after it has been shown that the ball $B[0; R_0]$ in $\mathbb{R}^d$ with centre 0 and radius $R_0 := 3\sqrt{c_0/\alpha_0}$ is a forwards absorbing set uniformly in all parameters for both the continuous time and discrete time cocycle systems under consideration. The convergence assertions require additional work.

6.1 Existence of an absorbing set

Write $x(t)$ for the solution $\phi(t, p, x_0)$ of (3), so

$$\frac{dx}{dt}(t) = A(\theta_t p)x(t) + f(\theta_t p, x(t)).$$

The following estimate will also be used to construct an absorbing set.

$$\begin{aligned}
\frac{d}{dt}|x(t)|^2 &= 2\left(\frac{dx}{dt}(t), x(t)\right) && (6)\\
&= 2\left(A(\theta_t p)x(t) + f(\theta_t p, x(t)), x(t)\right)\\
&= 2\left(A(\theta_t p)x(t)\right) + 2\left(f(\theta_t p, x(t)), x(t)\right)\\
&\le -2\alpha(\theta_t p)\,|x(t)|^2 + 2a(\theta_t p)\,|x(t)|^2 + 2c(\theta_t p)\\
&\le -2\left(\alpha(\theta_t p) - a(\theta_t p)\right)\,|x(t)|^2 + 2c_0\\
&\le -2\alpha_0\,|x(t)|^2 + 2c_0 && (7)
\end{aligned}$$

and so

$$|x(t)|^2 \leq |x_0|^2 e^{-2\alpha_0 t} + \frac{c_0}{\alpha_0}\left(1 - e^{-2\alpha_0 t}\right).$$

This implies that the ball $B[0; R_0]$ with radius $R_0 = 3\sqrt{c_0/\alpha_0}$ is a forwards absorbing and positively invariant set for all solutions of the differential equation (3) uniformly in $p \in P$. Note for later purposes that the ball $B[0; 2R_0/3]$ is also positively invariant for the differential equation.

The proof that $B[0; R_0]$ is a uniform forwards absorbing set for the numerical scheme (4) is more complicated. First we show that the inequality (7) implies

$$|x(t)| \leq |x_0| e^{-\alpha_0 t} + \sqrt{\frac{c_0}{\alpha_0}}\left(1 - e^{-\alpha_0 t}\right). \tag{8}$$

as long as $|x(t)| \geq \frac{1}{3} R_0 = \sqrt{\frac{c_0}{\alpha_0}}$. To see this note that (7) can be rewritten as

$$\frac{d}{dt}|x(t)|^2 \leq -2\alpha_0 |x(t)|^2 + 2c_0 \leq -2\alpha_0 |x(t)|^2 + 2\frac{c_0}{\sqrt{\frac{c_0}{\alpha_0}}}\,|x(t)|$$

and hence as

$$\frac{d}{dt}|x(t)| \leq -\alpha_0 |x(t)| + \sqrt{c_0 \alpha_0}$$

as long as $|x(t)| \geq \frac{1}{3} R_0$.

We note also by continuity that there exists a $T = T(c_0, \alpha_0) > 0$ such that $|x(t)| = |\phi(t, p, x_0)| \geq \frac{1}{3} R_0$ for all $t \in [0, T]$, x_0 with $|x_0| \geq \frac{2}{3} R_0$ and $p \in P$.

We fix an $R \gg R_0$ and let K_R be the constant in the local discretization error estimate for the ball $B[0; R]$. Then from (8) with $x(h) = \phi(h, p, x_0)$ and Assumption **N2** we have

$$\begin{aligned} |x_1| &\leq |\phi(h, p, x_0)| + |\phi(h, p, x_0) - x_1| \\ &\leq |x_0| e^{-\alpha_0 h} + \sqrt{\frac{c_0}{\alpha_0}}\left(1 - e^{-\alpha_0 h}\right) + K_R h \mu_R(h) \end{aligned} \tag{9}$$

for $x_0 \in A[R, 2R_0/3] := B[0; R] \backslash B[0; 2R_0/3]$ and $h \in (0, T]$. Now let $\delta_0 \in (0, 1] \cap (0, T]$ be such that

$$K_R h \mu_R(h) \leq \frac{1}{3} R_0, \qquad \frac{K_R h \mu_R(h)}{1 - e^{-\alpha_0 h}} \leq R - \frac{1}{3} R_0$$

for $h \in (0, \delta_0]$. Then from (9) for $x_0 \in A[R, 2R_0/3]$ and $h \in (0, \delta_0]$ we have

$$
\begin{aligned}
|x_1| &\leq |x_0| e^{-\alpha_0 h} + \sqrt{\frac{c_0}{\alpha_0}} \left(1 - e^{-\alpha_0 h}\right) + K_R h \mu_R(h) \\
&\leq R e^{-\alpha_0 h} + \frac{1}{3} R_0 \left(1 - e^{-\alpha_0 h}\right) + \left(R - \frac{1}{3} R_0\right) \left(1 - e^{-\alpha_0 h}\right) \leq R,
\end{aligned}
$$

so $x_1 \in B[0; R]$. In addition, for $x_0 \in B[0; 2R_0/3]$ we have $x(h) \in B[0; 2R_0/3]$, so

$$
|x_1| \leq |\phi(h, p, x_0)| + |\phi(h, p, x_0) - x_1| \leq \frac{2}{3} R_0 + K_R h \mu_R(h) \leq \frac{2}{3} R_0 + \frac{1}{3} R_0 = R_0
$$

for $h \in (0, \delta_0]$, so $x_1 \in B[0; R]$ here too. Hence, the ball $B[0; R]$ is positively invariant for the numerical scheme for $x_0 \in B[0; R]$ and $h \in (0, \delta_0]$. We can thus apply the inequality (9) iteratively as long as the $x_n \in A[R, 2R_0/3]$, that is we have

$$
|x_{n+1}| \leq |x_n| e^{-\alpha_0 h} + \sqrt{\frac{c_0}{\alpha_0}} \left(1 - e^{-\alpha_0 h}\right) + K_R h \mu_R(h) \tag{10}
$$

when x_0, x_1, $\ldots$, $x_n \in A[R, 2R_0/3]$ and $h \in (0, \delta_0]$.

Now further restrict the stepsize so that

$$
\frac{K_R h \mu_R(h)}{1 - e^{-\alpha_0 h}} \leq \frac{1}{2} R_0 - \frac{1}{3} R_0 = \frac{1}{2} R_0 - \frac{1}{2} \sqrt{\frac{c_0}{\alpha_0}}
$$

for all $h \in (0, \delta_1]$, where $\delta_1 \in (0, \delta_0)$. Using a similar argument as above for the ball $B[0; R]$, we can show that the ball $B[0; R_0]$ is positively invariant for the numerical scheme with stepsizes $h \in (0, \delta_1]$.

