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GLOBAL ATTRACTORS OF NON-AUTONOMOUS QUASI-HOMOGENEOUS DYNAMICAL SYSTEMS

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ABSTRACT. It is shown that non-autonomous quasi-homogeneous dynamical systems admit a compact global attractor. The general results obtained here are applied to differential equations both in finite dimensional spaces and in infinite dimensional spaces, such as ordinary differential equations in Banach space and some types of evolutional partial differential equations.

Krasovskii [19, 20], Zubov [26] and Coleman [11] showed that for homogeneous autonomous systems in a finite dimensional space the existence of power-low asymptotics is equivalent to asymptotic stability.

Filippov [13, 14] generalized this result for homogeneous differential inclusions. Ladis [22] showed that in the general case this result does not apply to periodic systems. For non-autonomous homogeneous systems (of order $k = 1$) uniform asymptotic stability is equivalent to exponential stability (see, for example [15]). Morozov [24] obtained a similar result for periodic differential inclusions.

In [3] the author established the connection between the uniform asymptotic stability and power-low (exponential) asymptotic of solutions of infinite dimensional homogeneous systems is studied.

The goal of the present paper is to prove that the quasi-homogeneous differential system

$$x' = f(x) + F(x, t),$$

(where $f(\lambda x) = \lambda^m f(x)$ for all $\lambda > 0$ and $x \in E, |F(x,t)||x|^{-m} \to 0$ as $|x| \to +\infty$) admits a compact global attractor, if the homogeneous differential system

$$x' = f(x)$$

is uniform asymptotic stable. This problem is studied and solved within the framework of general dynamical systems with infinite dimensional phase space. The general results obtained are applied to various differential equations both in finite dimensional spaces and in infinite dimensional spaces (such as ordinary differential equations in Banach space and some types of evolutional partial differential equations).

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1. Global attractors of dynamical systems

Let (X,ρ) be a complete metric space, $\mathbb{R}(\mathbb{Z})$ be a group of real (integer) numbers, $\mathbb{S}=\mathbb{R}$ or $\mathbb{Z}, \mathbb{S}_+=\{t\in\mathbb{S}: t\geq 0\}$ and $\mathbb{T}(\mathbb{S}_+\subseteq\mathbb{T})$ be a subgroup of group $\mathbb{S}$.

By $(X,\mathbb{T},\pi)$ we denote a dynamical system on the X and $xt=\pi(t,x)=\pi^t x$.

The dynamical system $(X,\mathbb{T},\pi)$ is called compact dissipative [15, 4, 5, 6], if there exists a nonempty compact set $K\subseteq X$ such that

$$\lim_{t\to+\infty}\rho(xt,K)=0 \tag{1.1}$$

for all $x\in X$, moreover equality (1.1) holds uniformly with respect to $x\in X$ on each compact set from X. In this case the set K is called an attractor of family of all the compacts $C(X)$ in the space X.

We assume

$$J=\Omega(K)=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi^\tau K},$$

then [16, 4, 5, 6] the set J does not depend of choice of attractor K and is characterized by the properties of dynamical system $(X,\mathbb{T},\pi)$. The set J is called [26] Levinson's center of dynamical system $(X,\mathbb{T},\pi)$.

Let us mention some facts, which we will need below.

We will say that the space X has property (S), if for any compact $K\subseteq X$ there exists a connected set $M\subseteq X$ such that $K\subseteq M$.

Theorem 1.1 ([16, 4, 5, 6]). *If $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J is its Levinson's center, then:*

1. *J is invariant, i.e. $\pi^t J=J$ for all $t\in T$*
2. *J is orbitally stable, i.e. for all $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\rho(x,J)<\delta$ implies $\rho(xt,J)<\varepsilon$ for all $t\geq 0$*
3. *J is an attractor for the family of all compact subsets of X*
4. *$J$is the maximal compact invariant set of $(X,\mathbb{T},\pi)$*
5. *The set J is connected, if the space X possesses the (S)-property.*

Let Y be a compact metric space and $(X,\mathbb{T}_1,\pi)((Y,\mathbb{T}_2,\sigma))$ be a dynamical system on the $X(Y)$ and $h: X\to Y$ be a homomorphism $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$, then the triple $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is called [7, 8, 9, 2] a non-autonomous dynamical system.

Let W and Y are two complete metric spaces, $(Y,\mathbb{S},\sigma)$ be a group dynamical system on Y and $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a skew product [25] (cocycle [18, 12]) over $(Y,\mathbb{S},\sigma)$ with the fibre W, i.e. φ is a continuous mapping $W\times Y\times\mathbb{T}$ into W, satisfying the following conditions: $\varphi(0,w,y)=w$ and $\varphi(t+\tau,w,y)=\varphi(t,\varphi(\tau,w,y),\sigma(\tau,y))$ for all $t,\tau\in\mathbb{T}, w\in W$ and $y\in Y$.

We denote by $X=W\times Y$ and define on X a dynamical system $(X,\mathbb{T},\pi)$ by equality $\pi=(\varphi,\sigma)$ i.e. $\pi(t,(w,y))=(\varphi(t,w,y),\sigma(t,y))$ for all $t\in\mathbb{T}$ and $(w,y)\in W\times Y$,then the triple $\langle (X,\mathbb{T},\pi),((Y,\mathbb{S},\sigma),h\rangle$, where $h=pr_2$, is a non-autonomous dynamical system.

For any two bounded subsets A and B from X by $\beta(A,B)$ we denote the semi-deviation A to B , i.e. $\beta(A,B)=\sup\{\rho(a,B): a\in A\}$ and $\rho(a,B)=\inf\{\rho(a,b): b\in B\}$.

The skew product over $(Y,\mathbb{S},\sigma)$ with the fibre W is called compact dissipative [5], if there exists a nonempty compact set $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t,y)M,K) : y \in Y\} = 0 \tag{1.2}$$

for all $M \in C(W)$, where $U(t,y) = \varphi(t,\cdot,y)$.

Lemma 1.2. *The skew product $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ over $(Y,\mathbb{T},\sigma)$ with fibre W is compact dissipative, if and only if the autonomous dynamical system $(X,\mathbb{T},\pi)$, $(X = W \times Y$ and $\pi = (\varphi,\sigma))$ is compact dissipative.*

By entire trajectory of semi-group dynamical system $(X,\mathbb{T},\pi)$ (of skew product $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ over $(Y,\mathbb{T},\sigma)$ with the fibre W), passing through point $x \in X$ ($(u,y) \in W \times Y$) is called a continuous mapping $\gamma : \mathbb{S} \to X(\nu : \mathbb{S} \to W)$ satisfying the following conditions : $\gamma(0) = x(\nu(0) = w)$ and $\gamma(t+\tau) = \pi^t\gamma(\tau)$ $\quad(\gamma(t+\tau) = \varphi(t,\nu(\tau),y\tau))$ for all $t \in \mathbb{T}$ and $\tau \in \mathbb{S}$.

Theorem 1.3 ([5]). *Let Y be a compact space, $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be compactly dissipative and K be a non-empty compact set, appearing in equality (1.2), then:*

1. *The set $I_y = \Omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$, and $\lim_{t\to+\infty} \beta(U(t,y^{-t})K,I_y) = 0$ for every $y \in Y$, where*
$$\Omega_y(M) = \bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t} U(\tau,y^{-\tau})M}$$
and $y^{-\tau} = \sigma(-\tau,y)$
2. *The equality $U(t,y)I_y = I_{yt}$ holds for all $y \in Y$ and $t \in \mathbb{T}$*
3. *The equality*
$$\lim_{t\to+\infty} \beta(U(t,y^{-t})M,I_y) = 0 \tag{1.3}$$
holds for all $M \in C(W)$ and $y \in Y$
4. $$\lim_{t\to+\infty} \sup\{\beta(U(t,y^{-t})M,I) : y \in Y\} = 0$$
wherever is $M \in C(W)$, where $I = \cup\{I_y : y \in Y\}$
5. *$I = pr_1 J$ and $I_y = pr_1 J_y$, where J is the Levinson's center of (X,T,π) and $J_y = J \cap X_y$*
6. *The set I is compact*
7. *The set I is connected, if the space $W \times Y$ has the property (S).*

Theorem 1.4. *Under the conditions of Theorem 1.3, the following affirmations take place:*

1. *$w \in I_y (y \in Y)$ if and only if, when there exists a whole trajectory $\nu : \mathbb{S} \to W$ of the skew product $\langle W,(Y,\mathbb{S},\sigma)\rangle$, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S}) = \{\nu(s)|s \in \mathbb{S}\}$ is relatively compact*
2. *$I_y(y \in Y)$ is connected, if the space W possesses the (S)-property .*

