UPPER SEMI-CONTINUITY OF ATTRACTORS OF SET-VALUED NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract: This is a systematic study into upper semi-continuity of compact global attractors in set-valued non-autonomous dynamical systems for small perturbations. For general non-autonomous dynamical systems we give the conditions of upper semi-continuity of attractors for small parameter. Applications of these results are given (quasi-homogeneous systems, dissipative differential inclusions).

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1. Introduction

The problem of upper semi-continuity of global attractors for small perturbations is well studied (see, for example, Hale Hal88 and references therein) for autonomous and periodical dynamical systems. In works Caraballo, Langa and Robinson [3], Caraballo and Langa [4] and Cheban [18] this problem was studied for non-autonomous and random dynamical systems.

Our paper is devoted to the systematical study of the problem of upper semi-continuity of compact global attractors and compact pullback attractors in abstract non-autonomous set-valued dynamical systems for small perturbations. Applications of obtained results are given for certain classes of evolutional equations (without uniqueness) and inclusions.

The paper is organized as follows. In section 2 we study some general properties of maximal compact invariant sets of dynamical systems. In particular, we prove that the compact global attractor and pullback attractor are maximal compact invariant sets (Theorem 3.10).

Section 3 contains the main results about upper semi-continuity of compact global attractors of abstract non-autonomous dynamical systems for small perturbations (Lemmas 4.3, 4.6 and Theorems 4.10, 4.13, 4.14 and 4.16).

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In section 4 we give the conditions of connectedness and component connectedness of global and pullback attractors (Theorem 5.6).

Section 5 is devoted to the application of the general results obtained in sections 2-4, to study of certain classes of non-autonomous differential equations (without uniqueness) and inclusions.

2. Global attractors in autonomous set-valued dynamical systems

Let (X,ρ) be a complete metric space, $\mathbb{S}$ is a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{S}_+ := \{s \in \mathbb{S} \ : \ s \geq 0\}$, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) is semigroup of additive group $\mathbb{S}$. If $A \subseteq X$ and $x \in X$ then will note $\rho(x,A)$ a distance from point x to set A i.e. $\rho(x,A) = \inf\{\rho(x,a) : a \in A\}$. Will note by $B(A,\varepsilon)$ the $\varepsilon-$neighborhood of set A i.e. $B(A,\varepsilon) = \{x \in X : \rho(x,A) < \varepsilon\}$. We will denote by $C(X)$ the family of all non-empty compact subsets of X. For every point $x \in X$ and number $t \in \mathbb{T}$ will put in correspondence closed compact subset $\pi(x,t) \in C(X)$ and so if $\pi(A,P) = \bigcup\{\pi(x,t) : x \in A, t \in P\}(P \subseteq \mathbb{T})$ then

(1) $\pi(x,0) = x$ for all $x \in X$;
(2) $\pi(\pi(x,t_1),t_2) = \pi(x,t_1+t_2)$ for all $x \in X$ and $t_1,t_2 \in \mathbb{T}$, if $t_2 \cdot t_2 > 0$;
(3) $\lim\limits_{x \to x_0, t \to t_0} \beta(\pi(x,t),\pi(x_0,t_0)) = 0$ for all $x_0 \in X$ and $t_0 \in \mathbb{T}$, where $\beta(A,B) = \sup\{\rho(a,B) : a \in A\}$ semi-distance of Hausdorff of set $A \subseteq X$ from set $B \subseteq X$.

In this case it is said Sibirskii and Shube [29] that it is defined set-valued semi-group dynamical system. Let $\mathbb{T} = \mathbb{S}$ and is fulfilled condition

4. If $p \in \pi(x,t)$ then $x \in \pi(p,-t)$ for all $x,p \in X$ and $t \in \mathbb{T}$

then it is said that is defined a set-valued dynamical system $(X,\mathbb{T},\pi)$ or dynamical systems without uniqueness.

Remark 2.1. *Later on by the set-valued dynamical system $(X,\mathbb{T},\pi)$ we will mean a semi-group dynamical system unless otherwise stated, i.e. we will consider, that $\mathbb{T} = \mathbb{S}_+$.*

Let $\mathfrak{M}$ is some family of subsets of X. We will call a dynamical system $(X,\mathbb{T},\pi)$ $\mathfrak{M}$-dissipative if there exists a bounded set $K \subseteq X$, such that for any $\varepsilon > 0$ and $M \in \mathfrak{M}$ exists $L = L(\varepsilon,M) > 0$ such that $\pi^t M \subseteq B(K,\varepsilon)$ for every $t \geq L(\varepsilon,M)$, where $\pi^t M = \{\pi(x,t) = xt : x \in M\}$. In addition we will call set K as attractor for family $\mathfrak{M}$. The most interesting cases for application are when $\mathfrak{M} = \{\{x\} : x \in X\}, \mathfrak{M} = C(X), \mathfrak{M} = \{B(x,\delta_x) : x \in X, \delta_x > 0$ is fixed $\}$ or $\mathfrak{M} = \mathbb{B}(X)$ (where $\mathbb{B}(X)$ is a family of all bounded subsets of X).

System $(X,\mathbb{T},\pi)$ is called:

(1) pointwise dissipative if there exists $K \in \mathbb{B}(X)$ such that

$$\lim_{t \to +\infty} \beta(xt,K) = 0 \tag{1}$$

for all $x \in X$;

(2) compact dissipative if equality (1) holds uniformly by x on compacts from X;
(3) locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that equality (1) holds uniformly by $x \in B(p, \delta_p)$;
(4) bounded dissipative if equality (1) holds uniformly by x on every bounded subset from X.

In the study of dissipative systems two different cases occur when K is compact and bounded (but not compact). According to this we will call the system $(X, \mathbb{T}, \pi)$ pointwise $k(b)-$ dissipative if $(X, \mathbb{T}, \pi)$ is pointwise dissipative and set K appearing in (1) is compact (bounded). Notions of compact $k(b)$ dissipativity and other types of dissipativity are defined analogously.

Let $(X, \mathbb{T}, \pi)$ be compact k dissipative and K is compact set being attractor for all compact subsets of X. We will set

$$J = \Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^{\tau} K}. \tag{2}$$

It can be shown Cheban and Fakeeh [6] that set J defined by equality (2), don't depends on the choice of attractor K, but is characterized only by properties of dynamical system $(X, \mathbb{T}, \pi)$ itself. Set J is called center of Levinson of compact dissipative system $(X, \mathbb{T}, \pi)$.

Will state some known facts, which will be necessary for us below.

Theorem 2.2. *(Cheban and Fakeeh* [6], [8]*) If $(X, \mathbb{T}, \pi)$ is compact dissipative dynamical system and J is its center of Levinson then :*
(1) *J is invariant, i.e. $\pi^t J = J$ for all $t \in \mathbb{T}$;*
(2) *J is orbital stable , i.e. for any $\varepsilon > 0$ exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\beta(xt, J) < \varepsilon$ for all $t \geq 0$;*
(3) *J is attractor of family of all compact subsets of X;*
(4) *J is maximal compact invariant set $(X, \mathbb{T}, \pi)$.*

Continuous single-valued mapping $\varphi_x : \mathbb{T} \to X$ is called motion of disperse dynamical system $(X, \mathbb{T}, \pi)$ starting from the point $x \in X$ if $\varphi_x(0) = x$ and $\varphi_x(t_2) \in \pi(\varphi_x(t_1), t_2 - t_1)$ for any $t_1, t_2 \in \mathbb{T}(t_2 > t_1)$.

The set of all motions $(X, \mathbb{T}, \pi)$ starting from point x is noted by Φ_x and $\Phi(\pi) = \bigcup\{\Phi_x : x \in X\}$.

A dynamical system $(X, \mathbb{T}, \pi)$ is called :
(1) locally completely continuous if for any point $p \in X$ exists $\delta_p > 0$ and $l_p > 0$ such that $\pi^{l_p} B(p, \delta_p)$ is relatively compact ;
(2) weakly dissipative, if there exists non-empty compact $K \subseteq X$ such that for any $x \in X$ and $\varphi_x \in \Phi_x$ be found $\tau = \tau(x, \varphi_x) > 0$ for which $\varphi_x(\tau) \in K$;
(3) trajectoricaly dissipative if there exists non-empty compact such that

$$\lim_{t \to +\infty} \rho(\varphi_x(t), K) = 0 \tag{3}$$

for all $x \in X$ and $\varphi_x \in \Phi_x$.

Let $(\Omega, \mathbb{T}, \sigma)$ is dynamical system with uniqueness (i.e. $\sigma(t, \omega)$ consist of unique point whatever are $\omega \in \Omega$ and $t \in \mathbb{T}$) and $(X, \mathbb{T}, \pi)$ is a set-valued dynamical system. Triple $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ will be called a non-autonomous set-valued dynamical system where h is homomorphism $(X, \mathbb{T}, \pi)$ on $(\Omega, \mathbb{T}, \sigma)$ i.e. h is continuous mapping of X on Y satisfying condition : $h(\pi(t, x)) = \sigma(t, h(x))$ for all $x \in X$ and $t \in \mathbb{T}$.

Non-autonomous set-valued dynamical system $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ will be called pointwise (compact, local, bounded) dissipative if such is $(X, \mathbb{T}, \pi)$.

We will call as a center of Levinson of non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi)$,
$(\Omega, \mathbb{T}, \sigma), h\rangle$ a center of Levinson of $(X, \mathbb{T}, \pi)$.

Everywhere below (in this section) we will assume that Ω is compact, (X, h, Ω) is locally-trivial finite-dimensional fibering Bourbaki [2] and $|\cdot|$ is a norm on (X, h, Ω) compatible with metric ρ on X (i.e. $\rho(x_1, x_2) =| x_1 - x_2 |$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$).

Denote by $\mathfrak{A}$ the family of all continuous and strictly increasing functions $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$ and $Im(a) := \{q \in \mathbb{R}_+ \ : \ \exists p \in \mathbb{R}_+$ such that $a(p) = q\}$.

