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Asymptotic Stability of autonomous and Non-Autonomous Discrete Linear Inclusions

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Abstract. The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions (both autonomous and non-autonomous) in Banach space. We establish the relation between absolute asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for compact (completely continuous) discrete linear inclusions these notions of stability are equivalent. We study this problem in the framework of non-autonomous dynamical systems (cocyles).

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1 Introduction

The aim of this paper is studying the problem of absolute asymptotic stability of the discrete linear inclusion (see, for example,[2, 18] and the references therein)

$$x_{t+1} \in F(x_t), \tag{1}$$

where $F(x) = \{A_1x, A_2x, ..., A_mx\}$ for all $x \in E$ (E is a Banach space) and A_i ($1 \le i \le m$) is a linear bounded operator acting on E.

The problem of asymptotic stability for the discrete linear inclusion arises in a number of different areas of mathematics: control theory – Molchanov [23]; linear algebra – Artzrouni [1], Beyn and Elsner [3], Bru, Elsner and Neumann [5], Daubechies and Lagarias [12], Elsner and Friedland [13], Elsner, Koltracht and Neumann [14], Gurvits [18], Vladimirov, Elsner and Beyn [31], Wirth [33, 34]; Markov Chains – Gurvits [15], Gurvits and Zaharin [16, 17]; iteration process – Bru, Elsner and Neumann [5], Opoitsev [24] and see also the bibliography therein.

Along with inclusion (1) we consider also the more general inclusions (non-autonomous case)

$$x_{t+1} \in F(t, x_t), \tag{2}$$

with $F(t, x) := \{A_1(t)x, A_2(t)x, ..., A_m(t)x\}$ and the operator-functions $A_i : \mathbb{Z}_+ \to [E]$ ($[E]$ is the space of all linear bounded operators $A : E \to E$).

We establish the relation between absolute asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for compact (completely continuous) discrete linear inclusions these notions of stability are equivalent.

We study this problem in the framework of non-autonomous dynamical systems (cocyles). We show that the problem of absolute asymptotic stability for the discrete linear inclusions is related with the compact global attractors of non-autonomous dynamical systems (both ordinary dynamical systems (with uniqueness) and set-valued dynamical systems). We plan to continue the studying of discrete inclusions (both linear and nonlinear) in the framework of non-autonomous dynamical systems. In our future publications we will give the proofs of the followings results:

(i) finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit nontrivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable;

(ii) discrete inclusion, defined by nonlinear (in particular, affine) contractive mappings $\{A_1, A_2, ..., A_m\}$ admits a compact global chaotic attractor,

amongst others. We consider that this method of studying discrete inclusions (both linear and nonlinear) is fruitful and it permits to obtain the new and nontrivial results.

This paper is organized as follows.

In Section 2 we give a new approach to the study of discrete linear inclusions (DLI) which is based on the non-autonomous dynamical systems (cocycles). The main result of this section is Theorem 2.6 which gives conditions for the asymptotic stability for finite-dimensional DLI.

In Section 3 we introduce the shift dynamical system on the space of continuous set-valued functions, set-valued cocycles and set-valued non-autonomous dynamical systems. They play a very important role in the study of of discrete linear inclusions. We show that every discrete linear inclusion generates a cocycle (Example 3.2).

Section 4 is dedicated to the study of non-autonomous discrete linear inclusions (Example 4.1). The main result of this section is Theorem 4.12 which establishes the equivalence between absolute asymptotic stability and uniform exponential stabilty for the compact (completely continuous) non-autonomous discrete linear inclusions on the arbitrary Banach space.

2 Autonomous discrete linear inclusions and cocycles

Let E be a real or complex Banach space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a semigroup of additive group $\mathbb{S}$. Consider a finite set of operators $: = \{A_i \mid 1 \le i \le m\}$, where $A_i \in [E]$.