To show that the ball $B[0; R_0]$ is absorbing, we further restrict the stepsize so that

$$
\frac{1}{2} \left(1 + e^{-\alpha_0 h}\right) \leq e^{-\frac{1}{4} \alpha_0 h}
$$

for all $h \in (0, \delta_2]$, where $\delta_2 \in (0, \delta_1]$. Then, if $x_0 \in A[R, R_0]$, from inequality (10) we have

$$
\begin{aligned}
|x_1| &\leq |x_0| e^{-\alpha_0 h} + \sqrt{\frac{c_0}{\alpha_0}} \left(1 - e^{-\alpha_0 h}\right) + K_R h \mu_R(h) \\
&\leq |x_0| e^{-\alpha_0 h} + \sqrt{\frac{c_0}{\alpha_0}} \left(1 - e^{-\alpha_0 h}\right) + \left(\frac{1}{2} R_0 - \sqrt{\frac{c_0}{\alpha_0}}\right) \left(1 - e^{-\alpha_0 h}\right)
\end{aligned}
$$

$$\leq \; |x_0| e^{-\alpha_0 h} + \frac{1}{2}|x_0| \left(1 - e^{-\alpha_0 h}\right)$$

$$\leq \; \frac{1}{2}\left(1 + e^{-\alpha_0 h}\right)|x_0| \; \leq \; e^{-\frac{1}{4}\alpha_0 h}|x_0|.$$

In particular, when $x_0, x_1, \ldots, x_j \in B[0; R] \setminus B[0; R_0]$ we can iterate the last inequality to obtain

$$|x_j| \leq e^{-j\frac{1}{8}\alpha_0 \delta}|x_0|$$

if we use variable stepsizes with $\delta/2 \leq h_j \; \delta$ with $\delta \in (0, \delta_2]$. Obviously, there exists a finite integer $J_R(x_0)$ such that $|x_j| \leq R_0$ for all $j \geq J_R(x_0)$, that is the ball $B[0; R_0]$ is absorbing for the numerical scheme for all stepsize sequences $\mathbf{h} \in \mathcal{H}^\delta$ with $\delta \in (0, \delta_2]$. Note that this holds uniformly in $p \in P$.

6.2 Upper semi–continuity of the pullback attractor component sets

Let $\psi(n, q, x)$ denote the numerical trajectory and Θ the shift operator on $\mathcal{Q}^\delta$. The mappings $q \mapsto \Theta q$ and $(q, x) \mapsto \psi(n, q, x)$ for each integer positive n are continuous.

The absorbing set $B = B[0; R_0]$ is compact absorbing set and forwards invariant uniformly in $q \in \mathcal{Q}^\delta$. Hence, by the cocycle property, the compact sets $\psi(n, \Theta_{-n}q, B)$ are nested with increasing n and the pullback attractor has component sets defined by

$$A_q^\delta := \bigcap_{n \geq 0} \psi(n, \Theta_{-n}q, B)$$

This means, in particular, that $H^*_{\mathbb{R}^d}(\psi(n, \Theta_{-n}q, B), A_q^\delta) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and pick $n_0 = n_0(\varepsilon, q) > 0$ so that

$$\psi(n_0, \Theta_{-n_0}q, B) \subset B\left(A_q^\delta, \varepsilon\right),$$

where $B\left(A_q^\delta, \varepsilon\right)$ is the ball of radius ε about A_q^δ.

Now the compact setvalued mappings $q \mapsto \psi(n, \Theta_{-n}q, B)$ are continuous in q with respect to the Hausdorff metric $H_{\mathbb{R}^d}$ for each fixed n. Fix $\varepsilon > 0$ and pick $n_0 = n_0(\varepsilon, \bar{q})$ from above. Then there exists $\delta(\varepsilon, \bar{q}) = \delta(\varepsilon, n_0(\varepsilon, \bar{q})) > 0$ such that

$$H_{\mathbb{R}^d}\left(\psi(n_0, \Theta_{-n_0}q, B), \psi(n_0, \Theta_{-n_0}\bar{q}, B)\right) < \varepsilon$$

for all $q \in \mathcal{Q}^\delta$ with $\rho_{\mathcal{Q}^\delta}(q, \bar{q}) < \delta(\varepsilon, \bar{q})$. In particular,

$$\psi(n_0, \Theta_{-n_0}q, B) \subset B\left(\psi(n_0, \Theta_{-n_0}\bar{q}, B), \varepsilon\right)$$

for all $q \in \mathcal{Q}^\delta$ with $\rho_{\mathcal{Q}^\delta}(q,\bar{q}) < \delta(\varepsilon,\bar{q})$. Hence we have

$$\begin{aligned} A_q^\delta \subset \psi(n_0,\Theta_{-n_0}q,B) &\subset B\left(\psi(n_0,\Theta_{-n_0}\bar{q},B),\varepsilon\right) \\ &\subset B\left(B\left(A_{\bar{q}}^\delta,\varepsilon\right),\varepsilon\right) = B\left(A_{\bar{q}}^\delta,2\varepsilon\right) \end{aligned}$$

that is $A_q^\delta \subset B\left(A_{\bar{q}}^\delta, 2\varepsilon\right)$ or equivalently $H^*_{\mathbb{R}^d}\left(A_q^\delta, A_{\bar{q}}^\delta\right) < 2\varepsilon$ for all $q \in \mathcal{Q}^\delta$ with $\rho_{\mathcal{Q}^\delta}(q,\bar{q}) < \delta(\varepsilon,\bar{q})$. This means the mapping $q \mapsto A_q^\delta$ is upper semi–continuous.

The proof for the mapping $p \mapsto A_p$ is essentially the same.

6.3 Upper semi–continuous convergence of the discretized pullback attractors

We will now prove the upper semi–continuous convergence of the discretized pullback attractor component sets to their continuous time counterparts. For the proof we need the following lemma on the convergence of the numerical trajectories to the corresponding continuous time trajectory with convergence of the maximum step size to zero. Its proof is given in the appendix.

