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Generalization of the second Bogolyubov's theorem for non-almost periodic systems

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Abstract

The article is devoted to the generalization of the second Bogolyubov's theorem to non-almost periodic dynamical systems. We prove the analog of the second Bogolyubov's theorem for recurrent or pseudorecurrent dynamical systems in Banach spaces. Namely, we obtain the relation between a recurrent dynamical system and its averaged dynamical system. We also study existence of recurrent and pseudorecurrent motions (including special cases of periodic, quasi-periodic and almost periodic motions) in related nonautonomous systems.

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1. Introduction

The problem of averaging in time is well studied for almost periodic systems in Banach spaces. A well-known result in this direction is the second Bogolyubov's theorem (see for example [1,2]) which affirms that the equation

$$\dot{x} = \varepsilon f(t,x) \tag{1}$$

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with almost periodic function f for sufficiently small ε admits a unique almost periodic solution in the neighborhood of hyperbolic stationary point x_0 of the "averaged" equation

$$\dot{x} = \varepsilon f_0(x), \tag{2}$$

where

$$f_0(x) = \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} f(s,x)\, \mathrm{d}s \tag{3}$$

and limit (3) holds uniformly with respect to (w.r.t.) $t \in \mathbb{R}$. The first Bogolyubov's theorem determines the closeness or nearness of the solutions on finite time intervals for the original equation (1) and the averaged equation (2). Note that periodic and quasi-periodic functions are special almost periodic functions.

In this paper, we generalize the second Bogolyubov's theorem for Eq. (1) to the case when function f is recurrent or pseudorecurrent (see definitions in Sections 4 and 6).

The paper is organized as follows. In Section 2, we study the existence of invariant integral manifolds of quasi-linear non-autonomous dynamical systems (Theorems 2.5 and 2.6). Results in this section are used in the following sections.

Section 3 contains the main results about generalization of the second Bogolyubov's theorem for non-almost periodic systems (Theorems 3.4 and 3.5).

In Section 4 we give conditions of existence of recurrent solutions of non-autonomous equations in a standard form, if corresponding averaging equation admits a hyperbolic stationary point (Theorem 4.7).

Section 5 is devoted to study of the existence of invariant torus and quasi-periodic solutions of quasilinear equations on the torus (Theorems 5.2, 5.3 and Corollary 5.4).

In Section 6 we discuss the existence of pseudorecurrent integral manifolds (Theorem 6.2).

2. Quasi-linear non-autonomous dynamical systems

Let Ω be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on Ω. Let E be a Banach space, and Y and W are two complete metric spaces. Denote $L(E)$ the space of all linear continuous operators on E and $C(Y,W)$ the space of all continuous functions $f : Y \to W$ endowed with compact-open topology, i.e., uniform convergence on compact subsets in Y. We use these notations for the rest of the paper. The results in this section will be used in later sections.

Consider the linear equation

$$\dot{x} = A(\omega t)x \quad (\omega \in \Omega,\ \omega t = \sigma(t,\omega)) \tag{4}$$

and the inhomogeneous equation

$$\dot{x} = A(\omega t)x + f(\omega t), \tag{5}$$

where $A \in C(\Omega, L(E))$ and $f \in C(\Omega, E)$.

Definition 2.1. Let $U(t,\omega)$ be the operator of Cauchy (solution operator) of the linear equation (4). Equation (4) is called hyperbolic if there exist positive numbers N, $v>0$ and continuous projection $P\in C(\Omega,L(E))$ (i.e. $P^2(\omega)=P(\omega)$ for all $\omega\in\Omega$) such that

(1) For all $t\in\mathbb{R}$ and $\omega\in\Omega$, $U(t,\omega)P(\omega)=P(\omega t)U(t,\omega)$;
(2) For all $t\geqslant\tau$ and $\omega\in\Omega$, $\|U(t,\omega)P(\omega)U^{-1}(\tau,\omega)\|\leqslant N\exp(-v(t-\tau))$;
(3) For all $t\leqslant\tau$ and $\omega\in\Omega$, $\|U(t,\omega)Q(\omega)U^{-1}(\tau,\omega)\|\leqslant N\exp(v(t-\tau))$, where $Q(\omega)=I-P(\omega)$.

Definition 2.2. The function $G:\mathbb{R}^2_*\times\Omega\to L(E)$ defined by

$$G(t,\tau,\omega)=\begin{cases} U(t,\omega)P(\omega)U^{-1}(\tau,\omega) & \text{for } t>\tau,\\ -U(t,\omega)Q(\omega)U^{-1}(\tau,\omega) & \text{for } t<\tau \end{cases} \tag{6}$$

is called the Green's function for hyperbolic linear equation (4), where $\mathbb{R}^2_*=\mathbb{R}^2\backslash\Delta_{\mathbb{R}^2}$, $\Delta_{\mathbb{R}^2}=\{(t,t)\,|\,t\in\mathbb{R}\}$ and P, Q are the projections from definition 2.1.

