Electronic Journal of Differential Equations, Vol. 2003(2003), No. 41, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

HETEROCLINIC POINTS OF MULTI-DIMENSIONAL DYNAMICAL SYSTEMS

DAVID CHEBAN, JINQIAO DUAN, & ANATOLY GHERCO

Abstract. The authors investigate dynamical behavior of multi-dimensional dynamical systems. These are the systems with a multi-dimensional independent "time" variable. Especially they consider the problem of concordance, in the sense of Shcherbakov, of limit points and heteroclinic or homoclinic points for multi-dimensional dynamical systems and solutions of the multi-dimensional non-autonomous differential equations.

Introduction

In the qualitative research of the non-autonomous ordinary differential equation $\frac{dy}{dt} = f(t, y)$, where the independent variable t is a real number, crucial concepts are stability, asymptotic stability, and concordance. See for example [10] among others. It is also interesting to investigate qualitative behavior of non-autonomous differential equations in total (or Jacobian) derivative in the form $y' = f(t, y)$, where the independent variable t takes values in a finite-dimensional vector space.

In this paper, we investigate dynamical behavior of multi-dimensional non-autonomous differential equations. It becomes necessary to generalize the concepts of unilateral stability, asymptotic stability, concordance and others. Because in multi-dimensional spaces different ways of approaching ∞ are possible, the above concepts allow various generalizations. These concepts are related to the concept of the limit at $+\infty$ or $-\infty$ as well as to the concept of limit set of a point (ω-limit set, α-limit set and limit set).

In an abstract dynamical system (X, S, π), where S is an arbitrary topological group or semigroup, a natural generalization of ω-limit set of a point is the concept of P-limit P_x of a point $x \in X$ [9], [6]: $P_x = \bigcap_{t \in P} \overline{xtP}$ (here $P \subset S$).

In the research of multi-dimensional differential equations, the concepts of the filter and the convergence on the filter are widely used (see [5], for example). However, the usage of the convergence on the filter does not allow to apply to the investigation of non-autonomous multi-dimensional differential equations, the theory of extensions of transformation groups and transformation semigroups. In the research on asymptotic recurrent solutions of equations in total derivatives [7], by methods of extensions of transformation semigroups, the generalization of the limit

1991 *Mathematics Subject Classification.* [2000]37B05, 37B55, 54H15, 35B15, 35B35.
Key words and phrases. Topological dynamics, transformation semigroup, nonautonomous dynamical system, limit set, heteroclinic point, almost periodicity, concordance, multi-dimensional differential equations.

Submitted December 03, 2002. Published April 15, 2003.

at ∞ through the addition operation "+" on the range of definition of the solutions of equation is used. This is similar to the one-dimensional case where the concept of the limit at ∞ is related to the ordering of the range of definition of the solutions by the relation $>$ by the rule $x > y$ iff $x = y + z$ for some positive number z.

Shcherbakov and Cheban [10, 11, 3, 12] have developed both the general theory of the concordance of points of dynamical systems (flows) , the concordance of points in limit sets, and the concordance of heteroclinic or homoclinic points; see also the book [4] ,where the concordance of heteroclinic or homoclinic points are fully explained.

In the present paper, we use, from our point of view, a natural generalization of the concepts of limit sets of a point through the consideration of whole classes of sequences with certain properties. It has enabled us to generalize some ideas in [10, 11, 3, 12, 4] for flows and for ordinary differential equations to bear on transformation semigroups (X, S, π) and their extensions $\varphi : (X, S, \pi) \to (Y, S, \eta)$, including the special case for $S = \mathbb{R}^n$. Note that transformation semigroups are naturally associated with autonomous differential equations and extensions of transformation semigroups are associated with non-autonomous differential equations. Thus as a generalization of a heteroclinic (homoclinic) point, the concept of $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point of certain type $(k_1, k_2, \dots, k_m)$ naturally appears. Here $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$, with $P_1, P_2, \dots, P_m \subset S$ and k_i is a natural number by which a certain property of recursivity (P-recurrence, $\mathcal{P}$-recurrence, P-almost periodicity) is numbered. We will also apply some obtained general results to multi-dimensional differential equations.

For $n = 1$ the basic results of [4] respect to the heteroclinic and homoclinic points of dynamical systems and the heteroclinic or homoclinic solutions of ordinary differential equations follow from our results. In addition to [4]:

1) we consider also the distal case (it is known [2] that the distal functions, generally speaking, are not almost periodic);

2)we further consider the case when the type of recursivity at $t \to +\infty$ and the type of recursivity at $t \to -\infty$ are different, while in [4] both type of recursivity are the same.

The main results of our paper are theorems 4.9–5.4. Next, we describe briefly the content of our paper.

In Section 1, we recall basic definitions from the theory of topological transformation semigroups.

In Section 2, we discuss basic results of the theory of concordance in the sense of Shcherbakov [10, 12, 1] for flows in the context of transformation semigroups. We also deduce some sufficient conditions for the P-concordance under which the uniform S-concordance follows (from a concordance on some sub-semigroup P of S the uniform concordance on semigroup S follows).

In Section 3, we investigate elementary properties of the limit set P_x of a point x of a transformation semigroup (X, S). In this section we also discuss another approach for the definition of limit sets of a point of transformation group (X, S), where S is the additive group of $\mathbb{R}^n$. We do it with the help of some classes of the sequences $N(P)$ and $N(\mathcal{P})$. Here $\mathcal{P}$ is a family of all sub-semigroups P of S consists of points $(s_1, s_2, \dots, s_n)$ for which k fixed components are nonnegative and the rest are non-positive ($k = 0, 1, \dots, n$ and for $P \in \mathcal{P}$, $N(P)$ is family of all sequences $\{s_k\}$ in P such that for all $p \in P\, \exists n_0 \in \mathbb{N}\, \forall k \in \mathbb{N}$, $k > n_0$, $s_k \in p + P$).

We introduce two types of limit sets $P_x^{N(P)}$ and $\mathcal{P}_x$ of a point x: $P_x^{N(P)} = \{y \mid y = \lim_{n\to+\infty} xf(n)\ (f \in N(P))\}$; $\mathcal{P}_x = \bigcup_{P\in\mathcal{P}} P_x^{N(P)}$. We will prove that $P_x^{N(P)} = P_x$.

In this section, we obtain sufficient conditions on the existence of a continuous map $h : P_y \to P_x$ so that we can establish the concordance of points of a limit set.

In Section 4, we introduce the concept $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point and concept of $\mathcal{P}$-concordance of $\mathcal{P}$-heteroclinic points. We also discuss the type $(k_1, k_2, \dots, k_m)$ of a $\mathcal{P}$-heteroclinic point (k_i is a natural number by which a certain property of recursivity as P-recurrence, $\mathcal{P}$-recurrence, P-almost periodicity is numbered).

In this section the problem of the concordance of the points of the limit sets and $\mathcal{P}$-concordance of $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) points of the defined type are considered, too.

In Section 5, the basic results from the previous section are applied to the multidimensional differential equations of the form $y'(t) = f(t, y(t))$, where $f \in C(\mathbb{R}^n \times \mathbb{R}^l, L(\mathbb{R}^n, \mathbb{R}^l))$ and $y'(t)$ denotes the Frechet derivative of $y \in C(\mathbb{R}^n, \mathbb{R}^l)$ at the point t. Note that $L(\mathbb{R}^n, \mathbb{R}^l)$ is the space of all linear operators $\mathbb{R}^n \to \mathbb{R}^l$ with the natural operator norm, $C(\mathbb{R}^n, \mathbb{R}^l)$ is the space of all continuous mappings $\mathbb{R}^n \to \mathbb{R}^l$ equipped with compact-open topology and $C(\mathbb{R}^n \times \mathbb{R}^l, L(\mathbb{R}^n, \mathbb{R}^l))$ is the space of all continuous mappings $\mathbb{R}^n \times \mathbb{R}^l \to L(\mathbb{R}^n, \mathbb{R}^l)$ equipped with compact-open topology.

1. Basic definitions and notation

We recall some basic concepts in topological dynamics [9] (see also [13, 6, 12, 1]). We denote the set of all natural or real numbers with symbol $\mathbb{N}$ or $\mathbb{R}$, respectively.

Let X be a Hausdorff topological space, S be a topological semigroup with the unit element e, and $\pi : X \times S \to X$ be a continuous map. Triple (X, S, π) is called a transformation semigroup if the following conditions are satisfied: 1) $\forall x \in X :\ \pi(x, e) = x$; 2) for all $x \in X$ $\forall s, t \in S :\ \pi(\pi(x, s), t) = \pi(x, st)$.

The space X is usually called the phase space of (X, S, π). Let (X, S, π) be a transformation semigroup, $A \subset X$, $P \subset S$, $s \in S$. Usually we shall write π^s for the map $X \to X$ defined by $\pi^s(x) = \pi(x, s)$ $(x \in X)$; $xs = \pi^s(x)$. The image of the set $A \times P$ by the mapping π is designated by AP.

As usual, if there is no misunderstanding, instead of (X, S, π) we write (X, S).

The set A is called P-invariant (in the transformation semigroup (X, S)) if $AP \subset A$. An S-invariant set is called invariant. The empty set is considered invariant.

A nonempty closed P-invariant set Y is called P-minimal (in the transformation semigroup (X, S)) if Y does not contain proper P-invariant nonempty closed subsets. A S-minimal set is called minimal.

Let (X, S, π) and (Y, S, ρ) be transformation semigroups. A continuous mapping φ of the space X to Y satisfying the condition $\varphi \circ \pi^s = \rho^s \circ \varphi$ $(s \in S)$ is called homomorphism of the transformation semigroup (X, S, π) on to the transformation semigroup (Y, S, ρ). We also say that the extension $\varphi : (X, S) \to (Y, S)$ of the transformation semigroups is given if φ is a homomorphism of the transformation semigroup (X, S) on to (Y, S).

Let (X, S) be a transformation semigroup and $[S]$ be a nonempty family of subsets of S, which we shall call $[S]$-admissible.

The transformation semigroup (X, S) is called $[S]$-recursive at a point $x \in X$, and the point x is called $[S]$-recursive if for an arbitrary neighborhood V of the point x there exists $A \in [S]$ such that $xA \subset V$.

Let $(X,\mathcal{U})$ be a uniform space. The transformation semigroup (X, S) is called $[S]$-recursive on the set M, and the set M is called $[S]$-recursive if for an arbitrary $\alpha \in \mathcal{U}$ there exists $A \in [S]$ such that $xA \subset x\alpha$ for all $x \in M$.