Theorem 2.3. *(Cheban and Fakeeh [6]) Let $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ be a non-autonomous set-valued dynamical system and Ω be compact, if $\mathbb{T} \subseteq \mathbb{Z}$. Suppose that there exist a positive number r and the function $V : X_r \to \mathbb{R}_+$ satisfying the following conditions:*

1. *$a(|x|) \leq V(x) \leq b(|x|)$ $(a, b \in \mathfrak{A}, \Im(a) = \Im(b))$ for all $x \in X_r$;*
2. *$V'_\pi(x) \leq -c(|x|)$ for all $x \in X_r$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and positive on $[r, +\infty)$, $V'_\pi := \bigcup\{V'_{\varphi_x}(x) \ : \ \varphi_x \in \Phi_x\}$ and $V'_{\varphi_x}(x) := \lim\limits_{t\downarrow 0} t^{-1}[V(\varphi_x(t)) - V(x)]$ for $\mathbb{T}_+ = \mathbb{R}_+$ and $V'_{\varphi_x}(x) := V(\pi(1, x)) - V(x)$ for $\mathbb{T}_+ = \mathbb{Z}_+$.*

Then the non-autonomous set-valued dynamical system $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ admits a compact global attractor.

3. Maximal compact invariant sets.

Let W be a complete metric space, Ω be a compact metric space and $(\Omega, \mathbb{T}, \sigma)$ be a dynamical system on Ω.

Definition 3.1. *Triplet $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ is said to be a set-valued cocycle over $(\Omega, \mathbb{T}, \sigma)$ with fiber W, where φ is a mapping from $\mathbb{T}_+ \times W \times \Omega$ onto $C(W)$ and possesses the following properties:*

1. *$\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;*
2. *$\varphi(t+\tau, u, \omega) = \varphi(t, \varphi(\tau, x, \omega), \omega t)$, where $\omega t := \sigma(t, \omega)$ and $\varphi(t, A, \omega) := \{\varphi(t, u, \omega) \ : \ u \in A\}$:*
3. $$\lim_{t\to t_0, u\to u_0, \omega\to\omega_0} \beta(\varphi(t, u, \omega), \varphi(t_0, u_0, \omega_0)) = 0$$

for all $(t_0, u_0, \omega_0) \in \mathbb{T}_+ \times W \times \Omega$.

We denote by $X = W \times \Omega, (X, \mathbb{T}_+, \pi)$ a set-valued dynamical system on the X defined by equality $\pi = (\varphi, \sigma)$, i.e. $\pi^t x := \{(v, \sigma(t, \omega)) \mid v \in \varphi(t, u, \omega)\}$ for every $t \in \mathbb{T}_+$ and $x = (u, \omega) \in X = W \times \Omega$. Then the triplet $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \sigma), h \rangle$ is a set-valued non-autonomous dynamical system (skew-product system), where $h = pr_2 : X \mapsto \Omega$.

Definition 3.2. *A family* $\{I_\omega \mid \omega \in \Omega\}(I_\omega \subset W)$ *of non-empty compact subsets of* W *is said to be a maximal compact invariant set of set-valued cocycle* φ*, if the following conditions are fulfilled:*

1. $\{I_\omega \mid \omega \in \Omega\}$ *is invariant, i.e.* $\varphi(t, I_\omega, \omega) = I_{\omega t}$ *for every* $\omega \in \Omega$ *and* $t \in \mathbb{T}_+$*;*
2. $I = \bigcup \{I_\omega \mid \omega \in \Omega\}$ *is relatively compact;*
3. $\{I_\omega \mid \omega \in \Omega\}$ *is maximal, i.e. if the family* $\{I'_\omega \mid \omega \in \Omega\}$ *is relatively compact and invariant, then* $I'_\omega \subseteq I_\omega$ *for every* $\omega \in \Omega$.

Lemma 3.3. *The family* $\{I_\omega \mid \omega \in \Omega\}$ *is invariant w.r.t. set-valued cocycle* φ *if and only if the set* $J = \bigcup \{J_\omega \mid \omega \in \Omega\}$ $(J_\omega = I_\omega \times \{\omega\})$ *is invariant w.r.t. dynamical system* $(X, \mathbb{T}_+, \pi)$ *that is* $\pi^t J = J$ *for all* $t \in \mathbb{T}_+$*, where* $\imath^t := \pi(t, \cdot)$.

Proof. Let the family $\{I_\omega \mid \omega \in \Omega\}$ be invariant, $J = \bigcup \{J_\omega \mid \omega \in \Omega\}$ and $J_\omega = I_\omega \times \{\omega\}$, then we have

$$\begin{aligned} \pi^t J &= \bigcup \{\pi^t J_\omega \mid \omega \in \Omega\} = \bigcup \{(\varphi(t, I_\omega, \omega), \omega t) \mid \omega \in \Omega\} \\ &= \bigcup \{I_{\omega t} \times \{\omega t\} \mid \omega \in \Omega\} = \bigcup \{J_{\omega t} \mid \omega \in \Omega\} = J \end{aligned} \tag{4}$$

for all $t \in \mathbb{T}_+$. From the equality (4) follows that the family $\{I_\omega \mid \omega \in \Omega\}$ is invariant w.r.t. set-valued cocycle φ if and only if a set J is invariant w.r.t. set-valued dynamical system $(X, \mathbb{T}_+, \pi)$. □

Theorem 3.4. *(Sibirskii and Shube* [29]*) Let* $(X, \mathbb{T}, \pi)$ *be a set-valued dynamical system. If* $y \in \pi(t, x)$*, then there exists* $\varphi \in \Phi_x$ *such that* $\varphi(t) = y$.

Theorem 3.5. *Let family of sets* $\{I_\omega | \omega \in \Omega\}$ *be maximal, compact and invariant with respect to set-valued cocycle* φ*. Then it is closed.*

Proof. We note that the set $J = \bigcup \{J_\omega \mid \omega \in \Omega\}$ $(J_\omega = I_\omega \times \{\omega\})$ is relatively compact and according to Lemma 3.3 it is invariant. Let $K = \overline{J}$, then K is compact. We shall show that K is invariant. Really, if $x \in K$, then there exists $\{x_n\} \subset J$ such that $x = \lim_{n \to +\infty} x_n$. Thus $x_n \in J = \pi^t J$ for all $t \in \mathbb{T}_+$, then for $t \in \mathbb{T}_+$ there exists $\bar{x}_n \in J$ and $\varphi_{\bar{x}_n} \in \Phi_{\bar{x}_n}$ such that $x_n = \varphi_{\bar{x}_n}(t)$. Since J is relatively compact it is possible to consider that the sequences $\{\bar{x}_n\}$ and $\{\varphi_{\bar{x}_n}\}$ are convergent, moreover the sequence $\{\varphi_{\bar{x}_n}\}$ converge uniformly on every compact from $\mathbb{T}$. We denote by $\bar{x} = \lim_{n \to +\infty} \bar{x}_n$ and $\varphi_x := \lim_{n \to +\infty} \varphi_{x_n}$, then $\varphi_{\bar{x}} \in \Phi_{\bar{x}}$, $\bar{x} \in \overline{J}, x = \varphi_{\bar{x}}(t) \in \pi^t \bar{x}$ and, consequently, $x \in \pi^t \overline{J}$ for all $t \in \mathbb{T}_+$, i.e. $\overline{J} = \pi^t \overline{J}$.

Let $I' = pr_1 K$, where by pr_1 we denote the first projection from $X = W \times \Omega$ to W, then we have $I' = \bigcup\{I'_\omega \mid \omega \in \Omega\}$, where $I'_\omega = \{u \in W \mid (u,\omega) \in K\}$ and $K_\omega = I'_\omega \times \{\omega\}$. Since the set K is invariant, then according to Lemma 3.3 the set I' is also invariant w.r.t. set-valued cocycle φ. The set I' is compact, because K is compact and $pr_1 : X \mapsto W$ is continuous. According to the maximality of the family $\{I_\omega \mid \omega \in \Omega\}$ we have $I'_\omega \subseteq I_\omega$ for every $\omega \in \Omega$ and, consequently, $I' \subseteq I$.

On the other hand $I = pr_1 \overline{J} = I'$ and, consequently, $I' = I$. Thus the set I is compact. The theorem is proved. □

Definition 3.6. *Let $\langle W, \varphi, (\Omega, \mathbb{S}, \sigma)\rangle$ be a set-valued cocycle. A family $\{I_\omega \mid \omega \in \Omega\}$ ($I_\omega \subset W$) of non-empty compact subsets of W is said to be a compact pullback attractor of the set-valued cocycle φ, if the following conditions are fulfilled:*

a. *$I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is relatively compact ;*
b. *I is invariant w.r.t. set-valued cocycle φ, i.e. $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{T}_+$ and $\omega \in \Omega$;*
c. *for every $\omega \in \Omega$ and $K \in C(W)$*

$$\lim_{t\to+\infty} \beta(\varphi(t, K, \omega^{-t}), I_\omega) = 0, \tag{5}$$

where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff and $\omega^{-t} := \sigma(-t, \omega)$.

Definition 3.7. *A family $\{I_\omega \mid \omega \in \Omega\}(I_\omega \subset W)$ of nonempty compact subsets of W is called a compact global attractor of set-valued cocycle φ, if the following conditions are fulfilled:*

a. *a family $\{I_\omega \mid \omega \in \Omega\}$ is compact and invariant;*
b. *for every $K \in C(W)$*

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, K, \omega), I) = 0, \tag{6}$$

where $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$.

We will say that the space X has the property (S), if for any compact $K \subseteq X$ there exists a connected set $M \subseteq X$ such that $K \subseteq M$.

