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# UNIFORM EXPONENTIAL STABILITY OF LINEAR PERIODIC SYSTEMS IN A BANACH SPACE

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Abstract. This article is devoted to the study of linear periodic dynamical systems, possessing the property of uniform exponential stability. It is proved that if the Cauchy operator of these systems possesses a certain compactness property, then the asymptotic stability implies the uniform exponential stability. We also show applications to different classes of linear evolution equations, such as ordinary linear differential equations in the space of Banach, retarded and neutral functional differential equations, some classes of evolution partial differential equations.

## Introduction

Let $A(t)$ be a $\tau$-periodic continuous $n \times n$ matrix-function. It is well-known that the following three conditions are equivalent:

(1) The trivial solution of equation

$$u' = A(t)u \tag{0.1}$$

is uniformly exponentially stable.

(2) The trivial solution of equation (0.1) is uniformly asymptotically stable.

(3) The trivial solution of equation (0.1) is asymptotically stable.

For equations in infinite-dimensional spaces the statements 1)-3) are not equivalent, as shown by the examples in [15, 26].

It is clear that in general for the infinite-dimensional case condition 1) implies 2) and 2) implies 3). In this article we show that if the Cauchy operator of equation (0.1) satisfies some compactness condition, then 3) implies 1) (see Theorem 2.5 below).

Applications to different classes of linear evolution equations (ordinary linear differential equations in a Banach space, retarded and neutral functional-differential equations, some classes of evolutionary partial differential equations) are given.

The exponential dichotomy of asymptotically compact cocycles was studied by R. Sacker and G. Sell [29]. The general case was studied by C. Chicone and Yu, Latushkin [14] (see also their references), Yu. Latushkin and R. Schnaubelt [25], and many other authors.

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## 1. Linear non-autonomous dynamical systems

Assume that $X$ and $Y$ are complete metric spaces, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}, \mathbb{T}_+ = \{t \in \mathbb{T} : t \geq 0\}, \mathbb{T}_- = \{t \in \mathbb{T} | t \leq 0\}$ and $\mathbb{C}$ be the set of complex numbers.

For a system $(X, \mathbb{T}_+, \pi)$, we defined the following concepts: (see [9,10])
**Point dissipative**, if there is $K \subseteq X$ such that for all $x \in X$

$$\lim_{t \to +\infty} \rho(xt, K) = 0, \tag{1.1}$$

where $xt = \pi^t x = \pi(t, x)$;
**Compactly dissipative**, if the equality (1.1) takes place uniformly with respect to $x$ on compacts of $X$;
**Locally dissipative**, if for any point $p \in X$ there is $\delta_p > 0$ such that the equality (1.1) takes place uniformly with respect to $x \in B(p, \delta_p) = \{x \in X : \rho(x, p) < \delta_p\}$.

Denote by $(X, \mathbb{T}_+, \pi)$ ($(Y, \mathbb{T}, \sigma)$) a semigroup (group) dynamical system on $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$, where $h$ is a homomorphism of $(X, \mathbb{T}_+, \pi)$ onto $(Y, \mathbb{T}, \sigma)$, is called a non-autonomous dynamical system.

A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is said to be point (compactly, locally) dissipative, if the autonomous dynamical system $(X, \mathbb{T}_+, \pi)$ is so.

Let $(X, h, Y)$ be a locally trivial Banach fibre bundle over $Y$ [1]. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is said to be linear if the mapping $\pi^t : X_y \to X_{yt}$ is linear for every $t \in \mathbb{T}_+$ and $y \in Y$, where $X_y = \{x \in X | h(x) = y\}$ and $yt = \sigma(t, y)$. Let $|\cdot|$ be some norm on $(X, h, Y)$ such that $|\cdot|$ is co-ordinated with the metric $\rho$ on $X$ (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$).

Let $E$ be a Banach space and $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E$ be a continuous mapping with properties: $\varphi(0, u, y) = u$ and $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $u \in E$, $y \in Y$ and $t, \tau \in \mathbb{T}_+$. A triplet $\langle E, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ is called a continuous cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $E$.

Let $[E]$ be a Banach space of the all linear continuous operators acting onto $E$ with the operator norm and $U : \mathbb{T}_+ \times Y \mapsto [E]$ be a mapping with properties: $U(0, y) = I$, $U(t + \tau, y) = U(t, \sigma(\tau, y))U(\tau, y)$ for all $y \in Y$ and $t, \tau \in \mathbb{T}_+$ and the mapping $\varphi(\cdot, u, \cdot) : \mathbb{T}_+ \times Y \to E$ ($\varphi(t, u, y) = U(t, y)u$) is continuous for every $u \in E$. A triplet $\langle [E], U, (Y, \mathbb{T}, \sigma) \rangle$ is called a $C_0-$ cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$.

The dynamical system $(X, \mathbb{T}_+, \pi)$ is called [17] a skew-product system if $X = E \times Y$ and $\pi = (\varphi, \sigma)$ (i.e. $\pi(t, (u, y)) = (\varphi(t, u, y), \sigma(t, y))$ for all $u \in E$, $y \in Y$ and $t, \tau \in \mathbb{T}_+$).