Remark 2.3. The Green's function satisfies the following conditions (see [2,3]):

(1) For every $t\neq\tau$ the function $G(t,\tau,\omega)$ is continuously differentiable and

$$\frac{\partial G(t,\tau,\omega)}{\partial t}=A(\omega t)G(t,\tau,\omega)\quad(\omega\in\Omega).$$

(2) $G(\tau+0,\tau,\omega)-G(\tau-0,\tau,\omega)=I$ $(\tau\in\mathbb{R},\ \omega\in\Omega)$.
(3) $\|G(t,\tau,\omega)\|\leqslant N\exp(-v|t-\tau|)$ $(t,\tau\in\mathbb{R},\ \omega\in\Omega)$.
(4) $G(0,\tau,\omega t)=G(t,t+\tau,\omega)$ $(t,\tau\in\mathbb{R},\ \tau\neq 0,\ \omega\in\Omega)$.

Theorem 2.4. *Suppose that the linear equation* (4) *is hyperbolic. Then for* $f\in C(\Omega,E)$, *the function* $\gamma(\omega)$ *defined by*

$$\gamma(\omega)=\int_{-\infty}^{+\infty}G(0,\tau,\omega)f(\omega\tau)\,\mathrm{d}\tau\quad(\omega\in\Omega) \tag{7}$$

is continuous, i.e., $\gamma\in C(\Omega,E)$, *and*:

(1) $\gamma(\omega t)=\varphi(t,\gamma(\omega),\omega)$ *holds for all* $\omega\in\Omega$ *and* $t\in\mathbb{R}_+$, *where* $\varphi(t,x,\omega)$ *is the unique solution of the corresponding inhomogeneous equation* (5) *with the initial condition* $\varphi(0,x,\omega)=x$;
(2) $\|\gamma\|\leqslant 2N/v\|f\|$, *where* $\|\gamma\|=\max_{\omega\in\Omega}|\gamma(\omega)|$.

Proof. The proof of this assertion is obtained by slight modification of arguments from [3, Chapter III] and we omit the details. □

Let us consider the following quasi-linear equation in Banach space E

$$\dot{x} = A(\omega t)x + f(\omega t) + F(\omega t, x), \tag{8}$$

where $A \in C(\Omega, L(E))$, $f \in C(\Omega, E)$ and $F \in C(\Omega \times E, E)$.

Theorem 2.5 (Invariant integral manifold). *Assume that there exist positive numbers $L < L_0 := v/2N$ and $r < r_0 := \gamma_0(v/2N - L_0)^{-1}$ such that*

$$\|F(\omega, x_1) - F(\omega, x_2)\| \leqslant L\|x_1 - x_2\| \tag{9}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in B[Q, r] = \{x \in E \mid \rho(x, Q) \leqslant r\}$, where $Q = \gamma(\Omega)$, $\gamma \in C(\Omega, E)$ is defined in (7) and $\gamma_0 = \max_{\omega \in \Omega} \|F(\omega, \gamma(\omega))\|$. Then there exists a unique function $u \in C(\Omega, B[Q, r])$ such that

$$u(\omega t) = \psi(t, u(\omega), \omega) \tag{10}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\psi(\cdot, x, \omega)$ is the unique solution of the quasi-linear equation (8) with the initial condition $\psi(0, x, \omega) = x$. Therefore, the graph of u is an invariant integral manifold for the quasi-linear equation (8).

Proof. Let $x = y + \gamma(\omega t)$. Then from Eq. (8) we obtain

$$\dot{y} = A(\omega t)y + F(\omega t, y + \gamma(\omega t)). \tag{11}$$

If $0 < r < r_0$ and $\alpha \in C(\Omega, B[Q, r])$, then the equality

$$(\Phi\alpha)(\omega) = \int_{-\infty}^{+\infty} G(0, \tau, \omega)F(\omega\tau, \alpha(\omega\tau) + \gamma(\omega\tau))\,\mathrm{d}\tau \tag{12}$$

defines a function $\Phi\alpha \in C(\Omega, E)$. In virtue of Theorem 2.4, we have

$$\begin{aligned} \|\Phi\alpha\| &\leqslant \frac{2N}{v} \max_{\omega\in\Omega} \|F(\omega, \alpha(\omega) + \gamma(\omega))\| \\ &\leqslant \frac{2N}{v} \max_{\omega\in\Omega} \|F(\omega, \alpha(\omega) + \gamma(\omega)) - F(\omega, \gamma(\omega))\| + \frac{2N}{v} \max_{\omega\in\Omega} \|F(\omega, \gamma(\omega))\| \\ &\leqslant \frac{2N}{v} L\|\alpha\| + \frac{2N}{v}\gamma_0 \leqslant \frac{2N}{v} Lr + \frac{2N}{v}\gamma_0 \leqslant \frac{2N}{v} L_0 r_0 + \frac{2N}{v}\gamma_0 = r_0 \end{aligned} \tag{13}$$

and consequently $\Phi(C(\Omega, B[Q, r_0])) \subseteq C(\Omega, B[Q, r_0])$.

Now we will show that the mapping $\Phi : C(\Omega, B[Q, r_0]) \to C(\Omega, B[Q, r_0])$ is Lipschitzian. In fact, according to Theorem 2.4 we have

$$\begin{aligned} \|\Phi\alpha_1 - \Phi\alpha_2\| &\leqslant \frac{2N}{v} \max_{\omega\in\Omega} \|F(\omega, \alpha_1(\omega) + \gamma(\omega)) - F(\omega, \alpha_2(\omega) + \gamma(\omega))\| \\ &\leqslant \frac{2N}{v} L \max_{\omega\in\Omega} \|\alpha_1(\omega) - \alpha_2(\omega)\|. \end{aligned} \tag{14}$$

We note that $(2N/\nu)L \leqslant (2N/\nu)L_0 < 1$. Thus, the mapping Φ is a contraction and, consequently by Banach fixed point theorem, there exists a unique function $\alpha \in C(\Omega, B[Q, r_0])$ such that $\Phi\alpha = \alpha$. To finish the proof of the theorem it is sufficient to put $u = \gamma + \alpha$. □

We now consider the perturbed quasi-linear equation

$$\dot{x} = A(\omega t)x + f(\omega t) + \varepsilon F(\omega t, x), \tag{15}$$

where $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ ($\varepsilon_0 > 0$) is a small parameter. We have a similar theorem.

