RECURRENT MOTIONS AND GLOBAL ATTRACTORS OF NONAUTONOMOUS LORENZ SYSTEMS

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Abstract. The article is devoted to the study of nonautonomous Lorenz systems. This problem is formulated and solved in the context of nonautonomous dynamical systems. First, we prove that such systems admit a compact global attractor and characterize its structure. Then, we obtain conditions of convergence of the nonautonomous Lorenz systems, under which all solutions approach a point attractor. Third, we derive a criterion for existence of almost periodic (periodic, quasi-periodic) and recurrent solutions of the systems. Finally, we prove a global averaging principle for nonautonomous Lorenz systems.

1. Introduction

The following n-dimensional systems of differential equations are called systems of hydrodynamic type or autonomous Lorenz systems ([30]):

$$u_i' = \Sigma_{j,k} b_{ijk} u_j u_k + \Sigma_j a_{ij} u_j + f_i,\ i = 1, 2, ..., n, \tag{1}$$

where $\Sigma b_{ijk} u_i u_j u_k$ is identically equal to zero, $\Sigma a_{ij} u_i u_j$ is negative definite, and f_i are constants. The well-known three-dimensional Lorenz system for geophysical flows or climate modeling [25] is a special case of this type of systems.

It is known that solutions of (1) imbed in some ellipsoid and do not leave it later, i.e. the autonomous system (1) is dissipative, and hence admits a compact global attractor.

In the vector-matrix form the system (1) may be written as:

$$u' = Au + B(u, u) + f, \tag{2}$$

where A is a positive definite matrix and $B : H \times H \to H$ (H is a n-dimensional real or complex Euclidian space) is a bilinear form satisfying the identity

$$Re\langle B(u, v), w\rangle = -Re\langle B(u, w), v\rangle \tag{3}$$

for every $u, v, w \in H$.

When f is not a constant vector but a bounded function of time t, it is known that the equation (2) also admits a compact global attractor [22].

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The aim of the present article is to study the *nonautonomous* version of the equation (2). Namely, in this case, the matrix A, the bilinear form B, and the function f all depend on time t. We will consider issues like compact global attractors, convergence, almost periodic (including periodic and quasi-periodic) solutions and recurrent solutions, and averaging principles.

This paper is organized as follows:

In Section 2 we introduce a class of nonautonomous Lorenz dynamical systems and establish its dissipativity (Theorem 2.2).

In Section 3 we prove that asymptotic compact Lorenz systems admit a compact global attractor (Theorem 3.7) and we characterize the structure of the global attractor. Furthermore, we obtain conditions for convergence of these systems (Theorem 3.9), under which each section of the global attractor contains a single point.

Section 4 is devoted to study of existence of almost periodic (periodic, quasi-periodic) and recurrent solutions of nonautonomous Lorenz systems (Corollaries 4.2 and 4.6).

In Section 5 we prove a uniform averaging principle for a class of nonautonomous dynamical systems (Theorem 5.3). With the help of this uniform averaging principle, we prove a global averaging principle for nonautonomous Lorenz systems on the semi-axis (Theorem 6.4) in Section 6.

2. Nonautonomous Lorenz systems

Let Ω be a compact metric space, $\mathbb{R} = (-\infty, +\infty), (\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω and H be a real or complex Hilbert space. We denote $L(H)$ $(L^2(H))$ the space of all linear (bilinear) forms on H. When W is some metric space, $C(\Omega, W)$ denotes the space of all continuous functions $f : \Omega \to W$, endowed with the topology of uniform convergence.

Let us consider the nonautonomous Lorenz system

$$u' = A(\omega t)u + B(\omega t)(u,u) + f(\omega t), \quad \omega \in \Omega, \tag{4}$$

where $\omega t := \sigma(t,\omega)$, $A \in C(\Omega, L(H)), B \in C(\Omega, L^2(H))$ and $f \in C(\Omega, H)$. Note that when the autonomous Lorenz system (2) is perturbed by periodic, quasi-periodic, almost periodic or recurrent forces, it can then be written as (4). Moreover, we assume that the following conditions are fulfilled:

(i) There exists $\alpha > 0$ such that

$$Re\langle A(\omega)u, u\rangle \le -\alpha |u|^2 \tag{5}$$

for all $\omega \in \Omega$ and $u \in H$, where $|\cdot|$ is a norm in H ;

(ii)

$$Re\langle B(\omega)(u,v), w\rangle = -Re\langle B(\omega)(u,w), v\rangle \tag{6}$$

for every $u, v, w \in H$ and $\omega \in \Omega$.

Remark 2.1. *a. It follows from (6) that*

$$Re\langle B(\omega)(u,v),v)\rangle = 0 \tag{7}$$

for every $u, v \in H$ and $\omega \in \Omega$.

b. From bilinearity and continuity, we obtain

$$|B(\omega)(u,v)| \le C_B|u||v| \tag{8}$$

for all $u, v \in H$ and $\omega \in \Omega$, where $C_B = \sup\{|B(\omega)(u,v)| : \omega \in \Omega,\ u, v \in H,\ |u| \le 1,$ and $|v| \le 1\}$.

We will call the system (4) with conditions (5) and (6) a nonautonomous Lorenz system or a nonautonomous system of hydrodynamic type.

We note that from the conditions (6) -(8) it follows that

$$|B(\omega)(x_1,x_1) - B(\omega)(x_2,x_2)| \le C_B(|x_1| + |x_2|)|x_1 - x_2| \tag{9}$$

for all $x_1, x_2 \in H$ and $\omega \in \Omega$.

Since the coefficients of (4) are locally Lipschitzian with respect to $u \in H$, through every point $x \in H$ passes a unique solution $\varphi(t,x,\omega)$ of equation (4) at the initial moment $t = 0$. And this solution is defined on some interval $[0, t_{(x,\omega)})$. Let us note that

$$\begin{aligned} w'(t) &= 2Re\langle\varphi'(t,x,\omega), \varphi(t,x,\omega)\rangle = 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + \\ &2Re\langle B(\omega t)(\varphi(t,x,\omega), \varphi(t,x,\omega)), \varphi(t,x,y)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &= 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &\le -2\alpha|\varphi(t,x,\omega)|^2 + 2\|f\||\varphi(t,x,\omega)|, \end{aligned} \tag{10}$$

where $\|f\| := \max\{|f(\omega)| : \omega \in \Omega\}$ and $w(t) = |\varphi(t,x,\omega)|^2$. Then

$$w' \le -2\alpha w + 2\|f\|w^{\frac{1}{2}}. \tag{11}$$

Thus

$$w(t) \le v(t) \tag{12}$$

for all $t \in [0, t_{(x,\omega)})$, where $v(t)$ is an upper solution of equation

$$v' = -2\alpha v + 2||f||v^{\frac{1}{2}}, \tag{13}$$

satisfying condition $v(0) = w(0) = |x|^2$. Hence

$$v(t) = [(|x| - \frac{||f||}{\alpha})e^{-\alpha t} + \frac{||f||}{\alpha}]^2 \tag{14}$$

and consequently

$$|\varphi(t,x,\omega)| \le (|x| - \frac{||f||}{\alpha})e^{-\alpha t} + \frac{||f||}{\alpha} \tag{15}$$

for all $t \in [0, t_{(x,\omega)})$. It follows from the inequality (15) that solution $\varphi(t,x,\omega)$ is bounded and therefore it may be extended to a global solution on $\mathbb{R}_+ = [0, +\infty)$.