Proof. To prove the first statement, we note that the continuous function $\nu : \mathbb{S} \to W$ is a whole trajectory of the skew product $\langle W,(Y,\mathbb{S},\sigma)\rangle$ if and only if, when $\gamma = (\nu, Id_Y)$ is a whole trajectory of semigroup dynamical system $(X,\mathbb{T},\pi)$ $(X = W \times Y, \pi = (\varphi,\sigma))$. According to Lemma 1.2 the dynamical system $(X,\mathbb{T},\pi)$ is compactly dissipative and by Theorem 1.1 the set J is compact and invariant and, consequently, the point $(w,y) = x \in J$ if and only if, through $(w,y) = x$ pass the whole trajectory $\gamma = (\nu, Id_Y)$ of dynamical system $(X,\mathbb{T},\pi)$ which is completely included in J, i.e. $\gamma(0) = (\nu(0),y) = (w,y)$ and $\gamma(s) \in J$ for all $s \in \mathbb{S}$. To prove the first statement of this theorem, we refer to Theorem 1.3 (item 5). To prove the second statement, we note that in conditions of Theorem 1.3, the set $I_y \neq \emptyset$ and compact. Since the space W possesses the (S)-property, then there is a connected compact set $V \supseteq I$.

According to (1.3) the equality

$$\lim_{t\to+\infty} \beta(U(t, y^{-t})V, I_y) = 0$$

takes place. We note, that $I_y \subseteq U(t, y^{-t})V$ for each $y \in Y$ and $t \in \mathbb{T}$, the mapping $U(t, y^{-y}) : W \to W$ is continuous and , consequently, the set $U(t, y^{-t})V$ is compact and connected. Now to complete the proof of Theorem 1.4 it is sufficient to refer to [23, Lemma 3.12]. □

The dynamical system $(X, \mathbb{T}, \pi)$ is called asymptotically compact [16, 4, 5, 23], if for every bounded positively invariant set $M \subseteq X$ there exists a nonempty compact set $K \subseteq X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t M, K) = 0.$$

The non-autonomous dynamical system $\langle(X, \mathbb{T}_1, \pi),((Y, \mathbb{T}_2, \sigma), h\rangle$ is called asymptotically compact if Y is compact and autonomous dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.

The nonempty set $J \subseteq X$ is called global attractor [16, 4, 5, 23, 1] of dynamical system $(X, \mathbb{T}, \pi)$, if J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$ and J attracts the every bounded subset $M \subseteq X$, i.e.

$$\lim_{t\to+\infty} \beta(\pi^t M, J) = 0.$$

Theorem 1.5 ([6]). *Let Y be a compact space and (X, h, Y) be a locally trivial Banach fibering, $|\cdot| : X \to \mathbb{R}_+$ be a norm on (X, h, Y) coordinated with the metric ρ on X (i.e. $\rho(x_1, x_2) = |x_1 - x_2|$ for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$) and $(X, \mathbb{T}, \pi)$ be asymptotically compact. If there are a number $r > 0$ and a function $V : X_r \to \mathbb{R}_+$ ($X_r = \{x \in X|\ |x| \geq r\}$) with the following properties:*

1. *The function $V$is bounded on bounded sets and for every $c \in \mathbb{R}_+$ the set $\{x \in X_r|V(x) \leq c\}$is bounded*
2. *$V'_\pi(x) \leq -c(|x|)$ for all $x \in X_r$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is positive on $[r, +\infty)$, $V'_\pi(x) = \lim_{t\to+0} \sup t^{-1}[V(xt) - V(x)]$ if $\mathbb{T} = \mathbb{R}_+$ and $V'_\pi(x) = V(x1) - V(x)$ if $\mathbb{T} = \mathbb{Z}_+$.*

Then the dynamical system $(X, \mathbb{T}, \pi)$ admits a compact global attractor.

2. Homogeneous dynamical systems. Method of Lyapunov functions

Let (X, h, Y) be a locally trivial Banach fibering, $|\cdot| : X \to \mathbb{R}_+$ is a norm on (X, h, Y) coordinated with the metric ρ on X.

The autonomous dynamical system $(X, \mathbb{R}_+, \pi)$ is called [3] homogeneous of order $m \in \mathbb{R}_+$, if for any $x \in X, t \in \mathbb{R}_+$ and $\lambda > 0$ the equality $\pi(t, \lambda x) = \lambda\pi(\lambda^{m-1}t, x)$ takes place.

The non-autonomous dynamical system $\langle(X, \mathbb{T}, \pi),((Y, \mathbb{T}, \sigma), h\rangle$ is called homogeneous of order $m = 1$ if the autonomous dynamical system $(X, \mathbb{T}, \pi)$ is homogeneous of order $m = 1$.

Theorem 2.1. *For an autonomous homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$, Assertions 1. and 2. are equivalent:*

(1) *There exist positive numbers a and b such that*

$$|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}} \tag{2.1}$$

for all $t \geq 0$ and $x \in X$

(2) *For all $k > m-1$ there exists a continuous function $V : X \to \mathbb{R}_+$ with the following properties:*
  (2.1) *$V(\lambda x) = \lambda^{k-m+1}V(x)$ for all $\lambda \geq 0$ and $x \in X$*
  (2.2) *$\alpha|x|^{k-m+1} \leq V(x) \leq \beta|x|^{k-m+1}$ for all $x \in X$, where α and β are certain positive numbers;*
  (2.3) *$V'_\pi(x) = -|x|^k$ for all $x \in X$, where $V'_\pi(x) = \frac{d}{dt}V(\pi(t,x)) \mid_{t=0}$ for $\mathbb{T} = \mathbb{R}_+$ and $V'_\pi(x) = V(\pi(1,x)) - V(x)$ for $\mathbb{T} = \mathbb{Z}_+$.*

Proof. We will show that from item 1 results item 2. Let a and b are positive numbers, such that the inequality (2.1) takes place , then for each $k > m-1$ we define the function $V : X \to \mathbb{R}_+$ by equality

$$V(x) = \int_0^{+\infty} |\pi(t,x)|^k dt. \tag{2.2}$$

First of all we note that by equality (2.2) it is defined correctly the function $V : X \to \mathbb{R}_+$ because the integral, which figures in the second number of (2.2) is convergent, moreover it is uniformly convergent with respect to x on every bounded set from X. Really, since

$$|\pi(t,x)|^k \leq (a|x|^{1-m} + bt)^{\frac{k}{1-m}}, \tag{2.3}$$

$$\int_0^{+\infty} |(a|x|^{1-m} + bt)^{\frac{k}{1-m}} dt = \frac{1}{b}\int_{a|x|^{1-m}}^{+\infty} \tau^{\frac{k}{1-m}} d\tau \tag{2.4}$$

and $\frac{k}{1-m} < -1$, then the integral (2.4) is convergent, moreover the convergence is uniform on every bounded set from X.

We will show that the function V, defined by equality (2.2), is our unknown function. The continuity of V results from the continuity of mapping $\pi : \mathbb{T}\times X \to X$ and uniform convergence of integral (2.4) with respect to x on every bounded set from X .

We now note that

$$\begin{aligned} V(\lambda x) = &\int_0^{+\infty} |\pi(t,\lambda x)|^k dt = \int_0^{+\infty} \lambda^k |\pi(\lambda^{m-1}t, x)|^k dt \\ =&\lambda^{k-m+1}\int_0^{+\infty} |\pi(\tau,x)|^k dt = \lambda^{k-m+1}V(x) \end{aligned}$$

for all $\lambda > 0$ and $x \in X$. It is not difficult to show that the function V is positive definite. Since

$$V(x) = \int_0^{+\infty} |\pi(t,x)|^k, dt \geq \int_0^{\varepsilon} |\pi(t,x)|^k, dt = |\pi(\xi,x)|^k \tag{2.5}$$

for all $\varepsilon > 0$, where $\xi \in [0,\varepsilon]$. Then passing to the limit in (2.5) as $\varepsilon \to 0$, we obtain

$$V(x) \geq |x|^k \tag{2.6}$$

for all $x \in X$. In virtue of (2.3) and (2.4) we have

$$V(x) \leq \beta|x|^{k-m+1} \tag{2.7}$$

for every $x \in X$, where

$$\beta = \frac{(m-1)a^{\frac{k-m+1}{1-m}}}{b(k-m+1)}.$$

From (2.4)-(2.7) results that the function V satisfies and the condition 2.2. Finally, we note that

$$\frac{d}{dt}V(\pi(t,x)) = -|\pi(t,x)|^k \tag{2.8}$$

and, consequently, $V'_\pi(x) = -|x|^k$ for all $x \in X$.

