GLOBAL ATTRACTORS FOR V-MONOTONE NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. This article is devoted to study of the compact global atrractors of V-momotone nonautonomous dynamical systems.We give a description of the structure of of compact global attractors of this classe of systems. Several applications of general results for different class of differential equations (ODEs, ODEs with impulse, some class of evolutionary partial differential equations) are given.

1. Introduction

The differential equations with monotone right hand side are one of the most studied class of nonlinear equations (see, for example, [4], [16], [20], [24], [25] and the literature quoted there).

By many authors it was studied the problem of existence of almost periodic solutions of monotone nonlinear almost periodic equation (see [12], [13], [15], [18], [19], [24], [25] and others).

Purpose of our article is the study of global attractors of general V- monotone nonautonomous dynamical systems and their applications to different class of differential equations (ODEs, ODEs with impulse, some class of evolution partial differential equations).

For autonomous equations analogical problem was studied before (see, for example, [2], [14],[23]),but for nonautonomous dynamical system this problem is considered in our paper for the first time.

2. Nonautonomous dynamical systems and skew-product flows

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Definition 2.1. *Let $\Theta = \{\theta_t\}_{t\in\mathbb{R}}$ be a group of mappings of Ω into itself, that is a continuous time autonomous dynamical system on a metric space Ω, and let $\mathbb{B}$ be a Banach space. Consider a continuous mapping $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{B} \to \mathbb{B}$ satisfying the properties*

$$\varphi(0, \omega, \cdot) = id_{\mathbb{B}} \quad \varphi(t + \tau, \omega, x) = \varphi(s, \theta_t\omega, \varphi(t, \omega, x))$$

for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in \mathbb{B}$. Such mapping φ (or more explicit $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{R}, \Theta)\rangle$) is called [1], [22] *a continuous cocycle or nonautonomous dynamical system (NDS) on $\Omega \times \mathbb{B}$.*

Example 2.2. *As an example, consider a parameterized differential equation*

$$\frac{dx}{dt} = F(\theta_t\omega, x) \quad (\omega \in \Omega)$$

on a Banach space $\mathbb{B}$ with $\Omega = C(\mathbb{R} \times \mathbb{B}, \mathbb{B})$. Define $\theta_t : \Omega \to \Omega$ by $\theta_t\omega(\cdot, \cdot) = \omega(t + \cdot, \cdot)$ for each $t \in \mathbb{R}$ and interpret $\varphi(t, \omega, x)$ as the solution of the initial value problem

$$\frac{d}{dt}x(t) = F(\theta_t\omega, x(t)), \quad x(0) = x. \tag{1}$$

Under appropriate assumptions on $F : \Omega\times\mathbb{B} \to \mathbb{B}$ (or even $F : \mathbb{R} \times\mathbb{B} \to \mathbb{B}$ with $\omega(t)$ instead of $\theta_t\omega$ in (1) to ensure forwards existence and uniqueness, (Θ, φ) generates a nonautonomous dynamical system on $\Omega \times \mathbb{B}$.

3. Attractors for nonautonomous dynamical systems

The usual concept of a global attractor for the autonomous semi-dynamical system π on the state space $X = \Omega \times \mathbb{B}$ can be used here.

Definition 3.1. *It is the maximal nonempty compact subset $\mathcal{A}$ of X =$\Omega \times \mathbb{B}$ which is π-invariant, that is*

$$\pi(t, \mathcal{A}) = \mathcal{A} \quad \text{for all} \quad t \in \mathbb{R}^+,$$

and attracts all compact subsets of $X = \Omega \times \mathbb{B}$, that is

$$\lim_{t\to\infty} \beta\left(\pi(t, \mathcal{D}), \mathcal{A}\right) = 0 \qquad \text{for all} \quad \mathcal{D} \in \mathcal{K}(\mathbb{X}),$$

where $C(X)$ is the space of all nonempty compact subsets of X and β is the Hausdorff semi-metric on $C(X)$.