By entire trajectory of semi-group dynamical system $(X, \mathbb{S}_+, \pi)$ (of set-valued cocycle $\langle W, \varphi, (\Omega, \mathbb{S}, \sigma)\rangle$ over $(\Omega, \mathbb{S}, \sigma)$ with fiber W), passing through the point $x \in X$ ($(u, \omega) \in W \times \Omega$) is called a continuous mapping $\gamma : \mathbb{S} \to X(\nu : \mathbb{S} \to W)$ satisfying the conditions : $\gamma(0) = x$ ($\nu(0) = w$) and $\gamma(t + \tau) \in \pi^t\gamma(\tau)$ ($\nu(t + \tau) \in \varphi(t, \nu(\tau), y\tau)$) for all $t \in \mathbb{S}_+$ and $\tau \in \mathbb{S}$.

Definition 3.8. *The set-valued cocycle φ is called compact dissipative if there exists a nonempty compact set $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(U(t, \omega)M, K) \mid \omega \in \Omega\} = 0$$

for all $M \in C(W)$.

Theorem 3.9. *(Cheban and Schmalfuss [14]) Let Ω be compact, $\langle W, \varphi, (\Omega, \mathbb{S}, \sigma)\rangle$ be a compact dissipative set-valued cocycle, then:*

(1) *$I_\omega = \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and*

$$\lim_{t\to+\infty} \beta(U(t,\omega)K, I_\omega) = 0 \tag{7}$$

for every $\omega \in \Omega$;

(2) *$U(t,\omega)I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{S}_+$;*

(3)

$$\lim_{t\to+\infty} \beta(U(t,\omega^{-t})M, I_\omega) = 0 \tag{8}$$

for all $M \in C(W)$ and $\omega \in \Omega$, where $\omega^{-t} := \sigma(-t,\omega)$;

(4)

$$\lim_{t\to+\infty} \sup\{\beta(U(t,\omega)M, I) : \omega \in \Omega\} = 0 \tag{9}$$

whatever is $M \in C(W)$, where $I = \bigcup\{I_\omega : \omega \in \Omega\}$;

(5) *$I = pr_1 J$ and $I_\omega = pr_1 J_\omega$, where J is center of Levinson of $(X, \mathbb{S}_+, \pi)$ and $J_\omega = J \bigcap X_\omega$;*

(6) *set I is compact ;*

(7) *set I is connected if one of the following two conditions:*
   a. *$\mathbb{S}_+ = \mathbb{R}_+$ and the spaces W and Y are connected;*
   b. *$\mathbb{S}_+ = \mathbb{Z}_+$ and the space $W \times Y$ has property (S) or it is connected and locally connected*

is fulfilled.

Theorem 3.10. *A family $\{I_\omega \mid \omega \in \Omega\}$ of nonempty compact subsets of W will be maximal compact invariant set w.r.t. set-valued cocycle φ, if one of the following two conditions is fulfilled:*

a. *$\{I_\omega \mid \omega \in \Omega\}$ is a compact pullback attractor w.r.t. set-valued cocycle φ;*

b. *$\{I_\omega \mid \omega \in \Omega\}$ is a compact global attractor w.r.t. set-valued cocycle φ.*

Proof. a. Let $\{I_\omega \mid \omega \in \Omega\}$ be a compact pullback attractor of set-valued cocycle φ. Since the family $\{I'_\omega \mid \omega \in \Omega\}$ is a compact and invariant set of the set-valued cocycle φ, then we have

$$\beta(I'_\omega, I_\omega) = \beta(\varphi(t, I'_{\omega^{-t}}, \omega^{-t}), I_\omega) \leq \beta(\varphi(t, K, \omega^{-t}), I_\omega) \to 0$$

as $t \to +\infty$, where $K = \overline{\bigcup\{I'_\omega \mid \omega \in \Omega\}}$, and, consequently, $I'_\omega \subseteq I_\omega$ for every $\omega \in \Omega$, i.e. $\{I_\omega \mid \omega \in \Omega\}$ is maximal.

b. Let the family $\{I_\omega \mid \omega \in \Omega\}$ be a compact global attractor w.r.t. set-valued cocycle φ, then according to Theorem 3.9 it is a compact pullback attractor and, consequently, the family $\{I_\omega \mid \omega \in \Omega\}$ is maximal compact invariant set of the set-valued cocycle φ. $\square$

Remark 3.11. *Family* $\{I_\omega \mid \omega \in \Omega\}$ $(I_\omega \subset W)$ *is a maximal compact invariant w.r.t. set-valued cocycle* φ *if and only if the set* $J = \bigcup\{J_\omega \mid \omega \in \Omega\}$, *where* $J_\omega = I_\omega \times \{\omega\}$, *is a maximal compact invariant in the dynamical system* $(X, \mathbb{S}_+, \pi)$.

4. Upper semi-continuity

Lemma 4.1. *Let* $\{I_\omega \mid \omega \in \Omega\}$ *be a maximal compact invariant set of the set-valued cocycle* φ, *then the function* $F : \Omega \mapsto C(W)$, *defined by the equality* $F(\omega) = I_\omega$ *is upper semi-continuous, i.e. for all* $\omega_0 \in \Omega$

$$\beta(F(\omega_k), F(\omega_0)) \to 0,$$

if $\rho(\omega_k, \omega_0) \to 0$.

Proof. Let $\omega_0 \in \Omega, \omega_k \to \omega_0$ and there exists $\varepsilon_0 > 0$ such that

$$\beta(F(\omega_k), F(\omega_0)) \geq \varepsilon_0,$$

then there exists $x_k \in I_{\omega_k}$ such that

$$\rho(x_k, I_{\omega_0}) \geq \varepsilon_0. \tag{10}$$

As set I is compact, without loss of generality we can suppose that the sequence $\{x_k\}$ is convergent. We denote by $x = \lim\limits_{k\to+\infty} x_k$, then in virtue of Theorem 3.5 the set $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is compact and hence there exists $\omega_0 \in \Omega$ such that $x \in I_{\omega_0} \subset I$.

On the other hand, according to the inequality (10) $x \notin I_{\omega_0}$. This contradiction shows that the function F is upper semi-continuous. □

Remark 4.2. *Lemma 4.1 was proved for the pullback attractors of non-autonomous quasi-liner differential equations in the work of Cheban, Schmalfuss and Kloeden* [12, p.13-14].

Lemma 4.3. *Let* Λ *be a compact metric space and* $\varphi : \mathbb{T}_+ \times W \times \Lambda \times \Omega \mapsto C(W)$ *verifies the following conditions :*

(1) φ *is* β*-continuous, i.e.*

$$\lim_{t\to t_0,\ u\to u_0,\ \lambda\to\lambda_0} \beta(\varphi(t, u, \lambda), \varphi(t_0, u_0, \lambda_0)) = 0;$$

(2) *for every* $\lambda \in \Lambda$ *the function* $\varphi_\lambda = \varphi(\cdot, \cdot, \lambda, \cdot) : \mathbb{T}_+ \times W \times \Omega \mapsto W$ *is a set-valued cocycle on* Ω *with the fiber* W*;*

(3) *the set-valued cocycle* φ_λ *admits a pullback attractor* $\{I^\lambda_\omega \mid \omega \in \Omega\}$ *for every* $\lambda \in \Lambda$*;*

(4) *the set* $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ *is precompact, where* $I^\lambda = \bigcup\{I^\lambda_\omega \mid \omega \in \Omega\}$,

then the following equality

$$\lim_{\lambda\to\lambda_0, \omega\to\omega_0} \beta(I^\lambda_\omega, I^{\lambda_0}_{\omega_0}) = 0 \tag{11}$$

takes place for every $\lambda_0 \in \Lambda$ *and* $\omega_0 \in \Omega$ *and*

$$\lim_{\lambda\to\lambda_0} \beta(I_\lambda, I_{\lambda_0}) = 0 \tag{12}$$

for every $\lambda_0 \in \Lambda$.

Proof. Let $\tilde{\Omega} = \Lambda \times \Omega$ and $\tilde{\sigma} : \mathbb{T} \times \tilde{\Omega} \mapsto \tilde{\Omega}$ be the mapping defined by the equality $\tilde{\sigma}(t, (\lambda, \omega)) = (\lambda, \sigma(t, \omega))$ for every $t \in \mathbb{T}, \lambda \in \Lambda$ and $\omega \in \Omega$. It is clear that the triplet $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ is a dynamical system on $\tilde{\Omega}$ and $\tilde{\varphi} : \mathbb{T}_+ \times W \times \tilde{\Omega} \mapsto C(W)$ $(\tilde{\varphi}(t, x, (\lambda, \omega)) := \varphi(t, x, \lambda, \omega))$ is the set-valued cocycle on $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ with fiber W. Under the conditions of Lemma 4.3 the set-valued cocycle $\tilde{\varphi}$ admits a maximal compact invariant set $\{I_{\tilde{\omega}} \mid \tilde{\omega} \in \tilde{\Omega}\}$ (where $I_{\tilde{\omega}} = I_{(\lambda,\omega)} = I^{\lambda}_{\omega}$) because

$$\bigcup\{I_{\tilde{\omega}} \mid \tilde{\omega} \in \tilde{\Omega}\} = \bigcup\{I^{\lambda}_{\omega} \mid \lambda \in \Lambda, \omega \in \Omega\} = \bigcup\{I^{\lambda} \mid \lambda \in \Lambda\}.$$

According to Lemma 4.1 the function $F : \tilde{\Omega} \mapsto C(W)$, defined by the equality $F(\lambda, \omega) = I^{\lambda}_{\omega}$ is upper semi-continuous and in particular the equality (12) takes place.