**Theorem 1.1 [12,13].** *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a linear non-autonomous dynamical system and the following conditions hold:*

1. *$Y$ is compact and minimal (i.e. $Y = H(y) = \overline{\{yt : t \in \mathbb{T}\}}$ for all $y \in Y$);*
2. *for any $x \in X$ there exists $C_x \geq 0$ such that $|xt| \leq C_x$ for all $t \in \mathbb{T}_+$;*
3. *the mapping $y \mapsto \|\pi^t_y\|$ is continuous, where $\|\pi^t_y\|$ is a norm of linear operator $\pi^t_y = \pi^t|_{X_y}$, for every $t \in \mathbb{T}_+$ or $(X, \mathbb{T}_+, \pi)$ is a skew-product dynamical system.*

*Then there exists $M \geq 0$ such that the inequality*

$$|\pi(t,x)| \leq M|x|$$

*holds for all $t \in \mathbb{T}_+$ and $x \in X$.*

**Lemma 1.2.** *Let $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ be a $C_0-$ cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$ and $Y$ be a compact, then the following assertions hold:*

1. *For every $\ell > 0$ there exists a positive number $M(\ell)$ such that $\|U(t,y)\| \leq M(\ell)$ for all $t \in [0,\ell]$ and $y \in Y$;*
2. *The mapping $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E \quad (\varphi(t,u,y) = U(t,y)u)$ is continuous;*
3. *There exist positive numbers $N$ and $\nu$ such that $\|U(t,y)\| \leq Ne^{\nu t}$ for all $t \in \mathbb{T}_+$ and $y \in Y$.*

*Proof.* Let $\ell > 0$ and $u \in E$, then there exists a positive number $M(\ell, u)$ such that $|U(t,y)u| \leq M(\ell,u)$ for all $(t,y) \in [0,\ell] \times Y$ because the mapping $(t,y) \to U(t,y)u$ is continuous. According to principle of uniformly boundedness there exists a positive number $M(\ell)$ such that $\|U(t,y)\| \leq M(\ell)$ for all $(t,y) \in [0,\ell] \times Y$.

Let now $(t_0, u_0, y_0) \in \mathbb{T}_+ \times E \times Y$ and $t_n \to t_0, u_n \to u_0$ and $y_n \to y_0$, then we have

$$\begin{aligned} &|\varphi(t_n,u_n,y_n) - \varphi(t_0,u_0,y_0)| \\ &\leq |\varphi(t_n,u_n,y_n) - \varphi(t_n,u_0,y_n)| + |\varphi(t_n,u_0,y_n) - \varphi(t_0,u_0,y_0)| \\ &\leq \|U(t_n,y_n)(u_n - u_0)\| + |(U(t_n,y_n) - U(t_0,y_0))u_0| \end{aligned} \tag{1.2}$$

In view of first statement of Lemma 1.2 there exists the positive number $M$ such that

$$\|U(t_n, y_n)\| \leq M \tag{1.3}$$

for all $n \in \mathbb{N}$. From inequalities (1.2) and (1.3) follows the continuity of mapping $\varphi : \mathbb{T}_+ \times E \times Y \to E \quad (\varphi(t,u,y) = U(t,y)u)$.

Denote by $a = \sup\{\|U(t,y)\| : (t,y) \in [0,1] \times Y\}$ and let $t \in \mathbb{T}_+, t = n + \tau(n \in \mathbb{N}, \tau \in [0,1))$, then we obtain

$$\|U(t,y)\| \leq \|U(n, y\tau)\| \|U(\tau,y)\| \leq a^{n+1} \leq Ne^{\nu t}$$

for all $t \in \mathbb{T}_+$ and $y \in Y$, where $N = a$ and $\nu = \ln a$.

**Theorem 1.3 [13].** *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system, $Y$ be a compact , then the following conditions are equivalent:*

1. *The non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is uniformly exponentially stable, i.e. there exist two positive constants $N$ and $\nu$ such that $|\pi(t,x)| \leq Ne^{-\nu t}|x|$ for all $t \in \mathbb{T}_+$ and $x \in X$.*
2. $\|\pi^t\| \to 0$ *as $t \to +\infty$, where* $\|\pi^t\| = \sup\{|\pi^t x| : x \in X, |x| \leq 1\}$.
3. *The non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is locally dissipative.*

## 2. Exponential stable linear periodic dynamical systems.

**Lemma 2.1 [15, Chapter 9].** *Let $m : \mathbb{T}_+ \mapsto \mathbb{T}_+$ be a positive and continuous function. If there exists a positive constant $M$ such that $m(t+s) \leq Mm(t)$ for all $s \in [0,1]$ and $t \in \mathbb{T}_+$, then $\int_0^{+\infty} m(t)dt < +\infty$ implies $m(t) \to 0$ as $t \to +\infty$.*

**Theorem 2.2.** *Let $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ be the $C_0-$ cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$ and $(Y, \mathbb{T}, \sigma)$ be a periodical dynamical system (i.e. there are $y_0 \in Y$ and $\tau \in \mathbb{T}$ $(\tau > 0)$ such that $Y = \{y_0 t : 0 \le t < \tau\}$). Then the following conditions are equivalent:*

(i)
$$\lim_{t\to+\infty} \|U(t, y_0)\| = 0. \tag{2.1}$$

(ii) *There exist positive constants $N$ and $\nu$ such that for all $t \in \mathbb{T}_+$ and $y \in Y$,*
$$\|U(t, y)\| \le N e^{-\nu t}. \tag{2.2}$$

