RECURRENT MOTIONS IN THE NONAUTONOMOUS NAVIER-STOKES SYSTEM

VENA PEARL BONGOLAN-WALSH, DAVID CHEBAN, AND JINQIAO DUAN

(Communicated by P. E. Kloeden)

ABSTRACT. We prove the existence of recurrent or Poisson stable motions in the Navier-Stokes fluid system under recurrent or Poisson stable forcing, respectively. We use an approach based on nonautonomous dynamical systems ideas.

1. **Introduction.** The mankind has been fascinated by time-periodic, quasi-periodic, almost periodic and recurrent motions for centuries. These motions have been observed in the solar system (e.g., the Earth rotates around the Sun), and in fluid systems, (e.g., vortices), and in other natural systems. In fact, in the atmospheric and oceanic flows, these motions (such as hurricanes) dominate some weather or climate evolution. From a dynamical system point of view, these motions are special invariant sets which provide a part of the skeleton for understanding the global dynamics. Therefore, the study of time-periodic, quasi-periodic, almost periodic and recurrent motions is not only physically useful but also mathematically interesting.

However, rigorous justification of the existence of such motions in fluid systems is a much recent issue. For example, time-periodic solutions for the two-dimensional (2D) Navier-Stokes fluid system under external time-periodic forcing are shown to exist by Foias and Prodi [13, 9], and Yudovic [23] in 1960s. Note that a time-periodic external forcing does not necessarily induce a time-periodic motion in a dynamical system, as in the resonance case of a periodically forced harmonic oscillator which only has unbounded, non-periodic motions.

For sufficiently small external nonautonomous forcing and/or for sufficiently large viscosity, Chepyzhov and Vishik [4] have investigated the existence of quasi-periodic and almost periodic solutions of the 2D Navier-Stokes fluid system. Duan and Kloeden [6, 7] have studied periodic, quasi-periodic, and almost periodic motions for large scale quasigeostrophic fluid flows under nonautonomous forcing.

In this paper, we investigate the existence of recurrent and Poisson stable motions (see definitions in the next section) of 2D Navier-Stokes fluid system under external forcing. We use a nonautonomous dynamical systems approach.

The problem of existence of recurrent and Poisson stable motions for ordinary differential equations was studied by Shcherbakov [17, 18].

2. **Nonautonomous dynamical systems.** We first introduce some basic concepts in autonomous dynamical systems.

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$, X be a metric space with metric ρ, and $(X, \mathbb{T}, \pi)$ be an autonomous dynamical system. That is, the continuous mapping $\pi : \mathbb{T} \times X \to X$ satisfies the identity property $\pi(0, x) = x$ and the flow or semi-flow property $\pi(t + \tau, x) = \pi(t, \pi(\tau, x))$. The solution mapping for the initial-value problem of an autonomous ordinary differential equation usually satisfies these properties.

The point $u \in X$ is called a stationary (τ-periodic, $\tau > 0, \tau \in \mathbb{T}$) point, if $ut = u$ ($u\tau = u$ respectively) for all $t \in \mathbb{T}$, where $ut := \pi(t, u)$.

The number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ shift (almost period) of point $u \in X$ if $\rho(u\tau, u) < \varepsilon$ (respectively $\rho(u(\tau + t), ut) < \varepsilon$ for all $t \in \mathbb{T}$).

The point $u \in X$ is called almost recurrent (almost periodic, in the sense of Bohr) if for any ε there exists a positive number l such that on any segment of length l, there is an ε shift (almost period) of point $u \in X$.

If a point $u \in X$ is almost recurrent and the hull $H(u) = \overline{\{ut \mid t \in \mathbb{T}\}}$ is compact, then u is called recurrent(in the sense of Birkhoff). The corresponding motion $\pi(t, u)$ is then called a recurrent motion. We identify the hull of the motion $\pi(t, u)$ with the hull of the point u, i.e., $H(u)$.

The $\omega-$limit set of $u \in X$ is defined as $\omega_u = \{v \mid \exists t_n \to +\infty$ such that $\pi(t_n, u) \to v\}$. Likewise, when $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, the $\alpha-$limit set of $u \in X$ is defined as $\alpha_u = \{v \mid \exists t_n \to -\infty$ such that $\pi(t_n, u) \to v\}$.

