RECURRENT MOTIONS AND GLOBAL ATTRACTORS OF NONAUTONOMOUS LORENZ SYSTEMS

DAVID CHEBAN AND JINQIAO DUAN

Abstract. This article is devoted to the study of dynamics of a nonautonomous Lorenz systems. These systems are formulated and investigated in the context of nonautonomous dynamical systems. First, we prove that such systems admit a compact global attractor and characterize its structure. Then, we obtain conditions of convergence, under which all solutions of the nonautonomous Lorenz systems approach a point attractor. Third, we derive a criterion for existence of almost periodic, quasi-periodic, periodic, and recurrent motions. Finally, we prove a global averaging principle for nonautonomous Lorenz systems.

1. Introduction

The following n-dimensional systems of differential equations are called *autonomous* Lorenz systems [25]:

$$u_i' = \Sigma_{j,k} b_{ijk} u_j u_k + \Sigma_j a_{ij} u_j + f_i, \ i = 1, 2, ..., n, \tag{1}$$

where $\Sigma b_{ijk} u_i u_j u_k$ is identically equal to zero, $\Sigma a_{ij} u_i u_j$ is negative definite, and f_i are constants. The well-known three-dimensional Lorenz system for geophysical flows or climate modeling [20] is a special case of this type of systems. In fact, the

three- dimensional Lorenz systemm is a three-mode t

truncation

of fluid equations for convection.

It is known that solutions of (1) imbed in some ellipsoid and do not leave it later, i.e. the autonomous system (1) is dissipative, and hence admits a compact global attractor.

In the vector-matrix form the system (1) may be written as:

$$u' = Au + B(u, u) + f, \tag{2}$$

where A is a positive definite matrix and $B : H \times H \to H$ (H is a n-dimensional real or complex Euclidean space) is a bilinear form satisfying the condition

$$Re\langle B(u, v), w\rangle = -Re\langle B(u, w), v\rangle \tag{3}$$

for every $u, v, w \in H$.

Date: October 11, 2013.

1991 *Mathematics Subject Classification.* primary:34C35, 34D20, 34D40, 34D45, 58F10, 58F12, 58F39; secondary: 35B35, 35B40.

Key words and phrases. Nonautonomous dynamical system, skew–product flow, global attractor, asymptotic stability, Lorenz systems, almost periodic solutions, global averaging principle.

When f is not constant but a bounded function of time t, it is known that the equation (2) also admits a compact global attractor [17].

The aim of the present article is to study the *nonautonomous* version of the equation (2). Namely, in this case, the matrix A, the bilinear form B, and the function f all depend on time t. This nonautonomous Lorenz system may arise from, for example, an $n-$mode Galerkin truncation (as in [20, 26]) of the *nonautonomous* Navier-Stokes equations. Note that the nonautonomous Navier-Stokes equations may arise, say, when homogenizing a time-dependent boundary condition or when reformulating the momentum equations along a known unsteady flow (moving the known unsteady flow to the zero flow). We also treat the evolution equation (2) as a model for developing nonautonomous dynamical systems ideas about almost periodic and recurrent motions, attractors and global averaging principle. Thus this class of systems are not only interesting for applications, but also interesting for the theory of nonautonomous differential equations. Moreover, this is a class of nonlinear nonautonomous dynamical systems that we have good understanding of asymptotic behavior and recurrent behavior. When these Lorenz systems are subject to small disturbances or perturbations, an averaging principle is desired.

We will consider issues like compact global attractors, convergence, almost periodic (including periodic and quasi-periodic) motions and recurrent motions, and averaging principles.

In the last 10-15 years, there have been interesting works on global attractors of nonautonomous systems; see [9, 10, 11, 24] and references therein. There rae also recent works on global attractors of stochastic orr random dynamical systems [1, 4, 5, 13, 22]. In this article we only consider the deterministic finite dimensional nonautonomous dynamical systems.

This paper is organized as follows:

In Section 2 we introduce a class of nonautonomous Lorenz dynamical systems and establish its dissipativity (Theorem 2.2).

In Section 3 we prove that asymptotic compact Lorenz systems admit a compact global attractor (Theorem 3.7) and we characterize the structure of the global attractor. Furthermore, we obtain conditions for convergence of these systems (Theorem 3.9), under which each section of the global attractor contains a single point.

Section 4 is devoted to study of existence of almost periodic (periodic, quasi-periodic) and recurrent solutions of nonautonomous Lorenz systems (Corollaries 4.2 and 4.6).

In Section 5 we prove a uniform averaging principle for a class of nonautonomous dynamical systems (Theorem 5.3). With the help of this uniform averaging principle, we prove a global averaging principle for nonautonomous Lorenz systems on the semi-axis (Theorem 6.4) in Section 6.