Thus we have proved the following theorem.

Theorem 2.2. *(Dissipativity) Let the conditions (5) and (6) are fulfilled. Then the following statements hold:*

(i)

$$|\varphi(t,x,\omega)| \le C(|x|), \tag{16}$$

for all $t \ge 0$, $\omega \in \Omega$ *and* $x \in H$, *where* $C(r) = r$ *if* $r \ge r_0 := \frac{\|f\|}{\alpha}$ *and* $C(r) = r_0$ *if* $r \le r_0$*;*

(ii)

$$\limsup_{t\to+\infty} \sup\{|\varphi(t,x,\omega)| \ : |x| \le r, \omega \in \Omega\} \le \frac{||f||}{\alpha} \tag{17}$$

for every $r > 0$.

The item (i) in this Theorem means that the nonautonomous Lorenz flow is bounded on bounded sets, while the item (ii) implies that the nonautonomous Lorenz system is dissipative, i.e., it admits a bounded absorbing set.

3. Nonautonomous dissipative dynamical systems and their attractors

Let Ω and W be two metric spaces and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on Ω. Let us consider a continuous mapping $\varphi : \mathbb{R}^+ \times W \times \Omega \to W$ satisfying the following conditions:

$$\varphi(0,\cdot,\omega) = id_W \quad \varphi(t+\tau,x,\omega) = \varphi(t,\varphi(\tau,x,\omega),\omega\tau)$$

for all $t,\tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in W$. Here $\omega\tau$ is the short notation for $\sigma_\tau(\omega) := \sigma(\tau,\omega)$. Such a mapping φ (or more explicitly $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$) is called a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fiber W; see [1, 28].

Example 3.1. *Let E be a Banach space and $C(\mathbb{R} \times E, E)$ be a space of all continuous functions $F : \mathbb{R} \times E \to E$ equipped by the compact-open topology. Let us consider a parameterized differential equation*

$$\frac{dx}{dt} = F(\sigma_t\omega, x), \quad \omega \in \Omega$$

on a Banach space E with $\Omega = C(\mathbb{R} \times E, E)$, where $\sigma_t\omega := \sigma(t,\omega)$. We will define $\sigma_t : \Omega \to \Omega$ by $\sigma_t\omega(\cdot,\cdot) = \omega(t+\cdot,\cdot)$ for each $t \in \mathbb{R}$ and interpret $\varphi(t,x,\omega)$ as solution of the initial value problem

$$\frac{d}{dt}x(t) = F(\sigma_t\omega, x(t)), \quad x(0) = x. \tag{18}$$

Under appropriate assumptions on $F : \Omega \times E \to E$ (or even $F : \mathbb{R} \times E \to E$ with $\omega(t)$ instead of $\sigma_t\omega$ in (18)) to ensure forward existence and uniqueness, then φ is a cocycle on $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ with fiber E. Note that $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ is a Bebutov's dynamical system (see for example [2],[13], [26],[28]*).*

Let φ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with the fiber E. Then the mapping $\pi : \mathbb{R}^+ \times E \times \Omega \to E \times \Omega$ defined by

$$\pi(t,x,\omega) := (\varphi(t,x,\omega), \sigma_t\omega)$$

for all $t \in \mathbb{R}^+$ and $(x,\omega) \in E \times \Omega$ forms a semi-group $\{\pi(t,\cdot,\cdot)\}_{t\in\mathbb{R}^+}$ of mappings of $X := \Omega \times E$ into itself, thus a semi-dynamical system on the state space X, which is called a skew-product flow [28]. The triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ (where $h := pr_2 : X \to \Omega$) is a nonautonomous dynamical system; see [3, 13].

A cocycle φ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W is called a compact (bounded) dissipative cocycle, if there is a nonempty compact set $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t,\omega)M, K) | \omega \in \Omega\} = 0 \tag{19}$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(W)$), where $C(W)(\ \mathcal{B}(W))$ denotes the family of all compact (bounded) subsets of W, β is the semidistance of Hausdorff and $U(t,\omega) := \varphi(t,\cdot,\omega)$. We can similarly define a compact or bounded dissipative skew-product system.

Lemma 3.2. *Let Ω be a compact metric space and $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W. In order for $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ to be compact (bounded) dissipative, it is necessary and sufficient that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is compact (bounded) dissipative.*

This assertion directly follows from the corresponding definitions (see for example [18],[13]).

We now define whole trajectories of the semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ (or whole trajectories of the cocycle $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W). A whole trajectory passes through the point $x \in X((u,y) \in W \times \Omega)$ is a continuous mapping $\gamma : \mathbb{R} \to X$ (or $\nu : \mathbb{R} \to W$) which satisfies the conditions : $\gamma(0) = x$ (or $\nu(0) = u$) and $\pi^t\gamma(\tau) = \gamma(t+\tau)$ (or $\nu(t+\tau) = \varphi(t, \nu(\tau), \omega\tau))$ for all $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$.

Moreover, for $M \subseteq W$, we denote by

$$\Omega_\omega(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M, \omega^{-\tau})} \tag{20}$$

for every $\omega \in \Omega$, where $\omega^{-\tau} := \sigma(-\tau, \omega)$. This formula is useful in the construction of global attractors. We recall the following result.

Theorem 3.3. ([11],[13]) *Let Ω be a compact metric space, $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a compact (bounded) dissipative cocycle and K be the nonempty compact set in the dissipation property (19). Then the following assertions hold:*

(i) *The set $I_\omega := \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and*

$$\lim_{t\to+\infty} \beta(U(t,\omega^{-t})K, I_\omega) = 0 \tag{21}$$

for every $\omega \in \Omega$;

(ii) *$U(t,\omega)I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$;*

(iii)

$$\lim_{t\to+\infty} \beta(U(t,\omega^{-t})M, I_\omega) = 0 \tag{22}$$

for all $M \in C(W)$ (respectively $M \in \mathcal{B}(X)$) and $\omega \in \Omega$;

(iv)

$$\lim_{t\to+\infty} \sup\{\beta(U(t,\omega^{-t})M, I) | \omega \in \Omega\} = 0 \tag{23}$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(X)$) , where $I = \cup\{I_\omega |\ \omega \in \Omega\}$;

(v) *$I_\omega := pr_1 J_\omega$ for all $\omega \in \Omega$, where J is a Levinson's centre of $(X, \mathbb{R}_+, \pi)$, and, hence, $I = pr_1 J$;*

(vi) *The set I is compact;*
(vii) *The set I is connected if the spaces W and Y are connected.*

Now we define the concept of compact global attractors. The family of compact sets $\{I_\omega | \omega \in \Omega\}$ ($I_\omega \subset W$ is nonempty compact for every $\omega \in \Omega$) is called (see, for example, [11] or [13]) the compact global attractor of cocycle φ if the following conditions are fulfilled:

(i) The set $I := \bigcup\{I_\omega| \quad \omega \in \Omega\}$ is precompact.
(ii) $\{I_\omega| \quad \omega \in \Omega\}$ is invariant w.r.t. the cocycle φ, i.e. $\varphi(t,\omega,I_\omega) = I_{\sigma_t\omega}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.
(iii) The equality $\lim_{t\to+\infty} \sup_{\omega\in\ \Omega} \beta(\varphi(t,K,\omega),I) = 0$ holds for every nonempty bounded set $K \subset W$.