We assume that the equality (12) is not true, then there exist $\varepsilon_0 > 0,\ \lambda_0 \in \Lambda,\ \lambda_k \to \lambda_0,\ \omega_k \in \Omega$ and $x_k \in I^{\lambda_k}_{\omega_k}$ such that

$$\rho(x_k, I_{\lambda_0}) \geq \varepsilon_0. \tag{13}$$

Without loss of generality we can suppose that $\omega_k \to \omega_0,\ x_k \to x_0$ because the sets Ω and $\bigcup\{I_{\lambda} \mid \lambda \in \Lambda\}$ are compact. According to the inequality 13 we have

$$\rho(x_0, I_{\lambda_0}) \geq \varepsilon_0.$$

On the other hand $x_k \in I^{\lambda_k}_{\omega_k}$ and from the equality 12 we have

$$x_0 \in I^{\lambda_0}_{\omega_0} \subset I_{\lambda_0}$$

and, consequently,

$$\varepsilon_0 \leq \rho(x_0, I_{\lambda_0}) \leq \beta(I^{\lambda_0}_{\omega_0}, I_{\lambda_0}) = 0.$$

This contradiction shows that the equality (12) takes place. □

Corollary 4.4. *Under the conditions of Lemma 4.3 the equality*

$$\lim_{\lambda \to \lambda_0} \beta(I^{\lambda}_{\omega}, I^{\lambda_0}_{\omega}) = 0$$

takes place for each $\omega \in \Omega$.

Remark 4.5. *The article of Caraballo and Langa* [4] *contains a statement close to Corollary 4.4 in the case when the non-perturbed cocycle* φ_{λ_0} *is autonomous, i.e. the mapping* $\varphi_{\lambda_0} : \mathbb{T}_+ \times W \times \Omega \to W$ *does not depend on* $\omega \in \Omega$.

Lemma 4.6. *Let the conditions of Lemma 4.3 and additionally the following condition:*

5. *for certain* $\lambda_0 \in \Lambda$ *the application* $F : \Omega \mapsto C(W)$*, defined by the equality* $F(\omega) = I^{\lambda_0}_{\omega}$ *is continuous, i.e.* $\alpha(F(\omega), F(\omega_0)) \to 0$ *if* $\omega \to \omega_0$ *for every* $\omega_0 \in \Omega$*, where* α *is the full metric of Hausdorff, i.e.* $\alpha(A, B) = \max\{\beta(A, B), \beta(B, A)\}$

be fulfilled.

Then the equality

$$\lim_{\lambda\to\lambda_0}\sup_{\omega\in\Omega}\beta(I^{\lambda}_{\omega},I^{\lambda_0}_{\omega})=0 \tag{14}$$

takes place.

Proof. Suppose that the equality (14) is not true, then there exist $\varepsilon_0>0,\ \lambda_k\to\lambda_0$ and $\omega_k\in\Omega$ such that

$$\beta(I^{\lambda_k}_{\omega_k},I^{\lambda_0}_{\omega_k})\ge\varepsilon_0 \tag{15}$$

for all $k\in\mathbb{N}$.

On the other hand we have

$$\begin{aligned}\varepsilon_0 &\le \beta(I^{\lambda_k}_{\omega_k},I^{\lambda_0}_{\omega_k})\le\beta(I^{\lambda_k}_{\omega_k},I^{\lambda_0}_{\omega_0})+\beta(I^{\lambda_0}_{\omega_0},I^{\lambda_0}_{\omega_k})\\ &\le\beta(I^{\lambda_k}_{\omega_k},I^{\lambda_0}_{\omega_0})+\alpha(I^{\lambda_0}_{\omega_k},I^{\lambda_0}_{\omega_0}).\end{aligned} \tag{16}$$

According to Lemma 4.3 (see the equality (11)) the equality

$$\lim_{k\to+\infty}\beta(I^{\lambda_k}_{\omega_k},I^{\lambda_0}_{\omega_0})=0 \tag{17}$$

takes place. Under condition 5. of Lemma 4.6 we have

$$\lim_{k\to+\infty}\alpha(I^{\lambda_0}_{\omega_k},I^{\lambda_0}_{\omega_0})=0. \tag{18}$$

From (16) - (18) passing to the limit as $k\to+\infty$ we obtain $\varepsilon_0\le 0$. This contradiction shows that the equality (14) takes place. □

Definition 4.7. *The family of set-valued cocycle $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is called collectively compact dissipative (uniformly collectively compact dissipative), if there exists a nonempty compact set $K\subseteq W$ such that*

$$\lim_{t\to+\infty}\sup\{\beta(U_\lambda(t,\omega)M,K)\mid\omega\in\Omega\}=0\qquad\forall\ \lambda\in\Lambda \tag{19}$$

$$(\textit{respectively}\ \lim_{t\to+\infty}\sup\{\beta(U_\lambda(t,\omega)M,K)\mid\omega\in\Omega,\ \lambda\in\Lambda\}=0)$$

for all $M\in C(W)$, where $U_\lambda(t,\omega)=\varphi_\lambda(t,\cdot,\omega)$.

Lemma 4.8. *The following conditions are equivalent:*

1. *the family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is collectively compact dissipative;*
2. (a) *every set-valued cocycle φ_λ $(\lambda\in\Lambda)$ is compact dissipative;*
   (b) *the set $\bigcup\{I^\lambda\mid\lambda\in\Lambda\}$ is precompact.*

Proof. According to the equality (19) every set-valued cocycle φ_λ $(\lambda\in\Lambda)$ is compact dissipative and $\bigcup\{I^\lambda\mid\lambda\in\Lambda\}\subseteq K$.

Suppose that the conditions a. and b. hold. Let $K=\overline{\bigcup\{I^\lambda\mid\lambda\in\Lambda\}}$, then the equality (19) takes place. □

Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued cocycles on $(\Omega,\mathbb{T},\sigma)$ with fiber W and $\tilde\Omega=\Omega\times\Lambda$. On $\tilde\Omega$ we define a dynamical system $(\tilde\Omega,\mathbb{T},\tilde\sigma)$ by equality $\tilde\sigma(t,(\omega,\lambda))=(\sigma(t,\omega),\lambda)$ for all $t\in\mathbb{T},\omega\in\Omega$ and $\lambda\in\Lambda$. By family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is generated a set-valued cocycle $\tilde\varphi$ on $(\tilde\Omega,\mathbb{T},\tilde\sigma)$ with fiber W, defined in the following way : $\tilde\varphi(t,w,(\omega,\lambda))=\varphi_\lambda(t,w,\omega)$ for all $t\in\mathbb{T}_+,w\in W,\omega\in\Omega$ and $\lambda\in\Lambda$, if the mapping $\tilde\varphi$ is β-continuous.

Lemma 4.9. *Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued cocycles on $(\Omega,\mathbb{T},\sigma)$ with fiber W and the mapping $\tilde\varphi$ is β-continuous. Then the following conditions are equivalent:*

1. *the family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly collectively compact dissipative;*
2. *the set-valued cocycle $\tilde\varphi$ is compact dissipative.*

Proof. This assertion follows from the equality

$$\sup\{\beta(\tilde U(t,\tilde\omega)M,K)\mid\tilde\omega\in\tilde\Omega\}=\sup\{\beta(U_\lambda(t,\omega)M,K)\mid\omega\in\Omega,\ \lambda\in\Lambda\},$$

where $\tilde U(t,\tilde\omega)=\tilde\varphi(t,\cdot,\tilde\omega)$, and from the corresponding definitions. □

Theorem 4.10. *Let Λ be a compact metric space and $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of uniformly collectively compact dissipative set-valued cocycles on $(\Omega,\mathbb{T},\sigma)$ with fiber W, then the following assertions take place:*

1. *every set-valued cocycle φ_λ $(\lambda\in\Lambda)$ is compact dissipative;*
2. *the family of compacts $\{I^\lambda_\omega\mid\omega\in\Omega\}=I^\lambda$ is a Levinson's centre (compact global attractor) of set-valued cocycle φ_λ, where $I^\lambda_\omega=I_{(\omega,\lambda)}$ and $I=\{I_{(\omega,\lambda)}\mid(\omega,\lambda)\in\tilde\Omega\}$ is a Levinson's centre of set-valued cocycle $\tilde\varphi$;*
3. *the set $\bigcup\{I^\lambda\mid\lambda\in\Lambda\}$ is precompact.*

Proof. Consider the set-valued cocycle $\tilde\varphi$ generated by the family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$. According to Lemma 4.9 $\tilde\varphi$ is compact dissipative and by virtue of the Theorem 3.9 the following assertions take place:

(1) $I_{\tilde\omega}=\Omega_{\tilde\omega}(K)\neq\emptyset$, is compact, $I_{\tilde\omega}\subseteq K$ and

$$\lim_{t\to+\infty}\beta(\tilde U(t,\tilde\omega^{-t})M,I_{\tilde\omega})=0 \tag{20}$$

for every $\tilde\omega\in\tilde\Omega$, where

$$\Omega_{\tilde\omega}(K)=\bigcap_{t\ge0}\overline{\bigcup_{\tau\ge t}\tilde U(\tau,\tilde\omega^{-\tau})K}, \tag{21}$$

$\tilde\omega^{-\tau}=\tilde\sigma(-\tau,\tilde\omega)$ and K is a nonempty compact appearing in the equality (19);

(2) $\tilde U(t,\tilde\omega)I_{\tilde\omega}=I_{\tilde\omega t}$ for all $\tilde\omega\in\tilde\Omega$ and $t\in\mathbb{T}_+$;

(3) the set $I=\bigcup\{I_{\tilde\omega}\mid\tilde\omega\in\tilde\Omega\}$ is compact.

To complete the proof we note that from the collective compact dissipativeness of the family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ results that every set-valued cocycle φ_λ will be compact dissipative. Let $\{I^\lambda_\omega \mid \omega\in\Omega\} = I^\lambda$ be a Levinson's centre of the set-valued cocycle φ_λ, then according to Theorem 3.9 the equality

$$I^\lambda_\omega = \bigcap_{t\ge 0}\overline{\bigcup_{\tau\ge t} U_\lambda(\tau,\omega^{-\tau})K}, \tag{22}$$

takes place.