Let us look at some classes of $[S]$-admissible sets. Let $P \subset S$, $\mathcal{P}$ be a nonempty family of subsets from S. A set $A \subset S$ is called P-extensive if $pP \cap A \neq \emptyset$ for all $p \in P$. The set P is called replete if for an arbitrary compact set $K \subset S$ there are such points p, $q \in P$ that $pKq \subset P$. A set $A \subset S$ is called $\mathcal{P}$-extensive if $P \cap A \neq \emptyset$ for arbitrary P from $\mathcal{P}$. A set $A \subset S$ is called P-sindetic if it contains the unit element and there is such a compact set $K \subset P$ that $pK \cap A \neq \emptyset$ for all $p \in P$. An S-sindetic set is called sindetic. P is called invariant if for all $s \in S$ $sP = Ps$.

If as class $[S]$ we take the class of all P-extensive ($\mathcal{P}$-extensive, P-sindetic, sindetic) subsets from S, in the above mentioned definitions the term "$[S]$-recursive" is changed for the term "P-recurrrent" ("$\mathcal{P}$-recurrent", "P-almost periodic","almost periodic").

It is clear that a point $x \in X$ is S-recurrent if for all $s \in S$ and an arbitrary neighborhood V of the point x we have the relation $xsS \cap V \neq \emptyset$; and a point x is almost periodic if for an arbitrary neighborhood V of the point x there exists a compact set $K \subset S$ such that $xsK \cap V \neq \emptyset$ for all $s \in S$.

Let (X, S) be a transformation semigroup with a uniform phase space with the uniformity $\mathcal{U}[X]$, $P \subset S$. Points x and y are called P-proximal if for an arbitrary $\alpha \in \mathcal{U}[X]$ there exists $s \in P$ such that $(xs, ys) \in \alpha$. Points which are not P-proximal are called P-distal.

A point $x \in M \subset X$ is called P-distal in the set M if for all $y \in M$, $y \neq x$, the points x and y are P-distal. The set M is called P-distal if its each point is P-distal in M.

The set $A \subset X$ is called $M \subset X$ if $A \subset \overline{M}$ and for any point $x \in A$ and for all $\alpha \in \mathcal{U}[X]$ $\exists \beta \in \mathcal{U}[X]$ such that $m \in M$ and $(x, m) \in \beta$ implies $(xs, ms) \in \alpha$ for all $s \in S$.

Let S be a topological group, $P \subset S$, $Q^* = P \cup P^{-1}$ and Q be the set of all possible finite products of elements from Q^*. If $S = \overline{Q}$, we say that S is topologically derived from the set P.

2. Concordance and its general properties

Let (X, S) and (Y, S) be transformation semigroups, $P \subset S$ and $x \in X$, $y \in Y$.

We will say that the point x is P-concordant with the point y if for each neighborhood V of the point x there is neighborhood U of the point y such that from $s \in P$ and $ys \in U$ it follows $xs \in V$.

Let $\mathcal{U}[X]$ ($\mathcal{U}[Y]$) be a uniformity on X (Y). We will say that the point x is uniformly P-concordant with the point y if for each index $\alpha \in \mathcal{U}[X]$ there is an index $\beta \in \mathcal{U}[Y]$ such that from $s, t \in P$ and $(ys, yt) \in \beta$ it follows $(xs, xt) \in \alpha$.

Theorem 2.1. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation semigroups, $x \in X$ and ψ be a restriction of φ on the set $\overline{xS}$. The following statements are true.*

1) *If the point x is S-concordant with $\varphi(x)$, then ψ is injective at the point x.*

2) *If ψ is injective at the point x and each net $\{s_\alpha\} \subset S$ with $\lim_\alpha \varphi(xs_\alpha) = \varphi(x)$ contains some sub-net $\{s_\beta\}$ for which $\{xs_\beta\}$ is convergent, then the point x is S-concordant with the point $\varphi(x)$.*
3) *Let X and Y be uniform spaces. If the point x is uniformly S-concordant with the point $\varphi(x)$, then the mapping ψ is an injection.*
4) *Let X and Y be uniform spaces. If the mapping ψ is injective, the set $\overline{\varphi(x)S}$ is compact and each net $\{s_\alpha\} \subset S$ for with $\lim_\alpha \varphi(xs_\alpha)$ exists contains some sub-net $\{s_\beta\}$ for which $\{xs_\beta\}$ is convergent, then the point x is uniformly S-concordant with the point $\varphi(x)$.*

Proof. We prove only the statement 4). Other statements may be proved similarly. We shall prove the statement 4) by the method of contradiction. We assume that the conditions of our statement are satisfied, but x is not uniformly S-concordant with the point $\varphi(x)$. There is an index ε such that for each $\delta \in \mathcal{U}[Y]$ and some $s_\delta, t_\delta \in S$ the relations are satisfied

$$(\varphi(x)s_\delta, \varphi(x)t_\delta) \in \delta, \tag{2.1}$$

$$(xs_\delta, xt_\delta) \notin \varepsilon. \tag{2.2}$$

By virtue of the compactness of the set $\overline{\varphi(x)S}$, without loss of generality we may suppose that $\lim_\delta \varphi(x)s_\delta = y$ and $\lim_\delta \varphi(x)t_\delta = z$. In that case from the relation (2.1) it follows $y = z$. According to the conditions of our statement we can consider that $\lim_\delta xs_\delta = x_1$ and $\lim_\delta xt_\delta = x_2$. Since x_1, $x_2 \in \psi^{-1}(y)$, then $x_1 = x_2$. It contradicts (2.2). The contradiction also proves the demanded proposition. □

Proposition 2.2. *Let (X, S) and (Y, S) be transformation semigroups, $[S]$ be a family of subsets of S and $x \in X$, $y \in Y$. The following statements are true.*

1) *If y is $[S]$-recursive and the point x is S-concordant with the point y, then each of points (y, x) and x is also $[S]$-recursive.*
2) *If x is uniformly S-concordant with y and the set yS is $[S]$-recursive, then each of sets $\overline{(y, x)S}$ and $\overline{xS}$ is also $[S]$-recursive.*

Proof. Let us prove, for example, the statement 2). To this end, it is sufficient to prove that set xS is $[S]$-recursive. Let α be an arbitrary index of the uniformity of space X and the index β of the uniformity of the space Y corresponds to α by virtue of the fact that the point x is uniformly S-concordant with the point y. There is a set $A \in [S]$ such that for arbitrary $s \in S$ and $a \in A$ the relation $(ys, ysa) \in \beta$ is satisfied. In that case the relation $(xs, xsa) \in \alpha$ is also satisfied for arbitrary $s \in S$ and $a \in A$. Hence the set xS is $[S]$-recursive. □

Proposition 2.3. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation semigroups, X and Y be uniform spaces, and φ be a uniformly continuous mapping. If the point $x \in X$ is S-concordant with $\varphi(x) = y$ and the point y is S-distal in the set $\overline{yS}$, then the point x is S-distal in the set $\overline{xS}$.*

Proof. Let $\alpha \in \mathcal{U}[Y]$ and $\beta \in \mathcal{U}[X]$ be such that $(\varphi \times \varphi)(\beta) \subset \alpha$. Suppose that the point x is S-proximal to a point $z \in \overline{xS}$. Then $(xs, zs) \in \beta$ for some $s \in S$. In that case $(\varphi(x)s, \varphi(z)s) \in (\varphi \times \varphi)(\beta) \subset \alpha$. It means the S-proximality of the points y and $\varphi(z) \in \overline{yS}$. Since the point y is S-distal in the set $\overline{yS}$, then $y = \varphi(z)$. By virtue of the theorem 2.1 $x = z$. The proof means the S-distality of point x in the set $\overline{xS}$. □

The following statement may be proved similarly.

Proposition 2.4. *Let X and Y be uniform spaces, φ be a uniformly continuous homomorphism of the transformation semigroup (X, S) to the transformation semigroup (Y, S). If the point $x \in X$ is uniformly S-concordant with the point $\varphi(x) = y$ and the set $\overline{yS}$ is S-distal, then the set $\overline{xS}$ is S-distal.*

Proposition 2.5. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation semigroups, $x \in X$, P be an invariant replete semigroup from S with the unit, the point $\varphi(x)$ is P-recurrent and the following condition is satisfied: From every net $\{s_\alpha\} \subset S$ there exists $\lim_\alpha \varphi(xs_\alpha)$ such that it is possible to select a sub-net $\{s_\beta\} \subset \{s_\alpha\}$ for which there exists $\lim_\beta xs_\beta$. If the point x is P-concordant with the point $\varphi(x)$, then the point x is S-concordant with the point $\varphi(x)$.*

Proof. By virtue of the proposition 2.2 the point x is P-recurrent. Therefore, $\overline{xP} = \overline{xS}$ by the proposition 3.2. The restriction ψ of the mapping φ on the set $\overline{xP}$ is injective at x by the theorem 2.1. Since $\overline{xP} = \overline{xS}$, in that case, by the theorem 2.1, the point x is S-concordant with the point $\varphi(x)$.

□

Proposition 2.6. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation semigroups, X and Y be uniform spaces; $x \in X$, P be an invariant replete semigroup from S with the unit, the point $\varphi(x)$ is P-recurrent and the following condition is satisfied: From every net $\{s_\alpha\} \subset S$ there exists $\lim_\alpha \varphi(xs_\alpha)$ such that it is possible to select a sub-net $\{s_\beta\} \subset \{s_\alpha\}$, for which there exists $\lim_\beta xs_\beta$. If the point x is uniformly P-concordant with the point $\varphi(x)$, then the point x is uniformly S-concordant with $\varphi(x)$.*

The proof is similar to the proof of the proposition 2.5.

Proposition 2.7. *Let (X, S) and (Y, S) be transformation semigroups with uniform phase spaces and X be complete. If $x \in X$ is uniformly S-concordant with $y \in Y$ and the set $\overline{yS}$ is compact, then the set $\overline{xS}$ is compact, too.*

Proof. Let $\{xs_n\}$ be an arbitrary net in xS and α be an arbitrary index of the uniformity of the space X and β be the index corresponding to it because the point x is uniformly S-concordant with y. From the net $\{ys_n\}$ it is possible to select a convergent subnet $\{ys_{n_k}\}$. Therefore, for β there is k_0 such that for arbitrary $l > k_0$ and $m > k_0$ the relation $(ys_{n_l}, ys_{n_m}) \in \beta$ is valid. In this case $(xs_{n_l}, xs_{n_m}) \in \alpha$ for arbitrary $l > k_0$ and $m > k_0$ and hence, the net $\{xs_{n_k}\}$ is a Cauchy net. Because from an arbitrary net in xS it is possible to select a Cauchy sub-net, then xS and, therefore, $\overline{xS}$ are totally bounded. By virtue of the completeness the set $\overline{xS}$ is compact.