The set I_ω will be called a *section* of the global attractor.

Corollary 3.4. *Under the conditions of Theorem 3.3, the cocycle φ admits a compact global attractor.*

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called asymptotically compact (see [18],[23], [29] and also [11],[13]) if for any positive invariant bounded set $A \subset X$ there is a compact $K_A \subset X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t A, K_A) = 0. \tag{24}$$

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called compact (completely continuous) if for every bounded set $A \subset X$ there exists a positive number $l = l(A)$ such that the set $\pi^l A$ is precompact.

It is easy to verify (see for example [13]) that every compact dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotically compact.

The cocycle $\langle W, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is called compact (asymptotically compact) if the associated skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ with $X = W \times Y$ and $\pi = (\varphi, \sigma)$ is compact (respectively asymptotic compact).

Let $(X, \mathbb{R}_+, \pi)$ be compact dissipative and K be a compact set, which attracts all compact subsets of X. Let

$$J = \Omega(K), \tag{25}$$

where $\Omega(K) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi^\tau K}$. The set J defined by the equality (25) does not depend on selection of the attracting set K, and is characterized only by the properties of the dynamical system $(X, \mathbb{R}_+, \pi)$ itself. The set J is called the Levinson's centre of the compact dissipative system $(X, \mathbb{R}_+, \pi)$.

Theorem 3.5. *(*[11],[13]*) Let (E, Ω, h) be a local-trivial Banach fibering, $\langle(E, \mathbb{R}_+, \pi)$, $(\Omega, \mathbb{R}, \sigma), h\rangle$ be a nonautonomous dynamical system and the dynamical system $(E, \mathbb{R}_+, \pi)$ be completely continuous. Then the following two statements are equivalent :*

(i) *There is a positive number r such that for any $x \in X$ there will be $\tau = \tau(x) \geq 0$ for which $|x\tau| < r$; here $x\tau := \pi(\tau, x)$.*

(ii) *Dynamical system* $(E, \mathbb{R}_+, \pi)$ *is compact dissipative and*

$$\lim_{t\to+\infty} \sup_{|x|\le R} \rho(xt, J) = 0 \tag{26}$$

for any $R > 0$, *where* J *is a Levinson's centre of dynamical system* $(E, \mathbb{R}_+, \pi)$, *that is, the nonautonomous system* $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ *admits a compact global attractor* J.

A dynamical system $(X, \mathbb{R}_+, \pi)$ satisfies conditions of Ladyzhenskaya (see [23] and also [13]) if for any bounded set $A \subset X$ there is a compact $K_A \subset X$ such that the equality (24) holds.

Theorem 3.6. *([11],[13]) Let* $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ *be a nonautonomous dynamical system and let* $(E, \mathbb{R}_+, \pi)$ *satisfy the condition of Ladyzhenskaya. Then the statements 1. and 2. of Theorem 3.5 are equivalent.*

Applying the above general theorems about nonautonomous dissipative systems to nonautonomous system constructed in the example 3.1, we will obtain series of facts concerning the nonautonomous Lorenz system (4). In particular, from Theorems 2.2, 3.3 and 3.6, we have the following results.

Theorem 3.7. *(Compact global attractor) Let* Ω *be a compact metric space,* $(\Omega, \mathbb{R}, \sigma)$ *be a dynamical system on* Ω *and the conditions (5) and (6) are fulfilled. If the cocycle* φ *generated by nonautonomous Lorenz system (4) is asymptotically compact, then for every* $\omega \in \Omega$, *there exists a non-empty compact connected set* $I_\omega \subset H$ *such that the following conditions hold:*

(i) *The set* $I = \cup\{I_\omega \ : \ \omega \in \Omega\}$ *is compact and connected in* H;
(ii)
$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(U(t, \omega^{-t})M, I) = 0$$
for any bounded set $M \subset H$, *where* $U(t, \omega) = \varphi(t, \cdot, \omega)$ *and* β *is the semi-distance of Hausdorff;*
(iii) $U(t, \omega)I_\omega = I_{\omega t}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$;
(iv) I_ω *consists of those and only those points* $x \in H$ *through which passing the bounded solutions (on* $\mathbb{R}$*) of the nonautonomous Lorenz system (4).*

This theorem states that $I = \cup\{I_\omega \ : \ \omega \in \Omega\}$ is the compact global attractor of the nonautonomous Lorenz system (4) and also characterizes the structure of the sections I_ω of the attractor.

Theorem 3.8. *(Flow estimate on sections of global attractor) Under conditions of Theorem 3.7*

$$|\varphi(t, x, \omega)| \le \frac{\|f\|}{\alpha} \tag{27}$$

for all $t \in \mathbb{R}$, $\omega \in \Omega$ *and* $x \in I_\omega$, *where* φ *is the cocycle generated by Lorenz nonautonomous system (4). This establishes the flow estimate on each section of the compact global attractor.*

Proof. According to Theorem 3.3 the set $J = \bigcup\{I_\omega \times \{\omega\} : \omega \in \Omega\}$ is a Levinson's centre of dynamical system $(X, \mathbb{R}_+, \pi)$ and according to (25) for any point $(u_0, y_0) =$

$z \in J$ there exists $t_n \to +\infty, u_n \in H$ and $\omega_n \in \Omega$ such that the sequence $\{u_n\}$ is bounded, $u_0 = \lim_{n\to+\infty} \varphi(t_n, u_n, \omega_n)$ and $\omega_0 = \lim_{n\to+\infty} \omega_n t_n$. From the inequality (15), it follows that $|u_0| \le \frac{\|f\|}{\alpha}$, i.e. $\varphi(t,x,\omega) \in I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, hence $|\varphi(t,x,\omega)| \le \frac{\|f\|}{\alpha}$ for any $t \in \mathbb{R}, x \in I_\omega$ and $\omega \in \Omega$. The theorem is proved. □

Theorem 3.9. *(Convergence Theorem) Let φ be the cocycle generated by the Lorenz nonautonomous system (4). Under conditions of Theorem 3.7 and further assume that $\alpha^{-2} C_B \|f\| < 1$. Then this cocycle φ is convergent, i.e. for any $\omega \in \Omega$ the set I_ω contains a single point u_ω.*

Proof. Let $\omega \in \Omega$ and $u_1, u_2 \in I_\omega$. We define $\psi(t) = \varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)$ and

$$w(t) = |\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|^2. \tag{28}$$

According to Theorem 3.8, the function $w(t)$ is bounded on $\mathbb{R}$. On the other hand, in view of (10) and (5), we have

$$w'(t) \le -2\alpha w(t) + 2Re\langle B(\omega t)(\psi(t), \varphi(t, u_2, \omega)), \psi(t)\rangle. \tag{29}$$

From the inequalities (9), (29) and Theorem 3.8, it follows that

$$w'(t) \le -2\alpha w(t) + 2C_B \frac{\|f\|}{\alpha} w(t).$$

Hence, $w(t) \le w(0) e^{-2(\alpha - C_B \frac{\|f\|}{\alpha})t}$, i.e.

$$|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)| \le |u_1 - u_2| e^{-(\alpha - \frac{\|f\|}{\alpha} C_B)t}$$

for all $t \ge 0$, $\omega \in \Omega$ and $u_1, u_2 \in I_\omega$. In particular,