Definition 2.1. *The discrete linear (autonomous) inclusion* $DLI(\mathcal{M})$ *is called (see, for example,[18]) the set of all sequences* $\{\{x_j\} \mid j \ge 0\}$ *of vectors in* E *such that*

$$x_j = A_{i_j} x_{j-1} \tag{3}$$

for some $A_{i_j} \in \mathcal{M}$, *i.e.* $x_j = A_{i_j} A_{i_{j-1}} ... A_{i_1} x_0$ *all* $A_{i_k} \in \mathcal{M}$.

We may consider this as a discrete control problem, where at each time j we may apply a control from the set $\mathcal{M}$, and $DLI(\mathcal{M})$ is the set of possible trajectories of the system. A basic issue for any control system concerns its stability. One of the more important type of stability is so called absolute asymptotic stability (AAS).

Definition 2.2. *$DLI(\mathcal{M})$ is called absolute asymptotic stable if for any of its trajectories $\{x_j\}$ we have $\lim_{j\to\infty} x_j = 0$.*

Let (X, ρ) be a complete metric space with metric ρ. Denote by $K(X)$ the family of all compact subsets of X. Consider the set-valued function $F : E \to K(E)$ defined by $F(x) := \{A_1x, A_2x, ..., A_mx\}$, then the discrete linear inclusion $DLI(\mathcal{M})$ is equivalent to difference inclusion

$$x_j \in F(x_{j-1}). \tag{4}$$

Denote by Φ_{x_0} the set of all trajectories of discrete inclusion (4) (or $DLI(\mathcal{M})$) issuing from the point $x_0 \in E$ and $\Phi := \bigcup\{\Phi_{x_0} \mid x_0 \in E^d\}$ the set of all trajectories of (4).

Below we will give a new approach to the study of discrete linear inclusions $DLI(\mathcal{M})$ (or difference inclusion (4)). Denote by $C(\mathbb{T}, X)$ the space of all continuous mappings $f : \mathbb{T} \to X$ equipped with the compact-open topology. This topology may be metrizied, for example, by the equality

$$d(f^1, f^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f^1, f^2)}{1 + d_n(f^1, f^2)},$$

where $d_n(f^1, f^2) := \max\{|f^1(t) - f^2(t)| \mid |t| \le n,\ t \in \mathbb{T}\}$, a complete metric is defined on $C(\mathbb{T}, X)$ which generates compact-open topology. Denote by $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [29, 30]) on $C(\mathbb{T}, X)$, i.e. $\sigma(t, f) := f_t$ and f_t is a $t \in \mathbb{T}$ shift of f ($f_t(s) := f(t + s)$ for all $s \in \mathbb{T}$).

Denote by $\Omega := \{f \in C(\mathbb{Z}_+, [E]) \mid f(\mathbb{Z}_+) \subseteq \mathcal{M}\}$. It is clear that Ω is an invariant (with respect to shifts) and closed subset of $C(\mathbb{Z}_+, [E])$ and, consequently, on the space Ω a dynamical system of shifts $(\Omega, \mathbb{Z}_+, \sigma)$ (induced by dynamical system of Bebutov $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$) is defined.

Notice that by Tihonoff's theorem (see, for example,[21]) the space Ω is compact in $C(\mathbb{Z}_+, [E])$.

We may now rewrite the equation (3) in the following way

$$x_{j+1} = \omega(j)x_j,\ (\omega \in \Omega) \tag{5}$$

where $\omega \in \Omega$ is an operator-function defined by the equality $\omega(j) := A_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$.

Denote by $\varphi(n, x_0, \omega)$ a solution of equation (5) issuing from the point $x_0 \in E$ at the initial moment $n = 0$. Notice that $\Phi_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and

$\Phi = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in E, \omega \in \Omega\}$, i.e. the $DLI(\mathcal{M})$ (or inclusion (4)) is equivalent to the family of linear non-autonomous equations (5) ($\omega \in \Omega$).

From the general properties of linear difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times \Omega \to E$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in E \times \Omega$;

(ii) $\varphi(n + \tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in E \times \Omega$;

(iii) the mapping φ is continuous;

(iv) $\varphi(n, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(n, x_1, \omega) + \mu\varphi(n, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in E$ and $\omega \in \Omega$.