We will prove now that from condition 2, it follows item 1. In fact, we denote by $\psi(t) = V(\pi(t,x))$, by condition 2.3 we will have

$$\psi'(t) = -|\pi(t,x)|^k \tag{2.9}$$

for all $t \geq 0$. From condition 2.2 we have $|\pi(t,x)|^{k-m+1} \geq \frac{1}{\beta}\psi(t)$ and, consequently,

$$\psi'(t) \leq -\frac{1}{\beta^{\frac{k}{k-m+1}}}\psi(t)^{\frac{k}{k-m+1}}$$

for all $t \geq 0$. If $x \neq 0$, then $\psi(t) = V(\pi(t,x)) > 0$ for all $t \geq 0$,therefore

$$V(\pi(t,x)) \leq (V^{-\frac{m-1}{k-m+1}}(x) + \frac{m-1}{k-m+1}\frac{1}{\beta^{\frac{k}{k-m+1}}}t)^{\frac{1}{1-m}} \tag{2.10}$$

for all $x \in X$ and $t \geq 0$. From condition 2.2 and inequality (2.10), it results that $|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}}$ for all $x \in X$ and $t \geq 0$, where

$$a = (\alpha\beta)^{\frac{m-1}{k-m+1}} \quad \text{and} \quad b = (\alpha)^{\frac{m-1}{k-m+1}}(\beta)^{\frac{k}{k-m+1}}\frac{m-1}{k-m+1}.$$

The proof of this theorem is complete. □

Corollary 2.2. *For an autonomous homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ the following assertions are equivalent:*

1. *The trivial motion of the dynamical system $(X, \mathbb{R}_+, \pi)$ is uniform asymptotical stable*
2. *There exist positive numbers a and b such that $|\pi(t,x)| \leq (a|x|^{1-m}+bt)^{\frac{1}{1-m}}$ for all $t \in \mathbb{R}_+$ and $x \in X$*
3. *For every number $k > m-1$ there exists a continuous function $V : X \to \mathbb{R}_+$ which possesses properties 2.1-2.3 from Theorem 2.1.*

This assertion directly follows from Theorem 2.1 and [12, theorem 1.2].

Theorem 2.3. *Let a non-autonomous system $\langle (X, \mathbb{T}, \pi), ((Y, \mathbb{T}, \sigma), h\rangle$ be homogeneous of order $m = 1$. Then the following two conditions are equivalent:*

1. *There exist positive numbers N and ν such that $|\pi(t,x)| \leq Ne^{-\nu t}|x|$ for all $x \in X$ and $t \geq 0$*
2. *For each $k > 0$ there exists a continuous function $V : X \to \mathbb{R}_+$ satisfying the following conditions*
   (a) *$V(\lambda x) = \lambda^k V(x)$ for all $x \in X$ and $\lambda > 0$*
   (b) *There exist positive numbers $\alpha \geq 1$ and β so that for every $x \in X$,*

$$\alpha|x|^k \leq V(x) \leq \beta|x|^k \tag{2.11}$$

   (c) *$V'_\pi(x) = -|x|^k$ for all $x \in X$.*

Proof. Let us show that under the conditions of this theorem, from assumption 1 follows 2. First of all let us show that the function V, defined by equality (2.2) is the unknown, in the case when $\mathbb{T} = \mathbb{R}_+$. We note that by equality (2.2) is correctly defined the function $V : X \to \mathbb{R}_+$, such that

$$|\pi(t, x)|^k \le N^k e^{-\nu k t} |x|^k \tag{2.12}$$

for all $t \ge 0, x \in X$ and, consequently, the integral in second member of equality (2.2) is convergent uniformly with respect to x on every bounded set from X. In particularly, the function $V$is continuous with respect to $x \in X$.

From (2.2),(2.6) and (2.12) follows that

$$\alpha|x|^k \le V(x) \le \beta|x|^k \tag{2.13}$$

for every $x \in X$, where $\beta = N^k(\nu k)^{-1}$ and $\alpha = 1$. From equality (2.4) we have that $V(\lambda x) = \lambda^k V(x)$ for every $x \in X$ and $\lambda > 0$. From (2.8) it follows that $V'_\pi(x) = -|x|^k$.

Let us show that the inverse implication takes place too. We now suppose that the conditions 2.1-2.3 of this theorem hold. One denote by $\psi(t) = V(\pi(t, x))$, then the equality (2.9) takes place for all $t \ge 0$. From condition 2.2 follows that $|\pi(t, x)|^k \ge \frac{1}{\beta}\psi(t)$ and, consequently,

$$\psi'(t) \le -\frac{1}{\beta}\psi(t) \tag{2.14}$$

for all $t \ge 0$. From (2.14) we have

$$V(\pi(t, x)) \le V(x)e^{-\frac{1}{\beta}t} \tag{2.15}$$

for any $t \ge 0$ and $x \in X$. According to (2.13) and (2.15) we have

$$|\pi(t, x)| \le Ne^{-\nu t}|x|$$

for any $t \ge$ and $x \in X$, where $N = (\frac{\beta}{\alpha})^{\frac{1}{k}}$ and $\nu = \frac{1}{\beta k}$.

If $\mathbb{T} = \mathbb{Z}_+$, then we will define the function $V : X \to \mathbb{R}_+$ by equality

$$V(x) = \sum_{n=1}^{+\infty} |\pi(n, x)|^k. \tag{2.16}$$

The series in second member of equality (2.16) is convergent uniformly with respect to x on every bounded subset from X, because under the conditions of this theorem we have

$$|\pi(n, x)|^k \le N^k e^{-\nu k n}|x|^k \tag{2.17}$$

for all $n \in \mathbb{Z}_+$ and $x \in X$ and, consequently, the series (2.16) is majored by the geometric series with the denominator $q = e^{-\nu k} < 1$. Thus the function V defined by equality (2.16) is continuous.

¿From the homogeneity of $(X, \mathbb{T}, \pi)$ of order $m = 1$ and equality (2.16) follows that $V(\lambda x) = \lambda^k V(x)$ for all $\lambda > 0$ and $x \in X$. According to (2.16) and (2.17) we have

$$\alpha|x|^k \le V(x) \le \beta|x|^k \tag{2.18}$$

for any $x \in X$, where $\alpha = 1$ and $\beta = N^k(1 - e^{-\nu k})^{-1}$. Finally, from equality (2.16) follows that $V'_\pi(x) = V(\pi(1, x)) - V(x) = -|x|^k$ for all $x \in X$.

We now will show that from the conditions 2.1-2.3 of theorem (in the case, when $\mathbb{T} = \mathbb{Z}_+$) follows the estimation (2.11). Really, we denote by $\psi(n) = V(\pi(n,x))$, then from the conditions 2.2-2.3 we have

$$\Delta\psi(n) = \psi(n+1) - \psi(n) = -|\pi(n,x)|^k \leq -\frac{1}{\beta}\psi(n). \tag{2.19}$$

We may assume without loss of generality that $\beta > 1$, then from (2.19) we have

$$V(\pi(n,x)) \leq \gamma^n V(x) \tag{2.20}$$

for all $x \in X$ and $n \in \mathbb{Z}_+$, where $\gamma = 1 - \beta^{-1}$. From (2.18) and (2.20) follows the inequality (2.11), if we suppose that $N = (\frac{\beta}{\alpha})^{\frac{1}{k}}$ and $\nu = -\frac{1}{k} ln(1 - \beta^{-1})$. The theorem is proved. $\square$

Corollary 2.4. *Let a non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), ((Y, \mathbb{T}, \sigma), h\rangle$ be homogeneous of order $m = 1$. Then the following three conditions are equivalent:*

1. *The zero section of fibering (X, h, Y) is uniform asymptotic stable, i.e.*
   - (1.a) *For all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|\pi(t,x)| < \varepsilon$ for all $t \geq 0$ and $|x| < \delta$*
   - (1.b) $\lim_{t\to+\infty} |\pi(t,x)| = 0$ *for all $x \in X$, moreover this equality takes place uniformly with respect to x on every bounded set from X*
2. *There exist positive numbers N and ν such that $|\pi(t,x)| \leq Ne^{-\nu t}|x|$ for all $t \geq 0$ and $x \in X$*
3. *For all $k > 0$ there exists a continuous function $V : X \to \mathbb{R}_+$ satisfying the conditions 2.1-2.3 of Theorem 2.3.*

This statements follow from Theorem 2.3 and [3, Theorem 1.1].