From (21) and (22) follows that $I^\lambda_\omega = \Omega_{\tilde\omega}(K) = I_{\tilde\omega}$ and, consequently, $I^\lambda = \bigcup\{I^\lambda_\omega \mid \omega\in\Omega\} \subseteq \bigcup\{I^\lambda_\omega \mid \omega\in\Omega,\ \lambda\in\Lambda\} = I$ for all $\lambda\in\Lambda$. Thus $\bigcup\{I^\lambda \mid \lambda\in\Lambda\}\subseteq I$ and, consequently, it is compact. The theorem is proved. □

Definition 4.11. *The family $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ of set-valued dynamical systems is called collectively (uniformly collectively) asymptotic compact if for every bounded positive invariant set $M\subseteq X$ there exists a nonempty compact K such that*

$$\lim_{t\to+\infty}\beta(\pi^t_\lambda M, K) = 0 \qquad \forall\ \lambda\in\Lambda \tag{23}$$

$$(\lim_{t\to+\infty}\sup_{\lambda\in\Lambda}\beta(\pi^t_\lambda M, K) = 0).$$

Definition 4.12. *The bounded set $K\subset X$ is called absorbing (uniformly absorbing) for the family $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ of set-valued dynamical systems if for any bounded subset $B\subset X$ there exists a number $L = L(\lambda,B) > 0$ $(L = L(B) > 0)$ such that $\pi^t_\lambda B\subseteq K$ for all $t\ge L(\lambda,B)$ $(t\ge L(B))$ and $\lambda\in\Lambda$.*

Theorem 4.13. *Let Λ be a complete metric space. If the family $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ of set-valued dynamical systems admits an absorbing bounded set $K\subset X$ and it is collectively asymptotic compact, then $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ admits a global compact attractor, i.e. there exists a nonempty compact set $K\subset X$ such that*

$$\lim_{t\to+\infty}\beta(\pi^t_\lambda B, K) = 0 \tag{24}$$

for all $\lambda\in\Lambda$ and bounded $B\subset X$.

Proof. Let the family $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ of set-valued dynamical systems be collectively asymptotic compact and a bounded M be its absorbing set. According to Theorem 1.1.1 Cheban and Fakeeh [6] the nonempty set $K = \Omega(M)$ is compact and the equality (24) takes place. The theorem is proved. □

Theorem 4.14. *Let Λ be a complete compact metric space. If the family $\{(X,\mathbb{T}_+,\pi_\lambda)\}_{\lambda\in\Lambda}$ of set-valued systems admits a uniformly absorbing bounded set $K\subset X$ and it is uniformly collectively asymptotic compact, then $\{(X,\mathbb{T}_+, pi_\lambda)\}_{\lambda\in\Lambda}$ admits a uniform compact global attractor, i.e. there exists a nonempty compact set $K\subset X$ such that*

$$\lim_{t\to+\infty}\sup_{\lambda\in\Lambda}\beta(\pi^t_\lambda B, K) = 0 \tag{25}$$

for all bounded $B\subset X$.

Proof. Consider the set-valued dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ on $\tilde{X} = X \times \Lambda$ defined by equality $\tilde{\pi}(t,(x,\lambda)) = (\pi_\lambda(t,x),\lambda)$ for all $t \in \mathbb{T}_+, x \in X$ and $\lambda \in \Lambda$. We note that under the conditions of Theorem 4.14 the bounded set $K \times \Lambda$ is absorbing for dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ if the set K is uniformly absorbing for the family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda\in\Lambda}$ and $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ is asymptotically compact. According to Theorem 2.2.5 (Cheban and Fakeeh [6]) the dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ admits a compact global attractor $\tilde{K} \subset \tilde{X} = X \times \Lambda$. To finish the proof it is sufficient to note that the set $K = pr_1\tilde{K} \subset X$ is compact and

$$\sup_{\lambda\in\Lambda} \beta(\pi^t_\lambda B, K) \leq \beta(\tilde{\pi}^t_\lambda B, K_0) \to 0$$

as $t \to +\infty$, where $K_0 = K \times \Lambda \supset \tilde{K}$, for all bounded subset $B \subset X$. □

Let φ be a set-valued cocycle on $(\Omega, \mathbb{T}, \sigma)$ with fiber W and $(X, \mathbb{T}_+, \pi)$ be a skew-product dynamical system, where $X = W \times \Omega$ and $\pi(t,(w,\omega)) = (\varphi(t,w,\omega), \omega t)$ for all $t \in \mathbb{T}_+, w \in W$ and $\omega \in \Omega$.

Definition 4.15. *The set-valued cocycle φ is called asymptotically compact (a family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is called collectively asymptotically compact) if a skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ (a family of skew-product dynamical systems $(X, \mathbb{T}_+, \pi_\lambda)_{\lambda\in\Lambda}$) is asymptotically compact.*

Theorem 4.16. *Let Ω and Λ be compact metric spaces, W be a finite-dimensional Banach space, $W_r := \{u \in W \ : \ |u| \geq r]\}$ $(r > 0)$ and $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued cocycles on $(\Omega, \mathbb{T}, \sigma)$ with fiber W. If there exist $r > 0$ and the function $V_\lambda : W \times \Omega \to \mathbb{R}_+$ for all $\lambda \in \Lambda$, with the following properties:*

1. *$a(|x|) \leq V_\lambda(x) \leq b(|x|)$ $(a, b \in \mathfrak{A},\ Im(a) = Im(b))$ for all $x \in X_r$ and $\lambda \in \Lambda$;*
2. *$V'_\lambda((u,\omega)) \leq -c(|u|)$ for all $u \in W_r$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and positive on $[r, +\infty)$, $V'_\lambda(u,\omega) := \limsup_{t\downarrow 0} t^{-1}[V_\lambda(\varphi(t,u,\omega)) - V_\lambda(u,\omega)]$ for $\mathbb{T}_+ = \mathbb{R}_+$ and $V'_\lambda(u,\omega) := V_\lambda(\varphi(1,u,\omega)) - V_\lambda((u,\omega))$ for $\mathbb{T}_+ = \mathbb{Z}_+$.*

Then every set-valued cocycle φ_λ $(\lambda \in \Lambda)$ admits a uniform compact global attractor I^λ $(\lambda \in \Lambda)$ and the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is precompact.

Proof. Let $X = W \times \Omega$ and $(X, \mathbb{T}, \pi_\lambda)$ be a skew-product dynamical system, generated by the set-valued cocycle φ_λ , then (X, h, Ω) , where $h = pr_2 : X \to \Omega$, is a trivial fibering with fiber W. Under the conditions of Theorem 4.16 and according to Theorem 2.3 the non-autonomous set-valued dynamical system $\langle (X, \mathbb{T}_+, \pi_\lambda), (\Omega, \mathbb{T}, \sigma), h\rangle$ admits a compact global attractor J^λ and according to Theorem 3.9 the set-valued cocycle φ_λ admits a compact global attractor $I^\lambda = \{I^\lambda_\omega \mid \omega \in \Omega\}$, where $I^\lambda_\omega = pr_1 J^\lambda_\omega$ and $J^\lambda_\omega = pr_2^{-1}(\omega) \bigcap J^\lambda$.

Let $\tilde{\Omega} = \Omega \times \Lambda$, $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ be a dynamical system on $\tilde{\Omega}$ defined by the equality $\tilde{\sigma}(t,(\omega,\lambda)) = (\sigma(t,\omega),\lambda)$ (for all $t \in \mathbb{T}, \omega \in \Omega$ and $\lambda \in \Lambda$), $\tilde{X} = W \times \tilde{\Omega}$ and $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ be an autonomous dynamical system defined by equality

$\tilde{\pi}(t,(u,\tilde{\omega})) = (\varphi_\lambda(t,u,\omega),(\omega t,\lambda))$ for all $\tilde{\omega} = (\omega,\lambda) \in \tilde{\Omega} = \Omega \times \Lambda$. Note that the triplet $(\tilde{X},h,\Omega)$, where $h = pr_2 : \tilde{X} \to \tilde{\Omega}$, is a trivial fibering with fiber W, $\langle(\tilde{X},\mathbb{T}_+,\tilde{\pi}),(\tilde{\Omega},\mathbb{T},\tilde{\sigma}),h\rangle$ is a non-autonomous set-valued dynamical system. The function $\tilde{V} : \tilde{X}_r = W_r \times \tilde{\Omega} \to \mathbb{R}_+$, defined by the equality $\tilde{V}(\tilde{x}) = V_\lambda(u,\omega)$ for all $\tilde{x} = (u,(\omega,\lambda)) \in \tilde{X}_r$ under the conditions of Theorem 4.16 verifies all the conditions of Theorem 2.3 and, consequently, the dynamical system $(\tilde{X},\mathbb{T}_+,\tilde{\pi})$ admits a compact global attractor. To finish the proof of the theorem it is sufficient to note that if the dynamical system $(\tilde{X},\mathbb{T}_+,\tilde{\pi})$ admits a compact global attractor $\tilde{J}$, then the family of set-valued cocycles $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly collectively compact dissipative and according to Theorem 4.10 the set $I = \bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is precompact, where $I^\lambda = \{I^\lambda_\omega \mid \omega \in \Omega\}$ is the compact global attractor of set-valued cocycle φ_λ. The theorem is proved. □

5. Connectedness

Definition 5.1. *We will say that the space W possesses the property (S) if for every compact $K \in C(W)$ there exists a compact connected set $V \in C(W)$ such that $K \subseteq V$.*

Remark 5.2. *1. It is clear that if space W possesses the property (S), then it is connected. The inverse statement generally speaking is not true.*

2. Every linear vectorial topological space W possesses the property (S), because the set $V(K) = \{\lambda x + (1-\lambda)y \mid x,y \in K, \lambda \in [0,1]\}$ is connected, compact and $K \subseteq V(K)$.

If $M \subseteq W$, then we denote by

$$\Omega_\omega(M) = \bigcap_{t\ge 0}\overline{\bigcup_{\tau\ge t}\varphi(\tau,M,\omega^{-\tau})}$$

for each $\omega \in \Omega$.