(iii) *There exists $p \ge 1$ such that for all $u \in E$,*
$$\int_0^{+\infty} |U(t, y_0)u|^p dt < +\infty. \tag{2.3}$$

*Proof.* We remark that from equality (2.1) follows the condition
$$\lim_{n\to+\infty} \sup_{0\le s\le\tau} \|U(s + n\tau, y_0)\| = 0. \tag{2.4}$$

In fact, by virtue of Lemma 1.2 there exists a positive constant $M$ such that
$$\|U(s, y)\| \le M \tag{2.5}$$
for all $s \in [0, \tau]$ and $y \in Y$. Therefore,
$$\|U(s + n\tau, y_0)\| = \|U(s, y_0)U(n\tau, y_0)\| \le M\|U(n\tau, y_0)\| \tag{2.6}$$
for all $0 \le s \le \tau$. Consequently, from (2.1) and (2.6) results the condition (2.4).

We will show that under the condition (2.4) the equality
$$\lim_{t\to+\infty} \sup_{y\in Y} \|U(t, y)\| = 0 \tag{2.7}$$
holds. In fact, let $y \in Y$ then there exists a number $s \in [0, \tau)$ such that $y = y_0 s$ and, consequently, for $t \in \mathbb{T}_+$ $\quad (t = n\tau + \bar{t}, \bar{t} \in [0, \tau))$ we obtain
$$\begin{aligned} \|U(t, y)\| &= \|U(t, y_0 s)\| = \|U(n\tau + \bar{t}, y_0 s)\| \\ &= \|U((n-1)\tau + \bar{t} + s, y_0\tau)U(\tau - s, y_0 s)\| \\ &\le M \max\{\sup_{0\le s\le\tau} \|U((n-1)\tau + s, y_0)\|, \sup_{0\le s\le\tau} \|U(n\tau + s, y_0)\|\}. \end{aligned} \tag{2.8}$$

From (2.4) and (2.8) results the equality (2.7). For finishing the proof that (i) implies (ii) is sufficient to apply Theorem 1.3 .

The fact that (ii) implies (iii) is obvious. Now we prove that (iii) implies (i). Indeed, let $u \in E$ and we consider the function $m(t) = |U(t, y_0)u|^p \quad (t \ge 0)$. We note that
$$\begin{aligned} m(t + s) &= |U(t + s, y_0)u|^p = |U(s, y_0 t)U(t, y_0)u|^p \\ &\le \|U(s, y_0 t)\|^p |U(t, y_0)u|^p \le M^p m(t) \end{aligned}$$

for all $t \in \mathbb{T}_+$ and $s \in [0,1]$, where $M = \sup_{0 \le s \le 1, y \in Y} \|U(s,y)\|$. By Lemma 2.1 $m(t) \to 0$ as $t \to +\infty$ and, consequently,

$$\lim_{t \to +\infty} |U(t,y_0)u|^p = 0 \tag{2.9}$$

for all $u \in E$. Let now $y \in Y$, then there exists $s \in [0,\tau)$ such that $y = y_0 s$ and for $t \geq \tau - s$ we have

$$U(t,y)u = U(t,y_0 s)u = U(t-\tau+s,y_0)U(\tau-s,y_0 s)u. \tag{2.10}$$

From equalities (2.9) and (2.10),

$$\lim_{t \to +\infty} |U(t,y)u|^p = 0 \tag{2.11}$$

for all $u \in E$ and $y \in Y$. According to Theorem 1.1 there exists a positive number $M$ such that $\|U(t,y)\| \le M$ for all $t \in \mathbb{T}_+$ and $y \in Y$. Let $t > 0$ and $u \in E$, then we obtain

$$\begin{aligned} t|U(t,y_0)u|^p &= \int_0^t |U(t,y_0)u|^p ds \le \int_0^t |U(t-s,y_0 s)|^p |U(s,y_0)u|^p ds \\ &\le M^p \int_0^t |U(s,y_0)u|^p ds \le M^p \int_0^{+\infty} |U(s,y_0)u|^p ds = C_u \end{aligned}$$

for all $t \geq 0$. By virtue of principle of uniformly boundedness there exists a positive number $C$ such that

$$t\|U(t,y_0)\|^p \le C$$

for all $t > 0$ and, consequently

$$\|U(t,y_0)\| \le C^{\frac{1}{p}} t^{-\frac{1}{p}} \to 0$$

as $t \to +\infty$. This completes the present proof.

**Remark 2.3.**

(1) Theorem 2.2 (the equivalence of assertions (ii) and (iii)) is a variant of the Datko-Pazy theorem (see [15-17,19]) for cocycle over periodic dynamical systems.
(2) Periodic, almost periodic and asymptotically almost periodic mild solutions of inhomogeneous periodic Cauchy problems considered recently by C. J. K. Batty, W.Hutter and F. Räbiger [2] and W. Hutter [23].

The operator $U(\tau,y_0)$ is called operator of monodromy for $\tau$- periodic cocycle $U(t,y)$. The number $0 \neq \lambda \in \mathbb{C}$ is called multiplicator of operator of monodromy $U(\tau,y_0)$ if there exists $u_0 \in E$ ($u_0 \neq 0$) such that $U(\tau,y_0)u_0 = \lambda u_0$ (or, what is the same, $U(t+\tau,y_0)u_0 = \lambda U(t,y_0)u_0$ for all $t \in \mathbb{T}_+$).