Lemma 7 *For fixed $t > 0$ and $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}} \in \mathcal{H}^\delta$ for some $\delta > 0$, let $N(t,\mathbf{h})$ be the positive integer such that*

$$h_{-1} + h_{-2} + \cdots + h_{-N(t,\mathbf{h})} \le t < h_{-1} + h_{-2} + \cdots + h_{-N(t,\mathbf{h})-1}$$

and consider a sequence (of stepsize sequences) with $\mathbf{h}^m \in \mathcal{H}^{\delta_m}$, where $\delta_m \to 0$ as $m \to \infty$. Then

$$\psi\left(N(t,\mathbf{h}^m),\Theta_{-N(t,\mathbf{h}^m)}(\mathbf{h}^m,p_m),x_m\right) \to \phi(t,\theta_{-t}p,x_0) \quad as \quad m \to \infty$$

for any sequence $x_m \to x_0 \in \mathbb{R}^d$ and $p_m \to p \in P$.

We suppose that the upper semi–continuous convergence assertion of Theorem 6 is not true. Then there exists an $\varepsilon_0 > 0$ and subsequences (for convenience we use the original index) $\mathbf{h}^m$ in $\mathcal{H}^{\delta_m}$ and $a_m \in A^{\delta_m}_{(p,\mathbf{h}^m)}$ for $m \in \mathbb{N}$ such that

$$\operatorname{dist}(a_m, A_p) \ge \varepsilon_0. \tag{11}$$

Note that the component sets A_p and A_q^δ are contained in the ball $B[0;R_0]$ where $R_0 = 3\sqrt{c_0/\alpha_0}$ for each $p \in P$. Then $a_m \in A^{\delta_m}_{(p,\mathbf{h}^m)} \subset B[0;R_0]$ for each for $m \in \mathbb{N}$ and $B[0;R_0]$ is compact, so there exists a convergent subsequence (again we use the orginal index) $a_m \to a_* \in B[0;R_0]$. Thus we have

$$\operatorname{dist}(a_*, A_p) \ge \varepsilon_0.$$

Now choose $t > 0$ sufficiently large so that by pullback attraction

$$\mathrm{dist}\left(\phi\left(t, \theta_{-t}p, B[0; R_0]\right), A_p\right) < \frac{1}{2}\,\varepsilon_0. \tag{12}$$

By the invariance property of a pullback attractor there exist $b_m \in A^{\delta_m}_{\Theta_{-N(t,\mathbf{h}^m)}(p,\mathbf{h}^m)}$ such that

$$\psi\left(N(t, \mathbf{h}^m), \Theta_{-N(t,\mathbf{h}^m)}(p, \mathbf{h}^m), b_m\right) = a_m. \tag{13}$$

Since the $A^{\delta_m}_{\Theta_{-N(t,\mathbf{h}^m)}(p,\mathbf{h}^m)} \subset B[0; R_0]$ for each for $m \in I\!N$, there exists a convergent subsequence (once again we use the orginal index) $b_m \to b_* \in B[0; R_0]$. By (13) and Lemma 7 we have

$$\phi\left(t, \theta_{-t}p, b_*\right) = a_*,$$

which contradicts (11) with respect (12). This contradiction proves the upper semi–continuous convergence of the numerical pullback attractor components.

6.4 Upper semi–continuous convergence of the discretized global attractors

Let $\mathcal{A} \in P \times I\!R^d$ be the global attractor of the continuous time skew product flow dynamical system $\pi(t, p, x) = (\theta_t p, \phi(t, p, x))$ and $\mathcal{A}^\delta \subset \mathcal{H}^\delta \times P \times I\!R^d$ the global attractor of the discrete time semi–dynamical system $\pi^\delta(n, \mathbf{h}, p, x) = (\Theta_n(\mathbf{h}, p), \psi^\delta(n, (\mathbf{h}, p), x))$ based on the numerical scheme, where we include the superscript δ on π^δ and ψ^δ for emphasis. Then $\mathcal{A}^\delta$ converges to $\mathcal{A}$ as $\delta \to 0$ uniformly in the sense that

$$\lim_{\delta\to 0}\ \sup_{(\mathbf{h}^\delta, p^\delta, x^\delta)\in\mathcal{A}^\delta} H^*_{P\times I\!R^d}\left(\left(p^\delta, x^\delta\right), \mathcal{A}\right) = 0. \tag{14}$$

Suppose that (14) is not true. Then there exist sequences $\delta_n \to 0$ and $\left(\mathbf{h}^{\delta_n}, p^{\delta_n}, x^{\delta_n}\right)$ $\subset \mathcal{A}^{\delta_n}$ and an $\varepsilon_0 > 0$ such that

$$H^*_{P\times I\!R^d}\left(\left(p^{\delta_n}, x^{\delta_n}\right), \mathcal{A}\right) \geq \varepsilon_0.$$

Let B be a absorbing compact set in $I\!R^d$ that is independent of δ. Since $\mathcal{A}^\delta \subset \mathcal{H}^\delta \times P \times B$ and $\mathcal{H}^\delta \times P \times B$ is compact, we can select a subsequence (we use the same index for convenience) such that $p^{\delta_n} \to p^0$ and $x^{\delta_n} \to x^0$ as $n \to \infty$. Hence

$$H^*_{P\times I\!R^d}\left(\left(p^0, x^0\right), \mathcal{A}\right) \geq \varepsilon_0. \tag{15}$$

On the other hand, since B is compact and the absorption is uniform in $p \in P$, there exists a $t > 0$ such that

$$H^*_{P\times I\!R^d}\left(\pi(t, P, B), \mathcal{A}\right) < \frac{1}{2}\,\varepsilon_0.$$

In addition, by invariance, $\pi^{\delta}(j, \mathcal{A}^{\delta}) = \mathcal{A}^{\delta}$ for all $j \in I\!N$. Now choose for j the integer $N(t, \mathbf{h}^{\delta_n})$, where $N(t, \mathbf{h})$ is defined in Lemma 7. Then we can find a $\left(\hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}, \hat{x}^{\delta_n}\right)$ $\in \mathcal{H}^{\delta} \times P \times B$ such that

$$\pi^{\delta_n}\left(N(t, \mathbf{h}^{\delta_n}), \hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}, \hat{x}^{\delta_n}\right) = \left(\mathbf{h}^{\delta_n}, p^{\delta_n}, x^{\delta_n}\right);$$

indeed we can define $\left(\hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}\right)$ by $\Theta_{-N(t,\mathbf{h}^{\delta_n})}(\mathbf{h}^{\delta_n}, p^{\delta_n})$. By a similar compactness argument, there is subsequence of this subsequence (again we use the original index) such that $\left(\hat{p}^{\delta_n}, \hat{x}^{\delta_n}\right) \to (\hat{p}^0, \hat{x}^0) \in P \times B$. By Lemma 7 we then have

$$x^{\delta_n} = \psi^{\delta_n}\left(N(t, \mathbf{h}^{\delta_n}), \left(\hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}\right), \hat{x}^{\delta_n}\right) \to \phi\left(t, \hat{p}^0, \hat{x}^0\right) = x^0$$

while

$$\Theta_{N(t,\mathbf{h}^{\delta_n})}\left(\hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}\right) = \left(\mathbf{h}^{\delta_n}, p^{\delta_n}\right) \to \left(\mathbf{0}, p^0\right) = \left(\mathbf{0}, \theta_t \hat{p}^0\right).$$