4. Global attractors of V- monotone NDS.

Let Ω be a compact topological space , (E, h, Ω) is locally trivial Banach stratification [3] and $|\cdot|$ is the norm on (E, h, Ω) co-ordinate with the metric ρ on E (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$).

Definition 4.1. *Let us remember* [8],[5],[6], *that the triplet $\langle (E, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is called by a (general) nonautonomous dynamical system, if $h : E \to \Omega$ is a homomorphism of the dynamical system $(E, \mathbb{T}_1, \pi)$ on $(\Omega, \mathbb{T}_2, \Theta)$, where $\mathbb{T}_1$ and $\mathbb{T}_2$ ($\mathbb{T}_1 \subseteq \mathbb{T}_2$) are two subsemigroups of group $\mathbb{T}$.*

Example 4.2. *Let $\mathbb{T}_2$ be a subsemigroup of $\mathbb{T}$, $(\Omega, \mathbb{T}_2, \Theta)$ be a dynamical system on Ω and $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}_2, \Theta)\rangle$ be a cocycle over $(\Omega, \mathbb{T}_2, \Theta)$ with the fiber $\mathbb{B}$, $X := \Omega \times \mathbb{B}$, $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be a subsemigroup of $\mathbb{T}_2$, $(X, \mathbb{T}_1, \pi)$ be a semi-group dynamical system on X defined by the equality $\pi = (\varphi, \theta)$ (i.e. $\pi(t, (\omega, u)) := (\varphi(t, \omega, u), \theta_t \omega)$ for all $t \in \mathbb{T}_1$ and $(\omega, u) \in X$), then the triple $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ ($h = pr_2$) will be a nonautonomous dynamical system, generated by cocycle φ.*

Definition 4.3. *The cocycle $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ we will define by a compact dissipative one , if there is a nonempty compact $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(\varphi(t,\omega)M, K) \mid \omega \in \Omega \} = 0 \qquad (2)$$

for any $M \in C(\mathbb{B})$, where $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$.

If $M \subseteq \mathbb{B}$, then suppose

$$\Omega_\omega(M) = \bigcap_{t\ge 0} \overline{\bigcup_{\tau \ge t} \varphi(\tau, \theta_{-\tau}\omega, M)}$$

for every $\omega \in \Omega$.

Definition 4.4. *We will say , that the space X possesses the (S)-property, if for any compact $K \subseteq X$ there is a connected set $M \subseteq X$ such that $K \subseteq M$.*

Theorem 4.5. [9] *Let Ω be a compact metric space, $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ be a compact dissipative cocycle and K is the nonempty compact, figuring in the equality (2) , then :*

1. $I_\omega = \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t}\omega)K, I_\omega) = 0$ for every $\omega \in \Omega$;

2. $\varphi(t, \omega)I_\omega = I_{\theta_t \omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$;

3. $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t})M, I_\omega) = 0$ for all $M \in C(\mathbb{B})$ and $\omega \in \Omega$;

4. $\lim_{t\to+\infty} \sup\{\beta(\varphi(t, \omega_{-t})M, I) \mid \omega \in \Omega \} = 0$ for any $M \in C(\mathbb{B})$, where $I = \cup\{I_\omega \mid \omega \in \Omega \}$;

5. $I_\omega = pr_1 I_\omega$ for all $\omega \in \Omega$, where J is a Levinson centre of $(X, \mathbb{T}^+, \pi)$, and, hence, $I = pr_1 J$;

6. the set I is compact;

7. the set I is connected if one of the next two conditions is fulfilled :

a. $\mathbb{T}^+ = \mathbb{R}^+$ and the spaces $\mathbb{B}$ and Ω are connected;

b. $\mathbb{T}^+ = \mathbb{Z}^+$ and the space $\Omega \times \mathbb{B}$ possesses the (S)-property or it is connected and locally connected.