**Remark 2.4.**

(a) Condition (2.1) and the equality

$$\lim_{n\to+\infty} \|U(n\tau, y_0)\| = 0. \tag{2.12}$$

are equivalent. We show that (2.12) implies (2.1) as follows. Let now $t = n\tau + s, 0 \le s < \tau$, then $U(t, y_0) = U(s + n\tau, y_0) = U(s, y_0)U(n\tau, y_0)$ and, consequently,

$$\|U(t, y_0)\| \le \max_{0\le s\le\tau} \|U(s, y_0)\|\|U(n\tau, y_0)\|. \tag{2.13}$$

From conditions (2.12) and (2.13) results (2.1).

(b) Condition (2.2) and the inequality

$$\|U(t, y_0)\| \le N_1 e^{-\nu_1 t} \quad (\forall t \in \mathbb{T}_+) \tag{2.14}$$

are equivalent, where $N_1$ and $\nu_1$ are some positive constants. Indeed, from (2.14), taking into account (2.10), we obtain (2.2).

(c) Condition (2.12) is satisfied if and only if $\sigma(U(\tau, y_0)) \subset \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\sigma(U(\tau, y_0))$ is a spectrum of operator of monodromy $U(\tau, y_0)$. In fact, from (2.2) results that $r_{U(\tau,y_0)} = \lim_{n\to+\infty} \sup(\|U(n\tau, y_0))\|)^{1/n} \le e^{-\nu} < 1$, because $U^n(\tau, y_0) = U(n\tau, y_0)$. If $\gamma = r_{U(\tau,y_0)} < 1$, then for all $\varepsilon > 0$ there exists a $n(\varepsilon) \in \mathbb{N}$ such that $(\|U(n\tau, y_0)\|)^{1/n} \le \gamma + \varepsilon$ for all $n \ge n(\varepsilon)$ and, consequently, $\|U(n\tau, y_0)\| \le (\gamma + \varepsilon)^n$ for all $n \ge n(\varepsilon)$. Thus $\|U(n\tau, y_0)\| \to 0$ as $n \to +\infty$.

A continuous mapping $P : E \to E$ is called [21] asymptotically compact if, for any nonempty bounded set $B \subset E$ for which $P(B) \subseteq B$, there is a compact set $K \subset \overline{B}$ such that $K$ attracts $B$, i.e. $\lim_{n\to+\infty} \sup_{x\in B} \rho(P^n x, K) = 0$, where $\rho(x, K) = \inf_{y\in K} |x - y|$.

**Theorem 2.5.** *Let $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ be a $C_0-$ cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$, $(Y, \mathbb{T}, \sigma)$ be a periodic dynamical system and $U(\tau, y_0)$ be asymptotically compact (i.e. if $k_n \to +\infty \quad (k_n \in \mathbb{N})$, the sequences $\{u_n\} \subseteq E$ and $\{U(k_n\tau, y_0)u_n\}$ are bounded; then the sequence $\{U(k_n\tau, y_0)u_n\}$ is precompact). Then the following conditions are equivalent*

(i) *Equality (2.1) holds.*

(ii) *For all $u \in E$,*

$$\lim_{t\to+\infty} |U(t, y_0)u| = 0\,. \tag{2.15}$$

*Proof.* It is evidently that (i) implies (ii). Now, under the conditions of Theorem 2.5 the mapping $P = U(\tau, y_0) : E \mapsto E$ is asymptotically compact because $P^n = U(n\tau, y_0)$. ¿From condition (2.15) according to uniform boundedness principle it follows that there is a positive constant $M$ such that $\|P^n\| \le M$ for all $n \in \mathbb{Z}_+$ and, consequently, the set $B = \cup\{P^n x : |x| \le 1, n \in \mathbb{Z}_+\}$ is bounded and $P(B) \subset B$. Since the mapping $P$ is asymptotically compact in virtue of Corollary 2.2.4 from [21] the set

$$\omega(B) = \cap_{n\ge0}\overline{\cup_{m\ge n} P^m(B)}$$

is nonempty, compact, and invariant and $\omega(B)$ attracts $B$.

Now we will prove that $\lim\limits_{n\to+\infty}\|P^n\|=0$. If we suppose the contrary, then there are $\varepsilon_0>0,\{x_n\}(|x_n|\le 1)$ and $n_k\to+\infty(\{n_k\}\subset\mathbb{Z}_+)$ such that

$$|P^{n_k}x_k|\ge\varepsilon_0. \tag{2.16}$$

Since $P$ is asymptotically compact without loss of generality we can suppose that the sequence $\{P^{n_k}x_k\}$ is convergent. Let $\bar{x}=\lim\limits_{k\to+\infty}P^{n_k}x_k$, then $\bar{x}\in\omega(B)$ and from (2.16) we have $|\bar{x}|\ge\varepsilon_0>0$. According to the invariance of the set $\omega(B)$ there exists a beside sequence $\{w_n\}_{n\in\mathbb{Z}}\subset\omega(B)$ such that: $w_0=\bar{x}$ and $P(w_n)=w_{n+1}$ for all $n\in\mathbb{Z}$. We note that

$$\inf_{n\in\mathbb{Z}_-}|w_n|=0. \tag{2.17}$$

Suppose that it is not true, then there is a positive number $\ell$ such that

$$|w_n|\ge\ell \tag{2.18}$$

for all $n\in\mathbb{Z}_-$. Let $p=\lim\limits_{k\to+\infty}w_{n_k}$ and $\{z_n\}\subseteq\alpha_{w_0}$, where

$$\alpha_{w_0}=\bigcap_{n\le 0}\overline{\bigcup_{m\le n}w_m},$$

be a beside sequence such that $z_0=p$ and $P(z_n)=z_{n+1}$ for all $n\in\mathbb{Z}$. ¿From the inequality (2.18) results that $|z_n|\ge\ell$ for all $n\in\mathbb{Z}$. On the other hand in view of (2.15) $\lim\limits_{n\to+\infty}|w_n|=\lim\limits_{n\to+\infty}|P^nw_0|=0$. The obtained contradiction proves the equality (2.17).