Theorem 2.6 (Invariant integral manifold and convergence). *Assume that there exist positive numbers r and L such that*

$$\|F(\omega, x_1) - F(\omega, x_2)\| \leqslant L\|x_1 - x_2\| \tag{16}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in B[Q, r]$. Then for sufficiently small ε there exists a unique function $u_\varepsilon \in C(\Omega, B[Q, r])$ such that

$$u_\varepsilon(\omega t) = \psi_\varepsilon(t, u_\varepsilon(\omega), \omega) \tag{17}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\psi_\varepsilon(\cdot, x, \omega)$ is the unique solution of Eq. (15) with the initial condition $\psi_\varepsilon(0, x, \omega) = x$. Moreover,

$$\lim_{\varepsilon \to 0} \max_{\omega \in \Omega} \|u_\varepsilon(\omega) - \gamma(\omega)\| = 0, \tag{18}$$

where $\gamma \in C(\omega, E)$ is defined in (7).

Proof. We can prove the existence of u_ε by slight modification of the proof of Theorem 2.4.

To prove (18) we note that

$$\|F(\omega, u_\varepsilon(\omega)\| \leqslant \|F(\omega, u_\varepsilon(\omega) - F(\omega, \gamma(\omega)\| + \|F(\omega, \gamma(\omega)\| \leqslant Lr + \gamma_0 \tag{19}$$

and

$$\begin{aligned} \|u_\varepsilon(\omega) - \gamma(\omega)\| &\leqslant \left\| \int_{-\infty}^{+\infty} \varepsilon G(0, \tau, \omega\tau) F(\omega\tau, u_\varepsilon(\omega\tau))\, d\tau \right\| \\ &\leqslant |\varepsilon| \frac{2N}{\nu} (Lr + \gamma_0) \end{aligned} \tag{20}$$

for all $\omega \in \Omega$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Passing to the limit in inequality (20) as $\varepsilon \to 0$ we obtain (18). □

3. Generalization of second Bogolyubov's theorem for non-almost periodic systems

In this section, we consider an analog of the second Bogolyubov's theorem for the non-autonomous system

$$\dot{x} = \varepsilon f(\omega t, x), \tag{21}$$

where $\varepsilon \in [0, \varepsilon_0]$ ($\varepsilon_0 > 0$) is a small parameter. We do *not* assume that f is almost periodic in time t. Suppose that the averaging

$$\bar{f}(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\omega t, x)\, \mathrm{d}t \tag{22}$$

exists uniformly w.r.t. $\omega \in \Omega$, and also uniformly w.r.t. x on every bounded subset of E.

Remark 3.1. Condition (22) is fulfilled if a dynamical system $(\Omega, \mathbb{R}, \sigma)$ is strictly ergodic, i.e. on Ω there exists a unique invariant measure μ w.r.t. $(\Omega, \mathbb{R}, \sigma)$.

Along with Eq. (21) we consider the averaged equation

$$\dot{x} = \varepsilon \bar{f}(x). \tag{23}$$

Setting slow time $\tau = \varepsilon t$ ($\varepsilon > 0$), Eqs. (21) and (23) can be written in the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f\left(\omega \frac{\tau}{\varepsilon}, x\right) \tag{24}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \bar{f}(x), \tag{25}$$

respectively.

Suppose that for certain point $x_0 \in E$

$$\bar{f}(x_0) = 0, \tag{26}$$

then Eq. (23) admits a stationary solution $\varphi_\varepsilon(t, x_0) \equiv x_0$.

Assume that the following conditions are fulfilled:

(i) Function $f \in C(\Omega \times B[x_0, r], E)$ where $B[x_0, r] = \{x \in E \mid \|x - x_0\| \leqslant r\}$ and $r > 0$, and F is bounded on $\Omega \times B[x_0, r]$. Limit (22) is uniform w.r.t. $(\omega, x) \in \Omega \times B[x_0, r]$ and functions $f'_x(\omega, x)$ and $\bar{f}'(x)$ are bounded on $\Omega \times B[x_0, r]$.

(ii) Functions $f(\omega, x)$ and $\bar{f}(x)$ are twice continuously differentiable w.r.t variable $x \in B[x_0, r]$.

(iii) Equality (22) can be twice differentiated, i.e., the following equalities:

$$\bar{f}^{(k)}(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f_x^{(k)}(\omega t, x)\, \mathrm{d}t \quad (k = 1, 2) \tag{27}$$

hold uniformly w.r.t $\omega \in \Omega$ and $x \in B[x_0, r]$.

We note that

$$\bar{f}(x + h) - \bar{f}(x) = \bar{f}'(x)h + R(x, h) \quad (x, x + h \in B[x_0, r]) \tag{28}$$

where $\|R(x, h)\| = o(\|h\|)$.