□

3. Limit sets

Let (X, S) be a transformation semigroup, $P \subset S$, $x \in X$. The P-limit of the point x is the set

$$P_x = \bigcap_{p \in P} \overline{xpP}.$$

It is clear that $y \in P_x$ if for an arbitrary neighborhood V of the point x and for all $p \in P$ $xpP \cap V \neq \emptyset$. It is also clear that $S_x s \subset S_x$ and $S_x \subset S_{xs}$ $(s \in S)$. If $Ss \subset sS$ then $S_x = S_{xs}$ $(s \in S)$.

The semigroup S is called directional if for arbitrary elements $s_1, s_2 \in S$ there are such elements $s_1', s_2' \in S$ that $s_1 s_1' = s_2 s_2'$.

Some properties of limit sets are given in the following theorem.

Theorem 3.1. *Let (X, S) be a transformation semigroup and $x \in X$. The following statements are true.*

1) *If S is a directional semigroup and the set $\overline{xS}$ is compact, then the set S_x is nonempty and compact.*
2) *If P is an invariant replete semigroup from S, then the set P_x is invariant.*
3) *If the set S_x is nonempty and stable in sense of Lyapunov relative to the set xS, then the set S_x is minimal.*
4) *If the set S_x is nonempty and the set $\overline{xS}$ is stable in sense of Lyapunov relative to the set xS, then S_x is the unique minimal subset in $\overline{xS}$.*

Proof. We shall prove the statement 1). From the condition that S is directional it follows that for arbitrary n elements $s_1, s_2, \dots, s_n \in S$ $\bigcap_{i=1}^n s_i S \neq \emptyset$. In that case $\bigcap_{i=1}^n \overline{xs_i S} \neq \emptyset$ for arbitrary n elements $s_1, s_2, \dots, s_n \in S$. Therefore, $\{\overline{xsS} \mid s \in S\}$ is a centered system of closed in $\overline{xS}$ sets. By virtue of the compactness of the set $\overline{xS}$ the relation $S_x = \bigcap_{s\in S} \overline{xsS} \neq \emptyset$ is correct. The compactness of the set S_x is obvious.

Let us now prove the statement 2). Let $s \in S$ and $p \in P$. There is an element $q \in P$ such that $sp = qs$. Therefore, $P_x s \subset \overline{xqPs} \subset \overline{xqPs} = \overline{xqsP} = \overline{xspP}$, i.e. $P_x s \subset \overline{xspP}$. That means $P_x s \subset P_{xs}$ for all $s \in S$. Let's prove for all $s \in S$ the inclusion $P_{xs} \subset P_x$. Let $s \in S$. Then $p_1 s p_2 \in P$ for some elements $p_1, p_2 \in P$. Let an element $q_1 \in P$ be such that $p_1 s = s q_1$. If $y \in P_{xs}$, then for an arbitrary element $p \in P$ $y \in \overline{xsq_1 p_2 pP} = \overline{xp_1 s p_2 Pp} \subset \overline{xPp} = \overline{xpP}$, i.e. $y \in \overline{xpP}$, therefore, $P_{xs} \subset \overline{xpP}$ and $P_{xs} \subset \bigcap_{p\in P} \overline{xpP} = P_x$, i.e. $P_{xs} \subset P_x$. So, $P_x s \subset P_{xs} \subset P_x$ that means for all $s \in S$ $P_x s \subset P_x$.

Let us prove the statement 3). Suppose the contrary that M is a nonempty closed invariant subset from S_x. Let's suppose that there is a point $y \in S_x \setminus M$. For M and y we shall select such an index α of uniformity of the space X that

$$M\alpha \cap y\alpha = \emptyset. \tag{3.1}$$

Let $z \in M$. For a point z and index α there is such an index β that from $(z, u) \in \beta$, $u \in xS$, it follows $(zs, us) \in \alpha$ for all $s \in S$. As $z \in \overline{xS}$, there is a point $z_0 \in z\beta \cap xS$. Then $S_x \subset S_{z_0}$ and $(zs, z_0 s) \in \alpha$ for all $s \in S$. Thus $z_0 s \in zs\alpha \subset M\alpha$ for all $s \in S$. Therefore, $z_0 S \subset M\alpha$. Then from (3.1) it follows that $z_0 S \cap y\alpha = \emptyset$. That means $y \notin S_{z_0}$. But this contradicts $y \in S_x \subset S_{z_0}$. This contradiction says that $S_x = M$.

Finally we prove the statement 4). From the statement 3) the minimality of the set S_x follows. Let's assume that M is minimal set from $\overline{xS}$, $M \neq S_x$. Then there is a point $y \in S_x \setminus M$. The further proof repeats the proof of the statement 3). □

Proposition 3.2. *Let (X, S) be a transformation semigroup, $x \in X$ and P be an invariant replete semigroup from S. If the point x is P-recurrent, then $\overline{xS} = \overline{xP}$.*

Proof. Since the point x is P-recurrent, then $x \in P_x$. Because the set P_x is invariant and is closed, then $\overline{xS} \subset P_x$. And as $P_x \subset \overline{xP} \subset \overline{xS}$, then $\overline{xS} = \overline{xP}$. □

Theorem 3.3. *Let (X, S) be a transformation semigroup, the space X be compact, $x \in X$; P be an invariant replete semigroup from S with the unit element (S is topologically derived from the semigroup P and $pP \subset Pp$ ($p \in P$) if S is a group). Then:*

1) *The point $x \in X$ is almost periodic iff it is P-almost periodic.*
2) *The point x is S-distal in the set $\overline{xS}$ iff it is P-distal in the set $\overline{xP}$. Therefore, the set $\overline{xS}$ is S-distal iff the set $\overline{xP}$ is P-distal.*
3) *The set $\overline{xS}$ is almost periodic iff the set $\overline{xP}$ is P-almost periodic.*

Proof. Let P be an invariant replete semigroup.

We shall prove the statement 1). Let x be an almost periodic point. Then the set $\overline{xS}$ is minimal [6]. Let $M \subset \overline{xS}$ be P-minimal set and $y \in M$. Then $M = \overline{yP}$ and $\overline{yS} = \overline{xS}$. Since the point y is P-recurrent, then $\overline{yP} = \overline{yS}$. Since $\overline{xS} = \overline{yS} = \overline{yP} = M$, i. e. $\overline{xS} = M$, then $\overline{xS}$ is P-minimal. Therefore x is P-almost periodic [6].

Let x be an P-almost periodic point. Since the point x is P-recurrent, then $\overline{xP} = \overline{xS}$. The set $\overline{xP}$ is P-minimal and any point $y \in \overline{xP}$ is P-almost periodic, too. Let $y \in \overline{xS}$. Then $y \in \overline{xP}$ and therefore $\overline{yP} = \overline{yS}$. Since $\overline{yP} = \overline{xP}$, then $\overline{yS} = \overline{yP} = \overline{xP} = \overline{xS}$, i. e. $\overline{yS} = \overline{xS}$. Therefore the set $\overline{xS}$ is minimal and hence the point x is almost periodic.

We shall prove the statement 2). Let the point x be distal in the set $\overline{xS}$. Then the point x is an almost periodic and for any $y \in \overline{xS}$ the point (x, y) is an almost periodic in $(X \times X, S)$. By statement 1) of our theorem the point x is an P-almost periodic and for any $y \in \overline{xP} \equiv \overline{xS}$ the point (x, y) is an P-almost periodic in $(X \times X, P)$. Let $y \in \overline{xP}$ and x and y be P-proximal. Then $xp = yp$ for any p from some minimal right ideal I of the Ellis enveloping semigroup of (X, P). Since the point (x, y) is an P-almost periodic, then $xu = yu$ for some idempotent $u \in I$. Then $x = xu = yu = y$, i. e. $x = y$. Therefore the point x is P-distal in the set $\overline{xP}$. Analogously we prove the inverse statement.

We shall prove the statement 3). Let the set $\overline{xS}$ be an almost periodic. Then it is distal and stable in the sense of Lyapunov relative to the set $\overline{xS}$. It is clear that $\overline{xS} = \overline{xP}$. By the statement 2) of our theorem the set $\overline{xP}$ is P-distal. The set $\overline{xP}$ is stable in the sense of Lyapunov relative to the set $\overline{xP}$, too. Therefore the set $\overline{xP}$ is an P-almost periodic.

Let the set $\overline{xP}$ be an P-almost periodic. Then it is P-distal and P-stable in the sense of Lyapunov relative to the set $\overline{xP}$. By the statement 2) of our theorem the set $\overline{xS}$ is distal. Since the set $\overline{xP}$ is stable in the sense of Lyapunov relative to the set $\overline{xP}$ and for any $y, z \in \overline{xS}$ $\overline{(y, z)S} = \overline{(y, z)P}$, then the set $\overline{xS}$ is stable in the sense of Lyapunov relative to the set $\overline{xS}$. Therefore the set $\overline{xS}$ is an almost periodic.

Anagously discussing and using the corollary 4 from [6] we shall prove our theorem in the case when S is topologically derived from the semigroup P. □

Theorem 3.4. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation semigroups, X and Y be uniform spaces; P be an invariant replete semigroup from S with the unit (S is topologically derived from the semigroup P and $pP \subset Pp$ ($p \in P$) if S is a group), $x \in X$, $y = \varphi(x)$, and the following conditions are satisfied:*

1) *The set $\overline{xP}$ is compact.*
2) *The set $\overline{yP}$ is minimal.*

3) *For arbitrary points* $z_1, z_2 \in \overline{xP}$, $\varphi(z_1) = \varphi(z_2)$, *and a net* $\{s_i\} \subset P$ *from* $\lim_i z_1 s_i = \lim_i z_2 s_i$ *it follows* $z_1 = z_2$.

If x *is* P*-concordant with* y*, then* x *is uniformly* S*-concordant with* y*.*

Proof. Since $\varphi(\overline{xP}) = \overline{\varphi(x)P}$, then the set $\overline{\varphi(x)P}$ is compact. Since $\varphi(x)$ is P-almost periodic, then by the proposition 2.2 x is P-almost periodic, too. By virtue of the theorem 3.3 $\overline{xP} = \overline{xS}$. If z_1, $z_2 \in \overline{xP}$, $\varphi(z_1) = \varphi(z_2)$, then by virtue of P-minimality of $\overline{xP}$ $x = \lim_i z_1 s_i$ for some net $\{s_i\} \subset P$ and then

$$\varphi(x) = \varphi(\lim_i z_1 s_i) = \lim_i \varphi(z_1)s_i = \lim_i \varphi(z_2)s_i = \varphi(\lim_i z_2 s_i),$$

i.e. $\varphi(x) = \varphi(\lim_i z_1 s_i) = \varphi(\lim_i z_2 s_i)$. Since by the theorem 2.1 the restriction ψ of mapping φ on the set $\overline{xP}$ is injective at x, then from here it follows that $\lim_i z_1 s_i = \lim_i z_2 s_i$ and $z_1 = z_2$ by the condition 3) of our proposition. We have proved that ψ is injective. Since $\overline{xP} = \overline{xS}$, then according to the theorem 2.1 x is uniformly S-concordant with the point y. □

Agreement. Further in the paper we assume that S is the additive group of $\mathbb{R}^n$ and $\mathcal{P}$ is the family of all $m = 2^n$ semigroups $P \subset \mathbb{R}^n$ consisting of points $(s_1, s_2, \dots, s_n)$ for which k, $k = 0, 1, \dots, n$, fixed components are nonnegative and the rest are non-positive. We consider also that the transformation groups have metric phase spaces with the metric ρ. Let (X, S) be a transformation group, $P \in \mathcal{P}$, $x, y \in X$. We denote:

$N(P)$ is the family of all sequences $\{s_k\}$ in P such that for all $p \in P\, \exists n_0 \in \mathbb{N}\, \forall k \in \mathbb{N}$, $k > n_0$, $s_k \in p + P$.