Definition 4.6. *A nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ is said to be uniformly stable in the positive direction on compacts of X* [7] *if, for arbitrary $\varepsilon > 0$ and $K \subseteq X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that inequality $\rho(x_1, x_2) < \delta$ ($h(x_1) = h(x_2)$) implies that $\rho(\pi^t x_1, \pi^t x_2) < \varepsilon$ for $t \in \mathbb{T}^+$.*

Definition 4.7. *A set $M \subset X$ is called minimal with respect to a dynamical system $(X, \mathbb{T}^+, \pi)$ if it is nonempty, closed and invariant and if no proper subset of M has these properties.*

Definition 4.8. *Denote by* $X\dot{\times}X = \{(x_1,x_2)\in X\times X \mid h(x_1)=h(x_2)\ \}$. *If there exists the function* $V : X\dot{\times}X \to \mathbb{R}_+$ *with the following properties:*

a. *V is continuous.*

b. *V is positive defined, i.e.* $V(x_1,x_2)=0$ *if and only if* $x_1=x_2$.

c. $V(x_1t,x_2t)\le V(x_1,x_2)$ *for all* $(x_1,x_2)\in X\dot{\times}X$ *and* $t\in\mathbb{T}_+$,
then the nonautonomous dynamical system $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ *is called (see* [12], [13] *and* [19], [25]*) V - monotone.*

Theorem 4.9. *Every V - monotone compact dissipative nonautonomous dynamical system* $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ *is uniformly stable in the positive direction on compacts from X.*

Corollary 4.10. *Let* $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ *be a V - monotone compact dissipative nonautonomous dynamical system and Ω be minimal, then:*

1. J is uniformly orbitally stable in the positive direction, i.e., for $\varepsilon>0$ *there is* $\delta(\varepsilon)>0$ *such that the inequality* $\rho(x,J_{h(x)})<\delta$ *implies that* $\rho(\pi^t x, J_{h(\pi^t x)})<\varepsilon$ *for* $t\ge 0$;

2. J is an attractor of compact sets from X, i.e., for $\varepsilon>0$ *and a compact* $K\subseteq X$, *there is* $L(\varepsilon,K)>0$ *such that* $\pi^t K_\omega\subseteq \tilde{B}(J_{\theta_t\omega},\varepsilon)$ *for* $\omega\in\Omega$ *and* $t\ge L(\varepsilon,K)$;

3. all motion on J can be continued to the left and J is bilaterally distal;

4. $J_\omega = X_\omega\bigcap J$ *for* $\omega\in\Omega$, *is a connected set if* X_ω *is connected, and for distinct* ω_1 *and* ω_2 *the sets* J_{ω_1} *and* J_{ω_2} *are homeomorphic;*

5. J is formed of recurrent trajectories, and two arbitrary points $x_1,x_2\in J_\omega$ $(\omega\in\Omega)$ *are mutually recurrent.*

Theorem 4.11. *Let* $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ *be a V - monotone compact dissipative nonautonomous dynamical system, Ω be minimal and J be its Levinson center, then*

$$V(x_1t,x_2t)=V(x_1,x_2) \tag{3}$$

for all $x_1,x_2\in J$ *such that* $h(x_1)=h(x_2)$.

Corollary 4.12. *Under the conditions of Theorem 4.11 if the nonautonomous dynamical system* $\langle (X,\mathbb{T}^+,\pi),(\mathbb{B},\mathbb{T},\Theta),h\rangle$ *is strict monotone, i.e.* $V(x_1t,x_2t)<V(x_1,x_2)$ *for all* $t>0$ *and* $(x_1,x_2)\in X\dot{\times}X$ $(x_1\ne x_2)$, *then* $J_\omega = J\bigcap X_\omega$ *consists a single point for all* $\omega\in\Omega$.

Theorem 4.13. *Let* $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ *be a $V-$monotone compact dissipative nonautonomous dynamical system with compact minimal base Ω and J be its Levinson's centre, then for every point* $x\in X_y$ *there exists a unique recurrent point* $p\in J_\omega$ *such that*

$$\lim_{t\to+\infty}\rho(xt,pt)=0, \tag{4}$$

i.e. every trajectory of this system is asymptotic recurrent.

Corollary 4.14. *Under the conditions of Theorem 4.13 the following assertions hold:*

a. $\omega-$limit set ω_x of every point $x \in X$ is a compact minimal set.

b. if $x_1, x_2 \in X_\omega$ ($\omega \in \Omega$) then $\omega_{x_1} = \omega_{x_2}$ or $\omega_{x_1} \bigcap \omega_{x_2} = \emptyset$.