Combining these results we have

$$\pi^{\delta_n}\left(N(t, \mathbf{h}^{\delta_n}), \hat{\mathbf{h}}^{\delta_n}, \hat{p}^{\delta_n}, \hat{x}^{\delta_n}\right) \to \left(\mathbf{0}, \pi\left(t, \hat{p}^0, \hat{x}^0\right)\right) = \left(\mathbf{0}, p^0, x^0\right)$$

and hence $H^*_{P \times I\!R^d}\left(\left(p^0, x^0\right), \mathcal{A}\right) < \frac{1}{2}\varepsilon_0$, which contradicts (15). Hence the original assertion (14) must be true.

This completes the proof of the main theorem, Theorem 6.

7 Singleton setvalued pullback attractor case

Let $x_1(t)$ and $x_2(t)$ be two solutions of the differential equation (3) with the same initial parameter p but different initial values in the positively absorbing ball $B[0; R_0]$ and write

$$\Delta(t) = x_1(t) - x_2(t), \qquad \Delta_{f,q}(t) = f(\theta_t p, x_1(t)) - f(\theta_t p, x_2(t)).$$

Let L_0 be the local Lipschitz constant of f in $B[0; R_0]$, which by assumption is uniform in $p \in P$, so

$$|f(p, x_1(t)) - f(p, x_2(t))| \leq L_0 \, |x_1(t) - x_2(t)|$$

or $|\Delta_{f,q}(t)| \leq L_0 |\Delta(t)|$ in $B[0; R_0]$. We assume that $L_0 \leq \alpha_0/2$. Then similarly to earlier (but now we do not use the inner product inequality on the f)

$$\frac{d}{dt}|\Delta(t)|^2 \;\; = \;\; 2\left(\frac{d}{dt}\Delta(t), \Delta(t)\right)$$

$$
\begin{aligned}
&= 2\left(A(\theta_t p)\Delta(t), \Delta(t)\right) + 2\left(\Delta_{f,q}(t), \Delta(t)\right) \\
&\leq -2\alpha_0 \, |\Delta(t)|^2 + 2|\Delta_{f,q}(t)||\Delta(t)| \\
&\leq -2\alpha_0 \, |\Delta(t)|^2 + 2L_0|\Delta(t)|^2 \\
&\leq -2\left(\alpha_0 - L_0\right) \, |\Delta(t)|^2 \leq -\alpha_0 \, |\Delta(t)|^2
\end{aligned}
$$

and so

$$
|\Delta(t)| \leq |\Delta(0)| e^{-(\alpha_0/2)t},
$$

which means the solution operator $x \mapsto \phi(t, p, x)$ is a contraction mapping on the ball $B[0; R_0]$ for each $t > 0$ and $p \in P$.

Now consider the numerical scheme. Let

$$
y_1 = x_1 + hF(h, p, x_1), \qquad y_2 = x_2 + hF(h, p, x_2)
$$

where $h \in [\delta/2, \delta]$ and $x_1,\, x_2 \in B[0; R_0]$. Write

$$
\Delta x = x_1 - x_2, \qquad \Delta y = y_1 - y_2, \qquad \Delta_F(h) = F(h, p, x_1) - F(h, p, x_2),
$$

so $|\Delta F| \leq L_{00}|\Delta x|$ in $B[0; R_0]$, where L_{00} is the local Lipschitz constant of F in x on $B[0; R_0]$, uniformly in $(h, p) \in [0, 1] \times P$. We assume the Lipschitz consistency condition **N4** here, so

$$
|\Delta_F(h) - A\Delta x - \Delta_{f,q}(h)| \leq \bar{\mu}_R(h)|\Delta x|
$$

where we omit the parameter p for convenience. Thus we have

$$
\begin{aligned}
|\Delta y|^2 = (\Delta y, \Delta y) &= (\Delta x + h\Delta_F(h), \Delta x + h\Delta_F(h)) \\
&= (\Delta x, \Delta x) + 2h(\Delta_F(h), \Delta x) + h^2(\Delta_F(h), \Delta_F(h)) \\
&= |\Delta x|^2 + 2h(A\Delta x + \Delta_{f,q}(h), \Delta x) + h^2|\Delta_F(h)|^2 \\
&\qquad +2h(\Delta_F(h) - A\Delta x - \Delta_{f,q}(h), \Delta x) \\
&\leq |\Delta x|^2 - 2h\alpha_0|\Delta x|^2 + 2h|\Delta_{f,q}(h)| \, |\Delta x| + h^2 L_{00}^2|\Delta x|^2 \\
&\qquad +2h|\Delta_F(h) - A\Delta x - \Delta_{f,q}(h)||\Delta x| \\
&\leq |\Delta x|^2(1 - 2h\alpha_0) + 2hL_0 \, |\Delta x|^2 + h^2 L_{00}^2|\Delta x|^2 \\
&\qquad +2h\bar{\mu}_R(h)|\Delta x|^2 \\
&\leq |\Delta x|^2 \left(1 - 2h(\alpha_0 - L_0) + h^2 L_{00}^2 + 2h\bar{\mu}_R(h)\right) \\
&\leq |\Delta x|^2 \left(1 - h\alpha_0 + h^2 L_{00}^2 + 2h\bar{\mu}_R(h)\right),
\end{aligned}
$$

where we have used the assumption that $L_0 \leq \alpha_0/2$. By further restricting h from above we can assure that

$$|\Delta y|^2 \leq |\Delta x|^2 (1 - h\alpha_0/2) \leq |\Delta x|^2 \gamma(\delta\alpha_0/4)$$

for $h \in [\delta/2, \delta]$ and δ sufficiently small. This means that the numerical solution satisfies the contractive condition

$$\left|\psi(n, (\bar{h}, p), x_0) - \psi(n, (\bar{h}, p), \bar{x}_0)\right| \leq |x_0 - \bar{x}_0| \gamma_0^n$$

for all x_0, $\bar{x}_0 \in B[0; R_0]$, $p \in P$ and stepsize sequence $\bar{h} \in \mathcal{H}^\delta$, where $\gamma_0 := \sqrt{\gamma(\delta\alpha_0/4)}$.