Remark 2.5. In the case, when the space E is finite dimensional Theorem 2.1 and Theorem 2.3 (for autonomous system with $\mathbb{T} = \mathbb{R}_+$) generalizes and make precise some results of Zubov (see, for example [26], theorems 36 and 37).

3. Differentiable homogeneous systems

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. We denote by $C(E)$ (or $C^1(E)$) the space of all continuous (differentiable continuous) functions defined on the space E wit values in some Banach space.

The function $f : E \times \Omega \to F$ is called homogeneous of order k with respect to the variable $u \in E$ if the equality $f(\lambda x, \omega) = \lambda^k f(x,\omega)$ holds for all $\lambda > 0$, $x \in E$ and $\omega \in \Omega$.

Lemma 3.1. *The following assertions take place.*

1. *Let Ω be a compact set, $f \in C(E\times\Omega)$, $f(0,\omega) = 0$ for all $\omega \in \Omega$, $f(\lambda u, \omega) = \lambda^m f(u,\omega)$ (for every $\lambda > 0$and $(u,\omega) \in E \times \Omega$)). Then there exists $M > 0$ such that $|f(u,\omega)| \leq M|u|^m$ for all $(u,\omega) \in E \times \Omega$*
2. *If the function $f \in C^1(E \times \Omega)$ is homogeneous (of order $m \geq 1$), then the function $D_u f(\cdot,\omega) : E \to L(E,F)$ $(\omega \in \Omega)$ will be homogeneous (of order $m-1$), where $L(E,F)$ is the space of all linear continuous operators $A : E \to F$*
3. *The function $f \in C^1(E)$ is homogeneous (of order m) if and only if f satisfies the equation $Df(x)x = mf(x)$ for all $x \in E$, where $Df(x)$ is the derivative of Frechet of function $f \in C^1(E)$ in the point x*

Proof. First of all we note that in conditions of Lemma 3.1 there exists $\delta_0 > 0$ such that $|f(u,\omega)| \leq 1$ for all $|u| \leq \delta$ and $\omega \in \Omega$. If we suppose that it is not so, then there exist $\delta_n \to 0, |u_n| < \delta_n$ and $\omega_n \in \Omega$ such that $|f(u_n,\omega_n)| > 1$. Since Ω is compact we may assume that the sequence $\{\omega_n\}$ is convergent. Let $\omega_n \to \omega_0$, then according to continuity of f one have $|f(0,\omega_0)| \geq 1$. On the other hand we have $f(0,\omega_0) = 0$. The obtained contradiction prove the required affirmation.

Thus there exists $\delta > 0$ such that $|f(u,\omega)| \leq 1$ for all $|u| \leq \delta$ and $\omega \in \Omega$ and, consequently, we have

$$|f(u,\omega)| \leq |f(\frac{|u|}{\delta}\frac{u\delta}{|u|},\omega)| = \frac{|u|^m}{\delta^m}|f(\frac{\delta u}{|u|},\omega)| \leq \frac{1}{\delta^m}|u|^m.$$

Let $f \in C^1(E)$ and f be homogeneous (of order m). Then

$$f(x+h) - f(x) = Df(x)h + r(x,h) \tag{3.1}$$

and $|r(x,h)| \to 0$ as $|h| \to 0$. We now will replace x by λx in equality (3.1), then we obtain

$$f(\lambda x + h) - f(\lambda x) = Df(\lambda x) + r(\lambda x, h)$$

and, consequently,

$$f(x+u) - f(x) = \lambda^{-m+1}Df(\lambda x)h + \lambda^{-m}r(\lambda x, \lambda u) \tag{3.2}$$

for all $\lambda > 0$, where $u = \lambda^{-1}h$. From (3.2) it follows that $\lambda^{-m+1}Df(\lambda x) = Df(x)$, i.e. $Df(\lambda x) = \lambda^{m-1}Df(x)$ for any $\lambda > 0$.

Finally, let us prove the third statement in the lemma. Let $f \in C^1(E)$ be a homogeneous function (of order m). Having differentiated the identity $f(\lambda x) = \lambda^m f(x)$ with respect to λ and taking $\lambda = 1$ one obtains the identity $Df(x)x = mx$. Let now $Df(x)x = mx$. We denote by $\varphi(\lambda) = \lambda^{-m}f(\lambda x)(\lambda > 0)$. Thus $Df(\lambda x)\lambda x = mf(\lambda x)$, then

$$\begin{aligned}\varphi'(\lambda) &= -\,m\lambda^{-m+1}f(\lambda x) + \lambda^{-m}Df(\lambda x)x\\ &= -\,m\lambda^{-m-1}f(\lambda x) + \lambda^{-m-1}mf(\lambda x) = 0\end{aligned}$$

for all $\lambda > 0$ and, consequently, $\varphi(\lambda) = \mathit{const}$ for all $\lambda > 0$. Thus $\varphi(1) = f(x)$, when $\varphi(\lambda) = f(x)$ for any $\lambda > 0$, i.e. $f(\lambda x) = \lambda^m f(x)$ for every $\lambda > 0$ and for all $x \in E$. The lemma is proved. □

The function $f \in C(E)$ is said to be regular, if for the differential equation

$$x' = f(x) \tag{3.3}$$

the conditions of existence, uniqueness on $\mathbb{R}_+$ and continuous dependence on initial data are fulfilled, i.e. by (3.3) it is generated a semigroup dynamical system $(E, \mathbb{R}_+, \pi)$, where $\pi(t,x)$ is the solution of (3.3) with initial condition $\pi(0,x) = x$.

Theorem 3.2. *Let $f \in C'(E)$ be regular and homogeneous (of order $m > 1$), then the following conditions are equivalent:*

(1) *The zero solution of (3.3) is uniform asymptotic stable*
(2) *For all sufficiently large k there exists a continuously differentiable function $V : E \to \mathbb{R}_+$ satisfying the following conditions:*
  (2.1) *$V(\lambda x) = \lambda^{k-m+1}V(x)$ for all $\lambda \geq 0$ and $x \in E$*
  (2.2) *There exist positive numbers α and β such that $\alpha|x|^{k-m+1} \leq V(x) \leq \beta|x|^{k-m+1}$ for all $x \in E$*
  (2.3) *$V'_\pi(x) = DV(x)f(x) = -|x|^k$ for any $x \in E$, where $DV(x)$ is a derivative of Frechet of function V in the point x.*

Proof. We suppose that the zero solution of (3.3) is uniform asymptotic stable. Denote by $(E, \mathbb{R}_+, \pi)$ a semigroup dynamical system, generated by equation (3.3). Thus, $f \in C^1(E)$ is homogeneous (of order $m > 1$), then according to [26, Theorem 3.4] the equality $\pi(t, \lambda x) = \lambda \pi(\lambda^{m-1} t, x)$ takes place for all $x \in E, \lambda \geq 0$ and $t \in \mathbb{R}_+$, i.e. a dynamical system $(E, \mathbb{R}_+, \pi)$ is homogeneous (of order $m > 1$). According to Theorem 2.1 and Corollary 2.2 by equality (2.2) it is defined a continuous function $V : E \to \mathbb{R}_+$, satisfying the condition 2.1-2.3. of Theorem 3.2. Let us show that in conditions of Theorem 3.2 function V will be continuously differentiable and that the equality $V'_\pi(x) = DV(x)x$ takes place. To this aim we will formally differentiate the equality (2.2) with respect to x, then we will have

$$DV(x)u = \int_0^{+\infty} k|\pi(t,x)|^{k-2} \operatorname{Re}\langle D_x\pi(t,x)u, \pi(t,x)\rangle dt. \tag{3.4}$$