5. On the structure of Levinson center of V-monotone NDS with minimal base

Definition 5.1. *(X, ρ) is called [18] a metric space with segments if for any $x_1, x_2 \in X$ and $\alpha \in [0,1]$, the intersection of $B[x_1, \alpha r]$ (the closed ball centered at x with radius αr, where $r = \rho(x_1, x_2)$) and $B[x_2, (1-\alpha) r]$ has a unique element $S(\alpha, x_1, x_2)$.*

Definition 5.2. *The metric space (X, ρ) is called [18] strict-convex if (X, ρ) is a metric space with segments, and for any $x_1, x_2, x_3 \in X$, $x_2 \neq x_3$, and $\alpha \in (0,1)$, the inequality $\rho(x_1, S(a, x_2, x_3)) < \max\{\rho(x_1, x_2), \rho(x_1, x_3)\}$ holds.*

Definition 5.3. *Let X be a strict metric-convex space. A subset M of X is said to be metric-convex if $S(\alpha, x_1, x_2) \in M$ for any $\alpha \in (0,1)$ and $x_1, x_2 \in M$.*

We note that every convex closed subset X of the Hilbert space H equipped with metric $\rho(x_1, x_2) = |x_1 - x_2|$ is strictly metric-convex.

Let $x \in X$ denote by Φ_x the family of all entire trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$ passing through point x for $t = 0$, i.e. $\gamma \in \Phi_x$ if and only if $\gamma : \mathbb{T} \to X$ is a continuous mapping with the properties: $\gamma(0) = x$ and $\pi^t \gamma(\tau) = \gamma(t + \tau)$ for all $t \in \mathbb{T}^+$ and $\tau \in \mathbb{T}$.

Theorem 5.4. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a V - monotone compact dissipative nonautonomous dynamical system, J is its Levinson center and the following conditions hold:*

1. $V(x_1, x_2) = V(x_2, x_1)$ for all $(x_1, x_2) \in X \dot{\times} X$.

2. $V(x_1, x_2) \leq V(x_1, x_3) + V(x_3, x_2)$ for all $x_1, x_2, x_3 \in X$ with condition $h(x_1) = h(x_2) = h(x_3)$.

3. the space (X_ω, V_ω) is strict metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X | h(x) = \omega\}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.
If $\gamma_{x_i} \in \Phi_{x_i}$ ($i = 1, 2$) and $x_1, x_2 \in I_\omega$ $(\omega \in \Omega)$, then the function $\gamma : \mathbb{T} \to X$ $(\gamma(t) = S(\alpha, \gamma_{x_1}(t), \gamma_{x_2}(t))$ for all $t \in \mathbb{T})$ defines an antier trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$.

We denote by $\mathcal{K} = \{a \in C(\mathbb{T}_+, \mathbb{R}_+) \mid a(0) = 0,\ a \text{ is strict increasing}\}$.

Theorem 5.5. *Under the conditions of Theorem 5.4 if in additionally the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ is bounded k - dissipative and there exists a function $a \in \mathcal{K}$ with property $\lim\limits_{t \to +\infty} a(t) = +\infty$ such that $a(\rho(x_1, x_2)) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$, then J_ω will be metric-convex for all $\omega \in \Omega$, where $J_\omega = J \bigcap X_\omega$ and J Levinson center of $(X, \mathbb{T}^+, \pi)$.*

6. Almost periodic solutions of V - monotone almost periodic dissipative systems.

Definition 6.1. *Let (X,ρ) be a metric space. A function $\phi : \mathbb{T} \to X$ is called almost periodic (in the sense of Bohr) if for every $\varepsilon > 0$ there exists a relatively dense subset A_ε of $\mathbb{T}$ such that*

$$\rho(\phi(t+\tau), \phi(t)) < \varepsilon$$

for all $t \in \mathbb{T}$ and $\tau \in A_\varepsilon$.