Let $A = \bar{f}'(x_0)$ and $B(h) = R(x_0, h)$. Then according to (26) and (28) we have

$$\bar{f}(x+h) = Ah + B(h) \tag{29}$$

It is clear (see [2, Chapter 7]) that the function $B(h)$ satisfies the condition of Lipschitz

$$\|B(h_1) - B(h_2)\| \leqslant L(r)\|h_1 - h_2\| \tag{30}$$

$(h_1, h_2 \in B[x_0, r])$ and $L(r) \to 0$ as $r \to 0$.

Eq. (21) can be rewritten in the following form:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \varepsilon Ah + \varepsilon g(\omega t, h), \tag{31}$$

where $h = x - x_0$ and

$$g(\omega, h) = f(\omega, x+h) - \bar{f}(x_0 + h) + B(h). \tag{32}$$

In Eq. (32) we make the following change of variable:

$$h = z - \varepsilon v(\omega, z, \varepsilon), \tag{33}$$

where

$$v(\omega, z, \varepsilon) = \int_0^{+\infty} V(\omega s, z)\exp(-\varepsilon s)\,\mathrm{d}s \tag{34}$$

and

$$V(\omega, z) = f(\omega, x_0 + z) - \bar{f}(x_0 + z). \tag{35}$$

Lemma 3.2 (Daleckij and Krein [2, p. 457]). *Let $\varphi : \mathbb{R}_+ \times \Lambda \to E$ be a function satisfying the following conditions*:

1. $M := \sup\{\|\frac{1}{t}\int_0^t \varphi(s, \lambda)\,\mathrm{d}s\| \,|\, t \geqslant 0,\ \lambda \in \Lambda\} < +\infty$.
2. $\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(s, \lambda)\,\mathrm{d}s = 0$ *uniformly w.r.t. variable* $\lambda \in \Lambda$.

Then the following equality:

$$\lim_{p\to 0} p \int_0^{+\infty} \varphi(s, \lambda)\exp(-ps)\,\mathrm{d}s = 0$$

takes place uniformly w.r.t. $\lambda \in \Lambda$.

Lemma 3.3. *The following equalities*:

$$\lim_{\varepsilon \downarrow 0} \varepsilon v(\omega, z, \varepsilon) = 0 \tag{36}$$

and

$$\lim_{\varepsilon \downarrow 0} \varepsilon v_z'(\omega, z, \varepsilon) = 0 \tag{37}$$

are fulfilled uniformly w.r.t. $\omega \in \Omega$ *and* $z \in B[0, r]$.

Proof. This assertion follows from Lemma 3.2. In fact, in virtue of (22), (27) and (34), the bounded functions $V(\omega s, z)$ and $V_z'(\omega s, z)$ satisfy the conditions of Lemma 3.2. □

From equality (27) it follows that for sufficiently small $\varepsilon > 0$ the operator $I - \varepsilon v_z'(\omega, z, \varepsilon)$ $(\omega \in \Omega,\ z \in B[0, r])$ is invertible and $(I - \varepsilon v_z'(\omega, z, \varepsilon))^{-1}$ is bounded, and, consequently, mapping (33) is invertible. According to Eq. (36) in the sufficiently small neighborhood of zero and for sufficiently small $\varepsilon > 0$, we can make the change of variable (33).

Note that

$$v(\omega t, z, \varepsilon) = \int_0^{+\infty} V(\omega(t+s), z, \varepsilon)) \exp(-\varepsilon s)\, ds$$

$$= \exp(\varepsilon t) \int_t^{+\infty} V(\omega s, z, \varepsilon)) \exp(-\varepsilon s)\, ds$$

and we find that

$$\frac{d}{dt} v(\omega t, z, \varepsilon) = \varepsilon v(\omega t, z, \varepsilon) - V(\omega t, z, \varepsilon) \tag{38}$$

and, consequently,

$$\frac{dh}{dt} = \frac{dz}{dt} - \varepsilon v_z' \frac{dz}{dt} - \varepsilon^2 v + \varepsilon V. \tag{39}$$

Using relation (33), (35) and (39) we reduce Eq. (31) to the form

$$\begin{aligned}(I - \varepsilon v_z'(\omega t, z, \varepsilon)) \frac{dz}{dt} &= \varepsilon[f(\omega t, x_0 + h, \varepsilon) - f(\omega t, x_0 + z, \varepsilon)] \\ &\quad + \varepsilon \bar{f}(x_0 + z) + \varepsilon^2 v(\omega t, z, \varepsilon) = \varepsilon(Az + B(z)) \\ &\quad + \varepsilon^2 v(\omega t, z, \varepsilon) + \varepsilon[f(\omega t, x_0 + z - \varepsilon v, \varepsilon) \\ &\quad - f(\omega t, x_0 + z, \varepsilon)]. \end{aligned} \tag{40}$$

After multiplication of the both sides of Eq. (40) by $(I - \varepsilon v_z'(\omega t, z, \varepsilon))^{-1}$ and introduction of the "slow" time $\tau = \varepsilon t$ we obtain

$$\frac{dz}{d\tau} = Az + F\left(\omega \frac{\tau}{\varepsilon}, z, \varepsilon\right), \tag{41}$$

where F possesses the following properties:

(a) F admits a bounded derivable $F_z'(\omega, z, \varepsilon)$ $(\omega \in \Omega,\ z \in B[0, r]$ and $\varepsilon \in [o, \varepsilon_0])$,
(b) $F(\omega, z, \varepsilon) = B(z) + O(z)$ uniformly w.r.t. $\omega \in \Omega$ and $z \in B[0, r]$,
(c) for every $M > 0$ and $\mu > 0$, there exists positive numbers $\varepsilon_0' \leqslant \varepsilon_0$ and β_0 such that for $0 < \varepsilon < \varepsilon_0'$, and $\|z\| < \beta_0$, the inequalities

$$\|F(\omega, z, \varepsilon)\| \leqslant M \tag{42}$$

and

$$\|F(\omega, z_1, \varepsilon) - F(\omega, z_2, \varepsilon)\| \leqslant \mu \|z_1 - z_2\| \tag{43}$$

take place for all $\omega \in \Omega$, $z_1, z_2 \in B[0, r]$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0'$.

Theorem 3.4 (Dynamics of the transformed system). *Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$, where $\sigma(A)$ is the spectrum of the operator $A = \bar{f}'(x_0)$. Then*

(1) *For the transformed equation (41), there exists a unique function $\tilde{u}_\varepsilon \in C(\Omega, B[0, \beta])$ such that*

$$\tilde{u}_\varepsilon(\omega\tau) = \psi_\varepsilon(\tau, \tilde{u}_\varepsilon(\omega), \omega) \tag{44}$$

for all $\tau \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\psi_\varepsilon(\cdot, x, \omega)$ is a unique solution of equation (41) *which initial condition $\psi_\varepsilon(0, x, \omega) = x$;*

(2)

$$\lim_{\varepsilon \to 0} \max_{\omega \in \Omega} \|\tilde{u}_\varepsilon(\omega)\| = 0. \tag{45}$$

Proof. This statement follows from Theorem 2.6. □

Theorem 3.5 (Analog of the second Bogolyubov's theorem). *Assume that the conditions* (i)–(iii) *and* (26) *are fulfilled and $\sigma(A) \cap i\mathbb{R} = \emptyset$, where $\sigma(A)$ is the spectrum of the operator $A = \bar{f}'(x_0)$. Then for sufficiently small $r_0 > 0$, there is ε_0' with $0 < \varepsilon_0' \leqslant \varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0'$, there exists a unique function $u_\varepsilon \in C(\Omega, B[x_0, r])$ such that*

$$u_\varepsilon(\omega t) = \psi_\varepsilon(t, u_\varepsilon(\omega), \omega) \tag{46}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$ and

$$\lim_{\varepsilon \to 0} \max_{\omega \in \Omega} \|u_\varepsilon(\omega) - x_0\| = 0, \tag{47}$$

where $\psi_\varepsilon(\cdot, x, \omega)$ is the unique solution of the non-autonomous equation (21) *with initial condition $\psi_\varepsilon(0, x, \omega) = x$, and x_0 is a stationary solution of the averaged equation* (23). *Note that the graph of u_ε is an invariant integral manifold for the non-autonomous equation* (21).

Proof. Under the conditions of the theorem and in virtue of Theorem 3.4 for Eq. (41), there exists a unique function $\tilde{u}_\varepsilon \in C(\Omega, B[0, r_0])$ with the properties (44) and (45). Denote by

$$u_\varepsilon(\omega) = x_0 + \tilde{u}_\varepsilon(\omega) - \varepsilon v(\omega, \tilde{u}_\varepsilon(\omega), \omega). \tag{48}$$

Then from equalities (34), (35), (43) and (47), we obtain equality (48) and the continuity of $u_\varepsilon : \Omega \to E$. Consequently, we have, $u_\varepsilon \in C(\Omega, B[x_0, r])$ for sufficiently small $\varepsilon > 0$. Equality (46) follows from equalities (44) and (48). The theorem is thus proved. □

4. Almost periodic and recurrent solutions

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$, $(X, \mathbb{T}, \pi)$ be a dynamical system, $x \in X$, $\tau, \varepsilon \in \mathbb{T}$, $\tau > 0$, $\varepsilon > 0$. We denote $\pi(x,t)$ by a short-hand notation xt.

The point x is called a stationary point if $xt = x$ for all $t \in \mathbb{T}$. The point x is called τ-periodic if $x\tau = x$.

The number τ is called ε-shift (ε-almost period) of a point x if $\rho(x\tau, x) < \varepsilon$ ($\rho(x(t+\tau), xt) < \varepsilon$ for all $t \in \mathbb{T}$).

The point x is called almost recurrent (almost periodic) if for any $\varepsilon > 0$ there exists positive number l such that on every segment of length l can be found an ε-shift (ε-almost period) of the point x.

A point x is called recurrent if it is almost recurrent and the set $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact.

Denote by $\mathfrak{M}_x = \{\{t_n\} | \{xt_n\}$ is convergent$\}$.

Theorem 4.1 (Scherbakov [4,5]). *Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be dynamical systems with $\mathbb{T}_1 \subset \mathbb{T}_2$. Assume that $h: X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$. If the point $x \in X$ is stationary (τ-periodic, quasi-periodic, almost periodic, recurrent), then the point $h(x) = y$ is also stationary (τ-periodic, quasi-periodic, almost periodic, recurrent) and $\mathfrak{M}_x \subset \mathfrak{M}_y$.*

Consider the following non-autonomous equation in Banach space E

$$w' = f(\omega t, w), \tag{49}$$

where $f \in C(\Omega \times E, E)$. Suppose that the function f is regular, i.e., for all $w \in E$ and $\omega \in \Omega$, Eq. (49) admits a unique solution $\varphi(t, w, \omega)$ defined on $\mathbb{R}_+$ with the initial condition $\varphi(0, w, \omega) = w$ and the mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \to E$ is continuous.