Now we will show that the integral figuring in the second hand of formula (3.4), is uniformly convergent with respect to x and u on the every ball from E and, consequently, the equality (3.4) really defines a derivative of function V. We note that operator-function $U(t,x) = D_x\pi(t,x)$ satisfies the operational equation

$$X' = \mathcal{A}(t,x)X, \tag{3.5}$$

where $\mathcal{A}(t,x) = D_x f(\pi(t,x))$, and initial condition $U(0,x) = I$ (I is the identity operator in E). According to Lemma 3.1 the function $\mathcal{A}(t,x) = D_x f(\pi(t,x))$ is homogeneous with respect to $\pi(t,x)$ of order $m-1$ and there exists a number $M > 0$ such that

$$\|\mathcal{A}(t,x)\| \leq M|\pi(t,x)|^{m-1} \tag{3.6}$$

for all $x \in E$ and $t \geq 0$. Since the zero solution of (3.3) is uniform asymptotic stable, in virtue of Corollary 2.2 there exist positive numbers a and b such that

$$|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{1/(1-m)} \tag{3.7}$$

for any $x \in E$ and $t \geq 0$ and, consequently, from (3.6)-(3.7) we have

$$\|\mathcal{A}(t,x)\| \leq M(a|x|^{1-m} + bt)^{-1}. \tag{3.8}$$

Since

$$\|U(t,x)\| \leq e^{\int_0^t \|\mathcal{A}(\tau,x)\|\, d\tau}, \tag{3.9}$$

from (3.8) and (3.9) we have

$$\|U(t,x)\| \leq (a|x|^{1-m} + bt)^{M/b}(a^{-1}|x|^{m-1})^{M/b} \tag{3.10}$$

for every $t \geq 0$ and $x \in E$. In virtue of inequality (3.10) we have

$$\begin{aligned} &|\pi(t,x)|^{k-2}|\operatorname{Re}\langle D_x\pi(t,x)u, \pi(t,x)\rangle| \\ &\leq \ |\pi(t,x)|^{k-1}\|D_x\pi(t,x)\||u| \\ &\leq \ (a|x|^{1-m} + bt)^{-\frac{k-1}{m-1} + \frac{M}{b}} a^{-M/b}|x|^{(m-1)m/b}|u| \end{aligned} \tag{3.11}$$

for all $t \geq 0$ and $x, u \in E$. From estimation (3.11) follows that for sufficiently large k the integral in second member of (3.4) is uniformly convergent with respect to x and u on the each ball from E and, consequently, the function V, defined by equality (2.2), in conditions of Theorem 3.2 is continuously differentiable. We now note, that

$$V'_\pi(x) = \frac{d}{dt}V(\pi(t,x))\Big|_{t=0} = DV(\pi(t,x))f(\pi(t,x))\Big|_{t=0} = DV(x)f(x)$$

for all $x \in E$. On the other hand, according to Theorem 2.1 $V'_\pi(x) = -|x|^k$. For finishing the proof of the theorem it is sufficient to notice that in conformity with Theorem 2.1 takes place the inverse statement, i.e. conditions 2.1-2.3 of Theorem 3.2 imply 1. This theorem is proved. □

Now let $f \in C^1(E \times F)$ and $\Phi \in C^1(F)$. Consider the differential system

$$\begin{aligned} u' &= f(u,\omega) \\ \omega' &= \Phi(\omega) \end{aligned} \quad (\omega \in \Omega) \tag{3.12}$$

where Ω is a certain differentiable compact submanifold in F, f and Φ are regular functions, i.e. for the differential system (3.12) and equation

$$\omega' = \Phi(\omega) \tag{3.13}$$

are fulfilled the conditions of existence, uniqueness and continuous dependence on initial data on the $\mathbb{R}_+$ for (3.12) and on the $\mathbb{R}$ for (3.13). We denote by $(\Omega, \mathbb{R}, \sigma)$ a group dynamical system, generated by equation (3.13), then the system (3.12) may be written in the form of non-autonomous equation

$$u' = f(u, \omega t), \quad (\omega \in \Omega) \tag{3.14}$$

where $\omega t = \sigma(t, \omega)$. Let $\varphi(t, u, \omega)$ be a solution of (3.14) satisfying the initial condition $\varphi(0, u, \omega) = u$, then it is not difficult to see that the triple $\langle E, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ is a skew product (cocycle) over $(\Omega, \mathbb{R}, \sigma)$ with the fibre E. It is easy to check that if a function f is homogeneous of order $m = 1$ with respect to $u \in E$, then takes place the equality $\varphi(t, \lambda u, \omega) = \lambda\varphi(t, u, \omega)$, for all $t \geq 0$, $\lambda > 0$, $u \in E$ and $\omega \in \Omega$.

Theorem 3.3. *Let the function $f \in C^1(E \times F)$ and $\Phi \in C^1(F)$ be regular, Ω is compact and the function f is homogeneous (of order $m = 1$) with respect to variable $u \in E$, then the following conditions are equivalent:*

1. *The trivial solution of (3.14) is uniform exponential stable, i.e. there exist positive numbers N and ν such that for all $u \in E, t \geq 0$ and $\omega \in \Omega$,*

$$|\varphi(t, u, \omega)| \leq N e^{-\nu t}|u| \tag{3.15}$$

2. *For all number $k > 1$ there exists a continuously differentiable function $V : E \times \Omega \to \mathbb{R}_+$ satisfying the following conditions:*
   - (2.1) *$V(\lambda u, \omega) = \lambda^k V(u, \omega)$ for all $u \in \Omega$ and $\lambda > 0$*
   - (2.2) *There exist positive numbers α and β such that $\alpha|u|^k \leq V(u, \omega) \leq \beta|u|^k$ for all $u \in E$ and $\omega \in \Omega$*
   - (2.3) *$V'_\pi(u, \omega) = \frac{d}{dt} V(\varphi(t, u, \omega), \omega t)\Big|_{t=0}$ $= D_u V(u, \omega) f(u, \omega) + D_\omega V(u, \omega)\Phi(\omega) = -|u|^k$ for all $u \in E$ and $\omega \in \Omega$, where $D_u V (D_\omega V)$ is a partial Frechet's derivative of function V with respect to variable $u \in E (\omega \in \Omega)$.*

Proof. Consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ generated by differential system (3.12), where $X = E \times \Omega, h = pr_2$ and $\pi = (\varphi, \sigma)$. The triple (X, h, Ω) is the trivial vectorial bundle with the fibre E, moreover the norm of element $x = (u, \omega) \in E \times \Omega$ is defined by equality $|x| = |u|$, where $|u|$ is a norm of element u in the space E. Then in conditions of theorem the zero section of non-autonomous dynamical system constructed above will satisfy conditions of Theorem 2.3. Thus to finish the proof of Theorem 3.3 it is sufficient to show that function $V : E \times \Omega \to \mathbb{R}_+$ from the Theorem 2.3 in conditions of Theorem 3.3 will be continuously differentiable and satisfies the condition

$$V'_\pi(u,\omega) = D_u V(u,\omega) f(u) + D_\omega V(u,\omega)\Phi(\omega)$$

for all $(u,\omega) \in E \times \Omega$. Let us show that a function $V : E \times \Omega \to \mathbb{R}_+$, defined by equality (2.2), i.e.

$$V(u,\omega) = \int_0^{+\infty} |\varphi(t,u,\omega)|^k dt \tag{3.16}$$

for all $(u,\omega) \in E \times \Omega$ (such that $|\pi(t,x)| = |(\varphi(t,u,\omega),\omega t)| = |\varphi(t,u,\omega)|$) is continuously differentiable. To this aim, we will formally differentiate the equality (3.16) with respect to $u \in E$, thus we obtain

$$D_u V(u,\omega)v = \int_0^{+\infty} k|\varphi(t,u,\omega)|^{k-2} \operatorname{Re}\langle D_u\varphi(t,u,\omega)v, \varphi(t,u,\omega)\rangle dt. \tag{3.17}$$

Let us show that integral in the second hand of (3.17) is uniformly convergent with respect to $\omega \in \Omega$ and u, v on every bounded set from E and since by (3.17) is really defined a derivative of Frechet of function V with respect to variable $u \in E$. First of all we note that operator-function $U(t,u,\omega) = D_u\varphi(t,u,\omega)$ satisfies the operational equation

$$X' = \mathcal{B}(t,u,\omega)X, \tag{3.18}$$

where $\mathcal{B}(t,u,\omega) = D_u f(\varphi(t,u,\omega),\omega t)$, and initial condition $U(0,u,\omega) = I$ (I is the identity operator in E). According to lemma 3.1 function $\mathcal{B}(t,u,\omega) = D_u f(\varphi(t,u,\omega),\omega t)$ is homogeneous with respect to $\varphi(t,u,\omega)$ of order $m = 1$ and there exists a number $M > 0$ such that

$$\|\mathcal{B}(t,u,\omega)\| \le M \tag{3.19}$$

for all $(u,\omega) \in E \times \Omega$ and $t \ge 0$. From (3.19), it follows that

$$\|U(t,u,\omega)\| \le e^{\int_0^t \|\mathcal{B}(\tau,u,\omega)\| d\tau} \le e^{Mt}$$

for all $t \ge 0$ and $(u,\omega) \in E \times \Omega$ and, consequently,

$$\begin{aligned} &|\varphi(t,u,\omega)|^{k-2} |\operatorname{Re}\langle D_u\varphi(t,u,\omega)v, \varphi(t,u,\omega)\rangle| \\ &\le |\varphi(t,u,\omega)|^{k-1} \|D_u\varphi(t,u,\omega)\| |v| \le M|v|N^{k-1}e^{-\nu(k-1)t}|u| \end{aligned} \tag{3.20}$$

for any $t \ge 0, u, v \in E$ and $\omega \in \Omega$. From inequality (3.20) follows that for $k > 1$ the integral in the second hand of (3.16) is convergent, moreover uniformly convergent with respect to $\omega \in \Omega$ and u, v on the every bounded subset from E. Thus, the function $V(u,\omega)$, defined by formula (3.16) is continuously differentiable with respect to variable $u \in E$ and by equality (3.17) it is defined its derivative of Frechet with respect to variable $u \in E$.