Definition 6.2. *A point $x \in X$ is said to be almost periodic if there is an entire trajectory $\gamma_x \in \Phi_x$ such that the function $\gamma_x : \mathbb{T} \to X$ is almost periodic.*

Definition 6.3. *The compact invariant set M of nonautonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is called* [19],[5] *distal on the invariant set M in the negative direction if $\inf\limits_{t\in\mathbb{T}_-} \rho(\gamma_{x_1}(t),\gamma_{x_2}(t)) > 0$ for all $x_1,x_2 \in M(h(x_1) = h(x_2)$ and $x_1 \neq x_2)$ and $\gamma_{x_i} \in \Phi_{x_i} (i=1,2)$, where Φ_x is a set of all entire trajectory of $(X,\mathbb{T}_+,\pi)$ passing through point $x \in X$.*

Lemma 6.4. [19] *Let Ω be a compact minimal set and $M \subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$, if nonautonomous dynamical system $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is distal on M in negative direction, then the mapping $\omega \longmapsto M_\omega := M \bigcap X_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 6.5. *Let $M \subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$, if nonautonomous dynamical system $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is uniformly stable in the positive direction on compacts from X, then $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is distal on the invariant set M in the negative direction .*

Corollary 6.6. *Under the conditions of Lemma 6.5 if Ω is a compact minimal set, then the mapping $\omega \longmapsto J_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 6.7. *Let (M,ρ) be a compact, strict metric-convex space and E be a compact subsemigroup of isometries of semigroup M^M (i.e. $E \subseteq M^M$ and $\rho(\xi x_1, \xi x_2) = \rho(x_1,x_2)$ for all $x_1,x_2 \in M$). Then there exists a common fixed point $\bar{x} \in M$ of E, i.e. $\xi(\bar{x}) = \bar{x}$ for all $\xi \in E$.*

Theorem 6.8. *Let $\langle (X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ be a V - monotone bounded k - dissipative NDS, J is its Levinson center and the following conditions hold:*

1. *$V(x_1,x_2) = V(x_2,x_1)$ for all $(x_1,x_2) \in X\dot{\times}X$.*

2. *$V(x_1,x_2) \leq V(x_1,x_3) + V(x_3,x_2)$ for all $x_1,x_2,x_3 \in X$ with condition $h(x_1) = h(x_2) = h(x_3)$.*

3. *the space (X_ω, V_ω) is strict metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X \mid h(x) = \omega\ \}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.*

Then the set-valued mapping $\omega \to J_\omega$ admits at least one continuous invariant section, i.e. there exists a continuous mapping $\nu : \Omega \to J$ with the properties: $h(\nu(\omega)) = \omega$ and $\nu(\theta(t,y)) = \pi(t,\nu(\omega))$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$.

Corollary 6.9. *Under the conditions of Theorem 6.8 the Levinson center of dynamical system $(X,\mathbb{T}_+,\pi)$ contains at least one stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point, if the minimal set Ω consists a stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point.*

7. Applications

7.1. **Finite-dimensional systems.** Denote by $\mathbb{R}^n$ a real $n-$dimensional Euclidean space with the scalar product $\langle , \rangle$ and the norm $|\cdot|$, generated by scalar product. Let $[\mathbb{R}^n]$ be a space of all the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, equipped with operational norm.

Theorem 7.1. *Let Ω be a compact minimal set, $F \in C(\Omega\times\mathbb{R}^n, \mathbb{R}^n)$, $W \in C(\Omega, [\mathbb{R}^n])$ and the following conditions hold:*

1. The matrix-function W is positively defined, i.e. $\langle W(\omega)u, u\rangle \in \mathbb{R}$ for all $\omega \in \Omega$, $u \in \mathbb{R}^n$ and there exists a positive constant a such that $\langle W(\omega)u, u\rangle \geq a|u|^2$ for all $\omega \in \Omega$ and $u \in \mathbb{R}^n$.