¿From the Contraction Mapping Principal we conclude that the original and numerical pullback attractors each consist of a single trajectory. The continuity of their component elements with respect to the parameter follows from Theorem 6 and the fact that upper semi–contiuity there reduces to continuty for the singleton setvalued mappings.

Theorem 8 *Let the Assumptions* **D1**–**D4** *and* **N1**–**N4** *hold and suppose that* $2L_0 \leq \alpha_0$ *and that* δ *is sufficiently small. Then the pullback attractors of Theorem 6 consist of singleton component sets, that is* $A_p = \{a^*(p)\}$ *and* $A^\delta_{(\mathbf{h},p)} = \{a^*_\delta(\mathbf{h}, p)\}$*, where the mappings* $p \mapsto a^*(p)$ *and* $(p, \mathbf{h}) \mapsto a^*_\delta(\mathbf{h}, p)$ *are continuous.*

These singleton valued pullback attractor–trajectories inherit the periodicty or almost periodicity of the differential equation and of the differential equation and stepsize sequence, respectively. This is formulated in the following theorem, the proof of which will be presented in the remainder of this section. The periodic case is straightforward, while the almost periodic case is considerably more complicated and requires the introduction of apropriate definitions and a number of auxiliary results.

A set $A \subset P$ is called minimal with respect to a dynamical system $(P, \mathbb{R}, \theta)$ if it is nonempty, closed and invariant and if no proper subset of A has these properties.

Theorem 9 *Suppose that the assumptions of Theorem 8 hold and that* P *is minimal. Then the singleton valued pullback attractor–trajectory* $A_p = \{a^*(p)\}$ *is periodic (resp., almost periodic) if* $p \in P$ *is periodic (resp., almost periodic), whereas the numerical singleton valued pullback attractor–trajectory* $A^\delta_{(\mathbf{h},p)} = \{a^*_\delta(\mathbf{h}, p)\}$ *is periodic (resp., almost periodic) if* $q = (\mathbf{h}, p) \in \mathcal{Q}^\delta$ *is periodic (resp., almost periodic).*

A sequence $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}}$ is m–periodic if $h_{n+m} = h_n$ for all $n \in \mathbb{Z}$ or, equivalently, if $\tilde{\theta}_m \mathbf{h} = \mathbf{h}$, where m is the smallest integer for which these equalities hold. Recall that we have defined a time sequence $\{t_n(\mathbf{h})\}_{n\in\mathbb{Z}}$ by $t_0(\mathbf{h}) = 0$, $t_n(\mathbf{h}) := \sum_{j=0}^{n-1} h_j$ and $t_{-n}(\mathbf{h}) := -\sum_{j=1}^{n} h_{-j}$ for $n \geq 1$ corresponding to a given sequence $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}}$.

Lemma 10 *Let* $\mathbf{h} \in \mathcal{H}^\delta$ *be* $m-$*periodic and let* $p \in P$ *be* $\tau-$*periodic with respect to* θ*, that is with* $\theta_\tau p = p$ *where* $\tau \in I\!R^+$*. Then the point* $(\mathbf{h}, p) \in \mathcal{Q}^\delta = \mathcal{H}^\delta \times P$ *is periodic with respect to* $\Theta = (\tilde{\theta}, \theta)$ *if and only if* $t_m(\mathbf{h})/\tau$ *is rational.*

Proof. Suppose that $t_m(\mathbf{h})/\tau = k/l$ for some k, $l \in I\!N$. Then $lt_m(\mathbf{h}) = k\tau$ and $\Theta_{lm}(\mathbf{h}, p) = (\mathbf{h}, \theta_{lt_m(\mathbf{h})}p) = (\mathbf{h}, \theta_{k\tau}p) = (\mathbf{h}, p)$. On the other hand, suppose that $\Theta_k(\mathbf{h}, p) = (\mathbf{h}, p)$ for some $k \in I\!N$. Then $(\tilde{\theta}_k\mathbf{h}, \theta_{t_k(\mathbf{h})}p) = (\mathbf{h}, p)$, which implies that $k = l_1 m$ and $t_k(\mathbf{h}) = l_1 t_m(\mathbf{h}) = l_1 t_m(\mathbf{h}) = l_2\tau$ where l_1, $l_2 \in I\!N$. Hence $t_m(\mathbf{h})/\tau = l_1/l_2$.

A subset $M \subseteq \mathbf{T}$ is called *relatively dense* in $\mathbf{T}$ if there exists a positive number $l \in \mathbf{T}$ such that for every $a \in \mathbf{T}$ the interval $[a, a+l] \bigcap \mathbf{T}$ of length l contains an element of M, that is $M \bigcap [a, a+l] \neq \emptyset$ for every $a \in \mathbf{T}$. Let (X, ρ) be a metric space. A function $\varphi : \mathbf{T} \to X$ is called *almost periodic* if for every $\varepsilon > 0$ there exists a relatively dense subset M_ε of $\mathbf{T}$ such that

$$\rho(\varphi(t+\tau), \varphi(t)) < \varepsilon$$

for all $t \in \mathbf{T}$ and $\tau \in M_\varepsilon$. A point $x \in X$ is said to be almost periodic with respect to a mapping $\pi : \mathbf{T} \times X \to X$ if the function $\pi(\cdot, x) : \mathbf{T} \to X$ is almost periodic. The following result can be found in [3, 18].

Theorem 11 *Let* $(X, \mathbf{T}, \pi)$ *be a dynamical system on a compact metric space* (X, ρ)*. Then a point* $x \in X$ *is almost periodic if and only if for every* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\rho\left(\pi(t+t_1, x), \pi(t+t_2, x)\right) < \varepsilon, \qquad \textit{for all } t \in \mathbf{T},$$

whenever $\rho\left(\pi(t_1, x), \pi(t_2, x)\right) < \delta$*.*

A sequence $\{c_n\}$ in $I\!R$ is said to be *almost periodic* if the function $\varphi : \mathbf{Z} \to I\!R$ defined by $\varphi(n) := c_n$ for $n \in \mathbf{Z}$ is almost periodic. A sequence $\{\tau_n\}$ in $I\!R$ will be called *regular* if it has the form

$$\tau_n = an + c_n, \qquad \text{for all } n \in \mathbf{Z},$$

where $a \in I\!R$ is a constant and$\{c_n\}$ is an almost periodic sequence; see Samoilenko and Trofimchiuk [19, 20].