Having formally differentiated the equality (3.16) with respect to variable $\omega \in F$ we will obtain

$$D_\omega V(u,\omega)w = \int_0^{+\infty} k|\varphi(t,u,\omega)|^{k-2} \operatorname{Re}\langle D_\omega\varphi(t,u,\omega)w, \varphi(t,u,\omega)\rangle dt \tag{3.21}$$

for all $(u,\omega) \in E \times \Omega$. Let us show that integral in the second member of (3.21) is uniformly convergent with respect to $\omega \in \Omega$ and $(u,w) \in E \times F$ on every bounded subset from $E \times F$. First of all we note that from (3.14) we have

$$\varphi(t,u,\omega) = u + \int_0^t f(\varphi(\tau,u,\omega),\omega\tau)d\tau$$

for all $t \geq 0$ and $(u,\omega) \in E \times F$ and, consequently,

$$\begin{aligned} &D_\omega \varphi(t,u,\omega) \\ &= \int_0^t D_u f(\varphi(\tau,u,\omega),\omega\tau) D_\omega \varphi(\tau,u,\omega) + D_\omega f(\varphi(\tau,u,\omega),\omega\tau) D\Phi(\omega\tau) d\tau. \end{aligned} \tag{3.22}$$

We denote by $\mathcal{V}(t,u,\omega)$ the operator-function $D_\omega \varphi(t,u,\omega)$, then from (3.22) follows that $\mathcal{V}(t,u,\omega)$ satisfies the operational equation

$$Y' = \mathcal{B}(t,u,\omega)Y + \mathcal{F}(t,u,\omega), \tag{3.23}$$

where $\mathcal{F}(t,u,\omega) = D_\omega f(\varphi(t,u,\omega),\omega t) D\Phi(\omega t)$, and initial condition

$$Y(0) = \mathcal{O} \tag{3.24}$$

($\mathcal{O}$ is a null operator, acting from F into F). ¿From equality (3.23) and condition (3.24) follows that

$$\mathcal{V}(t,u,\omega) = \int_0^t U(t,\tau,u,\omega)\mathcal{F}(\tau,u,\omega) d\tau \tag{3.25}$$

for all $t \geq 0$ and $(u,\omega) \in E \times \Omega$, where $U(t,\tau,u,\omega)$ is a solution of operational equation (3.18), satisfying the initial condition $U(\tau,\tau,u,\omega) = U(\tau,u,\omega)$. Therefore,

$$\|U(t,\tau,u,\omega)\| \leq e^{\int_\tau^t \|\mathcal{B}(\tau,u,\omega)\| d\tau} \leq e^{M(t-\tau)} \tag{3.26}$$

for any $t \geq \tau (t,\tau \in \mathbb{R}_+)$ and $(u,\omega) \in E \times \Omega$. Thus, a function $f(u,\omega)$ is homogeneous with respect to u (of order $m = 1$), then $D_\omega f(u,\omega)$ will be the same and, in virtue of Lemma 3.1 there exists a number $M > 0$ such that

$$\|D_\omega f(u,\omega)\| \leq M_1 |u|$$

for all $u \in E$ and $\omega \in \Omega$. Thus,

$$\|\mathcal{F}(t,u,\omega)\| = \|D_\omega f(\varphi(t,u,\omega),\omega t) D\Phi(\omega t)| \leq L|\varphi(t,u,\omega)| \tag{3.27}$$

for any $t \geq 0$ and $(u,\omega) \in E \times \Omega$, where $L = M_1 \max\{\|D\Phi(\omega)\| : \omega \in \Omega\}$. From (3.15),(3.25), (3.26) and (3.27) we have

$$\begin{aligned} &|\varphi(t,u,\omega)|^{k-2} |\operatorname{Re}\langle D_\omega \varphi(t,u,\omega) w\rangle| \\ &\leq |\varphi(t,u,\omega)|^{k-1} \|D_\omega \varphi(t,u,\omega)\| |w| \\ &\leq |\varphi(t,u,\omega)|^{k-1} |w| \int_0^t \|U(t,\tau,u,\omega)\| \cdot \|\mathcal{F}(\tau,u,\omega)\| d\tau \\ &\leq |\varphi(t,u,\omega)|^{k-1} |w| \int_0^t e^{M(t-\tau)} L |\varphi(\tau,u,\omega)| d\tau \\ &\leq N^{k-1} e^{-\nu(k-1)t} |u|^{k-1} |w| \int_0^t e^{M(t-\tau)} L N e^{-\nu\tau} |u| d\tau \\ &= N^k L e^{-\nu(k-1)t} \frac{(e^{Mt} - e^{-\nu t})}{M+\nu} |u|^k |w| \end{aligned}$$

for every $t \geq 0, u \in E$ and $\omega \in \Omega$. ¿From this inequality, it follows that for sufficiently large k the integral (3.21) is uniformly convergent with respect to $\omega \in \Omega$ and $(u,w) \in E \times F$ on the every bounded subset from $E \times F$. Therefore the function $V(u,\omega)$ defined by (3.16) is continuously differentiable with respect to $\omega \in \Omega$ and by (3.21) it is defined and it is Frechet derivative with respect to $\omega \in \Omega$. So, the function $V : E \times \Omega \to \mathbb{R}_+$ is continuously differentiable with respect to both $u \in E$

and $\omega \in \Omega$. Consequently, it is continuously differentiable in sense of Frechet. Note that

$$\begin{aligned}
&\frac{d}{dt}V((\varphi(t,u,\omega),\omega t)\\
&= D_uV(\varphi(t,u,\omega),\omega t)D_u\varphi(t,u,\omega) + D_\omega V(\varphi(t,u,\omega),\omega t)\frac{d}{dt}\omega t\\
&= D_uV(\varphi(t,u,\omega),\omega t)f(\varphi(t,u,\omega),\omega t) + D_\omega V(\varphi(t,u,\omega),\omega t)\Phi(\omega t)
\end{aligned}$$

for any $t \geq$ and $(u,\omega) \in E \times \Omega$. Consequently,

$$V'_\pi(u,\omega) = \frac{d}{dt}V(\varphi(t,u,\omega),\omega t)\Big|_{t=0} = D_uV(u,\omega)f(u,\omega) + D_\omega V(u,\omega)\Phi(u).$$

Thus, in Theorem 3.3, Conditions 2 follows from Condition 1.

For completing the proof of this theorem it is sufficient to remark that according to Theorem 2.3 conditions 2.1-2.3, Condition 1 follows from condition 2. $\square$

Remark 3.4. For finite dimensional systems the Theorem 3.2 generalizes and makes precise Theorem 38 in [26].

4. Global attractors of quasi-homogeneous systems

Let us consider the differential equation

$$x' = f(x), \tag{4.1}$$

where $f \in C(E)$ is regular and homogeneous of order $m \geq 1$. Along with (4.1) consider the perturbed equation

$$x' = f(x) + F(t,x) \tag{4.2}$$

with F a regular perturbation. Let us remember (see, for example [27, 14, 21], that equation (4.2) is called quasi-homogeneous if

$$\limsup_{|x|\to+\infty} \frac{|F(t,x)|}{|x|^m} = c, \tag{4.3}$$

where c is a certain nonnegative sufficiently small constant. Moreover, the limiting relation (4.3) takes place uniformly with respect to $t \in \mathbb{R}$.