(30)

$$|u_1 - u_2| \le |\varphi(t, \varphi(-t, u_1, \sigma(-t,\omega)), \omega) - \varphi(t, \varphi(-t, u_2, \sigma(-t,\omega)), \omega)| e^{-(\alpha - \frac{\|f\|}{\alpha} C_B)t}$$

for all $t \ge 0, \omega \in \Omega$ and $u_1, u_2 \in J_\omega$. Note that $|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|$ is bounded on $\mathbb{R}$. Thus from (26) it follows that $u_1 = u_2$, where $\varphi(-t, x, \omega) := u_{\sigma(-t,\omega)}$ for all $x \in I_\omega,\ t \ge 0$ and $\omega \in \Omega$. The theorem is proved. □

4. Almost periodic and recurrent solutions of nonautonomous Lorenz systems

In this section, we discuss almost periodic and recurrent solutions of nonautonomous Lorenz systems. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$ and $(X, \mathbb{T}, \pi)$ be a dynamical system. The point $x \in X$ is called a stationary (τ-periodic, $\tau > 0, \tau \in \mathbb{T}$) point, if $xt = x$ ($x\tau = x$ respectively) for all $t \in \mathbb{T}$, where $xt := \pi(t, x)$.

A number $\tau \in \mathbb{T}$ is called $\varepsilon > 0$ shift (almost period) of point $x \in X$ if $\rho(x\tau, x) < \varepsilon$ ($\rho(x(\tau + t), xt) < \varepsilon$, for all $t \in \mathbb{T}$, respectively).

A point $x \in X$ is called almost recurrent (almost periodic) if for any $\varepsilon > 0$, there exists a positive number l such that on any segment of length l, there is a ε shift (almost period) of point $x \in X$.

If a point $x \in X$ is almost recurrent and the set $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

The solution $\varphi(t,x,\omega)$ of nonautonomous Lorenz system (4) is called recurrent (almost periodic, quasi-periodic, periodic), if the point $(x,\omega)\in H\times\Omega$ is a recurrent (almost periodic, quasi-periodic, periodic) point of skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ $(X=H\times\Omega$ and $\pi=(\varphi,\sigma))$.

We note (see, for example, [26],[27] and [24]) that if $\omega\in\Omega$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of dynamical system $(\Omega,\mathbb{R},\sigma)$ and $h:\Omega\to X$ is a homomorphism of dynamical system $(\Omega,\mathbb{R},\sigma)$ onto $(X,\mathbb{R}_+,\pi)$, then the point $x=h(\omega)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X,\mathbb{R}_+,\pi)$.

Let $X=H\times\Omega$ and $\pi=(\varphi,\sigma)$, then mapping $h:\Omega\to X$ is a homomorphism of dynamical system $(\Omega,\mathbb{R},\sigma)$ onto $(X,\mathbb{R}_+,\pi)$ if and only if $h(\omega)=(u(\omega),\omega)$ for all $\omega\in\Omega$, where $u:\Omega\to H$ is a continuous mapping with the condition that $u(\omega t)=\varphi(t,u(\omega),\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}_+$.

Theorem 4.1. *Let Ω be a compact metric space, the cocycle φ, generated by the nonautonomous Lorenz system (4), is asymptotic compact and the conditions (5), (7)-(8) are fulfilled with $\frac{\|f\|_{C_B}}{\alpha^2}<1$. Then the set I_ω contains a unique point x_ω ($I_\omega=\{x_\omega\}$) for every $\omega\in\Omega$, the mapping $u:\Omega\to H$ defined by $u(\omega):=x_\omega$ is continuous and $u(\omega t)=\varphi(t,u(\omega),\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}_+$.*

Proof. According to Theorems 3.3 and 3.9, it is sufficient to show that the mapping $u:\Omega\to H$ defined above is continuous. Let $\omega\in\Omega,\{\omega_n\}\subseteq\Omega$ and $\omega_n\to\omega$. Consider the sequence $\{x_n\}:=\{x_{\omega_n}\}\subset I:=\bigcup\{I_\omega\mid\omega\in\Omega\}$. Since the set I is compact, then the sequence $\{x_n\}$ is precompact. Let x' be a limit point of this sequence, then there is a subsequence $\{x_{k_n}\}$ such that $x_{k_n}\to x'$. Let J be a Levinson's centre of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$, generated by the cocycle φ. Note that the point $(x_{k_n},\omega_{k_n})\in J_{\omega_{k_n}}:=I_{\omega_{k_n}}\times\{\omega_{k_n}\}\subseteq J$ and taking in the consideration that J is compact we obtain that $(x',\omega)\in J$. Thus $(x',\omega)\in J_\omega=I_\omega\times\{\omega\}$ and, consequently, $x'\in I_\omega=\{x_\omega\}$, i.e. the precompact sequence $\{x_n\}$ has a unique limit point x_ω. This means that the sequence $\{x_n\}$ converges to x_ω as $n\to+\infty$. The theorem is proved. ☐

Corollary 4.2. *Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal or periodic minimal) set of dynamical system $(\Omega,\mathbb{R},\sigma)$. Then under the conditions of Theorem 4.1, the nonautonomous Lorenz system (4) admits a compact global attractor I, and for all $\omega\in\Omega$, the section I_ω of the attractor contains a unique point x_ω through which passes a recurrent (almost periodic, quasi-periodic, or periodic) solution of equation (4).*

Let H be a d-dimensional complex Euclidean space, i.e. $H=\mathbb{C}^d$. Denote by $HC(\mathbb{C}^d\times\Omega,\mathbb{C}^d)$ the space of all continuous functions $f:\mathbb{C}^d\times\Omega\to\mathbb{C}^d$ holomorphic in $z\in\mathbb{C}^d$ and equipped with compact-open topology. Consider the differential equation

$$\frac{dz}{dt}=f(z,\sigma_t\omega),\quad(\omega\in\Omega)\tag{31}$$

where $f\in HC(\mathbb{C}^d\times\Omega,\mathbb{C}^d)$. Let $\varphi(t,\omega,z)$ be the solution of equation (31) passing through point z at $t=0$ and defined on $\mathbb{R}^+$. The mapping $\varphi:\mathbb{R}^+\times\Omega\times\mathbb{C}^d\to\mathbb{C}^d$ has the following properties (see, for example, [14] and [19]):

a) $\varphi(0, z, \omega) = z$ for all $z \in \mathbb{C}^d$.

b) $\varphi(t + \tau, z, \omega) = \varphi(t, \varphi(\tau, z, \omega), \sigma_\tau \omega)$ for all $t, \tau \in \mathbb{R}^+, \omega \in \Omega$ and $z \in \mathbb{C}^d$.

c) Mapping φ is continuous.

d) Mapping $U(t, \omega) := \varphi(t, \cdot, \omega) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for any $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