It is well known (see, for example, [6]) that the mapping φ satisfies the following conditions:

(a) $\varphi(0, w, \omega) = w$ for all $w \in E$ and $\omega \in \Omega$;
(b) $\varphi(t + \tau, w, \omega) = \varphi(t, \varphi(\tau, w, \omega), \omega\tau)$ for all $t, \tau \in \mathbb{T}_1$, $w \in E$ and $\omega \in \Omega$.

The solution $\varphi(t, w, \omega)$ of Eq. (49) is said to be stationary (τ-periodic, almost periodic, recurrent) if the point $x := (w, \omega) \in X := E \times \Omega$ is stationary (τ-periodic, almost periodic, recurrent) point of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$, where $\pi = (\varphi, \sigma)$, i.e. $\pi(t, (w, \omega)) = (\varphi(t, w, \omega), \omega t)$ for all $t \in \mathbb{R}_+$ and $(w, \omega) \in E \times \Omega$.

Lemma 4.2. *Suppose that $u \in C(\Omega, E)$ satisfies the condition*

$$u(\omega t) = \varphi(t, u(\omega), \omega) \tag{50}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$. Then the mapping $h : \Omega \to X$ defined by

$$h(\omega) = (u(\omega), \omega) \tag{51}$$

for all $\omega \in \Omega$ is a homomorphism from $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$.

Proof. This assertion follows from equalities (50) and (51). □

Remark 4.3. The function $u \in C(\Omega, E)$ with property (50) is called continuous invariant section (or integral manifold) for non-autonomous system (49).

Theorem 4.4. *If the function $u \in C(\Omega, E)$ satisfies condition (50) and the point $\omega \in \Omega$ is stationary (τ-periodic, almost periodic, recurrent), then the solution $\varphi(t, u(\omega), \omega)$ of the Eq. (49) also will be stationary (τ-periodic, almost periodic, recurrent).*

Proof. This statement follows from Theorem 4.1 and Lemma 4.2. □

Example 4.5. Consider the equation

$$u' = f(t, u), \tag{52}$$

where $f \in C(\mathbb{R} \times E, E)$; here $C(\mathbb{R} \times E, E)$ is the space of all continuous function $\mathbb{R} \times E \to E$) equipped with compact-open topology. Along with Eq. (52), we will consider the H-class of Eq. (52)

$$u' = g(t, u) \quad (g \in H(f)), \tag{53}$$

where $H(f) = \overline{\{f_\tau \mid \tau \in \mathbb{R}\}}$ and the over bar denotes the closure in $C(\mathbb{R} \times E, E)$ and $f_\tau(t, u) = f(t + \tau, u)$ for all $t \in \mathbb{R}$ and $u \in E$. Denote by $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ the Bebutov's dynamical system (see, for example, [4–6]). Here $\sigma(t, g) = g_t$ for all $t \in \mathbb{R}$ and $g \in C(\mathbb{R} \times E, E)$.

The function $f \in C(\mathbb{R} \times E, E)$ is called regular (see [6]) if for all $u \in E$ and $g \in H(f)$ Eq. (53) admits a unique solution $\varphi(t, u, g)$ defined on $\mathbb{R}_+$ with the initial condition $\varphi(0, u, g) = u$.

Let Ω be the hull $H(f)$ of a given regular function $f \in C(\mathbb{R} \times E, E)$ and denote the restriction of $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ on Ω by $(\Omega, \mathbb{R}, \sigma)$. Let $F : \Omega \times E \to E$ be a continuous mapping defined by $F(g, u) = g(0, u)$ for $g \in \Omega$ and $u \in E$. Then the equation (53) can be written in such form

$$u' = F(\omega t, u), \tag{54}$$

where $\omega = g$ and $\omega t = g_t$.

Lemma 4.6. *The following two conditions are equivalent*:

(1) *There exists a limit*

$$f_0(x) = \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} f(s, x)\, \mathrm{d}s \tag{55}$$

uniformly w.r.t. $t \in \mathbb{R}$ and x on every compact set $K \subset E$.

(2) *There exists a limit*

$$f_0(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^{T} g(s, x)\, \mathrm{d}s \tag{56}$$

uniformly w.r.t. $g \in H(f)$ and x on every compact set $K \subset E$.

Proof. Equality (55) follows from (56) because $f_t \in H(f)$ for all $t \in \mathbb{R}$ and

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f_t(s,x)\,\mathrm{d}s = \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} f(s,x)\,\mathrm{d}s.$$

Let $g \in H(f)$, then there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $g = \lim_{n\to+\infty} f_{t_n}$. From equality (55), it follows that for all $\varepsilon > 0$ and compact set $K \subset E$, there exists $L(\varepsilon, K) > 0$ such that

$$\left\| \frac{1}{T} \int_0^T f_{t_n}(s,x)\,\mathrm{d}s - f_0(x) \right\| < \varepsilon \tag{57}$$

for all $n \in \mathbb{N}$ and $T \geqslant L(\varepsilon, K)$. Passing to the limit in equality (57) as $n \to +\infty$ we obtain equality (56). □

Theorem 4.7 (Recurrent solutions). *Suppose that the following conditions are fulfilled*:

(1) *$f \in C(\mathbb{R} \times E, E)$ and there exist $x_0 \in E$ and $r > 0$ such that the function f is bounded on $\mathbb{R} \times B[x_0, r]$, i.e., there exists a positive number M such that*

$$\|f(t,x)\| \leqslant M \tag{58}$$

for all $t \in \mathbb{R}$ and $x \in B[x_0, r]$.