Along with (4.2) consider the family of equations

$$x' = f(x) + G(t,x), \tag{4.4}$$

where $G \in H(f) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}$ and the bar denotes closure in $C(\mathbb{R}\times E, E)$, $C(\mathbb{R}\times E, E)$ is equipped by topology of convergence on every compact from $\mathbb{R} \times E$. Let us denote by $(Y,\mathbb{R},\sigma)$ a dynamical system of translations on $Y = H(F)$, by $\varphi(t,u,f+G)$ the solution of (4.4), satisfying the condition $\varphi(0,u,f+G) = u$ and $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ (i.e. $h = pr_2$ and $\pi = (\varphi,\sigma)$) a non-autonomous dynamical system generated by (4.4).

Theorem 4.1. *Let $f \in C^1(E)$ be regular and homogeneous of order $m \geq 1$, and the zero solution of (4.1) be uniform asymptotic stable. If $f + F \in C(\mathbb{R}\times E, E)$ is regular, equation (4.2) is quasi-homogeneous and non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4.2) is asymptotically compact, then the following assertions take place:*

1. *The set $I_G = \{u \in E \mid \sup\{|\varphi(t,u,f+G)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact, and connected for each $G \in H(F)$*
2. *$\varphi(t,I_G,f+G) = I_{\sigma(t,f+G)}$ for all $t \in \mathbb{R}_+$ and $G \in H(F)$*

(3) *The set* $I = \bigcup\{I_G | G \in H(F)\}$ *is compact and connected*
(4) *For all* $G \in H(F)$ *and bounded subsets* $M \subseteq E$,

$$\lim_{t\to+\infty} \beta(\varphi(t, M, f + G_{-t}), I_G) = 0, \quad \lim_{t\to+\infty} \beta(\varphi(t, M, f + G), I) = 0\,.$$

Proof. Let $\langle(X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a non-autonomous dynamical system, generated by equation (4.2). Thus, $f \in C^1(E)$ and the zero solution of (4.1) is uniform asymptotic stable. Then according to Theorem 3.2, by equality (2.2), the function $V : E \to \mathbb{R}_+$ is defined, is continuously differentiable, and satisfying conditions 2.1-2.3 of Theorem 3.2. Let us define the function $\mathcal{V} : X \to \mathbb{R}_+$ as $\mathcal{V}(x) = V(u)$ for all $x = (u, G) \in E \times H(F)$. Note that

$$\begin{aligned} \mathcal{V}'_\pi(x) &= \frac{d}{dt}V(\varphi(t, u, f + G))\Big|_{t=0} = DV(\varphi(t, u, f + G))\frac{d}{dt}\varphi(t, u, f + G)\Big|_{t=0} \\ &= DV(u)(f(u) + G(0, u)) = DV(u)f(u) + DV(u)G(0, u) \\ &= -|u|^k + DV(u)G(0, u). \end{aligned} \tag{4.5}$$

In virtue of Condition 2.1 of Theorem 3.2 the function V is homogeneous of order $k - m + 1$, then according to Lemma 3.1 its Frechet derivative $DV(u)$ will be homogeneous of order $k - m$ and, consequently, there exists a number $L > 0$ such that

$$\|DV(u)\| \le L|u|^{k-m} \tag{4.6}$$

for all $u \in E$. From equality (4.3) follows that for all $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that

$$|G(t, u)| \le (c + \varepsilon)|u|^m \tag{4.7}$$

for all $|u| \ge r$, $t \in \mathbb{R}$ and $G \in H(F)$. From (4.6)-(4.7) we have

$$|DV(u)G(0, u)| \le L(c + \varepsilon)|u|^k \tag{4.8}$$

and, consequently, from (4.5),(4.8) we obtain

$$\mathcal{V}'_\pi(x) \le |u|^k(-1 + L(c + \varepsilon)) = -\gamma|x|^k$$

for each $x \in X_r$, where $\gamma = 1 - l(c + \varepsilon) > 0$ since C and ε are a sufficiently small positive numbers. To complete the proof of this theorem, we refer to Theorems 1.3-1.5. □

Theorem 4.2. *Let* $f \in C^1(E)$, $\Phi \in C^1(F)$ *and* $f + F \in C^1(E \times F, E)$ *be regular,* $\Omega \subseteq F$ *be a compact invariant set of dynamical system (3.13), the function* f *be homogeneous (of order* $m > 1$*), and the zero solution of (3.3) be uniform asymptotical stable. If*

$$|F(u, \omega)| \le c|u|^m$$

for all $|u| \ge r$ *and* $\omega \in \Omega$, *where* r *and* c *are certain positive numbers, and moreover* c *is sufficiently small and dynamical system*

$$\begin{aligned} u' &= f(u) + F(u, \omega) \\ \omega' &= \Phi(\omega) \end{aligned} \qquad (\omega \in \Omega)$$

is asymptotically compact, then the following assertions take place:

(1) *The set* $I_\omega = \{u \in E | \sup\{|\varphi(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ *is not empty, compact and connected for each* $\omega \in \Omega$, *where* $\varphi(t, u, \omega)$ *is a solution of equation* $u' = f(u) + F(u, \omega t)$ *satisfying the initial condition* $\varphi(0, u, \omega) = u$
(2) $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$
(3) *The set* $I = \bigcup\{I_\omega | \omega \in \Omega\}$ *is compact and connected*

(4) *For all $\omega \in \Omega$ and bounded subsets $M \subseteq E$,*

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \omega_{-t}), I_\omega) = 0\,, \tag{4.9}$$

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \omega), I) = 0 \tag{4.10}$$

The proof of the assertions in this theorem is carried out using the same scheme as in Theorem 4.1, therefore it is omitted.

Theorem 4.3. *Let $f \in C^1(E \times F, E), \Phi \in C^1(F)$ and $f + F \in C^1(E \times F, E)$ be regular, $\Omega \subseteq F$ is a compact invariant set of dynamical system (3.13), function f is homogeneous (of order $m = 1$) with respect to variable $u \in E$ and the zero solution of (3.14) is uniform asymptotic stable. If*

$$|F(u, \omega)| \le c|u| \tag{4.11}$$

for all $|u| \ge r$ and $\omega \in \Omega$, where r and c are certain positive numbers, and moreover c is sufficiently small and the dynamical system

$$\begin{aligned} u' &= f(u, \omega) + F(u, \omega) \\ \omega' &= \Phi(\omega) \end{aligned} \qquad (\omega \in \Omega) \tag{4.12}$$

is asymptotically compact. Then the following assertions take place:

(1) *The set $I_\omega = \{u \in E|\sup\{|\varphi(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi(t, u, \omega)$ is a solution of*

$$u' = f(u, \omega t) + F(u, \omega t) \tag{4.13}$$

satisfying the initial condition $\varphi(0, u, \omega) = u$

(2) *$\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$*

(3) *The set $I = \bigcup\{I_\omega|\omega \in \Omega\}$ is compact and connected*

(4) *The equalities (4.9) and (4.10) take place for every bounded subset $M \subseteq E$ and $\omega \in \Omega$.*

Proof. Let $\langle(X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a non-autonomous dynamical system, generated by equation (4.13) (or by differential system (4.12)). Since $f \in C^1(E\times F, E)$ and the zero solution of (3.14) is uniform asymptotic stable, then according to Theorem 3.3 by equality

$$V(u, \omega) = \int_0^{+\infty} |\varphi(t, u, \omega)|^k dt$$

it is defined a continuously differentiable function $V : X = E \times \Omega \to \mathbb{R}_+$, satisfying conditions 2.1-2.3 of Theorem 2.3. Let us remark that

$$\begin{aligned} V'_\pi(u, \omega) =& \frac{d}{dt} V((\varphi(t, u, \omega), \omega t)\Big|_{t=0} \\ =& \{D_u V(\varphi(t, u, \omega), \omega t) \frac{d}{dt}\varphi(t, u, \omega) + D_\omega V(\varphi(t, u, \omega), \omega t) \frac{d}{dt}\omega t\}\Big|_{t=0} \\ =& D_u V(u, \omega)[f(u, \omega) + F(u, \omega)] + D_\omega V(u, \omega)\Phi(\omega) \\ =& -|u|^k + D_u V(u, \omega)F(u, \omega). \end{aligned} \tag{4.14}$$

Since function $V(u, \omega)$ is homogeneous of order $k > 1$ with respect to $u \in E$, then in virtue of Lemma 3.1 there exists a number $L > 0$ such that

$$\|D_u V(u, \omega)\| \le L|u|^{k-1} \tag{4.15}$$

for all $u \in E$ and $\omega \in \Omega$. From (4.11) and (4.15) it follows that

$$|D_u V(u,\omega)F(u,\omega)| \leq cL|u|^k \tag{4.16}$$

for each $|u| \geq r$ and $\omega \in \Omega$ and, consequently, according to (4.14), (4.16) we have

$$V'_\pi(u,\omega) \leq -|u|^k + cL|u|^k = -\gamma|u|^k$$

for all $\omega \in \Omega$ and $|u| \geq r$, where $\gamma = 1 - cL > 0$ since a number c is sufficiently small. To complete the proof of this theorem, we use Theorems 1.3-1.5. $\square$

Remark 4.4. 1. In case when E is finite dimensional the dissipativity of (4.2) is established in [26] (Theorem 39 and corollary 1).
2. If E is a finite dimensional Hilbert space, then the non-autonomous dynamical system, generated by equation (4.2) will be asymptotically compact, if the function $F \in C(\mathbb{R} \times E, E)$ is bounded and uniformly continuous with respect to variable $t \in \mathbb{R}$ uniformly with respect to x on every compact from E.
3. If Ω is compact, then the non-autonomous dynamical system generated by equation (4.13) is asymptotically compact.
4. The conditions of asymptotical compactness of non-autonomous dynamical system generated by quasilinear equation in Banach space are given in [6].