Let W, Ω be two complete metric spaces and $(\Omega, \mathbb{Z}_+, \sigma)$ be a discrete semi-group dynamical system on Ω.

Definition 2.3. *Recall [29] that the triplet $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or shortly φ) is called a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W if φ is a mapping from $\mathbb{Z}_+ \times W \times \Omega$ to W satisfying the following conditions:*

1) $\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;

2) $\varphi(n+\tau, x, \omega) = \varphi(n, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x, \omega) \in W \times \Omega$;

3) the mapping φ is continuous.

If W is a real or complex Banach space and

4) $\varphi(n, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(n, x_1, \omega) + \mu\varphi(n, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in W$ and $\omega \in \Omega$, then the cocycle φ is called linear.

Definition 2.4. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle (respectively, linear cocycle) over $(Y, \mathbb{T}, \sigma)$ with the fiber W (or shortly φ). If $X := W \times Y, \pi := (\varphi, \sigma)$, i.e. $\pi((u, y), t) := (\varphi(t, x, y), \sigma(t, y))$ for all $(u, y) \in W \times Y$ and $t \in T$, then the dynamical system $(X, \mathbb{T}, \pi)$ is called [29] a skew product over $(Y, \mathbb{S}, \sigma)$ with the fiber W.*

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Denote by $\omega_x := \bigcap_{t\geq 0} \overline{\bigcup\{\pi(s, x) \ : \ s \geq t\}}$ and $\alpha_x := \bigcap_{t\leq 0} \overline{\bigcup\{\pi(s, x) \ : \ s \leq t\}}$ if $\mathbb{T} = \mathbb{S}$.

Let $\mathbb{T} = \mathbb{S}$, $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a linear cocycle (respectively, linear cocycle) over $(Y, \mathbb{T}, \sigma)$ with the fiber W and $(X, \mathbb{T}, \pi)$ be a skew-product dynamical system, generated by cocycle φ. Denote by $X^s := \{x \in X : \lim_{t\to+\infty} |\pi(t, x)| = 0\}$, $X^u := \{x \in X : \lim_{t\to-\infty} |\pi(t, x)| = 0\}$, $X^s_y := X^s \bigcap X_y$ and $X^u_y := X^u \bigcap X_y$, where $X_y := W \times \{y\}$.

From the above it follows that every $DLI(\mathcal{M})$ (respectively, inclusion (4)) generates in the natural way a linear cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is a solution of the equation (5) issuing from the point $x \in E$ at the initial moment $n = 0$. Thus we may study the inclusion (4) (respectively, $DLI(\mathcal{M})$) in the framework of the theory of linear cocycles with discrete time.

Definition 2.5. *A linear operator $A \in [E]$ is called asymptotically stable if $\sigma(A) \subseteq \mathbb{D}$, where $\sigma(A)$ is the spectrum of A and $\mathbb{D} := \{z|in\mathbb{C} \ : \ |z| < 1\}$ is a unit disk in $\mathbb{C}$.*

Theorem 2.6. *Let E be a finite-dimensional Banach space, $Dim(E) = n$, $A_i \in [E]$ $(i = 1, 2, ..., m)$ and $\mathcal{M} := \{A_1, A_2, ..., A_m\}$. Assume that the following conditions are fulfilled:*

1) every operator $A_j \in \mathcal{M}$ is invertible;

2) there exists $j \in \{1, 2, ..., m\}$ such that the operator A_j is asymptotically stable;

3) the discrete linear inclusion $DLI(\mathcal{M})$ has no nontrivial bounded on $\mathbb{Z}$ solutions.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable.