Let now $n_r\to-\infty$ and $|w_{n_r}|\to 0$, then $w_0=P^{-n_r}w_{n_r}$ for all $r\in\mathbb{N}$ and, consequently, $|w_0|=0$ because $|w_0|\le\|P^{-n_r}\||w_{n_r}|\le M|w_{n_r}|$. On the other hand $|w_0|=|\bar{x}|\ge\varepsilon_0>0$. The obtained contradiction finishes the proof of our assertion. The Theorem is proved.

**Remark 2.6.** C.Buşe wrote several papers [3-5] on evolutions periodic processes that are in the spirit of the current paper. In particularly, in [5] it is proved that a trivial solution of equation $u'(t)=A(t)u(t)$ with $p-$ periodic coefficients on a separable Hilbert space $H$ is uniformly exponentially stable if the mild solution $u_{\mu x}$ of a well-posed inhomogeneous Cauchy problem $u'(t)=A(t)u(t)+e^{i\mu t}x(t\ge 0),\mu\in\mathbb{R},u(0)=0$ satisfies the following condition $\sup\limits_{\mu\in\mathbb{R}}\sup\limits_{t>0}|u_{\mu x}(t)|<+\infty,\forall x\in H$.

## 3. Some classes of linear uniformly exponentially stable periodic differential equations.

Let $\Lambda$ be the complete metric space of linear operators that act on Banach space $E$ and $C(\mathbb{R},\Lambda)$ be the space of all continuous operator-functions $A:\mathbb{R}\to\Lambda$ equipped with open-compact topology and $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$ be the dynamical system of shifts on $C(\mathbb{R},\Lambda)$.

**3.1 Ordinary linear differential equations.** Let $\Lambda = [E]$ and consider the linear differential equation

$$u' = \mathcal{A}(t)u\,, \tag{3.1}$$

where $\mathcal{A} \in C(\mathbb{R}, \Lambda)$. Along with equation (3.1), we shall also consider its $H-$class, that is, the family of equations

$$v' = \mathcal{B}(t)v\,, \tag{3.2}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_s : s \in \mathbb{R}\}}, \mathcal{A}_s(t) = \mathcal{A}(t+s)$ $(t \in \mathbb{R})$ and the bar denotes closure in $C(\mathbb{R}, \Lambda)$. Let $\varphi(t, u, \mathcal{B})$ be the solution of equation (3.2) that satisfies the condition $\varphi(0, v, \mathcal{B}) = v$. We put $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y, \mathbb{R}, \sigma)$, then the triple $\langle [E], U, (Y, \mathbb{R}, \sigma)\rangle$ is the linear cocycle on $(Y, \mathbb{R}, \sigma)$, where $U(t, B) = \varphi(t, \cdot, B)$ for all $t \in \mathbb{R}$ and $B \in Y$.

**Lemma 3.1 [6,7].**

(i) *The mapping $(t, u, \mathcal{A}) \mapsto \varphi(t, u, \mathcal{A})$ of $\mathbb{R} \times E \times C(\mathbb{R}, [E])$ to $E$ is continuous, and*

(ii) *the mapping $\mathcal{U} : \mathcal{A} \to U(\cdot, \mathcal{A})$ of $C(\mathbb{R}, [E])$ to $C(\mathbb{R}, [E])$ is continuous, where $U(\cdot, \mathcal{A})$ is the Cauchy operator [12] of equation (3.1).*

**Theorem 3.2.** *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau-$ periodic (i.e. $\mathcal{A}(t+\tau) = \mathcal{A}(t)$ for all $t \in \mathbb{R}$), then the following conditions are equivalent:*

(1) *The trivial solution of (3.1) is uniformly exponentially stable, i.e. there exist positive numbers $N$ and $\nu$ such that $\|U(t, \mathcal{A})U(\tau, \mathcal{A})^{-1}\| \leq Ne^{-\nu(t-\tau)}$ for all $t \geq \tau$.*

(2) *There exist positive numbers $N$ and $\nu$ such that $\|U(t, \mathcal{B})U(\tau, \mathcal{B})^{-1}\| \leq Ne^{-\nu(t-\tau)}$ for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A}) = \{\mathcal{A}_s : s \in [0, \tau)\}$.*

(3) $\lim_{t \to +\infty} \|U(t, \mathcal{A})\| = 0$.

(4) *There exists $p \geq 1$ such that $\int_0^{+\infty} |U(t, \mathcal{A})u|^p dt < +\infty$ for all $u \in E$.*

*Proof.* Applying Theorem 2.2 to the cocycle $\langle [E], U, (Y, \mathbb{R}, \sigma)\rangle$, generated by equation (3.1) we obtain the equivalence of conditions 2), 3) and 4) According to Lemma 3 [7] the conditions 1) and 2) are equivalent. The theorem is proved.