A point $u \in X$ is called stable in the sense of Poisson in the positive (negative) direction if $u \in \omega_u$ ($u \in \alpha_u$). A point $u \in X$ is called Poisson stable if it is stable in the sense of Poisson in both the positive and negative directions (in this case $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$), i.e., $u \in \omega_u \cap \alpha_u$. The corresponding motion $\pi(t, u)$ is then called a Poisson stable motion. A recurrent motion is also a Poisson stable motion. Note that in the literature on topological dynamics, a Poisson stable point is sometimes called a recurrent point.

A motion $\pi(t, u)$ is called pre–compact, if the closure (under the topology of X) of its trajectory $\gamma(u)$, i.e., the hull $H(u) := \mathcal{C}l\gamma(u)$, is compact.

A set $E \subset X$ is called invariant if the trajectory $\gamma(u) \subset E$ whenever $u \in E$.

A set $E \subset X$ is called minimal if it is nonempty, closed and invariant, and it contains no proper subset with these three properties.

Denote by $C(\mathbb{R}, X)$ the set of all continuous functions $\phi : \mathbb{R} \to X$, equipped with the compact-open topology, i.e., uniform convergence on compact subsets of $\mathbb{R}$. This topology is metrizable as it can be generated by the complete metric

$$d(\phi_1, \phi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(\phi_1, \phi_2)}{1 + d_n(\phi_1, \phi_2)},$$

where $d_n(\phi_1, \phi_2) = \max_{|t| \leq n} \rho(\phi_1(t), \phi_2(t))$ with ρ the metric on X.

Let $\tau \in \mathbb{R}$ and ϕ_τ be the $\tau-$translation of function ϕ, i.e., $\phi_\tau(t) := \phi(t + \tau)$ for all $t \in \mathbb{R}$. Define a mapping $\sigma : \mathbb{R} \times C(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by $\sigma(\tau, \phi) := \phi_\tau$. The triplet $(C(\mathbb{R}, X), \mathbb{R}, \sigma)$ is a dynamical system and it is called a dynamical system of translations (shifts) or the Bebutov's dynamical system; see [20, 17, 18]. Note that Fu and Duan have shown that the Bebutov's shift dynamical system is a chaotic system [14].

A function $\phi \in C(\mathbb{R}, X)$ is called recurrent (Poisson stable) if it is recurrent (Poisson stable) under the Bebutov's dynamical system $(C(\mathbb{R}, X),$

$\mathbb{R}, \sigma$). The corresponding motion $\sigma(t, \phi)$ is also called recurrent (Poisson stable)
.

The set $H(\phi) := \mathcal{C}l\{\sigma(t, \phi) | t \in \mathbb{R}\}$ is called the hull of the function ϕ, where the closure $\mathcal{C}l$ is taken in the space $C(\mathbb{R}, X)$ with the compact-open topology.

We now discuss the difference between almost periodicity, recurrence and Poisson stability. It is known ([20, 21]) that an almost periodic point is recurrent, and a recurrent point is Poisson stable. However, the converse is not true. In fact, let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be the function defined by

$$\varphi(t) = \frac{1}{2 + \sin(t) + \sin(\pi t)}$$

for all $t \in \mathbb{R}$. It can be verified that this function is Poisson stable, but not recurrent, under the Bebutov's dynamical system, because it is not bounded.

An example of recurrent function which is not almost periodic can be found, for example, in Shcherbakov [16].

In general, there exists the following relation between almost periodicity (in the sense of Bohr) and recurrence (in the sense of Birkhoff). Let X be a complete metric space with metric ρ and $(X, \mathbb{R}, \pi)$ be a dynamical system. A point $x \in X$ is almost periodic if and only if the following conditions are fulfilled: (1) $x \in X$ is recurrent; (2) a dynamical system $(H(x), \mathbb{R}, \pi)$ is equicontinuous, i.e., for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x_1, x_2) < \delta$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $x_1, x_2 \in H(x)$ and $t \in \mathbb{R}$. Note that the hull $H(x) = Cl\{xt : t \in \mathbb{R}\}$.

Note that a function $\varphi \in C(\mathbb{R}, E)$ is almost periodic if and only if the hull of the function is compact in the uniform topology (with respect to sup-norm). However, in the present paper, we consider a compact-open topology for the Bebutov's dynamical system.

Now we consider basic concepts in nonautonomous dynamical systems.