The cocycle $\langle \mathbb{C}^d, \varphi, (\Omega, \mathbb{T}, \theta)\rangle$ is called (see [5],[10],[12],[13]) $\mathbb{C}$-analytic if the mapping $U(t, \omega) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Example 4.3. *Let $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ be a dynamical system of translations on $HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ (Bebutov's dynamical system (see, for example, [26] and [13])). Denote by F the mapping from $\mathbb{C}^d \times HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ to $\mathbb{C}^d$ defined by equality $F(z, f) := f(0, z)$ for all $z \in \mathbb{C}^d$ and $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$. Let Ω be the hull $H(f)$ of given function $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$, that is $\Omega = H(f) := \overline{\{f_\tau | \tau \in \mathbb{R}\}}$, where $f_\tau(t, z) := f(t + \tau, z)$ for all $t, \tau \in \mathbb{R}$ and $z \in \mathbb{C}^d$. Denote the restriction of $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ on Ω by $(\Omega, \mathbb{R}, \sigma)$. Then, under appropriate restriction on the given function $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$, the differential equation $\frac{dz}{dt} = f(z, t) = F(z, \sigma_t f)$ generates a $\mathbb{C}-$analytic cocycle.*

Theorem 4.4. *([12]) Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal, or periodic minimal) set of dynamical system $(\Omega, \mathbb{R}, \sigma)$, and let $\langle \mathbb{C}^d, \varphi, (\Omega, \mathbb{T}, \theta)\rangle$ be a $\mathbb{C}-$analytic cocycle admitting a compact global attractor $\{I_\omega | \quad \omega \in \Omega\}$. Then the following assertions hold:*

(i) *For every $\omega \in \Omega$, the set I_ω consists of a unique point $u(\omega)$.*
(ii) *$u(\sigma_t \omega) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.*
(iii) *The mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma := (u, Id_\Omega)$.*
(iv) *Every point $\gamma(\omega)$ is recurrent (almost periodic, quasi-periodic or periodic).*
(v) *The continuous invariant section ν is global uniformly asymptotically stable, i.e.*
  a. *The fact that for arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\rho(z, \nu(\omega)) < \delta$, implies $\rho(\varphi(t, \omega, z), \nu(\sigma_t \omega)) < \varepsilon$ for all $t \geq 0$ and $\omega \in \Omega$.*
  b.
$$\lim_{t \to +\infty} \rho(\varphi(t, z, \omega), u(\sigma_t \omega)) = 0$$
  for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$.

Theorem 4.5. *Let $H = \mathbb{C}^d$, Ω be a compact minimal set and the conditions (5), (7)-(8) are fulfilled. Then the nonautonomous Lorenz system admits a compact global attractor $\{I_\omega \mid \omega \in \Omega\}$ and the set I_ω contains a unique point x_ω ($I_\omega = \{x_\omega\}$) for every $\omega \in \Omega$, the mapping $u : \Omega \to H$ defined by equality $u(\omega) := x_\omega$ is continuous and $u(\omega t) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where φ is a cocycle generated by the nonautonomous Lorenz system.*

Proof. We note that under the conditions of Theorem 4.5 the right-hand side $f(\omega, z) := A(\omega)z + B(\omega)(z, z) + f(\omega)$ is $\mathbb{C}$-analytic because $D_z f(\omega, z)h = A(\omega)h + B(\omega)(h, z) + B(\omega)(z, h)$ for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$, where $D_z f(\omega, z)$ is a derivative of function $f(\omega, z)$ w.r.t. $z \in \mathbb{C}^d$. Now our statement directly results from Theorems 3.7 and 4.4. The proof is complete. ∎

Corollary 4.6. *(Amost periodic and recurrent motions) Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal or periodic minimal) set of dynamical system $(\Omega, \mathbb{R}, \sigma)$. Then under the conditions of Theorem 4.5, the nonautonomous Lorenz system (4) admits a compact global attractor I and for all $\omega \in \Omega$, the set I_ω contains a unique point x_ω through which passes a recurrent (almost periodic, quasi-periodic or periodic) solution of equation (4).*

5. Uniform averaging principle

Now we consider a uniform averaging principle for a general class of differential equations. In the next section, we apply this averaging principle to the nonautonomous Lorenz system (4).

Let $C(\mathbb{R} \times H, H)$ be the space of all continuous functions $f : \mathbb{R} \times H \to H$ equipped with compact open topology and let $\mathcal{F} \subseteq C(\mathbb{R} \times H, H)$. In Hilbert space H (with the norm $|\cdot|$ induced by the scalar product) we will consider the family of equations

$$x' = \varepsilon f(t, x), \quad f \in \mathcal{F}, \tag{32}$$

containing a small parameter $\varepsilon \in [0, \varepsilon_0]$ $(\varepsilon_o > 0)$.

We assume that on the set $\mathbb{R}_+ \times B[0, r]$, where $B[0, r] := \{x \in H \mid |x| \le r\}$ is a ball of radius $r > 0$ in H, the functions $f \in \mathcal{F}$ are uniformly bounded, i.e. there exists a positive constant M such that

$$|f(t, x)| \le M \tag{33}$$

for every $f \in \mathcal{F}$, $t \in \mathbb{R}_+$ and $x \in B[0, r]$, and satisfies the condition of Lipschitz

$$|f(t, x_1) - f(t, x_2)| \le L|x_1 - x_2| \ (x_1, x_2 \in B[0, r]) \tag{34}$$

with a constant $L > 0$ depending neither on $t \in \mathbb{R}_+$ nor on $f \in \mathcal{F}$.

Furthermore, we assume that the mean value of f is uniform with respect to (w.r.t.) $f \in \mathcal{F}$ and $x \in B[0, r]$

$$f_0(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t, x) dt, \tag{35}$$

i.e. for every $\varepsilon > 0$ there exists a $l = l(\varepsilon) > 0$ such that

$$|\frac{1}{T} \int_0^T f(t, x) dt - f_0(x)| < \varepsilon \tag{36}$$

for all $T \ge l(\varepsilon)$, $x \in B[0, r]$ and $f \in \mathcal{F}$, and the function f_0 does not depend on $f \in \mathcal{F}$.

Lemma 5.1. *The condition (35) holds if and only if there exists a decreasing continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying the condition $m(t) \to 0$ as $t \to 0$, such that*

$$|\frac{1}{T} \int_0^T f(t, x) dt - f_0(x)| \le m(T) \tag{37}$$

for all $T > 0$, $x \in B[0, r]$ and $f \in \mathcal{F}$. The function m depends neither on $x \in B[0, r]$ nor on $f \in \mathcal{F}$.

Proof. Denote by

$$k(T) := \sup_{f\in\mathcal{F}, x\in B[0,r]} |\frac{1}{T}\int_0^T f(t,x)dt - f_0(x)|. \tag{38}$$

The mapping k possesses the following properties:

(i) $0 \le k(T) \le 2M$, where $M := \sup\{|f(t,x)| \ : \ f \in \mathcal{F}, |x| \le r\}$;
(ii) $k(T) \to 0$ as $T \to +\infty$.

Let

$$c_n := \sup_{T\ge n} k(T),$$

then $c_o \ge c_1 \ge ... \ge c_n \ge ...$ and $c_n \to 0$ as $n \to +\infty$. Define now the function $m : \mathbb{R}_+ \to \mathbb{R}_+$ by the equality

$$m(t) := c_{n-1} + (t-n)(c_n - c_{n-1}) \quad (n \le t \le n+1, n = 0,1,...),$$

where $c_{-1} := c_0 + 1$. The lemma is proved. □

Lemma 5.2. *Let $\mathcal{F} \subseteq C(\mathbb{R}\times E, E)$ be a family of functions satisfying the condition (35), then for every $L > 0$*

$$l(\varepsilon) := \sup\{|\int_0^\tau f(\frac{t}{\varepsilon},x)dt - f_0(x)| \ : \ 0 \le \tau \le L, \ f \in \mathcal{F}, \ |x| \le r\} \to 0$$

as $\varepsilon \to 0$.