(2) *The functions $f \in C(\mathbb{R} \times E, E)$ and $f_0 \in C(E, E)$ are twice continuously differentiable w.r.t. variable $x \in B[x_0, r]$. Moreover, the function $f'_x(t,x)$ is bounded on $\mathbb{R} \times B[x_0, r]$, and $f'_0(x)$ is bounded on $B[x_0, r]$.*

(3) *Equality (55) can be twice differentiated, i.e. the following equalities*:

$$f_0^{(k)}(x) = \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} f_x^{(k)}(s,x)\,\mathrm{d}s \quad (k = 1, 2) \tag{59}$$

take place, uniformly w.r.t. $t \in \mathbb{R}$ and $x \in B[x_0, r]$.

(4) *$f_0(x_0) = 0$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$, where $A = f'_0(x_0)$ and $\sigma(A)$ is the spectrum of operator A.*

(5) *The function $f \in C(\mathbb{R} \times E, E)$ is stationary (τ-periodic, almost periodic, recurrent) w.r.t. $t \in \mathbb{R}$, and uniformly w.r.t. to x on every compact subset $K \subset E$.*

Then for sufficiently small $r_0 > 0$, there exists $0 < \varepsilon'_0 \leqslant \varepsilon_0$ such that for $0 < \varepsilon < \varepsilon'_0$ the equation

$$x' = \varepsilon f(t,x) \tag{60}$$

admits a unique stationary (τ-periodic, almost periodic, recurrent) solution $\varphi^\varepsilon(t)$ with the following properties:

(a) *$\|\varphi^\varepsilon(t) - x_0\| \leqslant r_0$ for all $t \in \mathbb{R}$.*

(b) $\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} \|\varphi^\varepsilon(t) - x_0\| = 0.$

(c) *$\mathfrak{M}_f \subset \mathfrak{M}_{\varphi^\varepsilon}$, where $\mathfrak{M}_f = \{\{t_n\} | \{f_{t_n}\}$ is convergent on $C(\mathbb{R} \times E, E)\}$ and $\mathfrak{M}_{\varphi^\varepsilon} = \{\{t_n\} | \{\varphi^\varepsilon_{t_n}\}$ is convergent on $C(\mathbb{R}, E)\}$.*

Proof. Note that conditions (1)–(4) imply conditions (i)–(iii). So the proof of the theorem follows from Theorems 3.5, 4.1, 4.4 and Lemma 4.6. □

Remark 4.8. Note that Theorem 4.7 is also true for Eq. (60) with non-recurrent function f. For example, if f is pseudorecurrent [4,5], i.e., if $H(f)$ is compact and every function $g \in H(f)$ is stable in the sense of Poisson. In this case we can affirm that the solution φ^{ε} will be also pseudorecurrent. See also Section 6 later in this paper.

5. Invariant torus and quasi-periodic solutions

Let $\mathscr{T}^m$ be an m-dimensional torus. We consider a non-autonomous dynamical system in Banach space E, with a driving system defined on the torus $\mathscr{T}^m$:

$$\begin{cases} x' = A(\omega)x + f(\omega) + F(\omega,x), \\ \omega' = \Phi(\omega), \end{cases} \tag{61}$$

where $\Phi \in C(\mathscr{T}^m, T\mathscr{T}^m)$, $T\mathscr{T}^m$ is a tangent space of the torus $\mathscr{T}^m$, $f \in C(\mathscr{T}^m, E)$, $A \in C(\Omega, L(E))$ and $F \in C(\mathscr{T}^m \times E, E)$.

We suppose that the second equation of system (61) generates an autonomous dynamical system $(\mathscr{T}^m, \mathbb{R}, \sigma)$ on the torus $\mathscr{T}^m$ and the equation

$$x' = A(\omega t)x + f(\omega t) + F(\omega t, x) \tag{62}$$

admits a unique solution $\varphi(t,x,\omega)$ defined on $\mathbb{R}_+$ and satisfying the initial condition $\varphi(0,x,\omega) = x$.

A function $\gamma \in C(\mathscr{T}^m, E)$ is called [3] an m-dimensional invariant torus of equation (62) (or system (61)) if

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{63}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \mathscr{T}^m$.

Applying the results from Sections 2–4, we have the following tests of existence of the invariant torus for Eq. (62).