Lemma 5.3. *(Cheban and Schmalfuss* [14]*) Following statements take place:*

1. *point $w \in \Omega_y(M)$ if and only if, there exists $t_n \to +\infty$, $\{x_n\} \subseteq M$ and $w_n \in U(t_n,y^{-t_n})x_n$ such that $w = \lim\limits_{n\to+\infty} w_n$;*
2. *$U(t,y)\Omega_y(M) \subseteq \Omega_{yt}(M)$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3. *if there exists non-empty compact $K \subset W$ such that*

$$\lim_{t\to+\infty}\beta(\varphi(t,M,y^{-t}),K) = 0, \tag{26}$$

then $\Omega_y(M) \neq \emptyset$, is compact,

$$\lim_{t\to+\infty}\beta(\varphi(t,M,y^{-t}),\Omega_y(M)) = 0, \tag{27}$$

and

$$U(t,y)\Omega_y(M) = \Omega_{yt}(M) \tag{28}$$

for all $y \in Y$ and $t \in \mathbb{T}_+$.

Lemma 5.4. *Suppose that the set-valued cocycle φ admits a compact pullback attractor $\{I_\omega \mid \omega \in \Omega\}$, then the following assertions take place:*

a. $\emptyset \neq \Omega_\omega(M) \subseteq I_\omega$ *for every* $M \in C(W)$ *and* $\omega \in \Omega$*;*
b. *the family* $\{\Omega_\omega(M) \mid \omega \in \Omega\}$ *is compact and invariant w.r.t. cocycle* φ *for every* $M \in C(W)$*;*
c. *if* $I = \bigcup\{I_\omega \mid \omega \in \Omega\} \subseteq M$*, then the following inclusion* $I_\omega \subseteq \Omega_\omega(M)$ *takes place for every* $\omega \in \Omega$.

Proof. The first and second assertions follow from the definition of pullback attractor and from the equalities (26)-(27).

Let I be a subset of M, then

$$I_\omega = \varphi(t, I_{\omega^{-t}}, \omega^{-t}) \subseteq \varphi(t, I, \omega^{-t}) \subseteq \varphi(t, M, \omega^{-t}) \tag{29}$$

and according to the equality (26) we have $I_\omega \subseteq \Omega_\omega(M)$ for each $\omega \in \Omega$. □

Lemma 5.5. *(Cheban and Fakeeh* [6]*, Fang* [22]*) Let X be a complete metric space and $f : X \to C(X)$ be a mapping such that $f(x)$ is a nonempty connected compact for all $x \in X$. Suppose that the mapping f is upper semi-continuous on X, i.e.*

$$\lim_{x \to x_0} \beta(f(x), f(x_0)) = 0$$

for every point $x_0 \in X$. If the space X is connected, then the set $f(X)$ is connected too.

Theorem 5.6. *Let W be a space with the property (S), the set-valued cocycle φ admits a compact pullback attractor $\{I_\omega \mid \omega \in \Omega\}$ and $\varphi(t, u, \omega)$ is a compact connected subset of W for all $t \in \mathbb{T}_+$ and $(u, \omega) \in W \times \Omega$, then:*

(1) *the set I_ω is connected for every $\omega \in \Omega$;*
(2) *if the space Ω is connected, then the set $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is also connected.*

Proof. 1. Since the equality (5) takes place and the space W possesses the property (S), then there exists a connected compact $V \in C(W)$ such that $I \subseteq V$ and

$$\lim_{t \to +\infty} \beta(\varphi(t, V, \omega^{-t}), I_\omega) = 0, \tag{30}$$

for every $\omega \in \Omega$. We shall show that the set I_ω is connected. If we suppose that it is not true, then there are $A_1, A_2 \neq \emptyset$, closes and $A_1 \bigsqcup A_2 = I_\omega$. Let $0 < \varepsilon_0 < d(A_1, A_2)$ and $L = L(\varepsilon_0) > 0$ be such that

$$\beta(\varphi(t, V, \omega^{-t}), I_\omega) < \frac{\varepsilon_0}{3} \tag{31}$$

for all $t \geq L(\varepsilon_0)$.

We note that in view of Lemma 5.5 the set $\varphi(t, V, \omega^{-t})$ is connected and according to the inclusion (29) and the inequality (31) the following condition

$$\varphi(t, V, \omega^{-t}) \bigcap (W \setminus [B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})]) \neq \emptyset$$

is fulfilled for every $t \geq L(\varepsilon_0)$ and $\omega \in \Omega$, where $B(A,\varepsilon) = \{u \in W | \rho(u,A) < \varepsilon\}$. Then there exists $t_n \to +\infty$ and $u_n \in W$ such that

$$u_n \in \varphi(t_n, V, \omega^{-t_n}) \bigcap (W \setminus [B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})]). \tag{32}$$

According to the equality (30) it is possible to suppose that the sequence $\{u_n\}$ is convergent. We denote by $u = \lim_{n\to+\infty} u_n$, then from Lemma 5.3 follows that $u \in \Omega_\omega(V)$. Since $I \subseteq V$, then according to Lemma 5.3 we have $u \in \Omega_\omega(V) \subseteq I_\omega \subseteq I$. On the other hand according to (32) we have $u \notin B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})$. This contradiction shows that the set I_ω is connected.

2. Let the space Ω be connected. According to Lemma 4.3 the function $F : \Omega \mapsto C(W)$, defined by equality $F(\omega) = I_\omega$ is upper semi-continuous and from the lemma 5.5 follows that the set $I = \bigcup \{I_\omega \mid \omega \in \Omega\} = F(\Omega)$ is connected. □

Corollary 5.7. *Let W be a metric space with the property (S), the set-valued cocycle φ admit a compact global attractor $\{I_\omega \mid \omega \in \Omega\}$ and $\varphi(t,u,\omega)$ is a compact connected subset of W for all $t \in \mathbb{T}_+$ and $(u,\omega) \in W \times \Omega$, then:*

(1) *the set I_ω is connected for every $\omega \in \Omega$;*
(2) *if the space Ω is connected, then the set $I = \bigcup \{I_\omega \mid \omega \in \Omega\}$ also is connected.*

Proof. This affirmation follows from Theorems 3.10, 5.6 and Lemma 4.1. □

6. Some applications

Here follow some examples of set-valued cocycles playing an important role in the study of differential equations (without uniqueness) and differential inclusions.

Example 6.1. *(Ordinary differential equations without uniqueness).* Let E^n be a n-dimensional real or complex Euclidean space with norm $|\cdot|$, Ω be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω. Will note by $C(\Omega \times E^n, E^n)$ a set of all continuous mappings $f : \Omega \times E^n \to E^n$ allotted with uniform convergence topology on compacts from $\Omega \times E^n$. Consider a differential equation

$$u' = f(\omega t, u), \ (\omega \in \Omega) \tag{33}$$

where $f \in C(\Omega \times E^n, E^n)$ and $\omega t := \sigma(t,\omega)$.

The function $f \in C(\Omega \times E^n, E^n)$ is called regular, if for any $v \in E^n$ and $\omega \in \Omega$ the equation (33) admits at least one solution $\varphi_{(v,\omega)}(t)$ defined on R_+ and passing through the point v when $t = 0$. Note by $\Phi_{(v,\omega)}$ a set of all solutions of equation (33), defined on R_+ and passing through the point v for $t = 0$.

Let $f \in C(\Omega \times E^n, E^n)$ be regular, denote by $\varphi(t,v,\omega) = \{\varphi_{(v,\omega)}(t) \mid \varphi_{(v,\omega)} \in \Phi_{(v,\omega)}\}$, then $\varphi : \mathbb{R}_+ \times E^n \times \Omega \to C(E^n)$ and from the general properties of solutions of differential equations (see, for example, Hartman [24]) follows that the following conditions are fulfilled:

a. $\varphi(0, v, \omega) = v$ for all $v \in E^n, \omega \in \Omega$;
b. $\varphi(t+\tau, v, \omega) = \varphi(t, \varphi(\tau, v, \omega), \omega\tau)$ for all $v \in E^n, \omega \in \Omega$ and $t, \tau \in \mathbb{R}_+$;
c. mapping $\varphi : \mathbb{R}_+ \times E^n \times \Omega \to C(E^n)$ is $\beta-$ continuous.

Then the triplet $\langle E^n, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ is a set-valued cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber E^n. Thus the differential equation (33) with regular right hand side $f \in C(\Omega \times E^n, E^n)$ naturally generates a set-valued cocycle.

Example 6.2. *(Differential inclusions).* Let $C_V(E^n)$ be a family of all convex compacts from E^n, and by $C(\Omega\times E^n, C_V(E^n))$ we denote the set of all continuous in Hausdorff's metric (see Sibirskii and Shube [29], Fillipov [21], Borisovich, Ghelman, Myshkis and Obukhovsky [1]) mappings $f : \Omega \times E^n \to C_V(E^n)$, allotted by uniform convergence topology on compacts. Consider the differential inclusion

$$u' \in f(\omega t, u), \ (\omega \in \Omega) \tag{34}$$

where $f \in C(\Omega \times E^n, C_V(E^n))$.

We will call the function $f \in C(\Omega \times E^n, C_V(E^n))$ regular, if for every inclusion (34) the conditions of existence and non-local extensibility to the right (i.e. for any $\omega \in \Omega$ and $v \in E^n$ there exists at least one solution $\varphi_{(v,\omega)}(t)$ of the inclusion (34) passing through the point v for $t = 0$ and defined on $\mathbb{R}_+$) are fulfilled.