$M_{x,y}^{N(P)} = \{\{s_k\} \mid \{s_k\} \in N(P) \wedge y = \lim_{k\to+\infty} x s_k\}$;

$M_x^{N(P)} = \bigcup\limits_{y\in X} M_{x,y}^{N(P)}$;

$P_x^{N(P)} = \{y \mid y = \lim_{k\to+\infty} x s_k\ (\{s_k\} \in N(P))\}$.

The set $P_x^{N(P)}$ can be naturally called $N(P)$-limit of the point x.

We denote $N(\mathcal{P})$ the family of all sequences in S which have property: for an arbitrary subsequence of each sequence from $N(\mathcal{P})$ there is their subsequence belonging to $N(P)$ for some $P \in \mathcal{P}$. As above:

$M_{x,y}^{N(\mathcal{P})} = \{\{s_k\} \mid \{s_k\} \in N(\mathcal{P}) \wedge y = \lim_{k\to+\infty} x s_k\}$;

$M_x^{N(\mathcal{P})} = \bigcup\limits_{y\in X} M_{x,y}^{N(\mathcal{P})}$;

$\mathcal{P}_x = \{y \mid y = \lim_{k\to+\infty} x s_k\ (\{s_k\} \in N(\mathcal{P}))\}$.

The set $\mathcal{P}_x$ can be naturally called the $N(\mathcal{P})$-limit of the point x.

Note the following properties of the introduced sets.

Proposition 3.5. 1) *For all* $P \in \mathcal{P}$ $N(P) \subset N(\mathcal{P})$.

2) *For arbitrary sequences* $\{s_k^i\} \in N(P^i)$ $(i = 1, 2, \dots, m)$ *the sequence* $\{p_k\}$ *which is defined by the rule*

$$p_k = \begin{cases} s_l^1 & \text{at } k = ml - m + 1 \\ s_l^2 & \text{at } k = ml - m + 2 \\ \dots & \dots \\ s_l^m & \text{at } k = ml \end{cases}$$

belongs to $N(\mathcal{P})$.

Proposition 3.6. $\mathcal{P}_x = \bigcup_{P\in\mathcal{P}} P_x^{N(P)}$.

Proposition 3.7. *If* $M_y^{N(\mathcal{P})} \subset M_x^{N(\mathcal{P})}$*, then for all* $P \in \mathcal{P}$ $M_y^{N(P)} \subset M_x^{N(P)}$.

Proposition 3.8. *For any* $P \in \mathcal{P}$ *the set* $N(P)$ *satisfies the following properties:*

1) *For all* $\{s_k\} \in N(P)$ $\forall p \in P$ $\{s_k + p\} \in N(P)$.
2) *Any subsequence of a sequence from* $N(P)$ *belongs to* $N(P)$.
3) *If* $\{s_k\}$, $\{t_k\} \in N(P)$*, then the sequence* $\{p_k\}$ *defined by the rule*

$$p_k = \begin{cases} s_l & \text{at } k = 2l-1 \\ t_l & \text{at } k = 2l \end{cases}$$

belongs to $N(P)$.

Proof. 1. Let $\{s_k\} \in N(P)$, $s, p \in P$. Since $\{s_k\} \in N(P)$, then for p there exists $n_0 \in \mathbb{N}$ such that for all $k > n_0$ $s_k \in p + P$. In this case for all $k > n_0$ $s_k + s \in (p+P) + s \subset p + P$. That means $\{s_k + s\} \in N(P)$.

2. Let $\{s_l\} \in N(P)$, $\{s_{l_k}\}$ be a subsequence of the sequence $\{s_l\}$ and $p \in P$. There is $n_0 \in \mathbb{N}$ such that for all $l > n_0$ $s_l \in p + P$. By virtue of the increase of the function $k \to l_k$ (from the definition of a subsequence) there is $k_0 \in \mathbb{N}$ such that $l_{k_0} > n_0$. Let $k > k_0$. Then $l_k > l_{k_0} > n_0$ and $s_{l_k} \in p + P$. Therefore, $\{s_{l_k}\} \in N(P)$.

3. Let $\{s_l\}, \{t_l\} \in N(P)$, $p \in P$ and

$$p_l = \begin{cases} s_k & \text{at } l = 2k-1 \\ t_k & \text{at } l = 2k. \end{cases}$$

There are $n_0, m_0 \in \mathbb{N}$ such that for all $l > n_0$ and all $m > m_0$ $s_l \in p + P$ and $t_m \in p + P$. Let $k_0 = \max\{n_0, m_0\}$. Then for all $k > k_0$ $p_k \in p + P$. Therefore, $\{p_l\} \in N(P)$. □

Proposition 3.9. *Let* $P \in \mathcal{P}$*, then:*

1) $P_x^{N(P)} = P_x$.
2) *If the set* $\overline{xP}$ *is compact, then the set* $P_x^{N(P)}$ *is nonempty,* P*-invariant and compact.*
3) $\mathcal{P}_x = \bigcup_{P\in\mathcal{P}} P_x$.

Proof. 1. To prove the statement 1) we shall conduct for the case when $P = \{(s_1, s_2, \dots, s_n) \mid s_1 \geq 0, s_2 \geq 0, \dots, s_n \geq 0\}$. Let $y \in P_x^{N(P)}$, $p \in P$. Then $y = \lim_{k\to+\infty} xs_k$ for some sequence $\{s_k\} \in N(P)$. For p there is $n_0 \in \mathbb{N}$ such that for all $k > n_0$ $s_k \in p + P$. In that case for all $k > n_0$ $xs_k \in x(p + P)$ and $y = \lim_{k\to+\infty} xs_k \in \overline{x(p+P)}$, i.e. $y \in \overline{x(p+P)}$. By virtue of the arbitrariness of p $y \in \bigcap_{p\in P} \overline{x(p+P)} = P_x$ and the inclusion $P_x^{N(P)} \subset P_x$ is proved.

Let $y \in P_x$. Then for all $p \in P$ $y \in \overline{x(p+P)}$. Let's designate $p_k = (k, k, \dots, k)$ ($k \in \mathbb{N}$) and let $\{V_k\}$ be a base of a system of neighborhoods at the point y, and $V_{k+1} \subset V_k$ for all $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ $\exists t_k \in P$ such that $x(p_k + t_k) \in V_k$. Therefore, $y = \lim_{k\to+\infty} x(p_k + t_k)$. Let's designate $s_k = p_k + t_k$ ($k \in \mathbb{N}$) and let's prove that $\{s_k\} \in N(P)$. Let $t \in P$, $t = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\alpha = \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. There is $k_0 \in \mathbb{N}$ such that $k_0 > \alpha$. And let $k > k_0$. Let's show that $s_k \in t + P$. Since $k > k_0 > \alpha$, then $k = \alpha_1 + q_1 = \alpha_2 + q_2 = \dots = \alpha_n + q_n$ for some positive numbers $q_1, q_2, \dots, q_n$. Let's suppose that $t_k = (l_1^k, l_2^k, \dots, l_n^k)$. Then $s_k = p_k + t_k = (k+l_1^k, k+l_2^k, \dots, k+l_n^k) = (\alpha_1+q_1+l_1^k, \alpha_2+q_2+l_2^k, \cdots, \alpha_n+q_n+l_n^k) = (\alpha_1, \alpha_2, \dots, \alpha_n) + (q_1 + l_1^k, q_2 + l_2^k, \cdots, q_n + l_n^k) \in t + P$, that is, $s_k \in t + P$ for an arbitrary $k > k_0$, therefore, $s_k \in t + P$, $y \in P_x^{N(P)}$ and the inclusion $P_x \subset P_x^{N(P)}$ is proved.

2. The statement 2) follows from the statement 1).
3. The statement 3) follows from the proposition 3.6. □

Proposition 3.10. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, $P \in \mathcal{P}$ and the set $\overline{xP}$ is compact. If $z \in P_y$ and $s \in P$ are such that the set $\varphi^{-1}(zs) \bigcap P_x$ is a singleton, then $M_{y,zs}^{N(P)} \subset M_x^{N(P)}$.*

Proof. Let $\{s_l\} \in M_{y,zs}^{N(P)}$. Then, according to the condition, $\{xs_l\}$ admits a convergent subsequence. Assume that $\{xs_{l_k}\}$ is an arbitrary convergent subsequence of the sequence $\{xs_l\}$, and let $\lim_{k\to+\infty} xs_{l_k} = x_0$. Since

$$\varphi(x_0) = \varphi(\lim_{k\to+\infty} xs_{l_k}) = \lim_{k\to+\infty} \varphi(x)s_{l_k} = \lim_{k\to+\infty} ys_{l_k} = zs,$$

i.e. $\varphi(x_0) = zs$, then $x_0 \in \varphi^{-1}(zs)$. Besides $x_0 \in P_x^{N(P)} = P_x$, that means $x_0 \in \varphi^{-1}(zs) \bigcap P_x$. Whereas the set $\varphi^{-1}(zs) \bigcap P_x$ is a singleton, we have proved that an arbitrary convergent subsequence of the sequence $\{xs_l\}$ has the same limit. In that case, by virtue of the compactness of the set $\overline{xP}$, the sequence $\{xs_l\}$ is convergent, therefore, $\{s_l\} \in M_x^{N(P)}$ and the proposition is proved. □

Proposition 3.11. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, $P \in \mathcal{P}$ and the set $\overline{xP}$ is compact. If for some $z \in P_y$, $\omega \in \overline{zP}$ and for all $s \in P$ the set $\varphi^{-1}(\omega s) \bigcap P_x$ is a singleton, then $M_{y,\omega s}^{N(P)} \subset M_x^{N(P)}$ for all $s \in P$.*

The proof is similar to the proof of the proposition 3.10.

Proposition 3.12. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, $P \in \mathcal{P}$ and the set $\overline{xP}$ is compact. If for all $z \in P_y$ the set $\varphi^{-1}(z) \bigcap P_x$ is a singleton, then $M_y^{N(P)} \subset M_x^{N(P)}$.*

The proof is similar to the proof of the proposition 3.10.