Let $(X, \mathbb{R}_+, \pi)$ and $(\Omega, \mathbb{R}, \sigma)$ be autonomous dynamical systems, and let $h : X \to \Omega$ be a homomorphism from $(X, \mathbb{R}_+, \pi)$ onto $(\Omega, \mathbb{R}, \sigma)$. Then the triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ is called a *nonautonomous* dynamical system (see [2],[3]). Since $(\Omega, \mathbb{R}, \sigma)$ is usually a simpler (*driving*) dynamical system, the homomorphism h may help understand the dynamical system $(X, \mathbb{R}_+, \pi)$. This definition is slightly more general than the definition used by some other authors.

Let Ω and E be two metric spaces and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on Ω. Consider a continuous mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \to E$ satisfying the following conditions:

$$\varphi(0, \cdot, \omega) = id_E, \quad \varphi(t + \tau, u, \omega) = \varphi(t, \varphi(\tau, u, \omega), \omega\tau)$$

for all $t, \tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $u \in E$. Such mapping φ, or more explicitly, $\langle E, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$, is called a cocycle on the driving dynamical system $(\Omega, \mathbb{R}, \sigma)$ with fiber E; see [1, 20]. This is a generalization of the semi-flow property.

In the following example, we will see that the solution mapping for the initial-value problem of a nonautonomous ordinary differential equation usually defines a cocycle.

Example 2.1. *Let E be a Banach space and $C(\mathbb{R} \times E, E)$ be a space of all continuous functions $F : \mathbb{R} \times E \to E$ equipped with the compact-open topology, i.e.,*

uniform convergence on compact subsets of $\mathbb{R} \times E$. *Let us consider a parameterized or nonautonomous differential equation*

$$\frac{du}{dt} + Au = F(\sigma_t\omega, u), \quad \omega \in \Omega,$$

on a Banach space E with $\Omega = C(\mathbb{R} \times E, E)$, where $\sigma_t\omega := \sigma(t, \omega)$ and the linear operator A is densely defined in E and such that the linear equation

$$u' + Au = 0$$

generates a c_0-semigroup of linear bounded operators

$$e^{-At} : E \to E, \ \varphi(t, x) := e^{-At}u.$$

We will define $\sigma_t : \Omega \to \Omega$ by $\sigma_t\omega(\cdot, \cdot) = \omega(t + \cdot, \cdot)$ for each $t \in \mathbb{R}$ and interpret $\varphi(t, x, \omega)$ as mild solution of the initial value problem

$$\frac{d}{dt}u(t) + Au(t) = F(\sigma_t\omega, u(t)), \quad u(0) = u. \tag{1}$$

Under appropriate assumptions on $F : \Omega \times E \to E$, or even $F : \mathbb{R} \times E \to E$ with $\omega(t)$ instead of $\sigma_t\omega$ in (1), to ensure forward existence and uniqueness of the solution, φ is a cocycle on $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ with fiber E, where $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ is a Bebutov's dynamical system which can be similarly defined as above; see, for example, [3], [16],[20].

Note that a cocycle induces a skew-product flow, which naturally defines a nonautonomous dynamical system. In fact, let φ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with the fiber E. Then the mapping $\pi : \mathbb{R}_+ \times E \times \Omega \to E \times \Omega$ defined by

$$\pi(t, u, \omega) := (\varphi(t, u, \omega), \sigma_t\omega)$$

for all $t \in \mathbb{R}_+$ and $(u, \omega) \in E \times \Omega$ forms a semi-group $\{\pi(t, \cdot, \cdot)\}_{t\in\mathbb{R}^+}$ of mappings of $X := \Omega \times E$ into itself, thus a semi-dynamical system on the (expanded) state space X. This semi-dynamical system is called a skew-product flow ([20]) and the triplet $\langle(X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ (where $h := pr_2 : X \to \Omega$) is a nonautonomous dynamical system.

3. **Recurrent motions.** We now formulate the two-dimensional Navier-Stokes system as a nonautonomous dynamical system:

$$u' + \sum_{i=1}^{2} u_i\partial_i u = \nu\Delta u - \nabla p + F(t) \tag{2}$$

$$div\ u = 0, \quad u|_{\partial D} = 0,$$

where $u = (u_1, u_2)$ is the velocity field, $F(x, y, t) = (F_1, F_2)$ is the external forcing, and D is an open bounded fluid domain with smooth boundary $\partial D \in C^2$.