Proof. Let $Q := \mathcal{M} \bigcup \mathcal{M}^{-1}$ (where $\mathcal{M}^{-1} := \{A^{-1} \ : A \in \mathcal{M}\}$, $Y = \Omega := C(\mathbb{Z}, Q)$ and $(Y, \mathbb{Z}, \sigma)$ be a group dynamical system of shifts on Y (see Section 2). It is easy to see that $Y = C(\mathbb{Z}, Q)$ is topologically isomorphic to $\Sigma_m := \{0, 1, ..., m-1\}^{\mathbb{Z}}$ and $(Y, \mathbb{Z}, \sigma)$ is dynamically isomorphic to the shift dynamical system on Σ_m (see, for example,[25, 32]) and, consequently, it possesses the following properties:

(i) Y is compact;

(ii) $Y = \overline{Per(\sigma)}$, where $Per(\sigma)$ is the set of all periodic points of the dynamical system $(Y, \mathbb{Z}, \sigma)$;

(iii) there exists a Poisson stable point $y \in Y$ (i.e. $y \in \omega_y = \alpha_y$) such that $Y = H(y) := \overline{\{\sigma(t, y) \ : \ t \in \mathbb{Z}\}}$.

Let $\langle E, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ be a cocycle generated by $DLI(\mathcal{M})$ (i.e. $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=1}^{n} \omega(k)$ $(\omega \in \Omega)$), $(X, \mathbb{Z}, \pi)$ be a skew-product system associated with the cocycle φ (i.e. $X := E \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$ $(h := pr_2 : X \to Y)$ be a linear non-autonomous dynamical system generated by the cocycle φ. According to Theorem D from [28] (see also [4, 27]) X^s and X^u are two fiber sub-bundles of fiber bundle (X, h, Y). In particular there exists a number $k \in \mathbb{Z}_+$ $(0 \le k \le dim(E) = n$, where $dim(E)$ is the dimension of the space $E)$ such that $dim(X^s_y) = k$ for all $y \in Y$. Denote by $\omega_0 : \mathbb{Z} \to \mathcal{M}$ the mapping defined by the quality $\omega_0(i) = A^i_j$ for all $i \in \mathbb{Z}$, where $A^i_j := A_j \circ A^{i-1}_j$ $(i \in \mathbb{Z})$. Since the operator A_j is asymptotically stable, then the fiber X_{ω_0} $(\omega_0 \in Y)$ is asymptotically stable, i.e. $X_{\omega_0} = X^s_{\omega_0}$. Now to finish the proof of the theorem it is sufficient to note that $k = dim(X^s_y) = dim(X^s_{\omega_0}) = dim(X_{\omega_0}) = n$ for all $y \in Y$. □

Remark 2.7. *This statement is true also without assumption 1), but the proof in this case is much more complicated. We will present it in a future publication.*

3 Dynamical system of translations, set-valued cocycles and non-autonomous dynamical systems

Let $\mathcal{E}$ be a real or complex Banach space with norm $|\cdot|$ and ρ be a distance on $\mathcal{E}$ generated by norm $|\cdot|$. We denote by $K(E)$ the family of all compacts of E, by $\rho(a, B) := \inf\{\rho(a, b) \ |b \in B\}$ $(a \in E$ and $B \in K(E))$ and by α the Hausdorff's distance distance on $K(E)$, i.e. $\alpha(A, B) := \max\{\beta(A, B), \beta(B, A)\}$ and $\beta(A, B) := \sup_{a \in A} \rho(a, B)$. Let $C(\mathbb{Z}_+ \times E, K(E))$ be the set of all continuous in Hausdorff's metric and bounded on every bounded set from $\mathbb{Z}_+ \times E$ mappings $F : \mathbb{R} \times E \to K(E)$ equipped with the distance

$$d(F_1, F_2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(F_1, F_2)}{1 + d_k(F_1, F_2)}, \tag{6}$$

where $d_k(F_1, F_2) := \sup\{\alpha(F_1(t,x), F_2(t,x)) \ : 0 \le t \le k, |x| \le k, (t,x) \in \mathbb{Z}_+ \times E\}$. The distance (6) defines on the space $C(\mathbb{Z}_+ \times E, K(E))$ the topology of convergence uniform on every bounded subset of $\mathbb{Z}_+ \times E$.