Proposition 3.13. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$ and for an arbitrary $P \in \mathcal{P}$ the set $\overline{xP}$ is compact. If for all $z \in \mathcal{P}_y$ the set $\varphi^{-1}(z) \bigcap \mathcal{P}_x$ is a singleton, then $M_y^{N(\mathcal{P})} \subset M_x^{N(\mathcal{P})}$.*

Proof. Let $\{s_l\} \in M_y^{N(\mathcal{P})}$ and $\lim_{l\to+\infty} ys_l = z$. It suffices to prove that the sequence $\{xs_l\}$ is convergent. We prove that $\{xs_l\}$ admits a convergent subsequence. Since $\{s_l\} \in N(\mathcal{P})$, then there exists a subsequence $\{s_{l_k}\}$ of the sequence $\{s_l\}$ belonging to the set $N(P)$ for some $P \in \mathcal{P}$. Because the set $\overline{xP}$ is compact, then the sequence $\{s_{l_k}\}$ admits a convergent subsequence. Suppose that $\{xs_{l_k}\}$ is an arbitrary convergent subsequence of the sequence $\{xs_l\}$ and $\lim_{k\to+\infty} xs_{l_k} = x_0$. Since

$$\varphi(x_0) = \varphi(\lim_{k\to+\infty} xs_{l_k}) = \lim_{k\to+\infty} \varphi(x)s_{l_k} = \lim_{k\to+\infty} ys_{l_k} = z \in \mathcal{P}_y,$$

i.e. $\varphi(x_0) = z$, then $x_0 \in \varphi^{-1}(z)$. By the definition of the set $N(\mathcal{P})$ there is $Q \in \mathcal{P}$ such that $\{s_{l_{k_i}}\} \in N(Q)$ for some subsequence $\{s_{l_{k_i}}\}$ of the sequence $\{s_{l_k}\}$.

Therefore, $x_0 \in Q_x^{N(Q)}$ and we have proved that $x_0 \in \varphi^{-1}(z) \bigcap \mathcal{P}_x$. Because the set $\varphi^{-1}(z) \bigcap \mathcal{P}_x$ is a singleton, then we have proved that an arbitrary convergent subsequence of the sequence $\{xs_l\}$ has the same limit. We call it A and we shall prove that $\lim_{l\to+\infty} xs_l = A$. Suppose the contrary. Then there exists a neighborhood U of the points A such that for all $k \in \mathbb{N}$ there exists $l_k > k$ for which $xs_{l_k} \notin U$. By the definition of $N(\mathcal{P})$ there exists $L \in \mathcal{P}$ such that $\{s_{l_{k_i}}\} \in N(L)$ for some subsequence $\{s_{l_{k_i}}\}$ of the sequence $\{s_{l_k}\}$. Since the set $\overline{xL}$ is compact then $\{xs_{l_{k_i}}\}$ contains a convergent (to A) subsequence. It contradicts the fact that for arbitrary $l_k > k$ $xs_{l_k} \notin U$. The contradiction thus proves the proposition. □

Proposition 3.14. *Let (X, S) and (Y, S) be transformation groups and $P \in \mathcal{P}$. If for $x, x_1 \in X$, $y, y_1 \in Y$, $s \in S$ $M_{y,y_1 s}^{N(P)} \subset M_x^{N(P)}$ and $M_{x,x_1}^{N(P)} \bigcap M_{y,y_1}^{N(P)} \neq \emptyset$, then $M_{y,y_1 s}^{N(P)} \subset M_{x,x_1 s}^{N(P)}$.*

Proof. Let $\{s_l\} \in M_{x,x_1}^{N(P)} \bigcap M_{y,y_1}^{N(P)}$ and $\{t_l\} \in M_{y,y_1 s}^{N(P)}$. We form a sequence $\{p_k\}$ according to the rule

$$p_k = \begin{cases} t_l & \text{at } k = 2l-1 \\ s_l + s\&\text{at } k = 2l. \end{cases}$$

It is clear that $\{p_k\} \in N(P)$. Since $\{t_l\} \in M_{y,y_1 s}^{N(P)}$ then $y_1 s = \lim_{l\to+\infty} yt_l$. Because $\{s_l\} \in M_{y,y_1}^{N(P)}$, then $y_1 = \lim_{l\to+\infty} ys_l$, therefore, $y_1 s = \lim_{l\to+\infty} y(s_l + s)$. From the definition of the sequence $\{p_k\}$ and the fact that $y_1 s = \lim_{l\to+\infty} yt_l = \lim_{l\to+\infty} y(s_l + s)$ it follows that $\{p_k\} \in M_{y,y_1 s}^{N(P)}$. Therefore, $\{p_k\} \in M_x^{N(P)}$ and $\lim_{k\to+\infty} xt_k = \lim_{k\to+\infty} xp_k = \lim_{k\to+\infty} x(s_k + s) = (\lim_{k\to+\infty} xs_k)s = x_1 s$, i.e. $\lim_{k\to+\infty} xt_k = x_1 s$. That means $\{t_k\} \in M_{x,x_1 s}^{N(P)}$ and the proposition is proved. □

Corollary 3.15. *Let (X, S) and (Y, S) be transformation groups and $P \in \mathcal{P}$. If for $x, x_1 \in X$, $y, y_1 \in Y$, $M_y^{N(P)} \subset M_x^{N(P)}$ and $M_{x,x_1}^{N(P)} \bigcap M_{y,y_1}^{N(P)} \neq \emptyset$, then $M_{y,y_1 s}^{N(P)} \subset M_{x,x_1 s}^{N(P)}$ for an arbitrary $s \in P$.*

Proposition 3.16. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If a point $z \in P_y$ is such that for arbitrary $s \in P$ $M_{y,zs}^{N(P)} \subset M_x^{N(P)}$, then there exists $u \in P_x$ and continuous mapping $h : zP \to uP$ for which $h(vs) = h(v)s$ for all $v \in zP$ and for all $s \in P$, and $h(z) = u$.*

Proof. If $z \in P_y$, then $z = \lim_{k\to+\infty} ys_k$ for some sequence $\{s_k\} \in M_{y,z}^{N(P)}$. According to the condition $\{s_k\} \in M_x^{N(P)}$, that means there exists $u = \lim_{k\to+\infty} xs_k \in P_x$. Let's designate $h : zP \to uP$ the mapping according to the rule: if $zs = \lim_{k\to+\infty} y\tau_k$ for some sequence $\{\tau_k\} \in M_y^{N(P)}$, then $h(zs) = \lim_{k\to+\infty} x\tau_k$ $(s \in P)$. Let's prove the correctness of the definition of the mapping h. By virtue of the theorem 3.1 for $s \in P$ $zs \in P_y$, therefore, there exists $\{\tau_k\} \in M_{y,zs}^{N(P)}$ for which $zs = \lim_{k\to+\infty} y\tau_k$. Since $M_{y,zs}^{N(P)} \subset M_x^{N(P)}$, then $\{\tau_k\} \in M_x^{N(P)}$, that means there exists $\lim_{k\to+\infty} x\tau_k$. Let's suppose further that

$$zs = \lim_{k\to+\infty} y\tau_k = \lim_{k\to+\infty} yt_k$$

for some sequences $\{\tau_k\}$, $\{t_k\} \in M_y^{N(P)}$. If we prove that $\lim\limits_{k\to+\infty} x\tau_k = \lim\limits_{k\to+\infty} xt_k$, then the correctness of the definition of mapping h will be proved, as $\lim_{k\to+\infty} x(s_k+$

$s) = us$ and as $\{\tau_k\}$ $\{s_k + s\}$ can act. We form a sequence $\{p_k\}$ according to the rule

$$p_k = \begin{cases} \tau_l & \text{at } k = 2l-1, \\ t_l & \text{at } k = 2l. \end{cases}$$

It is clear that $\{p_k\} \in M_{y,zs}^{N(P)}$ and, therefore, $\{p_k\} \in M_x^{N(P)}$ and there exists $\lim_{k\to+\infty} xp_k$. But then there are limits $\lim_{k\to+\infty} x\tau_k$ and $\lim_{k\to+\infty} xt_k$ and they are equal.

If $v \in zP$, then $v = zt$ for some $t \in P$. Since $z = \lim_{k\to+\infty} ys_k$, then $zt = \lim_{k\to+\infty} y(s_k + t)$, hence, $h(v) = \lim_{k\to+\infty} x(s_k + t) = ut$. Let $s \in P$. Then $vs = z(t+s)$ and $h(vs) = h(z(t+s)) = \lim_{k\to+\infty} x(s_k + t + s) = (\lim_{k\to+\infty} xs_k)(t+s) = u(t+s) = h(v)s$, i.e. $h(vs) = h(v)s$.

Let us prove the continuity of h. Take $p \in P$ and we shall prove the continuity of h at the point zp. Suppose that $\lim_{l\to+\infty} yt_l = zp$ for some sequence $\{t_l\}$ in P. Since for all $k \in \mathbb{N}$ $zt_k \in P_y$, then $zt_k = \lim_{k\to+\infty} y\tau_k^l$ for some sequence $\{\tau_k^l\} \in M_{y,zt_l}^{N(P)}$ $(l \in \mathbb{N})$. According to the condition $\{\tau_k^l\} \in M_x^{N(P)}$, that means there exists $\lim_{k\to+\infty} x\tau_k^l$ $(l \in \mathbb{N})$. Because $zt_l = \lim_{k\to+\infty} y\tau_k^l$, then $h(zt_l) = \lim_{k\to+\infty} x\tau_k^l$ $(l \in \mathbb{N})$. Besides $zt_l = \lim_{k\to+\infty} y(s_k + t_l)$ and $h(zt_l) = \lim_{k\to+\infty} x(s_k + t_l) = (\lim_{k\to+\infty} xs_k)t_l = ut_l$, i.e. $h(zt_l) = ut_l$ $(l \in \mathbb{N})$. Therefore, $\lim_{k\to+\infty} x\tau_k^l = ut_l$. In that case for all $l \in \mathbb{N}$ $\exists k_0^l \in \mathbb{N}$ such that for an arbitrary $k_l > k_0^l$

$$\rho(zt_l, y\tau_{k_l}^l) < \frac{1}{l} \quad \text{and} \quad \rho(ut_l, x\tau_{k_l}^l) < \frac{1}{l}.$$

Since for $k_l > k_0^l$

$$\rho(zp, y\tau_{k_l}^l) \le \rho(zp, zt_l) + \rho(zt_l, y\tau_{k_l}^l) < \rho(zp, zt_l) + \frac{1}{l},$$

then in the limit, as $l \to +\infty$, we shall receive $\lim_{l\to+\infty} \rho(zp, y\tau_{k_l}^l) = 0$. Whence it follow that $\lim_{l\to+\infty} y\tau_{k_l}^l = zp$. Therefore, $\{\tau_{k_l}^k\} \in M_{y,zp}^{N(P)}$. According to the condition $\{\tau_{k_l}^l\} \in M_x^{N(P)}$ and, therefore, there exists $\lim_{l\to+\infty} x\tau_{k_l}^l$. By the definition of h $\lim_{l\to+\infty} x\tau_{k_l}^l = h(zp) = up$. Therefore, for arbitrary $k_l > k_0^l$

$$\rho(up, ut_l) \le \rho(up, x\tau_{k_l}^l) + \rho(x\tau_{k_l}^l, ut_l) < \rho(up, x\tau_{k_l}^l) + \frac{1}{l}$$

and in the limit, as $l \to +\infty$, we shall receive $\lim_{l\to+\infty} \rho(up, ut_l) = 0$. Whence it follows that $\lim_{l\to+\infty} ut_l = up$. As $ut_l = h(zt_l)$ and $up = h(zp)$, from the last equality it follow that $\lim_{l\to+\infty} h(zt_l) = h(zp) = h(\lim_{l\to+\infty} zt_l)$. The continuity of h is proved. □

Proposition 3.17. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If a point $z \in P_y$ is such that for arbitrary $\omega \in \overline{zP}$ and $s \in P$ $M_{y,\omega s}^{N(P)} \subset M_x^{N(P)}$, then there is a continuous mapping $h : \overline{zP} \to P_x$ for which $h(vs) = h(v)s$ for all $v \in \overline{zP}$ and all $s \in P$.*

The proof is similar to the proof of the proposition 3.16.