Theorem 12 *Suppose that the time sequence* $\{t_n(\mathbf{h})\}_{n\in\mathbf{Z}}$ *corresponding to* $\mathbf{h} \in \mathcal{H}^\delta$ *is regular. In addition, suppose that dynamical system* $(P, I\!R, \theta)$ *is minimal and almost periodic, that is* P *is minimal and every point* $p \in P$ *is almost periodic. Then the point* $(\mathbf{h}, p) \in \mathcal{Q}^\delta$ *is almost periodic for the dynamical system* $(\mathcal{Q}^\delta, \mathbf{Z}, \Theta)$*.*

Proof. Since the point $p \in P$ is almost periodic for the dynamical system $(P, I\!R, \theta)$, then by Theorem 11 the point $p \in P$ will be almost periodic relatively to the discrete time dynamical system $(P, Z\!\!Z, \theta^{(a)})$, where $\theta^{(a)} = \{\theta_{an}\}_{n \in Z\!\!Z}$ and $t_n(\mathbf{h}) = an + c_n$ is the regularity representation of $\{t_n(\mathbf{h})\}_{n \in Z\!\!Z}$.

We apply Theorem 11 to the dynamical system $(P, I\!R, \theta)$. Given $\varepsilon > 0$, let $\delta(\varepsilon) \in (0, \varepsilon/3)$ be such that

$$\rho_P(\theta_t p_1, \theta_t p_2) < \frac{\varepsilon}{3} \tag{16}$$

for every $t \in I\!R$ and $p_1, p_2 \in P$ with $\rho_P(p_1, p_2) < \delta$. Then we use uniform continuity on the compact space P: given the above $\delta(\varepsilon) > 0$, let $\gamma(\varepsilon) \in (0, \delta(\varepsilon))$ be such that

$$\rho_P(\theta_s p, p) < \delta \tag{17}$$

for every $p \in P$ and $s \in I\!R$ with $|s| \leq \gamma$.

Now for this $\gamma(\varepsilon) > 0$ we denote by $\mathcal{M}_{\gamma(\varepsilon)}$ the relatively dense subset of $Z\!\!Z$ subset for which

$$\rho_{\mathcal{H}^\delta}(\tilde{\theta}_{n+m}\mathbf{h}, \tilde{\theta}_n \mathbf{h}) < \gamma(\varepsilon), \quad |c_{n+m} - c_n| < \gamma(\varepsilon), \quad \rho_P(\theta_{a(n+m)}p, \theta_{an}p) < \gamma(\varepsilon) \tag{18}$$

for all $m \in \mathcal{M}_\varepsilon$, $n \in Z\!\!Z$ and $p \in P$. From (16)–(18) we have

$$\begin{aligned}
\rho_{\mathcal{Q}^\delta}(\Theta_{n+m}(\mathbf{h}, p), \Theta_n(\mathbf{h}, p)) &= \rho_{\mathcal{H}^\delta}(\tilde{\theta}_{n+m}\mathbf{h}, \tilde{\theta}_n \mathbf{h}) + \rho_P(\theta_{t_{n+m}(\mathbf{h})}p, \theta_{t_n(\mathbf{h})}p) \\
&= \rho_{\mathcal{H}^\delta}(\tilde{\theta}_{n+m}\mathbf{h}, \tilde{\theta}_n \mathbf{h}) + \rho_P(\theta_{a(n+m)+c_{n+m}}p, \theta_{an+c_n}p) \\
&< \rho_{\mathcal{H}^\delta}(\tilde{\theta}_{n+m}\mathbf{h}, \tilde{\theta}_n \mathbf{h}) + \rho_P(\theta_{a(n+m)}(\theta_{c_{n+m}}p), \theta_{a(n+m)}(\theta_{c_n}p)) \\
&\quad\; + \rho_P(\theta_{a(n+m)}(\theta_{c_n}p), \theta_{an}(\theta_{c_n}p)) \\
&< \gamma(\varepsilon) + \frac{\varepsilon}{3} + \gamma(\varepsilon) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for all $n \in Z\!\!Z$ and $m \in \mathcal{M}_{\gamma(\varepsilon)}$. Hence the point $(\mathbf{h}, p) \in \mathcal{Q}^\delta$ is almost periodic for the dynamical system $(\mathcal{Q}^\delta, Z\!\!Z, \Theta)$.

Corollary 13 *Let $\mathbf{h} \in \mathcal{H}^\delta$ be m–periodic and let $p \in P$ be almost periodic for the dynamical system $(P, I\!R, \theta)$. Then the point $(\mathbf{h}, p) \in \mathcal{Q}^\delta$ is almost periodic for the dynamical system $(\mathcal{Q}^\delta, Z\!\!Z, \Theta)$. In particular, if $t_m(\mathbf{h})/\tau$ is irrational, then point $(\mathbf{h}, p)$ is almost periodic, but not periodic.*

As the final step in our proof of Theorem 9, we need the following lemma, which we prove directly here noting that the result also follows from Theorems 1 and 2 in [15].

Lemma 14 *Suppose that the assumptions of Theorem 8 hold and let* $\{a_\delta^*(q)\}_{q\in\mathcal{Q}^\delta}$ *denote the singleton valued pullback attractor for the numerical scheme (4). Then the function* $n \mapsto a_\delta^*(\Theta_n q)$ *for* $n \in \mathbb{Z}$ *is periodic (resp., almost periodic) if the point* q *is periodic (resp., almost periodic) for the dynamical system* $(\mathcal{Q}^\delta, \mathbb{Z}, \Theta)$.

Proof. Let $q \in \mathcal{Q}^\delta$ be m–periodic, that is $\Theta_m q = q$. Then $a_\delta^*(\Theta_{n+m} q) = a_\delta^*(\Theta_n \Theta_m q)$ $= a_\delta^*(\Theta_n q)$ for every $n \in \mathbb{Z}$. Hence $n \mapsto a_\delta^*(\Theta_n q)$ is periodic.