Let $f \in C(\Omega \times E^n, C_V(E^n))$ be regular. We set $\varphi(t, v, \omega) = \{\varphi_{(v,\omega)}(t) : \varphi_{(v,\omega)} \in \Phi_{(v,\omega)}\}$, where $\Phi_{(v,\omega)}$ is the set of all solutions of inclusion (34) defined on $\mathbb{R}_+$ and passing through the point v for $t = 0$. From the general properties of the differential inclusions Fillipov [21] follows that the following properties take place:

a. $\varphi(0, v, \omega) = v$ for all $v \in E^n, \omega \in \Omega$;
b. $\varphi(t+\tau, v, \omega) = \varphi(t, \varphi(\tau, v, \omega), \omega\tau)$ for all $v \in E^n, \omega \in \Omega$ and $t, \tau \in \mathbb{R}_+$;
c. mapping $\varphi : \mathbb{R}_+ \times E^n \times \Omega \to C(E^n)$ is β - continuous.

Then the triplet $\langle E^n, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ is a set-valued cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber E^n. Thus the differential inclusion (34) with regular right hand side $F \in C(\Omega \times E^n, C_V(E^n))$ naturally generates a set-valued cocycle.

Let $A \in C(E^n)$ and $|A| := \max\{|u| \mid u \in A\}$.

Lemma 6.3. *Let Y be a metric space and $F : Y \to C(E^n)$ be continuous with respect to Hausdorff metric (α-continuous). Then the function $m : Y \to \mathbb{R}_+$, defined by the equality $m(y) := |F(y)|$ (for all $y \in Y$) is continuous.*

Proof. Let $y \in Y$and $y_k \to y$. Consider the sequence $\{m(y_k)\}$. Since the set $F(y_k)$ is compact, then there exists a point $u_k \in F(y_k)$ such that $m(y_k) = |u_k|$. By the continuity of mapping F (with respect to Hausdorff metric) the set $F(\{y_k\}) := \bigcup_{k\in\mathbb{N}} F(y_k)$ is precompact in E^n. Let $\tilde{u}$ be a limit point of the sequence $\{u_k\}$ and $\tilde{m} := |\tilde{u}|$. We will show that $\tilde{m} = m(y)$. At first we note that by the α-continuity of the mapping $F : Y \to C(E^n)$ we have $\tilde{u} \in F(y)$ and, consequently, $\tilde{m} = |\tilde{u}| \le m(y)$. Now we will show that $\tilde{m} \ge m(y)$. Indeed, if we suppose that it is not true, then there exists a $\varepsilon_0 > 0$ such that $m(y) = \tilde{m} + 2\varepsilon_0$. For ε_0 there exists a number $k_0 \in \mathbb{Z}$ such that $m(y) - 3\varepsilon_0 = \tilde{m} - \varepsilon_0 < m(y_{k_0}) < \tilde{m} - \varepsilon_0$. Thus

$$|u| \le m(y_{k_0}) < m(y) - \varepsilon_0 \ (\forall u \in F(y_{k_0})). \tag{35}$$

On the other hand the mapping F is α-continuous and $y_k \to y$ and, consequently, there exists $u_{y_{k_0}} \in F(y_{k_0})$ such that $||u_{y_{k_0}}| - |u_y|| < \varepsilon_0$, where $u_y \in F(y)$ such that $|u_y| = m(y)$, and consequently,

$$|u_{y_{k_0}}| > m(y) - \varepsilon_0. \tag{36}$$

The inequality (35) and (36) are contradictory. The obtained contradiction proves the required statement. Since the sequence $\{m(y_k)\}$ is bounded and admits a unique limit point $m(y)$ it converges to $m(y)$. The Lemma is proved. □

6.1. **Quasi-homogeneous systems.** Let $G \subseteq E^l$, the function $f \in C(E^n \times G, C(E^n))$ is called (Cheban [16, 17]) homogeneous of order m with respect to variable $u \in E$ if the equality $f(\lambda u, \omega) = \lambda^m f(u, \omega)$ takes place for all $\lambda \ge 0, u \in E^n$ and $\omega \in G$.

The set-valued dynamical system $(X, \mathbb{R}_+, \pi)$ is said to be homogeneous of order $m \in \mathbb{R}_+$, if for any $x \in X, t \in \mathbb{R}_+$ and $\lambda > 0$ the equality $\pi(t, \lambda x) = \lambda\pi(\lambda^{m-1}t, x)$ takes place.

Theorem 6.4. *(Cheban [13]) Let X be a Banach space. For a set-valued homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ the following assertions are equivalent:*

1. *the trivial motion of $(X, \mathbb{R}_+, \pi)$ is asymptotically stable;*
2. *there exist positive numbers a and b such that*

$$|\pi(t, x)| \le (a|x|^{1-m} + bt)^{\frac{1}{1-m}} \tag{37}$$

for all $t \ge 0$ and $x \in X$.

Theorem 6.5. *For a set-valued homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ and th mapping $\pi : \mathbb{R}_+ \times X \to C(X)$ is α-continuous, then the following assertions are equivalent:*

1. *there exist positive numbers a and b such that (37) holds for all $t \ge 0$ and $x \in X$;*
2. *for all $k > m - 1$ there exists a continuous function $V : X \to \mathbb{R}_+$ with the following properties:*
   2.1. *$V(\lambda x) = \lambda^{k-m+1} V(x)$ for all $\lambda \ge 0$ and $x \in X$;*
   2.2. *$\alpha|x|^{k-m+1} \le V(x) \le \beta|x|^{k-m+1}$ for all $x \in X$, where α and β are certain positive numbers;*
   2.3. *$V'_\pi(x) \le -|x|^k$ for all $x \in X$, where $V'_\pi(x) = \frac{d}{dt}V(\pi(t, x)) \mid_{t=0}$ for $\mathbb{T} = \mathbb{R}_+$ and $V'_\pi(x) = V(\pi(1, x)) - V(x)$ for $\mathbb{T} = \mathbb{Z}_+$.*

Proof. We will show that from 1. results 2.. Let a and b are positive numbers, such that the inequality (37) takes place, then for each $k > m-1$ we define the function $V : X \to \mathbb{R}_+$ by equality

$$V(x) = \int_0^{+\infty} |\pi(t,x)|^k dt. \tag{38}$$

First of all we note that by equality (38) the function $V : X \to \mathbb{R}_+$ is defined correctly because the integral, which figures in the right hand side of equation (38) is convergent, moreover it is uniformly convergent w.r.t. x on every bounded set from X. Indeed, since

$$|\pi(t,x)|^k \leq (a|x|^{1-m} + bt)^{\frac{k}{1-m}}, \tag{39}$$

$$\int_0^{+\infty} |(a|x|^{1-m} + bt)^{\frac{k}{1-m}} dt = \frac{1}{b} \int_{a|x|^{1-m}}^{+\infty} \tau^{\frac{k}{1-m}} d\tau \tag{40}$$

and $\frac{k}{1-m} < -1$, then the integral (40) is convergent, moreover the convergence is uniform on every bounded set from X.

We will show that the function V, defined by equality (38), is our unknown function. Since the mapping $\pi : \mathbb{R}_+ \times X \to C(X)$ is α-continuous according to Lemma 6.3 the function $m : (t,x) \to |\pi(t,x)|$ is continuous. The continuity of V results from the continuity of m and uniform convergence of integral (40) w.r.t. x on every bounded set from X. We now note that

$$V(\lambda x) = \int_0^{+\infty} |\pi(t,\lambda x)|^k dt = \int_0^{+\infty} \lambda^k |\pi(\lambda^{m-1}t, x)|^k dt$$

$$= \lambda^{k-m+1} \int_0^{+\infty} |\pi(\tau,x)|^k dt = \lambda^{k-m+1} V(x)$$

for all $\lambda > 0$ and $x \in X$. It is not difficult to show that the function V is positive definite. Since

$$V(x) = \int_0^{+\infty} |\pi(t,x)|^k, dt = 0 \tag{41}$$

if and only if $|\pi(t,x)| = 0$ for all $t \in \mathbb{R}_+$ and, consequently, $x = 0$. Let now

$$\alpha := \min_{|x|=1} V(x) \quad \text{and} \quad \beta := \max_{|x|=1} V(x),$$

then from the condition 2.1 follow that $\alpha|x|^{k-m+1} \leq V(x) \leq \beta|x|^{k-m+1}$ for all $x \in X$.

Finally, we note that

$$\frac{d}{dt} V(\varphi_x(t)) \leq -|\varphi_x(t)|^k \tag{42}$$

for all $\varphi \in \Phi_x$ and, consequently, $V'_\pi(x) \leq -|x|^k$ for all $x \in X$.

We will prove now that from condition 2. results 1.. In fact, let $\varphi \in \Phi_x$ we denote by $\psi(t) = V(\varphi_x(t))$, then in virtue of the condition 2.3 we have

$$\psi'(t) \leq -|\varphi_x(t)|^k \tag{43}$$

for all $t \geq 0$.

From the condition 2.2 we have $|\varphi_x(t)|^{k-m+1} \geq \frac{1}{\beta}\psi(t)$ and, consequently,

$$\psi'(t) \leq -\frac{1}{\beta^{\frac{k}{k-m+1}}}\psi(t)^{\frac{k}{k-m+1}}$$

for all $t \geq 0$. If $x \neq 0$, then $\psi(t) = V(\varphi_x(t)) > 0$ for all $t \geq 0$,therefore

$$V(\varphi_x(t)) \leq (V^{-\frac{m-1}{k-m+1}}(x) + \frac{m-1}{k-m+1}\frac{1}{\beta^{\frac{k}{k-m+1}}}t)^{\frac{1}{1-m}} \tag{44}$$

for all $x \in X$ and $t \geq 0$.