2. Nonautonomous Lorenz systems

Let Ω be a compact metric space, $\mathbb{R} = (-\infty, +\infty)$, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω and H be a real or complex Hilbert space. We denote $L(H)$ $(L^2(H))$ the space of all linear (bilinear) endomorphisms on H. When W is some metric space, $C(\Omega, W)$ denotes the space of all continuous functions $f : \Omega \to W$, endowed with the topology of uniform convergence.

Let us consider the nonautonomous Lorenz system

$$u' = A(\omega t)u + B(\omega t)(u, u) + f(\omega t), \quad \omega \in \Omega, \tag{4}$$

where $\omega t := \sigma(t,\omega)$, $A \in C(\Omega, L(H)), B \in C(\Omega, L^2(H))$ and $f \in C(\Omega, H)$. Note that when the autonomous Lorenz system (2) is perturbed by periodic, quasi-periodic, almost periodic or recurrent forces, it can then be written as (4). Moreover, we assume that the following conditions are fulfilled:

(i) There exists $\alpha > 0$ such that

$$Re\langle A(\omega)u, u\rangle \le -\alpha|u|^2 \tag{5}$$

for all $\omega \in \Omega$ and $u \in H$, where $|\cdot|$ is a norm in H, generated by the scalar product $\langle\cdot,\cdot\rangle$;

(ii)

$$Re\langle B(\omega)(u,v), w\rangle = -Re\langle B(\omega)(u,w), v\rangle \tag{6}$$

for every $u, v, w \in H$ and $\omega \in \Omega$.

Remark 2.1. *a. It follows from (6) that*

$$Re\langle B(\omega)(u,v), v\rangle\rangle = 0 \tag{7}$$

for every $u, v \in H$ and $\omega \in \Omega$.

b. From bilinearity and continuity, we obtain

$$|B(\omega)(u,v)| \le C_B|u||v| \tag{8}$$

for all $u, v \in H$ and $\omega \in \Omega$, where $C_B = \sup\{|B(\omega)(u,v)| : \omega \in \Omega,\ u,v \in H,\ |u| \le 1,$ and $|v| \le 1\}$.

We will call the system (4) with conditions (5) and (6) a nonautonomous Lorenz system or a nonautonomous system of hydrodynamic type.

We note that from the conditions (6) -(8) it follows that

$$|B(\omega)(x_1,x_1) - B(\omega)(x_2,x_2)| \le C_B(|x_1| + |x_2|)|x_1 - x_2| \tag{9}$$

for all $x_1, x_2 \in H$ and $\omega \in \Omega$.

Since the coefficients of (4) are locally Lipschitzian with respect to $u \in H$, through every point $x \in H$ passes a unique solution $\varphi(t,x,\omega)$ of equation (4) at the initial moment $t = 0$. And this solution is defined on some interval $[0, t_{(x,\omega)})$. Let us note that

$$\begin{aligned} w'(t) &= 2Re\langle \varphi'(t,x,\omega), \varphi(t,x,\omega)\rangle = 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + \\ &\quad 2Re\langle B(\omega t)(\varphi(t,x,\omega), \varphi(t,x,\omega)), \varphi(t,x,y)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &= 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &\le -2\alpha|\varphi(t,x,\omega)|^2 + 2\|f\||\varphi(t,x,\omega)|, \end{aligned} \tag{10}$$

where $\|f\| := \max\{|f(\omega)| : \omega \in \Omega\}$ and $w(t) = |\varphi(t,x,\omega)|^2$. Then

$$w' \le -2\alpha w + 2\|f\|w^{\frac{1}{2}}. \tag{11}$$

Thus

$$w(t) \le v(t) \tag{12}$$

for all $t \in [0, t_{(x,\omega)})$, where $v(t)$ is a solution of equation

$$v' = -2\alpha v + 2||f||v^{\frac{1}{2}}, \tag{13}$$

satisfying condition $v(0) = w(0) = |x|^2$. Hence

$$v(t) = [(|x| - \frac{\|f\|}{\alpha})e^{-\alpha t} + \frac{\|f\|}{\alpha}]^2 \tag{14}$$

and consequently

$$|\varphi(t,x,\omega)| \le (|x| - \frac{\|f\|}{\alpha})e^{-\alpha t} + \frac{\|f\|}{\alpha} \tag{15}$$

for all $t \in [0, t_{(x,\omega)})$. It follows from the inequality (15) that solution $\varphi(t,x,\omega)$ is bounded and therefore it may be extended to a global solution on $\mathbb{R}_+ = [0, +\infty)$.

Thus we have proved the following theorem.