Denote by $(C(\mathbb{Z}_+ \times E, K(E)), \mathbb{Z}_+, \sigma)$ a dynamical system of translations on $C(\mathbb{Z}_+ \times E, K(E))$ (see, for example,[29,30]), where $\sigma(n, F)$ is an n-shift of function F with respect to variable $t \in \mathbb{Z}_+$, i.e $\sigma(n, F)(t, x) := F(t+n, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E$.

Definition 3.1. *The triplet $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ is said to be a set-valued cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber W, where φ is a mapping of $\mathbb{Z}_+ \times W \times Y$ onto $K(W)$ and possesses the properties:*

(i) $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;

(ii) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), yt)$ for all $t, \tau \in \mathbb{Z}_+$ and $(u, y) \in W \times Y$, where $yt := \sigma(t, y)$ and $\varphi(t, A, y) := \bigcup\{\varphi(t, u, y) \ : \ u \in A\}$;

(iii) $\lim\limits_{t\to t_0, u\to u_0, y\to y_0} \beta(\varphi(t, u, y), \varphi(t_0, u_0, y_0)) = 0$ for all $(t_0, u_0, y_0) \in \mathbb{Z}_+ \times W \times Y$.

Let $X := W \times Y$. We denote by $(X, \mathbb{Z}_+, \pi)$ the set-valued dynamical system on X defined by the equality $\pi := (\varphi, \sigma)$, i.e. $\pi^t x := \{(v, q) : v \in \varphi(t, u, y), \ q \in \sigma(t, y)\}$ for every $t \in \mathbb{Z}_+$ and $x = (u, y) \in X = W \times Y$. Then the triplet $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ is a set-valued non-autonomous dynamical system (a skew-product system), where $h = pr_2 : X \mapsto Y$.

Thus, if we have a set-valued cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ over the dynamical system $(Y, \mathbb{Z}_+, \sigma)$ with the fiber W, then it generates a set-valued non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ $(X := W \times Y)$, which is called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ over $(Y, \mathbb{Z}_+, \sigma)$.

Example 3.2. *(Difference inclusions).* Denote by $K(E)$ the family of all compact subsets of E. Let us consider the difference inclusion

$$u(t + 1) \in F(t, u(t)), \tag{7}$$

where $F \in C(\mathbb{Z}_+ \times E, K(E))$. Along with difference inclusion (7) we will consider the family of difference inclusions

$$v(t + 1) \in G(t, v(t)), \tag{8}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{Z}_+\}}$, $F_\tau(t, u) = F(t + \tau, u)$ and by bar the closure in $C(\mathbb{Z} \times E, C(E))$ is denoted.

We denote by $\varphi_{(v,G)}(n)$ a solution of inclusion (8) passing through the point v for $t = 0$ and defined for all $t \ge 0$. We set $\varphi(t, v, G) := \{\varphi_{(v,G)}(t) : \varphi_{(v,G)} \in \Phi_{(v,G)}\}$, where $\Phi_{(v,G)}$ is the set of all solutions of inclusion (8) , passing through the point v for $t = 0$. From the general properties of difference inclusions it follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times H(F) \to K(E)$ possesses the next properties :

1) $\varphi(0, v, G) = v$ for all $v \in E, G \in H(F)$;
2) $\varphi(t + \tau, v, G) = \varphi(t, \varphi(\tau, v, G), G_\tau)$ for all $v \in E, G \in H(F)$ and $t, \tau \in \mathbb{Z}_+$;
3) the mapping $\varphi : \mathbb{Z}_+ \times E \times H(F) \to K(E)$ is β-continuous.

Assume $Y = H(F)$ and denote by $(Y, \mathbb{Z}_+, \sigma)$ the disperse dynamical system of translations on Y. Then the triplet $\langle E, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ is a set-valued cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber E. Thus, non-autonomous difference inclusion (7) in a natural way generates a non-autonomous set-valued dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$, where $X = E \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$.