**Theorem 3.3.** *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau-$ periodic and $U(\tau, \mathcal{A})$ be asymptotically compact, then the following conditions are equivalent:*

(1) *The trivial solution of equation (3.1) is uniformly exponentially stable.*

(2) $\lim_{t \to +\infty} |U(t, \mathcal{A})u| = 0$ *for every* $u \in E$.

*Proof.* Applying Theorem 2.5 to non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (3.1), we obtain the equivalence of conditions 1) and 2). The theorem is proved.

**3.2 Partial linear differential equations.** Let $\Lambda$ be some complete metric space of linear closed operators acting into a Banach space $E$ (for example $\Lambda = \{A_0 + B | B \in [E]\}$, where $A_0$ is a closed operator that acts on $E$). We assume that the following conditions are fulfilled for equation (3.1) and its $H-$ class (3.2):

(a) for any $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (3.2) has exactly one mild solution defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, v, \mathcal{B}) = v$;

(b) the mapping $\varphi : (t, v, \mathcal{B}) \to \varphi(t, v, \mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$;

Under the assumptions above, (3.1) generates a linear cocycle $\langle [E], U, (Y, \mathbb{R}, \sigma)\rangle$, where $U(t, B) = \varphi(t, \cdot, B)$.

Applying the results from § 2 to this cocycle, we will obtain the analogous assertions for different classes of partial differential equations.

We will consider examples of partial differential equations which satisfy the above conditions a. and b.

**Example 3.1.** A closed linear operator $\mathcal{A} : D(\mathcal{A}) \to E$ with dense domain of definition $D(\mathcal{A})$ is said [22] to be a sectorial if one can find a $\theta \in (0, \frac{\pi}{2})$, an $M \geq 1$, and a real number $a$ such that the sector

$$S_{a,\theta} = \{\lambda : \theta \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ and $\|(\lambda I - \mathcal{A})^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\theta}$. If $A$ is a sectorial operator, then there exists $a_1 > 0$ such that $Re\sigma(A + a_1 I) > 0$ ($\sigma(A) = \mathbb{C} \setminus \rho(A)$). Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [14]

$$A_1^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha = D(A_1^\alpha)$, and let us equip the space $E^\alpha$ with the norm $|u|_\alpha = |A_1^\alpha u|, E^0 = E, X^1 = D(A)$. Then $E^\alpha$ is a Banach space with the norm $|\cdot|_\alpha$, and is densely continuously embedded in $E$. If the operator $A$ admits a compact resolvent, then the embedding $E^\alpha \to E^\beta$ is compact for $\alpha > \beta \geq 0$ [22]. An important class of a sectorial operators is formed by elliptic operators [22,24].

Consider the differential equation

$$u' = (\mathcal{A}_0 + \mathcal{A}(t))u, \tag{3.3}$$

where $\mathcal{A}_0$ is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $\mathcal{A} \in C(\mathbb{R}, [E])$. The results of [14] imply that equation (3.3) satisfies conditions a. and b.

Under the assumptions above, (3.3) generates a linear cocycle $\langle [E], U, (Y, \mathbb{R}, \sigma)\rangle$, where $Y = H(\mathcal{A})$ and $U(t, B) = \varphi(t, \cdot, B)$. Applying the results from § 2 to this system, we will obtain the following results.

**Theorem 3.4.** *Let $\mathcal{A}_0$ - be the sectorial operator and $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau$ -periodic, then the following conditions are equivalent:*

1. *The trivial solution of equation (3.3) is uniformly exponentially stable, i.e. there exist positive numbers $N$ and $\nu$ such that $\|U(t, \mathcal{A}_0 + \mathcal{A})U(\tau, \mathcal{A}_0 + \mathcal{A})^{-1}\| \leq Ne^{-\nu(t-\tau)}$ for all $t \geq \tau$.*
2. *There exist positive numbers $N$ and $\nu$ such that $\|U(t, \mathcal{A}_0 + \mathcal{B})U(\tau, \mathcal{A}_0 + \mathcal{B})^{-1}\| \leq Ne^{-\nu(t-\tau)}$ for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A})$.*
3. $\lim_{t\to+\infty} \|U(t, \mathcal{A}_0 + \mathcal{A})\| = 0$.
4. *There exists $p \geq 1$ such that $\int_0^{+\infty} |U(t, \mathcal{A}_0 + \mathcal{A})u|^p dt < +\infty$ for all $u \in E$.*
5. $\sigma(U(\tau, \mathcal{A}_0 + \mathcal{A})) \subset \mathbb{D}$.

**Theorem 3.5.** *Let $\mathcal{A}_0$ - be the sectorial operator with compact resolvent and $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau$ - periodic, then the following conditions are equivalent:*

1. *The trivial solution of equation (3.3) is uniformly exponentially stable.*
2. $\lim_{t\to+\infty} |U(t, \mathcal{A}_0 + \mathcal{A})u| = 0$ *for every* $u \in E$.
3. $|\lambda| < 1$ *for every multiplicator $\lambda$ of operator of monodromy $U(\tau, \mathcal{A}_0 + \mathcal{A})$.*

*Proof.* Since the sectorial operator $\mathcal{A}_0$ admits a compact resolvent, then in view of Lemma 7.2.2 [14] the operator $U(\tau, \mathcal{A}_0 + \mathcal{A})$ is compact and, consequently (see,for example [30, p.391-396]), every $0 \neq \lambda \in \sigma(U(\tau, \mathcal{A}_0 + \mathcal{A}))$ is a multiplicator for operator of monodromy $U(\tau, \mathcal{A}_0 + \mathcal{A})$. Applying Theorem 3.4 (see also Remark 2.3) to linear cocycle $\langle [E], U, (Y, \mathbb{R}, \sigma)\rangle$ generated by equation (3.3), we obtain the equivalence of conditions 1., 2. and 3. The theorem is proved.