The function $a_\delta^* : \mathcal{Q}^\delta \to \mathbb{R}^d$ defined by $q \mapsto a_\delta^*(q)$ for each $q \in \mathcal{Q}^\delta$ is continuous, hence uniformly continuous, on the compact space $\mathcal{Q}^\delta$. That is, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|a_\delta^*(q_1) - a_\delta^*(q_2)| < \varepsilon$ whenever $\rho_{\mathcal{Q}^\delta}(q_1, q_2) < \delta$. Now let the point q be almost periodic and for $\delta = \delta(\varepsilon) > 0$ denote by M_δ the relatively dense subset of $\mathbb{Z}$ such that $\rho_{\mathcal{Q}^\delta}(\Theta_{n+m} q, \Theta_n q) < \delta$ for all $m \in M_\delta$ and $n \in \mathbb{Z}$. From this and the uniform continuity we have

$$|a_\delta^*(\Theta_{n+m} q) - a_\delta^*(\Theta_n q)| < \varepsilon$$

for all $n \in \mathbb{Z}$ and $m \in M_{\delta(\varepsilon)}$. Hence $n \mapsto a_\delta^*(\Theta_n q)$ is almost periodic.

In conclusion, we can restate the assertions of Theorem 9 in more detail as follows.

Corollary 15 *Suppose that the assumptions of Theorem 8 hold and that* P *is minimal. In addition, let* $\{a_\delta^*(\mathbf{h}, p)\}_{(\mathbf{h},p)\in\mathcal{Q}^\delta}$ *denote the singleton valued pullback attractor for the numerical scheme (4).*
1. Let $\mathbf{h} \in \mathcal{H}^\delta$ *be* m*–periodic and* $p \in P$ *be* τ*–periodic. Then* $n \mapsto a_\delta^*(\Theta_n(\mathbf{h}, p)\}$ *is periodic if* $t_m(\mathbf{h})/\tau$ *is rational and almost periodic if* $t_m(\mathbf{h})/\tau$ *is irrational.*
2. Let the time sequence $\{t_n(\mathbf{h})\}_{n\in\mathbb{Z}}$ *corresponding to* $\mathbf{h} \in \mathcal{H}^\delta$ *be regular and let the point* $p \in P$ *be almost periodic. Then* $n \mapsto a_\delta^*(\Theta_n(\mathbf{h}, p)\}$ *is almost periodic.*

8 Appendix: Proof of Lemma 7

Consider an autonomous dynamical system on a compact metric space P described by a group $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ of mappings of P into itself such that the mapping $(t, p) \mapsto \theta_t p$ is continuous. Consider also an ordinary differential equation

$$\dot{x} = g(p, x)$$

on $\mathbb{R}^d$ with a unique solution $x(t; p, x_0)$ satisfying the initial value problem

$$\frac{d}{dt} x(t; p, x_0) = g(\theta_t p, x(t; p, x_0)), \qquad x(0; p, x_0) = x_0.$$

We assume that $(p, x) \mapsto g(p, x)$ is continuous on $P \times I\!\!R^d$ and locally Lipschitz in x uniformly in p, that is for each $R > 0$ there exists an L_R such that

$$|g(p,x) - g(p,y)| \leq L_R|x - y|, \qquad \forall x, y \in B[0; R].$$

In particular, then for a sequence of times t_n and time steps $h_n = t_{n+1} - t_n$ this gives in integral equation form

$$x(t_{n+1}; p, x_0) = x(t_n; p, x_0) + \int_{t_n}^{t_{n+1}} g(\theta_t p, x(t; p, x_0))\, dt.$$

In future we just write $x(t)$ for this solution. By the Mean Value Theorem there exists $\tau_n \in [0, 1]$ such that

$$x(t_{n+1}) = x(t_n) + h_n\, g(\theta_{t_n+\tau_n h_n} p, x(t_n + \tau_n h_n)).$$

The corresponding higher order scheme solution is

$$x_{n+1} = x_n + h_n\, F(h_n, \theta_{t_n} p, x_n),$$

where the increment function $F(h, p, x)$ is continuous and satisfies the consistency condition

$$F(0, p, x) = g(p, x), \qquad \forall p, x.$$

Thus

$$x(t_{n+1}) - x_{n+1} = x(t_n) - x_n + h_n \left[g(\theta_{t_n+\tau_n h_n} p, x(t_n + \tau_n h_n)) - F(h_n, \theta_{t_n} p, x_n) \right]$$

so the global discretization error $E_n := |x(t_n) - x_n|$ is estimated by

$$\begin{aligned}
E_{n+1} &\leq E_n + h_n \left| g(\theta_{t_n+\tau_n h_n} p, x(t_n + \tau_n h_n)) - F(h_n, \theta_{t_n} p, x_n) \right| \\
&\leq E_n + h_n \left| g(\theta_{t_n+\tau_n h_n} p, x(t_n + \tau_n h_n)) - g(\theta_{t_n+\tau_n h_n} p, x(t_n)) \right| \\
&\qquad + h_n \left| g(\theta_{t_n+\tau_n h_n} p, x(t_n)) - g(\theta_{t_n} p, x(t_n)) \right| \\
&\qquad + h_n \left| g(\theta_{t_n} p, x(t_n)) - g(\theta_{t_n} p, x_n) \right| \\
&\qquad + h_n \left| g(\theta_{t_n} p, x_n) - F(h_n, \theta_{t_n} p, x_n) \right| \\
&\leq E_n + h_n L_R \left| x(t_n + \tau_n h_n) - x(t_n) \right| + h_n \omega_g(\Delta_n p; R) \\
&\qquad + h_n L_R \left| x(t_n) - x_n \right| + h_n \omega_F(h_n; R) \\
&= (1 + h_n L_R) E_n + h_n L_R \left| \int_{t_n}^{t_n+\tau_n h_n} g(\theta_s p, x(s))\, ds \right| \\
&\qquad + h_n \omega_g(\Delta_n p; R) + h_n \omega_F(h_n p; R) \\
&\leq (1 + h_n L_R) E_n + h_n^2 L_R M_R + h_n \omega_g(\Delta_n p; R) + h_n \omega_F(h_n p; R)
\end{aligned}$$

where $M_R := \max_{p\in P, x\in B[0;R]} |g(p,x)|$ and $\omega_g(\delta;R)$ is the modulus of continuity of $g(\theta.p,x)$ uniformly in $p \in P$ and $x \in B[0;R]$ and $\omega_F(h_n p;R)$ is the modulus of continuity of $F(\cdot,p,x)$ uniformly in $p \in P$ and $x \in B[0;R]$ that is

$$\omega_g(\delta;R) := \sup_{0\le t\le\delta} \sup_{\substack{p\in P \\ x\in B[0;R]}} |g(\theta_t p, x) - g(p,x)|$$

and

$$\omega_F(\delta;R) := \sup_{0\le h\le\delta} \sup_{\substack{p\in P \\ x\in B[0;R]}} |F(h,p,x) - F(0,p,x)|.$$

Here $\omega_g(\delta;R) \to 0$ and $\omega_F(\delta;R) \to 0$ as $\delta \to 0$.