4 Non-stationary discrete linear inclusions

Example 4.1. Let $\mathcal{M} \subset [E]$ be a compact set and $F : \mathbb{Z}_+ \times E \to K(E)$ be the set-valued mapping defined by the equality $F(t, x) := \{A(t)x \ : \ A \in C(\mathbb{Z}_+, \mathcal{M})\}$ for all $t \in \mathbb{Z}_+$ and $x \in E$. It is easy to verify that the function $F : \mathbb{Z}_+ \times E \to K(E)$ is continuous, i.e. $F \in C(\mathbb{Z}_+ \times E, K(E))$. Consider the difference inclusion

$$x(t + 1) \in F(t, x(t)). \tag{9}$$

Note that the solution of inclusion (9) is a sequence $\{\{x(t)\} \mid t \in \mathbb{Z}_+\}$ of vectors in E such that $x(t + 1) = A_{i_t}(t)x(t)$ for some $A_{i_t}(t) \in \mathcal{M}$, i.e.

$$x(t) = A_{i_t}(t)A_{i_{t-1}}(t-1)...A_{i_1}(1)x(0) \ (A_{i_t}(t) \in \mathcal{M}).$$

Along with inclusion (9) we consider its H-class (see Example 3.2), i.e. the family of inclusions

$$x(t + 1) \in G(t, x(t)), \tag{10}$$

where $G \in H(F) := \overline{\{F_s \mid s \in \mathbb{Z}_+\}}$ and $F_s(t, x) := F(t+s, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E$.

Let Y be a compact metric space and (X, h, Y) be a locally trivial fiber bundle [20] with the fiber E, (X, ρ) be a complete metric space.

Definition 4.2. *$\langle (X, \mathbb{Z}_+, \pi),(Y, \mathbb{Z}_+, \sigma), h\rangle$ is said to be homogeneous if for any $x \in X$ and any $\gamma_x \in \Phi_x$ the function $\tilde{\gamma} : D(\gamma_x) \to X$ ($D(\gamma_x) := [r_x, +\infty)$ is the domain of the definition of γ_x, where $r_x \in \mathbb{Z}$) defined by $\tilde{\gamma}(t) := \lambda\gamma_x(t)$ is the motion of (X, Z_+, π) issuing from the point $\lambda x \in X$, i.e. $\tilde{\gamma} \in \Phi_{\lambda x}$.*

Remark 4.3. *1. Note that non-autonomous dynamical system from Example 3.2 is homogeneous if the set-valued mapping F which figures in this example is homogeneous, i.e. $F(t, \lambda x) = \lambda F(t, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E$.*

2. The non-autonomous dynamical system generated by discrete linear inclusion (9) is homogeneous, because the function $F(t, x) := \{A(t)x \ : \ A \in C(\mathbb{Z}_+, \mathcal{M})\}$ (for all $(t, x) \in \mathbb{Z}_+ \times E$) is homogeneous with respect to $x \in E$.

If $x \in X$, then we put $|x| := \rho(x, \theta_{h(x)})$, where θ_y $(y \in Y)$ is the null (trivial) element of the linear space X_y and $\Theta := \{\theta_y \mid y \in Y\}$ is the null (trivial) section of the vectorial bundle (X, h, Y). Let $A \in K(X)$, then we denote by $|A| := \max\{|a| \,:\, a \in A\}$. Denote by X^s a stable manifold of $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma) \rangle$, i.e. $X^s := \{x \mid x \in X, \lim\limits_{t \to +\infty} |\pi(t, x)| = 0\}$.