Proof. According to Lemma 5.1 there exists a decreasing continuous function m : $\mathbb{R}_+ \to \mathbb{R}_+$ with the condition $m(t) \to 0$ as $t \to 0$ and such that the inequlity (37) holds. Let $\nu \in (0,1)$, then

$$\begin{aligned} l(\varepsilon) &\le \sup\{|\int_0^\tau f(\frac{t}{\varepsilon},x)dt| \ : \ 0 \le \tau \le \varepsilon^\nu, \ f \in \mathcal{F}, \ |x| \le r\} + \\ &\sup\{|\int_0^\tau f(\frac{t}{\varepsilon},x)dt| \ : \ \varepsilon^\nu \le \tau \le L, \ f \in \mathcal{F}, \ |x| \le r\} = \\ &\sup\{\tau|\frac{\varepsilon}{\tau}\int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt| \ : \ 0 \le \tau \le \varepsilon^\nu, \ f \in \mathcal{F}, \ |x| \le r\} + \\ &\sup\{\tau|\frac{\varepsilon}{\tau}\int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt| \ : \ \varepsilon^\nu \le \tau \le L, \ f \in \mathcal{F}, \ |x| \le r\} \le \\ &m(0)\varepsilon^\nu + Lm(\varepsilon^{\nu-1}) \to 0 \end{aligned} \tag{39}$$

as $\varepsilon \to 0$. The lemma is proved. □

Under the assumptions above, it is expedient to consider along with equation (32) the *averaged* equation

$$x' = \varepsilon f_o(x). \tag{40}$$

From (35) we see that the function f_0 also satisfies the conditions (33) and (34). Let $\varphi(t,x)$ $(0 \le t \le T_0)$ be a solution of equation

$$y' = f_0(y). \tag{41}$$

taking values in $B[0,r]$ and passing through the point x at the initial moment $t = 0$. Then, as can easily be seen, the function $\varphi(t,x,\varepsilon) := \varphi(\varepsilon t, x)$ is the solution of equation (41) on the interval $0 \le t \le \frac{T_0}{\varepsilon}$. We will establish below a connection between $\varphi(t,x,\varepsilon)$ and the solution $\varphi(t,x,f,\varepsilon)$ of equation (32) with the initial condition $\varphi(0,x,f,\varepsilon)=x$.

More precisely, we will prove the following assertion.

Theorem 5.3. *(Uniform averaging principle) Suppose that on* $\mathbb{R}_+ \times B[0,r]$*, functions* $f \in \mathcal{F}$ *satisfy the conditions (33)-(35). Then for any* $\eta > 0$ *there exists an* $\varepsilon > 0$ *(*$0 < \varepsilon < \varepsilon_0$*) such that the estimate*

$$|\varphi(t,x,f,\varepsilon) - \varphi(t,x,\varepsilon)| \le \eta \ (0 \le t \le \frac{T_0}{\varepsilon})$$

holds uniformly w.r.t. $f \in \mathcal{F}$ *and* $x \in B[0,r]$.

Denote by $\mathcal{K}$ the family of all solutions (bounded by r) $x : [0,T_0] \to B[0,r]$ of the equation (41). Let us prove an auxiliary assertion.

Lemma 5.4. *Let* $\mathcal{F} \subseteq C(\mathbb{R} \times H, H)$ *be a family of functions satisfying the conditions (33)-(35). Then the equality*

$$\lim_{\varepsilon \to 0} \int_0^\tau f(\frac{s}{\varepsilon}, x(s))ds = \int_0^\tau f_0(x(s))ds \quad (0 < \tau \le T_0)$$

holds uniformly w.r.t. $x \in \mathcal{K}$, $\tau \in [0,T_0]$ *and* $f \in \mathcal{F}$.

Proof. Observe that

$$\lim_{\varepsilon \to 0} \int_0^\tau f(\frac{\tau}{\varepsilon}, x)d\tau = \tau f_0(x) \tag{42}$$

or, equivalently,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\tau} \int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt = f_0(x) \tag{43}$$

uniformly w.r.t. $x \in \mathcal{K}$, $\tau \in [0,T_0]$ and $f \in \mathcal{F}$. In fact, according to Lemma 5.2

$$|\frac{\varepsilon}{\tau} \int_0^{\frac{\varepsilon}{\tau}} f(t,x)dt - f_0(x)| \to 0$$

as $\varepsilon \to 0$ uniformly w.r.t. $x \in B[0,r]$, $f \in \mathcal{F}$ and $\tau \in [0,T_0]$. Let us note that the equality (43) is equivalent to (35). From (42) it follows that for any $\tau_1, \tau_2 \in [0,T_0]$ we have

$$\lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} f(\frac{\tau}{\varepsilon}, x)d\tau = \int_{\tau_1}^{\tau_2} f_0(x)d\tau$$

uniformly w.r.t. $x \in B[0,r]$, $\tau \in [0,T_0]$ and $f \in \mathcal{F}$. Hence for any $0 \le \tau_1 < \tau_2 < \ldots \tau_{n-1} < \tau_n = T_0$, $x_k \in B[0,r]$ $(k = 1,2,\ldots,n)$, we conclude that

$$\lim_{\varepsilon \to 0} \sum_1^n \int_{\tau_{k-1}}^{\tau_k} f(\tau, x_k, \varepsilon)d\tau = \sum_1^n \int_{\tau_{k-1}}^{\tau_k} f_0(x_k)d\tau \tag{44}$$

uniformly w.r.t. $x_1, x_2, \ldots, x_n \in B[0,r]$ and $f \in \mathcal{F}$.

If we introduce the step functions $\tilde{x}_n(\tau) := x(\tau_k)$ $(\tau_{k-1} \le \tau \le \tau_k;\ \tau_k - \tau_{k-1} = \frac{1}{n};\ k = 1,2,\ldots,n$ and $x \in \mathcal{K})$, then from the equality (44), we have the following relation

$$\lim_{\varepsilon\to 0}\int_0^\tau f(\frac{s}{\varepsilon},\tilde{x}_n(s))ds=\int_0^\tau f_0(\tilde{x}_n(s))ds. \tag{45}$$

Under our assumption the family of functions $\mathcal{K}$ is equicontinuous on $[0,T_0]$ and, consequently,

$$\sup_{x\in\mathcal{K}}\sup_{0\le\tau\le T_0}\|\tilde{x}_n(\tau)-x(\tau)\|\to 0 \tag{46}$$

as $n\to+\infty$. Using the condition of Lipschitz (34) for the family of functions $\mathcal{F}$ we obtain the estimate

$$|\int_0^\tau f(\frac{s}{\varepsilon},x(s))ds-\int_0^\tau f_0(x(s))ds|\le\int_0^\tau|f(\frac{s}{\varepsilon},x(s))-f(\frac{s}{\varepsilon},\tilde{x_n}(s))|ds+ \tag{47}$$
$$|\int_0^\tau[f(\frac{s}{\varepsilon},\tilde{x_n}(s))-f_0(\tilde{x_n}(s))]ds|+\int_0^\tau|f_0(x(s))-f_0(\tilde{x_n}(s))|ds\le$$
$$2LT_0\sup_{x\in\mathcal{K}}\sup_{0\le\tau\le T_0}|\tilde{x}_n(\tau)-x(\tau)|+|\int_0^\tau[f(\frac{s}{\varepsilon},\tilde{x_n}(s))-f_0(\tilde{x_n}(s))]ds|.$$