From the condition 2.2 and the inequality (44) results that $|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}}$ for all $x \in X$ and $t \geq 0$, where

$$a = (\alpha\beta)^{\frac{m-1}{k-m+1}} \quad \text{and} \quad b = (\alpha)^{\frac{m-1}{k-m+1}}(\beta)^{\frac{k}{k-m+1}}\frac{m-1}{k-m+1}.$$

The theorem is completely proved. □

Corollary 6.6. *Let X be a Banach space. For a set-valued homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ the following assertions are equivalent:*

1. *the trivial motion of the dynamical system $(X, \mathbb{R}_+, \pi)$ is uniform asymptotic stable;*
2. *there exist positive numbers a and b such that $|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}}$ for all $t \in \mathbb{R}_+$ and $x \in X$;*
3. *for every number $k > m-1$ there exists a continuous function $V : X \to \mathbb{R}_+$ which possesses properties 2.1-2.3 from the theorem 6.5.*

Proof. This assertion directly follows from Theorems 6.4 and 6.5. □

Theorem 6.7. *Let $f \in C^1(E^n, E^n)$, $\Phi \in C^1(G), \Omega \subseteq G$ be a compact invariant set of dynamical system*

$$\omega' = \Phi(\omega), \tag{45}$$

the function f be homogeneous (of order $m > 1$) and a zero solution of equation

$$u' = f(u) \tag{46}$$

be uniformly asymptotically stable. If $F \in C(E^n \times G, E^n)$ and

$$|F(u,\omega)| \leq c|u|^m$$

for all $|u| \geq r$ and $\omega \in \Omega$, where r and c are certain positive numbers, then there exists a positive number λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following assertions take place:

(1) *a set* $I^{\lambda}_{\omega} = \{u \in E^n \mid \sup\{|\varphi^{\lambda}_{(u,\omega)}(t)| : t \in \mathbb{R}\} < +\infty\}$ *is not empty, compact and connected for each* $\omega \in \Omega$*, where* $\varphi^{\lambda}_{(u,\omega)}(t)$ *is a solution (generally speaking not unique) of equation*

$$u' = f(u) + \lambda F(u, \omega t) \tag{47}$$

satisfying the initial condition $\varphi^{\lambda}_{(u,\omega)}(0) = u$*;*

(2) $\varphi_{\lambda}(t, I^{\lambda}_{\omega}, \omega) = I^{\lambda}_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$*, where* φ_{λ} *is a set-valued cocycle, generated by (47);*

(3) *the set* $I^{\lambda} = \bigcup\{I^{\lambda}_{\omega} \mid \omega \in \Omega\}$ *is compact and connected;*

(4) *the equalities*

$$\lim_{t\to+\infty} \beta(\varphi_{\lambda}(t, M, \omega_{-t}), I^{\lambda}_{\omega}) = 0 \tag{48}$$

and

$$\lim_{t\to+\infty} \beta(\varphi_{\lambda}(t, M, \omega), I^{\lambda}) = 0$$

take place for all $\lambda \in \Lambda, \omega \in \Omega$ *and bounded subset* $M \subseteq E$*.*

(5) *the set* $\bigcup\{I^{\lambda} \mid \lambda \in \Lambda\}$ *is compact;*

(6) *the equality*

$$\lim_{\lambda\to 0} \sup_{\omega\in\Omega} \beta(I^{\lambda}_{\omega}, 0) = 0$$

takes place.

Proof. Under the condition of Theorem 6.7 according to Theorems 6.5 and 5.7.2 from Cheban [19] by the equality

$$V(u) = \int_0^{+\infty} |\pi(t, u)|^k dt,$$

where $\pi(t, u)$ is a solution of equation (46) with condition $\pi(0, u) = u$, is defined a continuously differentiable function $V : E \to \mathbb{R}_+$, verifying the following conditions:

a. $V(\mu u) = \mu^{k-m+1} V(u)$ for all $\mu \geq 0$ and $u \in E^n$;

b. there exist positive numbers α and β such that $\alpha|u|^{k-m+1} \leq V(u) \leq \beta|u|^{k-m+1}$ for all $u \in E^n$;

c. $V'(u) = DV(u)f(u) = -|u|^k$ for all $u \in E^n$, where $DV(u)$ is a derivative of Frechet of function V in the point u.

Let us define a function $\mathfrak{V} : X \to \mathbb{R}_+$ ($X = E^n \times \Omega$) in the following way: $\mathfrak{V}(u, \omega) = V(u)$ for all $(u, \omega) \in X$. Note that

$$\mathfrak{V}'(u, \omega) = \frac{d}{dt} V(\varphi^{\lambda}_{(u,\omega)}(t))|_{t=0} = -|u|^k + DV(u)\lambda F(u, \omega)$$

and there exists $\lambda_0 > 0$ such that the inequality

$$\mathfrak{V}'(u, \omega) \leq -\nu|u|^k$$

takes place for all $\omega \in \Omega$ and $|u| \geq r$, where $\nu = 1 - \lambda_0 cL > 0$ (see the theorem 5.6.1 and the lemma 5.7.1 from Cheban [19]).

For finishing the proof of the theorem it is sufficient to refer to Theorem 4.16 and Lemma 4.6. □

Theorem 6.8. *Let $f \in C^1(E^n \times E^l, E^n), \Phi \in C^1(E^l)$ and $\Omega \subseteq G$ be a compact invariant set of dynamical system (45), the function f be homogeneous (of order $m = 1$) w.r.t. variable $u \in E$ and a zero solution of equation*

$$u' = f(u, \omega t) \quad (\omega \in \Omega) \tag{49}$$

be uniformly asymptotically stable. If $|F(u, \omega)| \leq c|u|$ for all $|u| \geq r$ and $\omega \in \Omega$, where r and c are certain positive numbers, then there exists a positive number λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following assertions take place:

1. *a set $I^\lambda_\omega = \{u \in E \mid \sup\{|\varphi^\lambda_{(u,\omega)}(t)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi^\lambda_{(u,\omega)}(t)$ is a solution (generally speaking not unique) of equation*

$$u' = f(u, \omega t) + \lambda F(u, \omega t) \tag{50}$$

   verifying the initial condition $\varphi^\lambda_{(u,\omega)}(0) = u$;

2. *$\varphi_\lambda(t, I^\lambda_\omega, \omega) = I^\lambda_{\sigma(t,\omega)}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where φ_λ is a set-valued cocycle, generated by the equation (50);*
3. *a set $I^\lambda = \bigcup\{I^\lambda_\omega \mid \omega \in \Omega\}$ is compact and connected;*
4. *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega^{-t}), I^\lambda_\omega) = 0$$

   and

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0 \tag{51}$$

   take place for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.

5. *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is precompact;*
6. *the equality*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I^\lambda_\omega, 0) = 0$$

   takes place.

Proof. The proof of this assertion is similar to the proof of Theorem 6.7 and it is based on Theorem 5.7.3 from Cheban [19]. □

6.2. **Dissipative differential inclusions.**

Theorem 6.9. *Let $f \in C^1(E^n \times \Omega, E^n)$ and there exist positive numbers a, b and $p \geq 2$ such that*

$$Re\langle f(u, \omega), u\rangle \leq -a|u|^p + b \tag{52}$$

for all $(u, \omega) \in E^n \times \Omega$, where $\langle \cdot, \cdot \rangle$ is a scalar product in the space E^n, then the equation (49) (the set-valued cocycle φ, generated by the inclusion (49)) admits a compact global attractor $\{I^0_\omega \mid \omega \in \Omega\}$.

Proof. Consider the function $V : E^n \times \Omega \to \mathbb{R}_+$, defined by the equality $V(u,\omega) := \frac{1}{2}|u|^2$, then it easy to verify that under the conditions of Theorem 6.9 we have

$$\frac{dV(\varphi_{(u,\omega)}(t))}{dt}|_{t=o} \le -a|u|^p + b$$

for all $(u,\omega) \in E^n \times \Omega$. To finish the proof of the theorem it is sufficient to apply Theorem 2.3. □

Theorem 6.10. *Let $f \in C(E^n \times \Omega, C(E^n))$ and there exist positive numbers a, b and $p \ge 2$ such that the inequality (52) holds. Suppose that $F \in C(E^n \times \Omega, C(E^n))$ and there exist positive numbers A and B such that*

$$Re\langle F(u,\omega), u\rangle \le A|u|^p + B \tag{53}$$

for all $(u,\omega) \in E^n \times \Omega$, then there exists a positive number $\lambda_0 < \frac{a}{A}$ such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following assertions take place:

1. *a set $I^\lambda_\omega = \{u \in E^n \mid \sup\{|\varphi^\lambda_{(u,\omega)}(t)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi^\lambda_{(u,\omega)}(t)$ is a solution of inclusion*

$$u' \in f(u,\omega t) + \lambda F(u,\omega t) \tag{54}$$

   verifying the initial condition $\varphi^\lambda_{(u,\omega)}(0) = u$;

2. *$\varphi_\lambda(t, I^\lambda_\omega, \omega) = I^\lambda_{\sigma(t,\omega)}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where φ_λ is a set-valued cocycle, generated by the inclusion (54);*
3. *a set $I^\lambda = \bigcup\{I^\lambda_\omega \mid \omega \in \Omega\}$ is compact and connected;*
4. *the equalities*

$$\lim_{t\to+\infty} \beta(\varphi_\lambda(t, M, \omega^{-t}), I^\lambda_\omega) = 0$$

   and

$$\lim_{t\to+\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0 \tag{55}$$

   take place for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.

5. *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*
6. *the equality*

$$\lim_{\lambda\to 0} \sup_{\omega\in\Omega} \beta(I^\lambda_\omega, 0) = 0$$

   takes place.

Proof. Consider the function $V : E^n \times \Omega \to \mathbb{R}_+$, defined by the equality $V(u,\omega) := \frac{1}{2}|u|^2$, then it easy to verify that under the conditions of Theorem 6.10 we have

$$\frac{dV(\varphi^\lambda_{(u,\omega)}(t))}{dt}|_{t=o} \le |u|^p(-a + \lambda_0 A + \frac{b + \varepsilon_0 B}{|u|^p})$$

for all $(u,\omega) \in E^n_r \times \Omega$, where $r > r_0 := \frac{b+\varepsilon_0 B}{a-\varepsilon_0 A}$. To finish the proof of the theorem it is sufficient to apply Theorem 4.16. □

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