The functional setting of the problem is well known [5, 22]. We denote by H and V the closures of the linear space $\{u|u \in C_0^\infty(D)^2, \ div\ u = 0\}$ in $L^2(D)^2$ and $H_0^1(D)^2$, respectively. We also denote by P the corresponding orthogonal projection $P : L^2(D)^2 \to H$. We futher set

$$A := -\nu P\Delta, \ B(u, v) := P(\sum_{i=1}^{2} u_i\partial_i v).$$

The Stokes operator A is self-adjoint and positive with domain $\mathcal{D}(A)$ dense in H. The inverse operator A^{-1} is compact. We define the Hilbert spaces $\mathcal{D}(A^\alpha)$, $\alpha \in$

$(0, 1]$ as the domains of the powers of A in the standard way. Note that $V := \mathcal{D}(A^{1/2})$, with norm $|u|_{\mathcal{D}(A^{1/2})} = |\nabla u|$.

Applying the orthogonal projection P, we rewrite (2) as an evolution equation in H

$$u' + Au + B(u, u) = \mathcal{F}(t), \ \mathcal{F}(t) := PF(t). \tag{3}$$

We suppose that $\mathcal{F} \in C(\mathbb{R}, H)$. Let $(C(\mathbb{R}, H), \mathbb{R}, \sigma)$ be the Bebutov's dynamical system; see for example, [17],[18] and [20]. We denote $\Omega := H(\mathcal{F}) = \overline{\{\mathcal{F}_\tau | \ \tau \in \mathbb{R}\}}$, where $\mathcal{F}_\tau(t) := \mathcal{F}(t + \tau)$ for all $t \in \mathbb{R}$ and the over bar denotes the closure in the compact-open topology in $C(\mathbb{R}, H)$. Note that Ω is an invariant, closed subset of $C(\mathbb{R}, H)$. Moreover, $(\Omega, \mathbb{R}, \sigma)$ is the Bebutov's dynamical system of translations on Ω.

Along with the equation (3), we consider the following family of equations

$$u' + Au + B(u, u) = \tilde{\mathcal{F}}(t), \tag{4}$$

where $\tilde{\mathcal{F}} \in \Omega = H(\mathcal{F})$. Let $f : \Omega \to H$ be a mapping defined by

$$f(\omega) = f(\tilde{\mathcal{F}}) := \tilde{\mathcal{F}}(0),$$

where $\omega = \tilde{\mathcal{F}} \in \Omega$. Thus, the Navier-Stokes system can be written as a nonautonomous system

$$u_t + Au + B(u, u) = f(\omega t), \tag{5}$$
$$u(0) = u_0, \tag{6}$$

for $\omega \in \Omega = H(\mathcal{F})$ and $u_0 \in H$. Here $\omega t = \sigma_t(\omega)$. Note that Ω is compact minimal if and only if $\mathcal{F}$ is recurrent. This nonautonomous system is well-posed in H for $t > 0$.

The solution $\varphi(t, u, \omega)$ of the nonautonomous Navier-Stokes system (5) is called recurrent (Poisson stable, almost periodic, quasi periodic), if the point $(u, \omega) \in H \times \Omega$ is a recurrent (Poisson stable, almost periodic, quasi periodic) point of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$, where $X = H \times \Omega$ and $\pi = (\varphi, \sigma)$.

As is known (in, for example, [15] and [12]), if $\omega \in \Omega$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of dynamical system $(\Omega, \mathbb{R}, \sigma)$ and $h : \Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$, then the point $u = h(\omega)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{R}_+, \pi)$.

It is known ([4]) if the forcing function $\mathcal{F}(t)$ is bounded with respect to time in H, such as in the case of a recurrent function, then every solution of the Navier-Stokes system (3) is bounded on $\mathbb{R}_+$.

Theorem 3.1. (*Recurrent motion*)
If the forcing $\mathcal{F}(t)$ is recurrent in $C(\mathbb{R}, H)$, then the 2D Navier-Stokes system (3) has at least one recurrent solution in $C(\mathbb{R}, H)$.

Proof. The 2D Navier-Stokes system (3) is well-posed in H. Moreover, the solution is in $H_0^1(D)$ (in fact, smooth) after a short transient time due to regularizing effect; see for example, [5, 8, 22]. Since $H_0^1(D)$ is compactly embedded in H, the solution is actually compact in H after a short transient time.