Theorem 5.1 (Invariant torus). *Suppose that Eq. (4) is hyperbolic and there exist positives numbers $0 < L < L_0 := \nu/2N$ and $0 < r < r_0 := \nu_0(\nu/2N - L_0)^{-1}$ such that the function $F \in C(\mathscr{T}^m \times E, E)$ satisfies condition (9). Then Eq. (62) admits an m-dimensional invariant torus.*

Theorem 5.2 (Invariant torus for perturbed system). *Suppose that there exist positives numbers r and L such that condition (16) is fulfilled. Then for sufficient small $\varepsilon \geqslant 0$ there exists an m-dimensional invariant torus u_ε for the perturbed equation*

$$x' = A(\omega t)x + f(\omega t) + \varepsilon F(\omega t, x) \quad (\omega \in \mathscr{T}^m) \tag{64}$$

and

$$\lim_{\varepsilon \to 0} \max_{\omega \in \Omega} \|u_\varepsilon(\omega) - u_0(\omega)\| = 0.$$

Theorem 5.3 (Unique invariant torus). *Let* $\Omega = \mathscr{T}^m$. *Assume the conditions of Theorem* 3.5 *are satisfied. Then, for a sufficient small* $r_0 > 0$, *there exists* $0 < \varepsilon_0' < \varepsilon_0$ *such that for all* $0 < \varepsilon < \varepsilon_0'$ *there exists a unique m-dimensional invariant torus* u_ε *for Eq.* (61) *and*

$$\lim_{\varepsilon \to 0} \max_{\omega \in \Omega} \|u_\varepsilon(\omega) - x_0\| = 0.$$

We have the following corollary for quasi-periodic non-autonomous dynamical systems, i.e., the driving system defined on the torus $\mathscr{T}^m$ is quasi-periodic in time.

Corollary 5.4 (Compact minimal invariant torus). *Suppose that the conditions of Theorem* 5.1 (*respectively*, *Theorems* 5.2 *or* 5.3) *are fulfilled and the dynamical system* $(\mathscr{T}^m, \mathbb{R}, \sigma)$ *generated by the second equation of system* (61) *is compact minimal and contains only quasi-periodic motions*, *then Eq.* (62) (*respectively*, *Eqs.* (64) *or* (21)) *admits an m-dimensional invariant torus* u_ε *which is compact minimal and contains only quasi-periodic motions.*

6. Pseudorecurrent solutions

An autonomous dynamical system $(\Omega, \mathbb{T}, \sigma)$ is said to be pseudorecurrent if the following conditions are fulfilled:

(a) Ω is compact;
(b) $(\Omega, \mathbb{T}, \sigma)$ is transitive, i.e. there exists a point $\omega_0 \in \Omega$ such that $\Omega = \overline{\{\omega_0 t | t \in \mathbb{T}\}}$;
(c) every point $\omega \in \Omega$ is stable in the sense of Poisson, i.e.

$$\mathfrak{N}_\omega = \{\{t_n\} \mid \omega t_n \to \omega \text{ and } |t_n| \to +\infty\} \neq \emptyset.$$

Lemma 6.1. *Let* $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h \rangle$ *be a non-autonomous dynamical system and the following conditions are fulfilled*:

(1) $(\Omega, \mathbb{T}_2, \sigma)$ *is pseudorecurrent*;
(2) $\gamma \in C(\Omega, X)$ *is an invariant section of the homomorphism* $h : X \to \Omega$.

Then the autonomous dynamical system $(\gamma(\Omega), \mathbb{T}_2, \pi)$ *is pseudorecurrent.*

Proof. It is evident that the space $\gamma(\Omega)$ is compact, because Ω is compact and $\gamma \in C(\Omega, X)$. We note that on the space $\gamma(\Omega)$, by the homomorphism $\gamma : \Omega \to \gamma(\Omega)$, we have a dynamical system $(\gamma(\Omega), \mathbb{T}_2, \hat{\pi})$, namely $\hat{\pi}^t \gamma(\omega) := \gamma(\omega t)$ for all $t \in \mathbb{T}_2$ and $\omega \in \Omega$, then $\hat{\pi}^t \gamma(\omega) = \pi^t \gamma(\omega)$ for all $t \in \mathbb{T}_1 \subseteq \mathbb{T}_2$ and $\omega \in \Omega$. Now we will show that $\gamma(\Omega) = \overline{\{\gamma(\omega_0)t \mid t \in \mathbb{T}_2\}}$. In fact, let $x \in \gamma(\Omega)$. Then there exists a unique point $\omega \in \Omega$ such that $x = \gamma(\omega)$. Let $\{t_n\} \subset \mathbb{T}_2$ be a sequence such that $\omega_0 t_n \to \omega$. Then $x = \gamma(\omega) = \lim_{n \to +\infty} \gamma(\omega_0 t_n) = \lim_{n \to +\infty} \gamma(\omega) t_n)$ and, consequently, $\gamma(\Omega) \subset \overline{\{\gamma(\omega_0)t \mid t \in \mathbb{T}_2\}}$. The inverse inclusion is trivial. Hence, $\gamma(\Omega) = \overline{\{\gamma(\omega_0)t \mid t \in \mathbb{T}_2\}}$. To finish the proof of the lemma it is sufficient to note that $\mathfrak{N}_\omega \subseteq \mathfrak{N}_{\gamma(\omega)}$ for every point $\omega \in \Omega$ and, consequently, every point $\gamma(\omega)$ is Poisson stable. The lemma is proved. □

Lemma 6.1 implies that the conditions of Theorem 5.1 (respectively, Theorems 5.2 or 5.3) are satisfied. Therefore, we have the following result.

Theorem 6.2 (Pseudorecurrent integral manifold). *Assume the driving dynamical system* $(\Omega, \mathbb{T}, \sigma)$ *is pseudorecurrent, and assume the conditions in Lemma* 6.1 *are satisfied. Then Eq.* (62) (*respectively, Eqs.* (64) *or* (21)) *admits a pseudorecurrent integral manifold.*

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