From (44) - (47) immediately we obtain the results in the lemma. □

Proof. of Theorem 5.2. Now we will prove Theorem 5.3. Denote by $\psi(\tau,x,f,\varepsilon)$ (respectively $\bar{\psi}(\tau,x)$) a unique solution of equation

$$x'=f(\frac{\tau}{\varepsilon},x) \tag{48}$$

(respectively (41)) passing through point $x\in B[0,r]$ at the moment $\tau=0$ and defined on $[0,\frac{T_0}{\varepsilon}]$. The functions $\psi(\tau,x,f,\varepsilon)$ and $\bar{\psi}(\tau,x)$ satisfy the integral equations

$$\psi(\tau,x,f,\varepsilon)=x+\int_0^\tau f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon))ds$$

and

$$\bar{\psi}(\tau,x)=x+\int_0^\tau f_0(\bar{\psi}(s,x))ds,$$

respectively. Using the condition of Lipschitz (34), we obtain the estimate

$$|\psi(\tau,x,f,\varepsilon)-\bar{\psi}(\tau,x)|=|\int_0^\tau[f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon))-f_0(\bar{\psi}(s,x))]ds|\le$$
$$\int_0^\tau|f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon))-f(\frac{s}{\varepsilon},\bar{\psi}(s,x))|ds+|\int_0^\tau[f(\frac{s}{\varepsilon},\bar{\psi}(s,x))-f_0(\bar{\psi}(s,x))]ds|\le$$
$$L\int_0^\tau|\psi(s,x,f,\varepsilon)-\bar{\psi}(s,x)|ds+c(\varepsilon),$$

where

$$c(\varepsilon):=\sup_{0\le\tau\le T_0,x\in\mathcal{K}}|\int_0^\tau[f_0\frac{s}{\varepsilon},x(s))-f_0(x(s))]ds|.$$

According to the Gronwall inequality (see, for example,[16] or [19]), we can now conclude that

$$|\psi(\tau,x,f,\varepsilon)-\bar{\psi}(\tau,x)|\le\exp(2L\tau)c(\varepsilon)$$

and it remains only to note that in virtue of Lemma 5.4, $c(\varepsilon)\to 0$ as $\varepsilon\to 0$ and

$$|x(t,\varepsilon)-y(\varepsilon t)|=|\psi(\tau,x,f,\varepsilon)-\bar{\psi}(\tau,x)|\le\exp(2L\tau)c(\varepsilon)=\exp(2L\varepsilon t))c(\varepsilon)$$

for all $t\in[0,\frac{T_0}{\varepsilon}]$. The theorem is thus proved. □

In the next section, we will also need the following lemma.

Lemma 5.5. *Let $\mathcal{F}$ be a transitive subset of $C(\mathbb{R} \times H, H)$, i.e. there exists a function $g \in \mathcal{F}$ such that $\mathcal{F} = H(g)$, the hull of g. Then the following two assertions are equivalent:*

(i) *There exists $f_0 \in C(H, H)$ such that*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t, x) dt = f_0(x)$$

uniformly w.r.t. $f \in \mathcal{F}$ and $x \in B[0, r]$;

(ii) *There exists $f_0 \in C(H, H)$ such that*

$$\lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} g(\tau, x) d\tau = f_0(x)$$

uniformly w.r.t $t \in \mathbb{R}$ and $x \in B[0, r]$.

Proof. It is evident that (i) implies (ii) because $g_t \in \mathcal{F}$ for all $t \in \mathbb{R}$ and, consequently,

$$\frac{1}{T} \int_t^{t+T} g(\tau, x) d\tau = \frac{1}{T} \int_0^T g(t + \tau, x) d\tau \to f_0(x)$$

as $T \to +\infty$ uniformly w.r.t $t \in \mathbb{R}$ and $x \in B[0, r]$.

Let now $\varepsilon > 0$ and $f \in \mathcal{F} = H(g)$, then there exists a sequence $\{t_n\} \subset \mathbb{R}$ and $L(\varepsilon) > 0$ such that $g_{t_n} \to f$ and

$$|\frac{1}{T} \int_0^T g(\tau + t_n, x) d\tau - f_0(x)| < \varepsilon \tag{49}$$

for all $T > L(\varepsilon)$. Passing to limit as $n \to +\infty$ in the inequality (49) we obtain

$$|\frac{1}{T} \int_0^T f(\tau, x) d\tau - f_0(x)| \leq \varepsilon$$

for all $T > L(\varepsilon)$. From the latter inequality, the required statement immediately follows. This proves the lemma. ☐

Remark 5.6. *All the results of this section are true for arbitrary Banach space too, not only for Hilbert space.*

6. Global averaging principle for the nonautonomous Lorenz systems

Now we consider a global averaging principle for the nonautonomous Lorenz systems. Let Ω be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω. We consider the "perturbed" nonautonomous Lorenz equation

$$\frac{dx}{dt} = \varepsilon A(\omega t)x + \varepsilon B(\omega t)(x, x) + \varepsilon f(\omega t), \tag{50}$$

where $\varepsilon \in [0, \varepsilon_0]$ ($\varepsilon_0 > 0$) is a small parameter. Suppose that the conditions (5)–(8) are fulfilled and the following averaging values exist uniformly w.r.t. $\omega \in \Omega$:

$$\overline{A} = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} A(\omega t) dt, \tag{51}$$

$$\overline{B} = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} B(\omega t) dt, \tag{52}$$

and

$$\overline{f} = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(\omega t) dt. \tag{53}$$

Remark 6.1. *The conditions* (51)$-$(53) *are fulfilled if a dynamical system* $(\Omega, \mathbb{R}, \sigma)$ *is strictly ergodic, i.e. there exists on* Ω *a unique invariant measure* μ *w.r.t.* $(\Omega, \mathbb{R}, \sigma)$*.*

Along with equation (50), we will also consider the averaged equation

$$\frac{dx}{dt} = \varepsilon \overline{A} x + \varepsilon \overline{B}(x, x) + \varepsilon \overline{f}. \tag{54}$$

If we introduce the "slow time" $\tau := \varepsilon t$ ($\varepsilon > 0$), then the equations (50) and 54) can be written as

$$\frac{dx}{d\tau} = A(\omega \frac{\tau}{\varepsilon}) x + B(\omega \frac{\tau}{\varepsilon})(x, x) + f(\omega \frac{\tau}{\varepsilon}) \tag{55}$$

and

$$\frac{dx}{d\tau} = \overline{A} x + \overline{B}(x, x) + \overline{f}. \tag{56}$$