Since the Navier-Stokes system (3) is nonautonomous, its solution operator $\phi(t, u_0, g)$ does not define a usual dynamical system or a semiflow in H. However, we can define an associated semiflow in an expanded phase space, i.e., a

skew-product flow on $H \times H(\mathcal{F})$:

$$\pi : R_+ \times H \times H(\mathcal{F}) \to H \times H(\mathcal{F}),$$

$$\pi \; : \; (t, (u_0, \tilde{\mathcal{F}})) \to (\phi(t, u_0, \tilde{\mathcal{F}}), \tilde{\mathcal{F}}_t).$$

Since $\mathcal{F}(t)$ is recurrent, $H(\mathcal{F})$ is compact and minimal. Combining with the above compact solution $\phi(t, u_0, \mathcal{F})$ in H, we conclude that the skew-product flow π has a pre-compact motion. Thus by a recurrence theorem due to Birkhoff and Bebutov (see [20]), the ω−limit set $\omega_{(u_0, \mathcal{F})}$ contains a compact minimal set $M \subset \omega_{(u_0, \mathcal{F})}$. Let $(\tilde{u}, \tilde{\mathcal{F}}) \in M$. Since $\mathcal{F} \in H(\mathcal{F})$ and $H(\mathcal{F})$ is a compact minimal set, we see that $H(\tilde{\mathcal{F}}) = H(\mathcal{F})$. Thus there exists a sequence $t_n \to +\infty$ such that $\tilde{\mathcal{F}}_{t_n} \to \mathcal{F}$. Since the sequence $\{\phi(t_n, \tilde{u}, \tilde{\mathcal{F}})\}$ is pre-compact, then without loss of generality we can suppose, that the sequence $\{\phi(t_n, \tilde{u}, \tilde{\mathcal{F}})\}$ is convergent. Denote by $u := \lim_{n \to \infty} \phi(t_n, \tilde{u}, \tilde{\mathcal{F}})$, then the point $(u, \mathcal{F}) \in M$ is recurrent and, consequently the solution $\phi(t, u, \mathcal{F})$ of equation (3) is recurrent. This completes the proof. $\square$

Remark 3.2. *This result is actually true for a more general evolution equation, as long as it generates a cocycle and the solution operator is compact (or asymptotically compact).*

We now assume that the external forcing $\mathcal{F}(t) \in C(\mathbb{R}, H)$ is Poisson stable. We will show that the Navier-Stokes system (3) has at least one Poisson stable motion or solution.

We define $\mathfrak{N}_{\mathcal{F}} := \{\{t_n\} | \; \mathcal{F}t_n \to \mathcal{F}, t_n \to +\infty\}$. Let $\langle(X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be a nonautonomous dynamical system. A point $u \in X$ is called weakly regular ([16]), if for every $\{t_n\} \in \mathfrak{N}_y$ with $y = h(u)$, the sequence $\{ut_n\}$ is pre-compact in X. We need the following result.

Theorem 3.3. *(Shcherbakov* [16]*)*
Let $\langle(X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be a general nonautonomous dynamical system. Let $y \in \Omega$ be Poisson stable under the dynamical system $(\Omega, \mathbb{R}, \sigma)$ and there exists a weakly regular solution u_0 of the abstract operator equation

$$h(u) = y.$$

Then this equation admits at least one Poisson stable solution in X under the dynamical system $(X, \mathbb{R}_+, \pi)$.

We have the following result.

Theorem 3.4. *(Poisson stable motion)*
Let the forcing $\mathcal{F}(t)$ be Poisson stable in $C(\mathbb{R}, H)$. Assume that the Navier-Stokes system (3) has one bounded solution in $H_0^1(D) \cap H$ for $t > 0$. Then the Navier-Stokes system (3) has at least one Poisson stable solution in $C(\mathbb{R}, H)$.

Proof. We again consider the skew-product flow on $H \times H(\mathcal{F})$:

$$\pi : R_+ \times H \times H(\mathcal{F}) \to H \times H(\mathcal{F}),$$

$$\pi(t, (u_0, \tilde{\mathcal{F}})) \to (\phi(t, u_0, \tilde{\mathcal{F}}), \tilde{\mathcal{F}}_t).$$

Since $\mathcal{F}(t)$ is Poisson stable, $\mathfrak{N}_{\mathcal{F}} \neq \emptyset$. Let $\{t_n\} \in \mathfrak{N}_{\mathcal{F}}$. Then the sequence $\{\mathcal{F}_{t_n}\}$ is convergent. By the assumption, the solution $\phi(t, u_0, \mathcal{F})$ is bounded in H_0^1 on R_+. Consequently, due to the compact embedding of H_0^1 in H, the sequence $\{\pi(t_n, u_0, \mathcal{F}))\}$ is pre-compact in H. We argue that $(\phi(t_n, u_0, \mathcal{F}), \mathcal{F}_{t_n})$ is

pre-compact. Then by Theorem 3.2 we conclude that the skew-product flow π has at least one Poisson stable motion, and hence the Navier-Stokes system (3) has at least one Poisson stable solution. This completes the proof.