Proposition 3.18. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If $M_y^{N(P)} \ne \emptyset$ and $M_y^{N(P)} \subset M_x^{N(P)}$, then there is a continuous mapping $h : P_y \to P_x$ for which $h(vs) = h(v)s$ for all $v \in P_y$ and forall s in P.*

The proof is similar to the proof of proposition 3.16.

Proposition 3.19. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If a point $z \in P_y$ is such that for all s in P the set $\varphi^{-1}(zs) \bigcap P_x$ is a singleton, then there exists $u \in P_x \cap \varphi^{-1}(z)$ and a continuous mapping $h : zP \to uP$ for which $h(vs) = h(v)s$ and $\varphi(h(v)) = v$ for all v in zP and all s in P, and $h(z) = u$.*

The proof follows from the propositions 3.10 and 3.16.

Proposition 3.20. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If a point $z \in P_y$ is such that for all ω in $\overline{zP}$ and for all sin P the set $\varphi^{-1}(\omega s) \bigcap P_x$ is a singleton, then there is a continuous mapping $h : \overline{zP} \to P_x$ for which $h(vs) = h(v)s$ and $\varphi(h(v)) = v$ for all v in $\overline{zP}$ and for all s in P.*

The proof follows from the propositions 3.11 and 3.17.

Proposition 3.21. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If for all z in P_y the set $\varphi^{-1}(z) \bigcap P_x$ is a singleton, then there is a continuous mapping $h : P_y \to P_x$ for which $h(vs) = h(v)s$ and $\varphi(h(v)) = v$ for all $v \in P_y$ and all $s \in P$.*

The proof follows from the propositions 3.12 and 3.18.

4. Concordance of points of limit sets

Proposition 4.1. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If $z \in P_y$ is such that for an arbitrary $s \in P$ $M^{N(P)}_{y,zs} \subset M^{N(P)}_x$, then there is a point $u \in P_x$ such that for all s in P us is P-concordant with zs.*

Proof. Let u and $h : zP \to uP$ be from the proposition 3.16. And let $s \in P$ and $V(us)$ be a neighborhood of us. Since h is continuous at zs and $h(zs) = us$, then there is a neighborhood $U(zs)$ of zs such that from $(zs)p \in U(zs)$ it follows $h((zs)p) \in V(xs)$ $(p \in P)$. Because $h((zs)p) = usp$, we have received that from $zsp \in U(zs)$ it follows $usp \in V(us)$ $(p \in P)$. The proof also means P-concordance of the points us and zs. □

Proposition 4.2. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If $z \in P_y$ is such that for arbitrary $s \in P$ the set $\varphi^{-1}(zs) \bigcap P_x$ is a singleton, then there is a point $u \in P_x \bigcap \varphi^{-1}(z)$ such that for all s in P, us is P-concordant with zs.*

The proof is similar to that of the proposition 4.1, by using the proposition 3.19.

Proposition 4.3. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If $z \in P^{N(P)}_y$ is such that for arbitrary $\omega \in \overline{zP}$ and $s \in P$ $M^{N(P)}_{y,\omega s} \subset M^{N(P)}_x$, then for an arbitrary $\omega \in \overline{zP}$ there is a point $u \in P_x$ which is P-concordant with ω. In addition, if the set P_y is compact, then u is uniformly P-concordant with ω.*

Proof. Let $h : \overline{zP} \to P_x$ be the mapping from the proposition 3.17, $\omega \in \overline{zP}$ and $u = h(\omega)$. And let $V(u)$ be a neighborhood of the point u. Since h is continuous at ω, there is a neighborhood $U(\omega)$ of point ω such that from $\omega s \in U(\omega)$ it follows $h(\omega s) \in V(u)$ $(s \in P)$. As $h(\omega s) = us$, we have proved the P-concordance of points

u and ω. Let set P_y be compact. Then mapping h is uniformly continuous. Let α be an arbitrary index and β be an index corresponding to α by virtue of the uniform continuity of h. Suppose that for $s, t \in P$ $(\omega s, \omega t) \in \beta$. Then $(us, ut) = (h(\omega s), h(\omega t)) \in \alpha$, therefore, u is uniformly P-concordant with ω. □

Proposition 4.4. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If $z \in P_y$ is such that for arbitrary $\omega \in \overline{zP}$ and $s \in P$ the set $\varphi^{-1}(\omega s) \bigcap P_x$ is a singleton, then for an arbitrary $\omega \in \overline{zP}$ there is $u \in P_x \bigcap \varphi^{-1}(\omega)$ such that u is uniformly P-concordant with ω.*

The proof repeats the proof of the proposition 4.3, using the proposition 3.20.

Proposition 4.5. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and $P \in \mathcal{P}$. If $M_y^{N(P)} \neq \emptyset$ and $M_y^{N(P)} \subset M_x^{N(P)}$. Then for an arbitrary $\omega \in P_y$ there is $u \in P_x$ such that u is P-concordant with ω. In addition, if the set P_y is compact, then u is uniformly P-concordant with ω.*

The proof repeats the proof of the proposition 4.3 using the proposition 3.18.

Proposition 4.6. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. If for any $z \in P_y$ the set $\varphi^{-1}(z) \bigcap P_x$ is a singleton, then for an arbitrary $\omega \in P_y$ there is $u \in P_x \bigcap \varphi^{-1}(\omega)$ such that u is uniformly P-concordant with ω.*

The proof repeats the proof of the proposition 4.4, using the proposition 3.21.

Theorem 4.7. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact. And let $z \in P_y \bigcap P_z$. If the following conditions are valid:*

1) $M_y^{N(P)} \subset M_x^{N(P)}$;
2) *There exists $\{t_l\}$ in $N(P)$ such that* $\lim_{l\to+\infty} \rho(yt_l, zt_l) = 0$,

then there is $u \in P_x \bigcap P_u$ such that u is uniformly P-concordant with z and $\lim_{l\to+\infty} \rho(xt_l, ut_l) = 0$.

Proof. Let $h : P_y \to P_x$ be the continuous mapping from the proposition 3.18 and $h(z) = u$. Since $z \in P_z$, then $z = \lim_{l\to+\infty} zs_l$ for some sequence $\{s_l\} \in N(P)$. Then $u = h(z) = h(\lim_{l\to+\infty} zs_l) = \lim_{l\to+\infty} h(z)s_l = \lim_{l\to+\infty} us_l$, i.e. $u = \lim_{l\to+\infty} us_l$. Therefore, $u \in P_x \bigcap P_u$. According to the proposition 4.5, u is uniformly P-concordant with z. By the method of contrary we shall prove that $\lim_{l\to+\infty} \rho(xt_l, ut_l) = 0$. Suppose that for some $\varepsilon > 0$ and arbitrary $l_0 \in \mathbb{N}$ there is a natural number $l > l_0$ for which

$$\rho(xt_l, ut_l) \geq \varepsilon. \tag{4.1}$$

By virtue of the compactness of the set $\overline{yP}$ we consider that the sequences $\{yt_l\}$ and $\{zt_l\}$ are convergent. Let $y_1 = \lim_{l\to+\infty} yt_l$ and $z_1 = \lim_{l\to+\infty} zt_l$. By virtue of the condition 2) of our proposition, $y_1 = z_1$. Since, on the one hand, according to the definition of the mapping h, $h(y_1) = \lim_{l\to+\infty} xt_l$ and, on the other hand,

$$h(y_1) = h(\lim_{l\to+\infty} yt_l) = h(\lim_{l\to+\infty} zt_l) = \lim_{l\to+\infty} h(z)t_l = \lim_{l\to+\infty} ut_l,$$

i.e. $h(y_1) = \lim_{l\to+\infty} ut_l$, $\lim_{l\to+\infty} xt_l = \lim_{l\to+\infty} ut_l$. This equality contradicts the (4.1). The theorem is proved. □

Theorem 4.8. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, $P \in \mathcal{P}$ is such that the set $\overline{xP}$ is compact, and $z \in P_y \bigcap P_z$. Assume the following conditions are valid:*

1) *For all ω in P_y the set $\varphi^{-1}(\omega) \bigcap P_x$ is a singleton.*
2) $\exists \{t_l\} \in N(P)$ $\lim_{l \to +\infty} \rho(yt_l, zt_l) = 0$.

Then there is $u \in P_x \bigcap P_u \bigcap \varphi^{-1}(z)$ such that u is uniformly P-concordant with z. In addition, $\lim_{l \to +\infty} \rho(xt_l, ut_l) = 0$.

The proof follows from the proposition 3.12 and the theorem 4.7.

Let (X, S) be a transformation group, and $x \in X$. The point x will be called $\mathcal{P}$-heteroclinic if for all P in $\mathcal{P}$ $\exists x^{(P)} \in P_x \bigcap P_{x^{(P)}}$ $\exists \{s_l\} \in N(P)$ such that

$$\lim_{l \to +\infty} \rho(xs_l, x^{(P)} s_l) = 0.$$

If in the definition of $\mathcal{P}$-heteroclinic point all points $x^{(P)}$ $(P \in \mathcal{P})$ coincide, then $\mathcal{P}$-heteroclinic point will be called $\mathcal{P}$-homoclinic.