Definition 4.4. *Let W be a Banach space. The cocycle $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma), h \rangle$ is said to be homogeneous if the skew-product set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ $(X := W \times Y, \pi := (\varphi, \sigma))$ also is homogeneous.*

Theorem 4.5. *[8] Let Y be a compact metric space and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$ be a homogeneous set-valued non-autonomous dynamical system. Then the following assertions are equivalent:*

(i) the trivial section Θ of fibering (X, h, Y) is uniformly asymptotically stable, i.e. $\lim\limits_{t \to \infty} \|\pi^t\| = 0$, where $\pi^t := \pi(t, \cdot) : X \to K(X)$, $\|\pi^t\| := \sup\{|\pi^t x| \ : x \in X, |x| \le 1\}$ and $|A| := \sup\{|a| \ : a \in A\}$;

(ii) the trivial section Θ of fibering (X, h, Y) is uniformly exponentially stable, i.e. there are two positive constants $\mathcal{N}$ and ν such that $|\pi(t, x)| \le \mathcal{N} e^{-\nu t}$ for all $x \in X$ and $t \in \mathbb{Z}_+$.

Definition 4.6. *A set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ is called compact (completely continuous) if for any bounded set $A \subseteq X$ there exists a positive number $l \in \mathbb{Z}_+$ such that the set $\pi^t A$ is relatively compact. A non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$ is called compact if the system $(X, \mathbb{T}_1, \pi)$ is so.*

Denote by $K(X)$ $(B(X))$ the family of all compact (bounded) subsets of X and $B(M, \delta) := \{x \in X \mid \rho(x, M) < \delta\}$.

Definition 4.7. *A system $(X, \mathbb{Z}_+, \pi)$ is called [6]:*

– pointwise dissipative if there exists $K_0 \in C(X)$ such that for all $x \in X$

$$\lim_{t \to \infty} \beta(xt, K_0) = 0; \tag{11}$$

– compactly dissipative if equality (11) holds uniformly w.r.t. x on compacts from X;

– locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that equality (11) holds uniformly w.r.t. $x \in B(p, \delta_p)$.

Theorem 4.8. *[6] Let $(X, \mathbb{Z}_+, \pi)$ be a pointwise dissipative compact dynamical system, then it is locally dissipative.*

Theorem 4.9. *Let Y be a compact metric space and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$ be a compact, homogeneous set-valued non-autonomous dynamical system. Then the following assertions are equivalent:*

1) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$ is convergent, i.e.

$$\lim_{t \to \infty} |\pi(t, x)| = 0, \tag{12}$$

for all $x \in X$*;*

2) the trivial section Θ *of fibering* (X, h, Y) *is uniformly exponentially stable.*

Proof. To prove this affirmation obviously it is sufficient to show that 1) implies 2), because the implication 2) $\to$ 1) is trivial. Since the space Y is compact and the fibering (X, h, Y) is locally trivial, then the trivial section Θ of (X, h, Y) is compact. Taking into account this fact and the equality (12) we obtain the pointwise dissipativity of dynamical system $(X, \mathbb{Z}_+, \pi)$. Now to finish the proof it is sufficient to apply Theorem 4.8. □

Definition 4.10. *Following [18] the inclusion (9) is said to be absolute asymptotic stable (AAS) if for any trajectory* $\{x(t)|\ t \in \mathbb{Z}_+\}$ *of any inclusion (10)* $\lim_{t\to+\infty} x(t) = 0$.

Theorem 4.11. *[6] Let* $(X, \mathbb{Z}_+, \pi)$ *be a completely continuous (compact) and trajectory dissipative set-valued dynamical system, then it is locally dissipative.*

Theorem 4.12. *Let* $\mathcal{M} \subset [E]$ *be compact and every operator* $A \in \mathcal{M}$ *be compact too. Then the following two statements are equivalent:*

1) the inclusion (9) is absolute asymptotic stable;

2) the inclusion (9) is uniformly exponentially stable, i.e. there are positive numbers N *and* ν *such that* $|x(t)| \le Ne^{-\nu t}|x(0)|$ *for all* $t \in \mathbb{Z}_+$*, where* $\{x(t)\ |t \in \mathbb{Z}_+\}$ *is an arbitrary solution (trajectory) of arbitrary inclusion (10).*