$\square$

Remark 3.5. *This theorem also holds for a more general evolution equation, as long as it generates a cocycle, admits at least one bounded (on $\mathbb{R}$) solution, and the solution operator is compact (or asymptotically compact).*

REFERENCES

[1] L. Arnold, *Random Dynamical Systems*. Springer, New York, 1998.

[2] I. U. Bronshteyn, Nonautonomous Dynamical Systems (Russian). Kishinev, Shtiintsa, 1984.

[3] D. N. Cheban, Global Attractors of Nonautonomous Dynamical Systems (Russian). Kishinev, State University of Moldova, 2002.

[4] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of mathematical physics*. American Mathematical Society, Providence, RI, 2002.

[5] P. Constantin and C. Foias, *Navier-Stokes Equations*. Univ. of Chicago Press, Chicago, 1988.

[6] J. Duan, Time-Periodic Quasigeostrophic Motion under Dissipation and Forcing, *Appl. Math. and Comput.* **102** (1999), 121-127.

[7] J. Duan and P. E. Kloeden, Dissipative Quasigeostrophic Motion under Temporally Almost Periodic Forcing, *J. Math. Anal. Appl.* **236** (1999), 74-85.

[8] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*. Cambridge Univ. Press, England, 2001.

[9] C. Foias and G. Prodi, Sur le comportement global des solutions non-stationnaires des quations de Navier-Stokes en dimension 2. (French) *Rend. Sem. Mat. Univ. Padova* **39** (1967), 1–34.

[10] I. Kukavica, Interior Gradient Bounds for the 2D Navier-Stokes System, *Discrete and Continuous Dynamical Systems* **7** (2001), 873-882.

[11] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows, Nauka, Moscow (1970); English transl. Gordon and Breach, New York, 1969.

[12] B. M. Levitan and V. V. Zhikov, Almost Periodic Functions and Differential Equations. Cembridge Univ. Press, Cambridge, 1982.

[13] G. Prodi, Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale. (Italian) *Rend. Sem. Mat. Univ. Padova* **30** (1960), 1–15.

[14] X. Fu and J. Duan, Infinite-dimensional linear dynamical systems with chaoticity, *J. Nonlinear Sci.*, **9** (1999), 197–211.

[15] B. A. Scherbakov, Poisson's Stability of Motions of Dynamical Systems and Sobilutions of Differential Equations. Kishinev, Shtiintsa, 1985. (in Russian)

[16] B. A. Scherbakov, Topological Dynamics and Poisson's Stability of Solutions of Differential Equations. Kishinev, Shtiintsa, 1972. (in Russian)

[17] B. A. Shcherbakov, Recurrent solutions of differential equations and general theory of dynamical systemss. *Differential Equations (English Transl.)* 3 (1967), pp.758-763.

[18] B. A. Shcherbakov, Poisson stable solutions of differential equations and topological dynamics. *Differential Equations (English Transl.)* 5 (1969), pp. 1608–1616.

[19] B. A. Shcherbakov and N. S. Fal'ko, The minimality of sets and the Poisson stability of motion in homomorphic dynamical systems. *Differential Equations (English Transl.)* 13(1977), N0. 6, pp. 755-758.

[20] G. R. Sell, *Topological Dynamics and Ordinary Differential Equations.* Van Nostrand–Reinbold, London, 1971.
[21] K. S. Sibirsky, *Introduction to Topological Dynamics.* Norordhoh, Leyden, 1975.
[22] R. Temam, *Navier–Stokes Equation: Theory and Numerical Analysis.* North-Holland, Amsterdam, 1979.
[23] V. I. Yudovic, Periodic motions of a viscous incompressible fluid, *Soviet Math.-Doklady* **1** (1960), 168-172.

(V. P. Bongolan-Walsh) Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
  E-mail address, V. P. Bongolan-Walsh: `bongven@iit.edu`

(D. Cheban) State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova
  E-mail address, D. Cheban: `cheban@usm.md`

(J. Duan) Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA

(J. Duan) Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026 CHINA
  E-mail address, J. Duan: `duan@iit.edu`