**3.3 Linear functional-differential equations.** Let $r > 0, C([a,b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with sup-norm. If $[a,b] = [-r, 0]$, then we put $\mathcal{C} = C([-r,0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}, \alpha \geq 0$ and $u \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma+\alpha]$ we define $u_t \in \mathcal{C}$ by equality $u_t(\theta) = u(t+\theta), -r \leq \theta \leq 0$. Denote by $\mathfrak{A} = \mathfrak{A}(C, \mathbb{R}^n)$ the Banach space of all linear continuous operators acting from $\mathcal{C}$ into $\mathbb{R}^n$, equipped by operator norm. Consider the equation

$$u' = \mathcal{A}(t)u_t\,, \tag{3.4}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$. We put $H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = \mathcal{A}(t+\tau)$ and the bar denotes the closure in the topology of uniform convergence on compacts of $\mathbb{R}$.

Along with equation (3.4) we also consider the family of equations

$$u' = \mathcal{B}(t)u_t\,, \tag{3.5}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi_t(v, \mathcal{B})$ be a solution of equation (3.5) with condition $\varphi_0(v, \mathcal{B}) = v$ defined on $\mathbb{R}_+$. We put $Y = H(\mathcal{A})$ and denote by $(Y, \mathbb{R}, \sigma)$ the dynamical system of shifts on $H(\mathcal{A})$. Let $X = C \times Y$ and $\pi = (\varphi, \sigma)$ the dynamical system on $X$, defined by the equality $\pi(\tau, (v, \mathcal{B})) = (\varphi_\tau(v, \mathcal{B}), \mathcal{B}_\tau)$. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ ($h = pr_2 : X \to Y$) is linear. The following assertion takes place.

**Lemma 3.6 [12].** *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R}, \mathfrak{A})$, then the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (3.4) is completely continuous, i.e. for every bounded set $A \subset X$ there exists a positive number $\ell$ such that $\pi^\ell A$ is precompact.*

**Theorem 3.7.** *Let $\mathcal{A}$ be $\tau-$ periodic. Then the following assertions are equivalent:*

(1) *The trivial solution of equation (3.4) is uniformly exponentially stable.*
(2) $\lim_{t \to +\infty} |U(t, \mathcal{A})u| = 0$ *for every* $u \in E$.
(3) $|\lambda| < 1$ *for every multiplicator $\lambda$ of operator of monodromy $U(\tau, \mathcal{A})$.*

*Proof.* Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be the linear non-autonomous dynamical system, generated by equation (3.4). According to Lemma 3.6 this system is completely continuous and, consequently, there exists a number $k \in \mathbb{N}$ such that $U^k(\tau, y_0) = U(k\tau, y_0)$ is precompact. By virtue of theory of Riesz-Schauder (see for example [30, p.391-395]) every $0 \neq \lambda \in \sigma(U(\tau, \mathcal{A}))$ is a multiplicator of operator of monodromy $U(\tau, \mathcal{A})$. To finish the proof it is sufficient to refer to Theorems 2.2, 2.5 and Remark 2.3.

Consider the neutral functional differential equation

$$\frac{d}{dt} Du_t = \mathcal{A}(t)u_t\,, \tag{3.6}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ and $D \in \mathfrak{A}$ is nonatomic at zero operator [20, p.67]. As well as in the case of equation (3.4), the equation (3.6) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = C \times Y, Y = H(\mathcal{A})$ and $\pi = (\varphi, \sigma)$. The following statement holds.

**Lemma 3.8 [12].** *Let $H(\mathcal{A})$ be compact and the operator $D$ is stable, i.e. the zero solution of homogeneous difference equation $Dy_t = 0$ is uniformly asymptotically stable. Then the linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, generated by equation (3.6), is asymptotically compact.*

**Theorem 3.9.** *Let $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ be $\tau-$ periodic and $D$ is stable, then the following assertions are equivalent:*

(1) *The trivial solution of equation (3.6) is uniformly exponentially stable;*
(2) $\lim_{t\to+\infty} |U(t, \mathcal{A})u| = 0$ *for every $u \in E$;*
(3) $|\lambda| < 1$ *for every multiplier $\lambda$ of operator of monodromy $U(\tau, \mathcal{A})$.*

*Proof.* Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be the linear non-autonomous dynamical system, generated by equation (3.6). According to Lemma 3.8 this system is asymptotically compact. . According to results of [20, Chapter 12] every $0 \neq \lambda \in \sigma(U(\tau, y_0))$ is a multiplier of operator of monodromy $U(\tau, y_0)$. To finish the proof of Theorem 3.8 it is sufficient to refer to Theorems 2.2, 2.5 and Remark 2.3. The theorem is proved.