5. Quasilinear parabolic equations

Let $A : D(A) \to E$ be a linear closed operator with dense domain. Operator A is called sectorial [17] if for some $\varphi \in (0, \frac{\pi}{2})$, some $M \geq 1$, and some real a, the sector

$$S_{a,\varphi} = \{\lambda : \varphi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. If A is a sectorial operator, then there exists an $a_1 > 0$ such that $\operatorname{Re}\sigma(A + a_1 I) > 0$ ($\sigma(A) = \mathbb{C} \setminus \rho(A)$). Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [17]

$$A_1^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda$$

which is linear, bounded, and one-to-one. Set $E^\alpha = D(A_1^\alpha)$, and let us equip the space E^α with the scalar product $\langle x, y\rangle_\alpha = \langle A_1^\alpha x, A_1^\alpha y\rangle (x, y \in E^\alpha), E^0 = E, X^1 = D(A)$ and $|\cdot|_\alpha = \sqrt{<\cdot,\cdot>_\alpha}$. Then E^α is a Hilbert space with the scalar product $\langle x, y\rangle_\alpha$, and is densely continuously embedded in E. If the operator A admits a compact resolvent, then the embedding $E^\alpha \to E^\beta$ is compact for $\alpha > \beta \geq 0$ [17].

Consider the system of differential equations

$$\begin{aligned} u' + Au &= F(u,\omega) \\ \omega' &= \Phi(\omega) \end{aligned} \quad (\omega \in \Omega) \tag{5.1}$$

where $F \in C^1(E^\alpha \times F, E)$ and $\Phi \in C^1(F)$.

Theorem 5.1. *Let $F \in C^1(E^\alpha \times F, E)$ and $\Phi \in C^1(F)$ be regular, Ω be a compact invariant set of dynamical system (3.13). If the trivial solution of equation of*

$$u' + Au = 0$$

is uniform asymptotical stable and there exists a sufficiently small constant $c \geq 0$ such that

$$|F(u,\omega)| \leq c|u|$$

for all $|u| \geq r$ and $\omega \in \Omega$, where r is a sufficiently large positive number, then the following assertions take place:

(1) *The set $I_\omega = \{u \in E^\alpha | \sup\{|\varphi(t,u,\omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact in E and connected for each $\omega \in \Omega$, where $\varphi(t,u,\omega)$ is a solution of $u' + Au = F(u, \omega t)$ satisfying the initial condition $\varphi(0,u,\omega) = u$*
(2) *$\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$*
(3) *The set $I = \bigcup\{I_\omega | \omega \in \Omega\}$ is compact in E and connected*
(4) *The equalities (4.9) and (4.10) take place with respect to the semi-deviation in the space E for every bounded subset $M \subseteq E^\alpha$ and $\omega \in \Omega$.*

This assertion follows from Theorem 4.3 and compact of embedding $i : E^\alpha \to E$.

Remark 5.2. For quasilinear parabolic equations a statement analogous to Theorem 4.1 holds.
The notion of asymptotical compactness used in this paper is equivalent to the respective definition of Ladyzhenskaya (it follows from [10, Lemma 3.3]).

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References

[1] A. V. Babin and M. I. Vishik *Attractors of Evolution Equations*, Nort-Holland, Amsterdam, London, New York, Tokyo, 1992
[2] I. U. Bronshtein, *Nonautonomous Dynamical Systems* [in Russian], Shtiintsa, Kishinev, 1984.
[3] D. N. Cheban *The asymptotics of solutions of infinite dimensional homogeneous dynamical systems*, Matematicheskie Zametki [Mathematical notes], 1998, v. 63, No 1, p. 115-126.
[4] D. N. Cheban, D. S. Fakeeh, *Global attractors of the dynamical systems without uniqueness*, "Sigma", Kishinev, 1994.
[5] D. N. Cheban, *Global attractors of the infinite-dimensional nonautonomous dynamical systems, I*, Izvestiya AN RM. Matematika, No. 3 (25) 1997, p. 42-55.
[6] D. N. Cheban, *Global attractors of the infinite-dimensional nonautonomous dynamical systems, II.* Izvestiya AN RM. Matematika, No. 2 (27) 1998, p. 25-28.
[7] D. N. Cheban, *On the stability of Levinson's center of nonautonomous dissipative dynamical systems*, Differentsial'nye Uravneniya [Differential Equations], 1984, v. 20, No 11, p. 2016-2018.
[8] D. N. Cheban, *Nonautonomous dissipative dynamical systems*, Soviet Math. Dokl., v. 33, N1, 1986, p. 207-210.
[9] D. N. Cheban, *Nonautonomous dissipative dynamical systems. Thesis of doctor of physical-mathematical sciences*, Minsk, 1991.
[10] D. N. Cheban *Global attractors of infinite-dimensional systems, I*, Bulletin of Academy of sciences of Republic of Moldova, Mathematics, 1994, N2 (15), p. 12-21.
[11] C. Coleman *Growth and decay estimates near non-elementary points*, Canad. J. Math., 1970, v. 22, No 6, p. 1156-1167.
[12] H. Crauel, F. Flandoli, *Attractors for random dynamical systems*, Probability Theory and Related Fields, 1994, 100, p. 365-393.
[13] A. F. Filippov *Differential Equations with Discontinuous Right-Hand Side*, Differentsial'nye Uravneniya [Differential Equations], 1979, v. 15, No 6, p. 1018-1027
[14] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Side* [in Russian], Nauka, Moscow, 1985.
[15] A. Halanay, *Teoria Calitativa a Ecuatiilor Differentiale*, Bucuresti, 1963.
[16] J. Hale, *Asymptotic behavior of dissipative systems*, Mathematical surveys and Monographs, 25, American Math. Soc., Providence, RI, 1988.
[17] D. Henry, *Geometric Theory of Semilinear Parabolic Equations* Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[18] P. E. Kloeden and B. Schmalfuss, *Lyapunov functions and attractors under variable time-step discretization*, Discrete and Continuous Dynamical Systems, 1996, v.2, No2, p. 163-172.
[19] N. N. Krasovskii, *Problems in the Theory of the Stability of Motion* [in Russian], Fizmatgiz, Moscow, 1959.
[20] N. N. Krasovskii, *On first approximation stability*, Prikl.Mat.Mekh [J.Appl.Math.Mekh.], 1955, 19, No 5, p. 516-530.
[21] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometric methods of nonlinear analysis*, Nauka, Moscow, 1975.
[22] N. N. Ladis, *The asymptotic of solutions to quasihomogeneous differential systems*, Differentsial'nye Uravneniya [Differential Equations], 1973, v. 9, No 12, p. 2257-2260.
[23] O. A. Ladyzhenskaya, *On finding minimal global attractors for Navier-Stokes and other partial differential equations*, Uspekhi matematicheskikh nauk, 1987, vol.42, issue 6 (258), p. 25-60.
[24] M. V. Morozov, *Some properties of periodic differential inclusions* Izv. Vyssh. Uchebn. Zaved., Mat. [Iz.VUZ], 1993, No 4(371), p. 95-100.
[25] G. R. Sell, *Lecture on Topological Dynamics and Differential equations*, Van-Nostrand-Rein hold, Prinston, NY, 1971.
[26] V. V. Zhikov, *On stability and instability of Levinson's center*, Differentsial'nye Uravneniya [Differential Equations], 1972, v. 8, No12, p. 2167-2070.
[27] V. I. Zubov, *Stability of Motion* [in Russian], Vysshaya Shkola, Moscow, 1973.

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