2. The function $t \to W(\theta_t\omega)$ is differentiable for every $\omega \in \Omega$ and $\dot{W}(\omega) \in C(\Omega, [\mathbb{R}^n])$, where $\dot{W}(\omega) = \frac{d}{dt}W(\theta_t\omega)|_{t=0}$.

3. $\langle \dot{W}(\omega)(u - v) + W(\omega)(F(\omega, u) - F(\omega, v)), u - v\rangle \leq 0$ for all $\omega \in \Omega$ and $u, v \in \mathbb{R}^n$.

4. There exist a positive constant r and the function $c : [r, +\infty) \to (0, +\infty)$ such that $\langle \dot{W}(\omega)u + W(\omega)F(\omega, u), u\rangle \leq -c(|u|)$ for all $|u| > r$.

Then the equation

$$u' = F(\theta_t\omega, u) \tag{5}$$

generates a cocycle φ on $\mathbb{R}^n$ which admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\}$ with the following properties:

a. I_ω is a nonvoid, compact and convex subset of $\mathbb{R}^n$ for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\theta_t\omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}_+$.

e. $\lim\limits_{t\to+\infty} \beta(\varphi(t, \theta_t\omega)M, I_\omega) = 0$ for all $M \in C(\mathbb{R}^n)$ and $\omega \in \Omega$;

f. $\lim\limits_{t\to+\infty} \sup\{\beta(\varphi(t, \theta_t\omega)M, I) \mid \omega \in \Omega \} = 0$ for any $M \in C(\mathbb{R}^n)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega \}$.

g. $I = \{I_\omega \mid \omega \in \Omega \}$ is a uniform forward attractor ,i.e.

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, \omega)M, I_{\theta_t\omega}) = 0$$

for any $M \in C(\mathbb{R}^n)$.

h. The equation (5) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution, if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Example 7.2. *In quality of example which illustrates this theorem we can consider the following equation*

$$u' = g(u) + f(\theta_t\omega),$$

where $f \in C(\Omega, \mathbb{R})$ *and*

$$g(u) = \begin{cases} (u+1)^2 & : \quad u < -1 \\ 0 & : \quad |u| \leq 1 \\ -(u-1)^2) & : \quad u > 1. \end{cases}$$

Example 7.3. *We consider the equation*

$$x'' + p(x)x' + ax = f(\theta_t \omega),$$

where $p \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\Omega, \mathbb{R})$ *and* a *is a positive number. Denote by* $y = x' + F(x)$, *where* $F(x) = \int_0^x p(s)ds$, *then we obtain the system*

$$\begin{cases} x' = y - F(x) \\ y' = -ax + f(\theta_t \omega). \end{cases} \tag{6}$$

Theorem 7.4. *Suppose the following conditions hold:*

1. $p(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. *There exist positive numbers* r *and* k *such that* $p(x) \geq k$ *for all* $|x| \geq r$.

Then the nonautonomous dynamical system, generated by (6) is compact dissipative and $V-$ *monotone.*

7.2. **Evolution equations with monotone operators.** Let H be a real Hilbert space with inner product $\langle , \rangle$, $|\cdot| = \sqrt{\langle , \rangle}$ and $\mathbb{B}$ be a reflexive Banach space contained in H algebraically and topologically. Furthermore, let $\mathbb{B}$ be dense in H in which case H can be identified with a subspace of the dual $\mathbb{B}'$ of $\mathbb{B}$ and $\langle , \rangle$ can be extended by continuity to $\mathbb{B}' \times \mathbb{B}$.

We consider the initial value problem

$$u'(t) + Au(t) = f(\theta_t \omega) \tag{7}$$

$$u(0) = u, \tag{8}$$

where $A : \mathbb{B} \to \mathbb{B}'$ is a (generally nonlinear) bounded,

$$|Au|_{\mathbb{B}'} \leq C|u|_{\mathbb{B}}^{p-1} + K, u \in \mathbb{B}, p > 1,$$

coercive,

$$\langle Au, u \rangle \geq a|u|_{\mathbb{B}}^p, u \in \mathbb{B}, a > 0,$$

monotone,

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \geq 0, u_1, u_2 \in \mathbb{B},$$

and hemicontinuous (see [20]).