**Remark 3.10.**

(1) The equivalence of conditions 1. and 3. in Theorem 3.5 (Theorem 3.7, Theorem 3.9) was proved in [22, p.219] (resp. in [20, p.233], [20, p.365]).

(2) All the statements from § 3 hold also for difference equations and can be proved in the same way.

## References


[1] N. Bourbaki, *Variétés différentielles et analitiques. Fascicule de résultats*, Hermann, Paris, 1971.

[2] C. J. K. Batty, W. Hutter and F. Räbiger, *Almost periodicity of mild solutions of inhomogeneous periodic Cauchy problems*, Preprint (http://michelangelo.mathematik.uni-tuebingen./tuebingerichte/1998.html).

[3] C. Buşe, *Asymptotic stability and Perron condition for periodic evolution families on the half line.*, Preprint in Evolution Equations and Semigroups (http://math.lsu.edu/tiger/evolve/archiv/98-00013ps).

[4] C. Buşe and M. Giurgiulescu, *A new proof for a Barbashin's theorem in the periodic case*, Preprint in Evolution Equations and Semigroups (http://math.lsu.edu/tiger/evolve/archiv/99-00099ps).

[5] C. Buşe, *An exponential stability criterion for q-periodic evolution families on Hilbert spaces*, Preprint in Evolution Equations and Semigroups (http://math.lsu.edu/tiger/evolve/archiv/00-00001ps).

[6] D. N. Cheban, *Theory of linear differential equations (selected topics)*, Shtiintsa, Kishinev, 1980 (in Russian).

[7] D. N. Cheban, *Test of the convergence of nonlinear systems by the first approximation*, Differential equations and their invariants. Shtiintsa, Kishinev, (1986), p.144-150.

[8] D. N. Cheban, *Global attractors of infinite-dimensional systems,I*, Bulletin of Academy of sciences of Republic of Moldova. Mathematics, (1994), No2 (15), p. 12-21.

[9] D. N. Cheban and D. S. Fakeeh, *Global attractors of the dynamical systems without uniqueness*, Sigma, Kishinev, 1994 (in Russian).

[10] Cheban D.N., *Global attractors of the infinite-dimensional nonautonomous dynamical systems,I.*, Izvestiya AN RM. Matematika, No. 3 (25) (1997) p. 42-55.

[11] D. N. Cheban, *Bounded solutions of linear almost periodic differential equations*, Izvestiya: Mathematics, (1998), v.62, No. 3, pp. 581-600.

[12] D. N. Cheban, *Relations between the different type of stability of the linear almost periodical systems in the Banach space.*, Electronic Journal of Differential Equations. vol. 1999 (1999), No.46, pp. 1-9. (http://ejde.math.unt.edu or http://ejde.math.swt.edu).

[13] D. N. Cheban, *Uniform exponential stability of linear almost periodical systems in Banach space.*, Electronic Journal of Differential Equations, vol. 2000 (2000), No. 29, pp. 1-18 (http://ejde.math.unt.edu or http://ejde.math.swt.edu).

[14] C. Chicone and Yu. Latushkin, *Evolution Semigroups in Dynamicals Systems and Differential Equations*, Amer. Math. Soc., Providence, RI, 1999.

[15] Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C.V. van Duijn and B. de Pagter, *One-Parameter Semigroups. CWI Monograph 5*, North-Holland, Amsterdam. New York. Oxford. Tokyo, 1987.

[16] R. Datko, *Extending a theorem of A. M. Lyapunov to Hilbert space*, J. Math. Anal. Appl. 32 (1970), pp. 610-616.

[17] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach space*, SIAM, J. Math. Anal. 3 (1973), pp. 428-445.

[18] Yu. L. Daletskii and M. G. Krein, *Stability of solutions of differential equations in Banach spaces. Translations of Mathematical Monographs, vol.43*, Amer. Math. Soc., Providence, 1974.

[19] K. J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, Berlin, 1999.

[20] J. Hale, *Theory of functional differential equations.*, Mir, Moscow, 1984 (in Russian).

[21] J. Hale, *Asymptotic behavior of dissipative systems.*, American Mathematical Society, Providence, Rhode Island, 1988.

[22] D. Henry, *Geometric theory of semilinear parabolic equations*, Mir, Moscow, 1985 (in Russian).

[23] W. Hutter, *Hyperbolicity of almost-periodic evolution families*, Preprint (http://michelangelo.mathematik.uni-tuebingen.de/tuebingerichte/1999.html).

[24] A. F. Izé, *On a topological method for the analysis of the asymptotic behavior of dynamical systems and processes.*, Complex analysis, functional analysis and approximation theory, Nort-Holland 1986, pp. 109-128.

[25] Yu. Latushkin and R. Schnaubelt, *Evolution semigroups, Translation Algebras, and Exponential Dichotomy of cocycles*, J. Diff. Eqns. 1999, v.159, pp. 321-369.

[26] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis, I.*, Ann. of Math., 1958, v.67, No.2, pp. 517-573.

[27] A. Pazy, *Semigrops of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sci. 44, Springer-Verlag, Berlin - New-York, 1983.

[28] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems I.*, J. Diff. Eqns. 1974, v.15, pp. 429-458.

[29] R. J. Sacker and G. R. Sell, *Dichotomies for linear evolutionary equations in Banach spaces.*, J. Diff. Eqns. 1994, v.113, pp. 17-67.

[30] K. Yosida, *Functional Analysis*, Mir, Moscow, 1967 (in Russian).



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