It is clear that the point x is $\mathcal{P}$-heteroclinic if there is a point $z \in \bigcap_{P \in \mathcal{P}} (P_x \bigcap P_z)$ such that for all P in $\mathcal{P}$ there exists $\{s_l\} \in N(P)$ with $\lim_{l \to +\infty} \rho(xs_l, zs_l) = 0$.

It is convenient to designate symbolically $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point $(x, \{x^{(P)}\})$ ((x, z)), where $x^{(P)}$ (z) is a point from P_x in the definition of $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point. The symbol $(x, \{x^{(P)}\})$ ((x, z)) will be also called $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point.

Further we consider reasonable to assign standard number (type of a point) to some properties which can have points $x \in X$ under the following scheme:

$1 \iff x$ is P-recurrent;
$2 \iff x$ is $\mathcal{P}$-recurrent;
$3 \iff x$ is P-almost periodic;
$4 \iff x$ is P-distal in the set $\overline{xP}$;
$5 \iff \overline{xP}$ is P-distal;
$6 \iff xP$ is P-almost periodic;
$3^0 \iff x$ is almost periodic;
$4^0 \iff x$ is S-distal in the set $\overline{xS}$;
$5^0 \iff \overline{xS}$ is S-distal;
$6^0 \iff xS$ is almost periodic.

Let $M \subset \{1, 2, 3, 4, 5, 6, 3^0, 4^0, 5^0, 6^0\}$ and $(k_1, k_2, \dots, k_m)$ be a vector with components from M. We say that the $\mathcal{P}$-heteroclinic point $(x, \{x^{(P_i)}\})$ has the type $(k_1, k_2, \dots, k_m)$ if the point $x^{(P_i)}$ has the type k_i $(i = 1, 2, \dots, m)$. The type of a $\mathcal{P}$-homoclinic point is defined similarly.

Let $(x, \{x^{(P)}\})$ be a $\mathcal{P}$-heteroclinic point in the transformation group (X, S) and $(y, \{y^{(P)}\})$ be a $\mathcal{P}$-heteroclinic point in the transformation group (Y, S). We say that $(x, \{x^{(P)}\})$ is $\mathcal{P}$-concordant with $(y, \{y^{(P)}\})$ if for all P in $\mathcal{P}$ $x^{(P)}$ is uniformly P-concordant with $y^{(P)}$. The $\mathcal{P}$-concordance of $\mathcal{P}$-homoclinic points is defined similarly.

Theorem 4.9. *Let (X, S) and (Y, S) be transformation groups, $x \in X$, $y \in Y$, and for all P in $\mathcal{P}$ the set $\overline{yP}$ is compact. Then:*

1) *If y is a $\mathcal{P}$-heteroclinic point and for all P in $\mathcal{P}$ $M_y^{N(P)} \subset M_x^{N(P)}$, then x is $\mathcal{P}$-heteroclinic and it is $\mathcal{P}$-concordant with y.*

2) *If y is a $\mathcal{P}$-homoclinic point and $M_y^{N(\mathcal{P})} \subset M_x^{N(\mathcal{P})}$, then x is $\mathcal{P}$-homoclinic and it is $\mathcal{P}$-concordant with y.*

Proof. The statement 1) follows from the theorem 4.7. We now prove the statement 2). Suppose that the point (y, z) is $\mathcal{P}$-homoclinic. If $M_y^{N(\mathcal{P})} \subset M_x^{N(\mathcal{P})}$, then according to the proposition 3.7, for all P in $\mathcal{P}$ $M_y^{N(P)} \subset M_x^{N(P)}$. Then, since the point (y, z) is $\mathcal{P}$-heteroclinic, according to the statement 1), for an arbitrary $P \in \mathcal{P}$ there is a point $x^{(P)}$ such that the point $(x, \{x^{(P)}\})$ is $\mathcal{P}$-heteroclinic. The statement 2) will be proved if we shall find a point u such that for P in $\mathcal{P}$ $u = x^{(P)}$. Since the point (y, z) is $\mathcal{P}$-homoclinic, then by virtue of the definition of the mapping $h : P_y \to P_x$ in the proof of the proposition 3.18 it is possible to select a point $x^{(P)}$ as follows: if $z = \lim_{l\to+\infty} yt_l$ for $\{t_l\} \in M_y^{N(P)}$, then $x^{(P)} = \lim_{l\to+\infty} xt_l$. Since $z \in \bigcap_{P\in\mathcal{P}} P_y$, then $z = \lim_{l\to+\infty} ys_l^{(P)}$ for some sequence $\{s_l^{(P)}\} \in M_y^{N(P)}$ $(P \in \mathcal{P})$. From the sequences $\{s_l^{(P)}\}$ we make a sequence $\{t_l\} \in N(\mathcal{P})$ according to the statement 2) of the proposition 3.5. Since for $P \in \mathcal{P}$ $\lim_{l\to+\infty} ys_l^{(P)} = z$ then $\lim_{l\to+\infty} yt_l = z$. Therefore, $\{t_l\} \in M_y^{N(\mathcal{P})}$, hence, $\{t_l\} \in M_x^{N(\mathcal{P})}$. In that case there exists $\lim_{l\to+\infty} xt_l$. We call this limit u. It is clear that $\lim_{l\to+\infty} xs_l^{(P)} = u$. Since $z = \lim_{l\to+\infty} ys_l^{(P)}$, then that means for all $P \in \mathcal{P}$ $x^{(P)} = \lim_{l\to+\infty} xs_l^{(P)}$. But in that case $u = \lim_{l\to+\infty} xs_l^{(P)} = x^{(P)}$ for all $P \in \mathcal{P}$. The theorem is thus proved. □

Theorem 4.10. *Let the conditions of the theorem* 4.9 *be valid. If the point y is $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) of the type $(k_1, k_2, \dots, k_m)$, where $k_1, k_2, \dots, k_m \in \{1, 2, 3, 6\}$, then the point x is $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) of type $(k_1, k_2, \dots, k_m)$, too.*

The proof follows from the theorem 4.9 and the proposition 2.2 while taking into consideration the corresponding definitions.

Theorem 4.11. *Let X be a complete space and let the conditions of the theorem* 4.9 *be valid. If $(y, \{y^{(P)}\})$ is $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) point of the type $(k_1, k_2, \dots, k_m)$, where $k_1, k_2, \dots, k_m \in \{1, 2, 3, 4, 5, 6\}$, then for all $P \in \mathcal{P}$ there is a point $x^{(P)}$ such that:*

1) *the set $\overline{x^{(P)}P}$ is compact;*
2) *the point $(x, x^{(P)})$ is $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) of type $(k_1, k_2, \dots, k_m)$.*

The proof follows from the theorem 4.9 and the propositions 2.2, 2.3, 2.4, 2.7 with taking in consideration the corresponding definitions.

Theorem 4.12. *If in the conditions of the theorem* 4.11 *the type $(k_1, \dots, k_j, \dots, k_m)$ of y contains $k_j \in \{3, 4, 5, 6\}$, then the type of x is $(k_1, \dots, k_j^0, \dots, k_m)$.*

The proof follows from the theorems 4.11 and 3.3.

Theorem 4.13. *Let $\varphi : (X, S) \to (Y, S)$ be an extension of transformation groups, $x \in X$, $y = \varphi(x)$, and for all $P \in \mathcal{P}$ the set $\overline{xP}$ is compact. Then:*

1) *If the point $(y, \{y^{(P)}\})$ is $\mathcal{P}$-heteroclinic and for all $P \in \mathcal{P}$ and for all $z \in P_y$ the set $\varphi^{-1}(z) \bigcap P_x$ is a singleton, then there are $x^{(P)} \in \varphi^{-1}(y^{(P)})$ such that the point $(x, \{x^{(P)}\})$ is $\mathcal{P}$-heteroclinic and it is $\mathcal{P}$-concordant with the point $(y, \{y^{(P)}\})$.*

2) *If* (y,ω) *is* $\mathcal{P}$*-homoclinic and for all* $z\in\mathcal{P}_y$ *the set* $\varphi^{-1}(z)\bigcap\mathcal{P}_x$ *is a singleton, then there is* $u\in\varphi^{-1}(\omega)$ *such that the point* (x,u) *is* $\mathcal{P}$*-homoclinic and it is* $\mathcal{P}$*-concordant with* (y,ω).

The proof follows from the proposition 3.13 and the theorems 4.8 and 4.9.

Theorem 4.14. *If in the conditions of the theorem* 4.13 y *is* $\mathcal{P}$*-heteroclinic (*$\mathcal{P}$*-homoclinic) of the type* $(k_1,k_2,\dots,k_m)$*, where* $k_1,k_2,\dots,k_m\in\{1,2,3,4,5,6\}$*, then* x *is* $\mathcal{P}$*-heteroclinic (*$\mathcal{P}$*-homoclinic) of the type* $(k_1,k_2,\dots,k_m)$.

The proof follows from the theorem 4.13 and the propositions 2.2, 2.3, 2.4 taking into account of the corresponding definitions.

Theorem 4.15. *If in the conditions of the theorem* 4.14 *the type* $(k_1,\dots,k_j,\dots,k_m)$ *of* y *contains* $k_j\in\{3,4,5,6\}$*, then the type of* x *is* $(k_1,\dots,k_j^0,\dots,k_m)$.

The proof follows from the theorems 4.14 and 3.3

5. Heteroclinic solutions of multi-dimensional non-autonomous differential equations

As in the last section, S is the additive group of $\mathbb{R}^n$ and $\mathcal{P}$ is the family of all $m=2^n$ semigroups $P\subset\mathbb{R}^n$ consisting of points $(s_1,s_2,\dots,s_n)$ for which k, $k=0,1,\dots,n$, fixed components are nonnegative and the rest are nonpositive.

$L(S,\mathbb{R}^l)$ is the space of all linear operators $S\to\mathbb{R}^l$ with the natural operator norm.

$C(S,\mathbb{R}^l)$ is the space of all continuous mappings $S\to\mathbb{R}^l$ equipped with compact-open topology.

$C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))$ is the space of all continuous mappings $S\times\mathbb{R}^l\to L(S,\mathbb{R}^l)$ equipped with compact-open topology.

It is clear $C(S,\mathbb{R}^l)$, $C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))$ and $C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))\times C(S,\mathbb{R}^l)$ are metric spaces (the metric is denoted by the symbol ρ).