The nonlinear "elliptic" operator

$$Au = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \phi(\frac{\partial u}{\partial x_i}) \quad \text{in } D \subset \mathbb{R}^n$$

$$u = 0 \text{ on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^n$, $\phi(\cdot)$ is a increasing function satisfying

$$\phi|_{[-1,1]} = 0,\ c|\xi|^p \leq \sum_{i=1}^{n} \xi_i \phi(\xi_i) \leq C|\xi|^p \text{ (for all } |\xi| \geq 2 \text{)},$$

provides an example with $H = L^2(D), \mathbb{B} = W_0^{1,p}(D),\ \mathbb{B}' = W^{-1,p'}(D),\ p' = \frac{p}{p-1}$.

The following result is established in [20] (Ch.2 and Ch.4). If $x \in H$ and $f \in C(\Omega, \mathbb{B}'),\ p' = \frac{p}{p-1}$, then there exists a unique solution $\varphi \in C(\mathbb{R}_+, H)$ of (7) and (8).

We denote by $\varphi(\cdot, \omega, u)$ the unique solutions of (7) and (8). According to [21] $\varphi(\cdot, \omega, u)$ is a continuous cocycle on H.

Theorem 7.5. *Suppose that the operator A satisfies the conditions above and the cocycle φ, generated by equation (7), is asymptotic compact, then it admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\ \}$ possessing the following properties:*

a. I_ω is a nonvoid, compact and convex subset of H for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\sigma_t \omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.

e. $\lim_{t \to +\infty} \beta(\varphi(t, \theta_{-t}\omega)M, I_\omega) = 0$ for all $M \in C(H)$ and $\omega \in \Omega$;

f. $\lim_{t \to +\infty} \sup\{\beta(\varphi(t, \theta_t \omega)M, I)|\ \omega \in \Omega\ \} = 0$ for any $M \in C(H)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$.

g. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is a uniform forward attractor ,i.e.

$$\lim_{t \to +\infty} \sup_{\omega \in \Omega} \beta(\varphi(t, \omega)M, I_{\theta_t \omega}) = 0$$

for any $M \in C(H)$.

h. The equation (7) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution, if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Remark 7.6. *If the injection of $\mathbb{B}$ into H is compact, then the cocycle φ generated by equation (7), evidently, is asymptotic compact.*

Example 7.7. *A typical example of equation of type (7) is the equation*

$$\frac{\partial}{\partial t} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \varphi(\frac{\partial u}{\partial x_i}) + f(\theta_t \omega),\ u|_{\partial D} = 0 \tag{9}$$

with "nonlinear Laplacian" $Au = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \phi(\frac{\partial u}{\partial x_i})$, where $\phi(\cdot)$ is a increasing function satisfying the condition

$$c|\xi|^p \leq \sum_{i=1}^{n} \xi_i \phi(\xi_i) \leq C|\xi|^p$$

for all $|\xi| \geq 2$ *and* $\phi|_{[-1,1]} = 0$, *provides an example with* $H = L^2(D), \mathbb{B} = W_0^{1,p}(D)$, $\mathbb{B}' = W^{-1,p'}(D), p' = \frac{p}{p-1}$. *It is possible to verify (see, for example,* [20],[4] *and* [2]*) that the "nonlinear Laplacian" verifies all the conditions of Theorem 7.5 and, consequently, (9) admits a compact global attractors with the properties a.-h.. We note that the attractor of equation (9) is not trivial, i.e. the set* I_ω *is not a single point set at least for certain* $\omega \in \Omega$.

Remark 7.8. *If the operator* $A = \sum_{i=1}^n \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i})$ *is uniformly elliptic, i.e.* $c|\xi|^p \leq \sum_{i=1}^n \xi_i \varphi(\xi_i) \leq C|\xi|^p$ *(for all* $\xi \in \mathbb{R}^n$*), then the set* I_ω *is a single point set for all* $\omega \in \Omega$ *(for autonomous system see* [23]*, Ch.III), because in this case the nonautonomous dynamical system generated by equation (9) is strict monotone.*

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