Theorem 2.2. *(Dissipativity) Let the conditions (5) and (6) are fulfilled. Then the following statements hold:*

(i)

$$|\varphi(t,x,\omega)| \le C(|x|), \tag{16}$$

for all $t \ge 0,\ \omega \in \Omega$ *and* $x \in H$, *where* $C(r) = r$ *if* $r \ge r_0 := \frac{\|f\|}{\alpha}$ *and* $C(r) = r_0$ *if* $r \le r_0$*;*

(ii)

$$\limsup_{t\to+\infty} \sup\{|\varphi(t,x,\omega)| \ : |x| \le r, \omega \in \Omega\} \le \frac{||f||}{\alpha} \tag{17}$$

for every $r > 0$.

The item (i) in this Theorem means that the nonautonomous Lorenz flow is bounded on bounded sets, while the item (ii) implies that the nonautonomous Lorenz system is dissipative, i.e., it admits a bounded absorbing set.

3. Nonautonomous attractors and their structure

Let Ω and W be two metric spaces and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on Ω. Let us consider a continuous mapping $\varphi : \mathbb{R}^+ \times W \times \Omega \to W$ satisfying the following conditions:

$$\varphi(0,\cdot,\omega) = id_W \quad \varphi(t+\tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \omega\tau)$$

for all $t, \tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in W$. Here $\omega\tau$ is the short notation for $\sigma_\tau(\omega) := \sigma(\tau, \omega)$. Such a mapping φ (or more explicitly $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$) is called a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fiber W; see [1, 23].

Example 3.1. *Let* E *be a Banach space and* $C(\mathbb{R} \times E, E)$ *be a space of all continuous functions* $F : \mathbb{R} \times E \to E$ *equipped by the compact-open topology. Let us consider a parameterized differential equation*

$$\frac{dx}{dt} = F(\sigma_t\omega, x), \quad \omega \in \Omega$$

on a Banach space E *with* $\Omega = C(\mathbb{R} \times E, E)$, *where* $\sigma_t\omega := \sigma(t,\omega)$. *We will define* $\sigma_t : \Omega \to \Omega$ *by* $\sigma_t\omega(\cdot,\cdot) = \omega(t+\cdot,\cdot)$ *for each* $t \in \mathbb{R}$ *and interpret* $\varphi(t,x,\omega)$ *as solution of the initial value problem*

$$\frac{d}{dt}x(t) = F(\sigma_t\omega, x(t)), \quad x(0) = x. \tag{18}$$

Under appropriate assumptions on $F : \Omega \times E \to E$ *(or even* $F : \mathbb{R} \times E \to E$ *with* $\omega(t)$ *instead of* $\sigma_t\omega$ *in (18)) to ensure forward existence and uniqueness, then* φ *is a cocycle on* $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ *with fiber* E.

Let φ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with the fiber E. Then the mapping $\pi : \mathbb{R}^+ \times E \times \Omega \to E \times \Omega$ defined by

$$\pi(t, x, \omega) := (\varphi(t, x, \omega), \sigma_t \omega)$$

for all $t \in \mathbb{R}^+$ and $(x, \omega) \in E \times \Omega$ forms a semi-group $\{\pi(t, \cdot, \cdot)\}_{t \in \mathbb{R}^+}$ of mappings of $X := \Omega \times E$ into itself, thus a semi-group dynamical system on the state space X, which is called a skew-product flow [23]. The triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ (where $h := pr_2 : X \to \Omega$) is a nonautonomous dynamical system; see [3, 9].

A cocycle φ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W is called a compact (bounded) dissipative cocycle, if there is a nonempty compact set $K \subseteq W$ such that

$$\lim_{t \to +\infty} \sup\{\beta(U(t, \omega)M, K) \mid \omega \in \Omega\} = 0 \tag{19}$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(W)$), where $C(W)(\mathcal{B}(W))$ denotes the family of all compact (bounded) subsets of W, β is the semi-distance of Hausdorff and $U(t, \omega) := \varphi(t, \cdot, \omega)$. We can similarly define a compact or bounded dissipative skew-product system.

Lemma 3.2. *Let Ω be a compact metric space and $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ be a cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W. In order for $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ to be compact (bounded) dissipative, it is necessary and sufficient that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is compact (bounded) dissipative.*

This assertion directly follows from the corresponding definitions (see for example [14],[9]).

We now define whole trajectories of the semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ (or whole trajectories of the cocycle $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W). A whole trajectory passes through the point $x \in X((u, y) \in W \times \Omega)$ is a continuous mapping $\gamma : \mathbb{R} \to X$ (or $\nu : \mathbb{R} \to W$) which satisfies the conditions : $\gamma(0) = x$ (or $\nu(0) = u$) and $\pi^t \gamma(\tau) = \gamma(t + \tau)$ (or $\nu(t + \tau) = \varphi(t, \nu(\tau), \omega\tau)$) for all $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$.

Moreover, for $M \subseteq W$, we denote by

$$\Omega_\omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M, \omega^{-\tau})} \tag{20}$$

for every $\omega \in \Omega$, where $\omega^{-\tau} := \sigma(-\tau, \omega)$. This formula is useful in the construction of global attractors. We recall the following result.