A multi-dimensional non-autonomous differential equation for the function $y\in C(S,\mathbb{R}^l)$ is the equation $y'(t)=f(t,y(t))$, where $f\in C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))$, and $y'(t)$ denotes the Frechet derivative of y at the point t. The equation $y'(t)=f(t,y(t))$ is then written more concisely in the form

$$y'=f(t,y). \tag{5.1}$$

The mapping $\sigma_1: C(S,\mathbb{R}^l)\times S\to C(S,\mathbb{R}^l)$ with $\sigma_1(f,s)=f_s$, where $f_s(p)=f(s+p)\,(p\in S)$, defines a topological transformation group $(\,C(S,\mathbb{R}^l),\,S,\,\sigma_1\,)$. Similarly, the mapping $\sigma_2: C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))\times S\to C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))$ with $\sigma_2(f,s)=f_s$, where $f_s(p,t)=f(s+p,t)\,(p\in S)$, defines a topological transformation group $(\,C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l)),\,S,\,\sigma_2\,)$. The topological transformation group $(\,C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))\times C(S,\mathbb{R}^l),\,S,\,\sigma\,)$ is defined as the direct product $(\,C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l)),\,S,\,\sigma_2\,)$ and $(\,C(S,\mathbb{R}^l),\,S,\,\sigma_1\,)$, where $\sigma=\sigma_2\times\sigma_1$.

If $f\in C(S\times\mathbb{R}^l,L(S,\mathbb{R}^l))$ and $\varphi\in C(S,\mathbb{R}^l)$, then the topological transformation groups $(\overline{fS},\,S,\,\sigma_2\,)$, $(\,\overline{\varphi S},\,S,\,\sigma_1\,)$ and $(\,\overline{(f,\varphi)S},\,S,\,\sigma\,)$ are well-defined as well. The projector $q:\overline{(f,\varphi)S}\to\overline{fS}$ define the extension $q:(\,\overline{(f,\varphi)S},\,S,\,\sigma\,)\to(\overline{fS},\,S,\,\sigma_2\,)$ of the transformation groups.

A solution φ of (5.1) is said to be compact if the set $\overline{\varphi(S)}$ is compact.

If φ is some compact solution of (5.1) and $(g, \psi) \in \overline{(f, \varphi)S}$, then ψ is compact solution of the equation $y' = g(t, y)$, as known from [7]. Therefore we use this extension q naturally in the investigation of the equation (5.1).

It is clear that for the compact solution φ of (5.1) any of the following conditions is sufficient for the set $\overline{\varphi S}$ to be compact [7]:

1) the map φ is uniformly continuous;
2) the set $f(S, \overline{\varphi(S)})$ is bounded;
3) the set $\overline{fS}$ is compact.

It is also clear, if the set $\overline{fS}$ is compact and solution φ of (5.1) is compact, then the set $\overline{(f, \varphi)S}$ is compact.

We determine properties of the "right-hand" f of (5.1), of the solutions φ of (5.1) and of the pairs (f, φ) through properties as the points f, φ and (f, φ) (or the sets $\overline{fS}$, $\overline{\varphi S}$ and $\overline{(f, \varphi)S}$) of dynamical systems $(\overline{fS}, S, \sigma_2)$, $(\overline{\varphi S}, S, \sigma_1)$ and $(\overline{(f, \varphi)S}, S, \sigma)$ accordingly [8].

For example, let $P \in \mathcal{P}$ and h be f, φ or (f, φ). Then we define the following concepts.

The function h will be called Poisson P-stable (Poisson stable), if the point h is a P-recurrent ($\mathcal{P}$-recurrent) point of $(\overline{hS}, S)$.

The function h will be called P-recurrent in sense Birkhoff (weakly P-distal, P-distal, P-almost periodic in sense Bohr), if the point h is P-almost periodic (the point h is P-distal in the set $\overline{hP}$, the set $\overline{hP}$ is P-distal, the set $\overline{hP}$ is P-almost periodic) in $(\overline{hS}, S)$.

We use standard number to denote some properties for the function h under the following scheme:

1 $\Longleftrightarrow$ h is Poisson P-stable;
2 $\Longleftrightarrow$ h is Poisson stable;
3 $\Longleftrightarrow$ h is P-recurrent in sense Birkhoff;
4 $\Longleftrightarrow$ h is weakly P-distal;
5 $\Longleftrightarrow$ h is P-distal;
6 $\Longleftrightarrow$ h is P-almost periodic in sense Bor;
3^0 $\Longleftrightarrow$ h is S-recurrent in sense Birkhoff;
4^0 $\Longleftrightarrow$ h is weakly S-distal;
5^0 $\Longleftrightarrow$ h is S-distal;
6^0 $\Longleftrightarrow$ h is S-almost periodic in the sense of Bohr.

The $\mathcal{P}$-heteroclinic, $\mathcal{P}$-homoclinic functions, its type $(k_1, k_2, \dots, k_m)$ and $\mathcal{P}$-concordance of the $\mathcal{P}$-heteroclinic ($\mathcal{P}$-homoclinic) functions are determined as above.

Theorem 5.1. *Let $P \in \mathcal{P}$, let the set $\overline{fP}$ be compact and φ be compact solution of* (5.1)*. If for any $g \in P_f$ the equation*

$$y' = g(t, y) \tag{5.2}$$

in the set P_φ there is one solution ψ, then ψ is uniformly P-concordant with g.

The proof follows from the proposition 4.6.

Theorem 5.2. *Let $P \in \mathcal{P}$, let the set $\overline{fP}$ be compact, φ be a compact solution of* (5.1) *and let $h \in P_f \cap P_h$. If for any $g \in P_f$ the equation* (5.2) *in the set P_φ has only one solution and there is $\{t_k\} \in N(P)$ for which $\lim_{k \to +\infty} \rho(f_{t_k}, g_{t_k}) = 0$, then the equation $y' = h(t, y)$ in the set P_φ has only one Poisson P-stable solution ψ. Moreover, ψ is uniformly P-concordant with h and $\lim_{k \to +\infty} \rho(\varphi_{t_k}, \psi_{t_k}) = 0$.*

The proof follows from the theorem 4.7.

Theorem 5.3. *Let φ be a compact solution of* (5.1) *and the set $\overline{fS}$ be compact. If the function $(f, \{h^{(P)}\})$ is $\mathcal{P}$-heteroclinic and for any $P \in \mathcal{P}$ and for any $g \in P_f$ the equation* (5.2) *in the set P_φ has only one solution, then for any $P \in \mathcal{P}$, the equation $y' = h^{(P)}(t, y)$ has a solution $\psi^{(P)}$ such that the function $(\varphi, \{\psi^{(P)}\})$ is $\mathcal{P}$-heteroclinic and is $\mathcal{P}$-concordant with $(f, \{h^{(P)}\})$. If, in addition, the type of $(f, \{h^{(P)}\})$ is $(k_1, k_2, \dots, k_m)$, where $k_1, k_2, \dots, k_m \in \{1, 2, 3, 4, 5, 6\}$, then the type of $(\varphi, \{\psi^{(P)}\})$ is $(k_1, k_2, \dots, k_m)$. If k_j in the type $(k_1, \dots, k_j, \dots, k_m)$ of $(f, \{h^{(P)}\})$ belongs to $\{3, 4, 5, 6\}$, then the type of $(\varphi, \{\psi^{(P)}\})$ is $(k_1, \dots, k_j^0, \dots, k_m)$.*

The proof follows from the theorems 4.13–4.15.

Theorem 5.4. *Let φ be a compact solution of* (5.1) *and the set $\overline{fS}$ be compact. If the function (f, h) is $\mathcal{P}$-homoclinic and for any $g \in \mathcal{P}_f$ the equation* (5.2) *in the set $\mathcal{P}_\varphi$ has only one solution, then the equation $y' = h(t, y)$ has a solution ψ such that the function (φ, ψ) is $\mathcal{P}$-homoclinic and is $\mathcal{P}$-concordant with (f, h). If, in addition, the type of (f, h) is $(k_1, k_2, \dots, k_m)$, where $k_1, k_2, \dots, k_m \in \{1, 2, 3, 4, 5, 6\}$, then the type of (φ, ψ) is $(k_1, k_2, \dots, k_m)$. If k_j in the type $(k_1, \dots, k_j, \dots, k_m)$ of $(f, \{h^{(P)}\})$ belongs to $\{3, 4, 5, 6\}$, then the type of $(\varphi, \{\psi^{(P)}\})$ is $(k_1, \dots, k_j^0, \dots, k_m)$.*

The proof follows from the theorems 4.13–4.15.

Acknowledgment. The research described in this publication was made possible in part by Award No. MM1-3016 of the Moldovan Research and Development Association (MRDA) and the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF).

References

[1] Bronshtein I. U. Extensions of minimal transformation groups, Kishinev: Shtiintsa, 1975 (in Russian).

[2] Bronshtein I.U. On the theory of distal minimal sets and distal functions. *Dokl. Akad. Nauk SSSR (Soviet Math. Dokl.)*, 172, No. 2, 255–257, 1967.

[3] Cheban D. N. On the comparability of the points of dynamical systems with regard to the character of recurrence property in the limit. *Mathematicheskie nauki*, Kishinev: Shtiintsa, 1: 66-71, 1977.

[4] Cheban D. N. Asymptotically almost periodic solutions of differential equations, Kishinev, State University of Moldova, 2002 (in Russian).

[5] Gaishun I. V. Total integrable multidimensional differential equations, Minsk: Nauka i tekhnika, 1983 (in Russian).

[6] Gherco A. I. Recursiveness, minimal sets and stability in the sense of Lyapunov in transformation semigroups. *Mat. Issled.* (Kishinev), 7, 2: 62 - 80, 1972.

[7] Gherco A. I. Asymptotically recurrent solutions of β-differential equations. *Matematicheskie Zametki*, 67, N. 6: 837–851, 2000 Translated in *Mathematical notes*, Vol. 67, No. 6, pp. 707–717, 2000.

[8] Gherco A. I. Poisson stability of mappings with respect to a semigroup. *Buletinul A.Ş. a R.M. Matematica*, N 1 (26): 95–102, 1998.

[9] Gottschalk W. H., Hedlund G. A. Topological dynamics, Amer. Math. Soc. Coll. Publ., 36, 1955.

[10] Shcherbakov B. A. Topological dynamics and Poisson stability of solutions of differential equations, Kishinev: Shtiintsa, 1972 (in Russian).

[11] Shcherbakov B. A., Cheban D. N. Poisson asymptotic stability of motions of dynamical systems and their comparability with regard to the recurrence property in the limit. *Differentsial'nye uravneniya*, 13 (5): 898 - 906, 1977.

[12] Shcherbakov B. A. Poisson stability of motions of dynamical systems and of solutions of differential equations, Kishinev: Shtiintsa, 1985 (in Russian).
[13] Sell G. R. Topological Dynamics and Ordinary Differential Equations. Van Nostrand–Reinhold, London, 1971.
[14] Sibirskii K. S. Introduction to topological dynamics, Kishinev, 1970 (in Russian).

David Cheban
State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova
E-mail address: `cheban@usm.md`

Jinqiao Duan
Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
Dept. of Mathematics, The University of Science and Technology of China, Hefei 230026, China
E-mail address: `duan@iit.edu`

Anatoly Gherco
State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova
E-mail address: `gerko@usm.md`