Now we consider an interval $[0,T]$ and restrict to stepsizes $h_n \in [\delta/2, \delta]$ for some $\delta > 0$. Note then that $t_{n+1} = \sum_{j=0}^{n} h_j$ then satifies $n\delta/2 \le t_n \le n\delta$ with $t_n \le T$, which means $n\delta \le 2T$ for these choices of n. The above difference inequality thus satisfies

$$E_{n+1} \le (1 + L_R\delta)E_n + \delta\left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right)$$

and hence with $E_0 = 0$ yields

$$\begin{aligned} E_n &\le \delta\left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{(1+L_R\delta)^n - 1}{(1+L_R\delta) - 1} \\ &\le \left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{1}{L_R} e^{L_R n\delta} \\ &\le \left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{1}{L_R} e^{2L_R T}, \end{aligned}$$

that is

$$|x(t_n) - x_n| \le \left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{1}{L_R} e^{2L_R T}.$$

Hence for $t \in (t_n, t_{n+1})$, we have

$$\begin{aligned} |x(t) - x_n| &\le |x(t) - x(t_n)| + |x(t_n) - x_n| \\ &\le \left|\int_{t_n}^{t} g(\theta_s, x(s))\, ds\right| + \left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{1}{L_R} e^{2L_R T} \\ &\le \delta M_R + \left(\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta\right) \frac{1}{L_R} e^{2L_R T}. \end{aligned}$$

Let us now consider variable parameters and initial values. let $p_j \to p$ in P and $x_{0j} \to x_0$ in $I\!R^d$. Let $x(t;p,x_0)$ and $x_n(p,x_0)$, etc, denote the corresponding solutions. By continuity in initial conditions and parameters uniformly on a compact time interval $[0,T]$, we have

$$x(t;p_j,x_{0j}) \to x(t;p,x_0),$$

as $j \to \infty$ for $t \in [0,T]$.

Combining all of these partial results for $t \in (t_n, t_{n+1})$ we obtain

$$\begin{aligned} |x_n(p_j,x_{0j}) - x(t;p,x_0)| &\leq |x_n(p_j,x_{0j}) - x(t;p_j,x_{0j})| + |x(t;p_j,x_{0j}) - x(t;p,x_0)| \\ &\leq \delta M_R + (\omega_g(\delta;R) + \omega_F(\delta;R) + L_R M_R \delta) \frac{1}{L_R} e^{2L_R T} \\ &\qquad + |x(t;p_j,x_{0j}) - x(t;p,x_0)| \end{aligned}$$

which converges to zero as the maximum stepsize δ converges to zero and j tends to ∞.

References

[1] Arnold, L., *Random Dynamical Systems.* Springer–Verlag, Heidelberg, 1998.

[2] Aulbach, B., and Garay, B.M., Discretization of semilinear equations with an exponential dichotomy, *Computers Math. Applic.* **28** (1994), 23–35.

[3] Bronshteyn, I.U., *Extensions of Minimal Transformation Groups.* Noordhoff, Leyden, 1979.

[4] Cheban, D.N., Global attractors of infinite–dimensional nonautonomous dynamical systems. *Izvestiya Akad Nauk RM. Mathematika* **25** (3) (1997), 42–57.

[5] Cheban, D.N., and Fakeeh, D.S., *Global Attractors of Dynamical Systems without Uniqueness.* Sigma, Kishinev, 1994.

[6] D. Cheban, P. E. Kloeden and B. Schmalfuß, *The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems . DANSE Preprint* 27/2000, FU Berlin.

[7] Crauel, H., and Flandoli, F., Attractors for random dynamical systems, *Probab. Theory. Relat. Fields* **100** (1994), 1095–1113.

[8] Flandoli, F., and Schmalfuß, B., Weak solutions and attractors of the 3D Navier Stokes equation with nonregular force, *J. Dynamics & Differential Equations* (1998), to appear.

[9] Flandoli, F., and Schmalfuß, B., Random attractors of the 3D Navier Stokes equation with multiplicative white noise, *Stochastics & Stochastics Reports* **59** (1996), 21–45.

[10] Hale, J., *Asymptotic Behavior of Dissipative Dynamical Systems*, Amer. Math. Soc., Providence, RI, 1988.

[11] Kloeden, P.E., and Lorenz, J., Stable attracting sets in dynamical systems and in their one-step discretizations, *SIAM J. Numer. Analysis* **23** (1986), 986-995.

[12] Kloeden, P.E., and Schmalfuß, B., Lyapunov functions and attractors under variable time–step discretization, *Discrete & Conts. Dynamical Systems* **2** (1996), 163–172.

[13] Kloeden, P.E., and Schmalfuß, B., Nonautonomous systems, cocycle attractors and variable time–step discretization, *Numer. Algorithms* **14** (1997), 141–152.

[14] Krasnosel'skii, M.A., *The Operator of Translation along Trajectories of Differential Equations*, Translations of Mathematical Monographs, Volume 19. American Math. Soc., Providence, R.I., 1968.

[15] Sacker, R.J., and Sell, G.R., *Lifting Properties in Skew–Product Flows with Applications to Differential Equations.* Memoirs oft he American Math. Soc., Volume 190 Providence, R.I., 1977.

[16] Schmalfuß, B., The stochastic attractor of the stochastic Lorenz system,in *Nonlinear Dynamics: Attractor Approximation and Global Behaviour,* Proc. ISAM 92 (Editors: N. Koksch, V. Reitmann and T. Riedrich), TU Dresden, 1992, 185–192.

[17] Sell, G.R., *Lectures on Topological Dynamics and Differential Equations.* Van Nostrand–Reinbold, London, 1971.

[18] Sibirsky, K.S., *Introduction to Topological Dynamics.* Noordhoff, Leyden, 1975.

[19] Samoilenko, A.M., and Trofimuchiuk, S.I., Unbounded functions with almost periodic differences, *Ukrainian Math. J.,* **43** (1991), 1409-1413.

[20] Samoilenko, A.M., and Trofimuchiuk, S.I., On the space of sectionally continuous almost periodic functions and almost periodic sets on the line, *Ukrainian Math. J.,* **43** (1991), 1613–1619.

[21] Stuart, A.M., and Humphries, A.R., *Numerical Analysis and Dynamical Systems,* Cambridge University Press, Cambridge 1996.

[22] Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* 2nd edition, Springer Verlag, New York,1997.