Proof. Denote by $\Omega := H(F)$ the closure (in the space $C(\mathbb{Z}_+ \times E, C(E))$) of family of translations $\{F_s \mid s \in \mathbb{Z}_+\}$ of function $F(t, x) := \{At)x\ :\ A \in C(\mathbb{Z}_+, \mathcal{M})\}$, $(\Omega, \mathbb{Z}_+, \sigma)$ the dynamical system of translations on Ω, $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ the cocycle generated by non-stationary discrete linear inclusion (10). Finally, by $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ we denote the non-autonomous dynamical system system, generated by cocycle φ ($X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to \Omega$). Note that this dynamical system possesses the following properties:

1) the set $\Omega = H(A)$ is compact, according to theorem of Tihonoff;

2) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is homogeneous;

3) the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is compact. In fact, let $A \subset E$ be a bounded subset of E , $c := \sup\{|a|\ : a \in A\}$ and $l \in \mathbb{N}$. We will show that the set $\varphi(l, A, \Omega)$ is relatively compact. Consider a sequence $\{y_k\} \subseteq \varphi(l, A, \Omega)$, then there are $\{x_k\} \subseteq A$, $\{\omega_k\} = \{G_k\} \subseteq \Omega$ ($G_k \in H(F)$) and $B^k_i \in H(A_i)$ ($H(A_i)$ is a closure of the set of translations $\{A_i(t + s) \mid s \in \mathbb{Z}_+\}$ in the space $C(\mathbb{Z}_+, [E])$ of all continuous mappings $f : \mathbb{Z}_+ \to [E]$ equipped with compact-open topology) such that

$$y_k = B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1)x_k.$$

Under the conditions of Theorem the operators $\{B^k_{i_s}(s)\}$ ($1 \le s \le l$ and $k \in \mathbb{N}$) are compact. Without loss of generality we may suppose that the sequences

$\{B^k_{i_s}\} \subset C(\mathbb{Z}_+, [E])$ are convergent as $k \to \infty$ (in the space $C(\mathbb{Z}_+, [E])$). Let $B_{i_s}(t) := \lim_{k\to\infty} B^k_{i_s}(t)$ (for any $t \in \mathbb{Z}_+$), then this operator will be compact and, consequently, the operator $B(t) := B_{i_l}(t)B_{i_{l-1}}(t)...B_{i_1}(t)$ will be so too. Since the sequence $\{x_k\} \subseteq A$ is bounded, then the sequence $\{B(l)x_k\}$ is relatively compact. For simplicity we may suppose that this sequence converges and denote by $y := \lim_{k\to\infty} B(l)x_k$, then we have

$$
\begin{aligned}
&|y_k - y| \le |B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1)x_k - B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k| + \\
&|B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k - y| \le \|B^k_{i_l}(l)B^k_{i_{l-1}}(l-1)...B^k_{i_1}(1) - \\
&B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)\| \cdot c + |B_{i_l}(l)B_{i_{l-1}}(l-1)...B_{i_1}(1)x_k - y|
\end{aligned} \tag{13}
$$

for all $k \in \mathbb{N}$. Passing to limit in the relation (13) we obtain $y = \lim_{n\to\infty} y_k$ and the required statement is proved.

4) the dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{T}_+, \sigma), h\rangle$ is convergent.

In fact, from condition 1. it follows that the skew-product set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := E^n \times Y$ and $\pi := (\varphi, \sigma)$) is trajectory dissipative and by Theorem 4.8 the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is locally dissipative and, in particular, we have $\lim_{t\to+\infty} \sup_{|x|\le 1} |\pi(t, x)| = 0$.

Note that the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$, generated by cocycle φ is homogeneous and compact. Now to finish the proof of Theorem it is sufficient to apply Theorem 4.9. □

Remark 4.13. *1. Note that a similar result has been proved for reflexive Banach spaces in [33, 34] for arbitrary bounded sets of bounded operators.*

2. Theorem 4.12 is true also for non-autonomous nonlinear homogeneous inclusions, i.e. if the operators $A \in \mathcal{M}$ are continuous and homogeneous, but in general not linear.

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