Remark 6.2. *a. From the conditions (7) and (52) it follows that*

$$Re\langle \overline{B}(u, v), v \rangle = 0 \tag{57}$$

for all $u, v \in H$*;*

b. From the inequality (5) it follows that

$$Re\langle \overline{A} x, x)\rangle \leq -\alpha |x|^2 \tag{58}$$

for all $x \in H$*.*

Theorem 6.3. *Assume the conditions enumerated above are all satisfied. Then for all* $T > 0$ *and* $\rho \geq r_0 := \frac{\|f\|}{\alpha} > 0$*, the solution for the nonautonomous Lorenz equation (50) approaches the solution of the averaged Lorenz equation (54) in the following sense:*

$$\max\{|\varphi(t, x, \omega, \varepsilon) - \overline{\varphi}(t, x, \varepsilon)| : \ 0 \leq t \leq T/\varepsilon, \ |x| \leq \rho, \ \omega \in \Omega\} \to 0 \tag{59}$$

as $\varepsilon \to 0$*, where* $\varphi(t, x, \omega, \varepsilon)$ *(respectively* $\overline{\varphi}(t, x, \varepsilon)$*) is a solution of equation (50) (respectively (54)), passing through point* x *at the initial moment* $t = 0$*.*

Proof. According to Theorem 2.2, we have $|\varphi(t,x,\omega,\varepsilon)| \le \rho$ and $\bar{\varphi}(t,x,\varepsilon)| \le \rho$ for all $t \ge 0$, $|x| \le \rho, \omega \in \Omega$ and $\varepsilon \in (0,\varepsilon_0]$. If we take $\mathcal{F} := \{F_\omega \mid \omega \in \Omega\} \subset C(\mathbb{R}\times H, H)$, where $f_\omega(t,x) := A(\omega t)x + B(\omega t)(x,x) + f(\omega t)$ for all $t \in \mathbb{R}$ and $x \in H$, then the relation (59) follows from Theorem 5.3. This completes the proof. □

Theorem 6.4. *(Global averaging principle for nonautonomous Lorenz systems) Let φ_ε be a cocycle generated by the equation (50). Assume the conditions enumerated above are all satisfied. If the cocycle $\varphi_\varepsilon(\varepsilon \in [0,\varepsilon_0])$ is asymptotically compact, then the following assertions hold:*

(i) *The averaged equation (56) admits a compact global attractor $\overline{I} \subset H$;*
(ii) *For every $\varepsilon \in (0,\varepsilon_0]$ the equation (50) has a compact global attractor $\{I^\varepsilon_\omega \mid \omega \in \Omega\}$;*
(iii) *The set $I = \cup\{I^\varepsilon \mid \varepsilon \in [0,\varepsilon_0]\}$ is bounded, where $I^0 = \overline{I}$ and $I^\varepsilon = \cup\{I^\varepsilon_\omega \mid \omega \in \Omega\}$;*
(iv)

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega} \beta(I^\varepsilon_\omega, \overline{I}) = 0 \tag{60}$$

and, in particular,

$$\lim_{\varepsilon\to 0} \beta(I^\varepsilon, \overline{I}) = 0.$$

Proof. The first three statements of the theorem follow from Theorems 2.2, 3.7 and Remark 6.2. Now we will prove the fourth statement of the theorem. To this end, we will use the same arguments as in [20, 8]. Let $\lambda > 0$ and $B(\overline{I},\lambda) = \{x \in H \mid \rho(x,\overline{I}) < \lambda\}$. According to orbital stability of the set $\overline{I}$ (see, for example, [18, Ch.I] or Theorem 1.2.4 from [13]), for given λ there exists $\delta = \delta(\lambda) > 0$ (we may consider $\delta(\lambda) < \lambda/2$) such that

$$\overline{\varphi}(t, B(\overline{I},\delta)) \subset B(\overline{I},\lambda/2) \tag{61}$$

for all $t \ge 0$. In virtue of boundedness of the set $I = \cup\{I^\varepsilon \mid 0 \le \varepsilon \le \varepsilon_0\}$ we may choose $\rho \le r_0$ such that $I \subset B(0,\rho) = \{x \in H \mid |x| < \rho\}$. Since $\overline{I}$ is a compact global attractor of the system (56), then for the closed ball $B[0,\rho] := \{x \in H| \, |x| \le \rho\}$ and the number $\delta > 0$ there exists $T = T(\rho,\delta) > 0$ such that

$$\overline{\varphi}(t, B[0,\rho]) \subset B(\overline{I},\delta/2), \quad t \ge T. \tag{62}$$

Let $x \in B[0,\rho]$. Then in virtue of Theorem 6.3 for the numbers $\rho \ge r_0$ and $T(\rho,\delta) > 0$ there exists $\mu = \mu(\rho,\delta) > 0$ such that $0 < \varepsilon \le \mu$, $m(\varepsilon) < \lambda/2$ (see (59)), i.e.

$$|\varphi(t,x,\omega,\varepsilon) - \overline{\varphi}(t,x)| < \delta/2 \tag{63}$$

for all $x \in B[0,\rho]$, $\omega \in \Omega$, $t \in [0,T/\varepsilon]$ and $0 < \varepsilon \le \mu$. According to (62) we have $\overline{\varphi}(T/\varepsilon,x,\omega,\varepsilon) \in B(\overline{I},\delta/2)$. Thus, taking into account (63), we obtain $\varphi(T/\varepsilon,x,\omega,\varepsilon) \in B(\overline{I},\delta)$. Let us take the initial point $x_1 := \varphi(T/\varepsilon,x,\omega,\varepsilon)$ and we will repeat for this point the same reasoning as above. Taking into consideration the equality $\varphi(t,x,\sigma(T/\varepsilon,\omega),\varepsilon) = \varphi(t+T/\varepsilon,x,\omega,\varepsilon)$, we will have

$$|\varphi(t+T/\varepsilon,x,\omega,\varepsilon)| = |\overline{\varphi}(t,x_1)| < \delta/2 \tag{64}$$

for all $t \in [0,T/\varepsilon]$, $x \in B[0,\rho]$ and $\omega \in \Omega$, where $x_1 = \varphi(T/\varepsilon,x,\omega,\varepsilon)$.

By the inequality (64) we obtain again $x_2 := \varphi(2T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \delta)$ and, consequently,

$$\varphi(t + T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \lambda/2 + \delta/2) \subset B(\overline{I}, \lambda).$$

If we continue this process and later (in virtue of uniformity w.r.t. $|x| \le \rho$ and $\omega \in \Omega$ of the estimation (63) it is possible), we will obtain

$$\varphi(t, x, \omega, \varepsilon) \in B(\overline{I}, \lambda) \tag{65}$$

for all $t \ge T/\varepsilon$, $x \in B[0, \rho]$, $\omega \in \Omega$ and $o \le \varepsilon \le \mu$ and, consequently,

$$\varphi(t, x, \sigma(-t, \omega), \varepsilon) \in B(\overline{I}, \lambda)$$

for all $t \ge T/\varepsilon$ and $|x| \le \rho$. Since $I = \cup\{I^\varepsilon \mid 0 \le \varepsilon \le \varepsilon_0\} \subseteq B(0, \rho)$, then according to Theorem 3.3

$$I^\varepsilon_\omega = \bigcap_{t \ge 0} \overline{\bigcup_{\tau \ge t} \varphi(\tau, B[0, \rho], \sigma(-\tau, \omega), \varepsilon)}.$$

Therefore, from (65) we have $I^\varepsilon_\omega \subset B(\overline{I}, \lambda)$ for all $\omega \in \Omega$ and $0 < \varepsilon < \mu$. Note that λ is arbitrarily chosen. Hence from the last inclusion we obtain the equality (60). The theorem is proved. □

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