Theorem 3.3. *([7],[9]) Let Ω be a compact metric space, $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ be a compact (bounded) dissipative cocycle and K be the nonempty compact set in the dissipation property (19). Then the following assertions hold:*

(i) *The set $I_\omega := \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and*

$$\lim_{t \to +\infty} \beta(U(t, \omega^{-t})K, I_\omega) = 0 \tag{21}$$

for every $\omega \in \Omega$;

(ii) *$U(t, \omega)I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$;*

(iii)

$$\lim_{t \to +\infty} \beta(U(t, \omega^{-t})M, I_\omega) = 0 \tag{22}$$

for all $M \in C(W)$ (respectively $M \in \mathcal{B}(X)$) and $\omega \in \Omega$;

(iv)

$$\lim_{t\to+\infty} \sup\{\beta(U(t,\omega^{-t})M, I) | \omega \in \Omega\} = 0 \quad (23)$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(X)$) , where $I = \cup\{I_\omega \ : \ \omega \in \Omega\}$;

(v) *$I_\omega := pr_1 J_\omega$ for all $\omega \in \Omega$, where J is a Levinson's centre of $(X, \mathbb{R}_+, \pi)$, and, hence, $I = pr_1 J$;*

(vi) *The set I is compact;*

(vii) *The set I is connected if the spaces W and Ω are connected.*

Note that a Levinson's centre is defined in [12, 15]. Now we define the concept of compact global attractors. The family of compact sets $\{I_\omega | \omega \in \Omega\}$ $(I_\omega \subset W$ is nonempty compact for every $\omega \in \Omega)$ is called the compact global attractor of cocycle φ if the following conditions are fulfilled [7]:

(i) The set $I := \bigcup\{I_\omega | \quad \omega \in \Omega\}$ is precompact, i.e. its closure is a compact set.

(ii) $\{I_\omega | \quad \omega \in \Omega\}$ is invariant w.r.t. the cocycle φ, i.e. $\varphi(t, \omega, I_\omega) = I_{\sigma_t \omega}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

(iii) The equality $\lim\limits_{t\to+\infty} \sup\limits_{\omega\in\ \Omega} \beta(\varphi(t, K, \omega), I) = 0$ holds for every nonempty bounded set $K \subset W$.

The set I_ω will be called a *section* of the global attractor.

Corollary 3.4. *Under the conditions of Theorem 3.3, the cocycle φ admits a compact global attractor.*

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called asymptotic compact (see [14]) if for any positively invariant bounded set $A \subset X$, there is a compact $K_A \subset X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t A, K_A) = 0. \quad (24)$$

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called compact

or completely continuous, if for every bounded set $A \subset X$ there exists a positive number $l = l(A)$ such that the set $\pi^l A$ is precompact, i.e., the closure of this set is compact.

It is easy to verify (see for example [9]) that every compact dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotic compact.

The cocycle $\langle W, \varphi, (Y, \mathbb{R}, \sigma \rangle$ is called compact (asymptotic compact, respectively) if the associated skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ with $X = W \times Y$ and $\pi = (\varphi, \sigma)$ is compact (asymptotic compact, respectively).

Let $(X, \mathbb{R}_+, \pi)$ be compact dissipative and K be a compact set, which attracts all compact subsets of X. Let

$$J = \Omega(K), \quad (25)$$

where $\Omega(K) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi^\tau K}$. The set J defined by the equality (25) does not depend on selection of the attracting set K, and is characterized only by the properties of the dynamical system $(X, \mathbb{R}_+, \pi)$ itself. The set J is called the Levinson's centre of the compact dissipative system $(X, \mathbb{R}_+, \pi)$.

Theorem 3.5. *([7],[9]) Let (E, Ω, h) be a local-trivial Banach fibering, $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ be a nonautonomous dynamical system and the dynamical system $(E, \mathbb{R}_+, \pi)$ be completely continuous. Then the following two statements are equivalent :*

(i) *There is a positive number r such that for any $x \in X$ there will be $\tau = \tau(x) \geq 0$ for which $|x\tau| < r$; here $x\tau := \pi(\tau, x)$.*

(ii) *Dynamical system* $(E, \mathbb{R}_+, \pi)$ *is compact dissipative and*

$$\lim_{t\to+\infty} \sup_{|x|\le R} \rho(xt, J) = 0 \tag{26}$$

for any $R > 0$*, where* J *is a Levinson's centre of dynamical system* $(E, \mathbb{R}_+, \pi)$*, that is, the nonautonomous system* $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ *admits a compact global attractor* J*.*

A dynamical system $(X, \mathbb{R}_+, \pi)$ satisfies conditions of Ladyzhenskaya (see [18] and also [9]) if for any bounded set $A \subset X$ there is a compact $K_A \subset X$ such that the equality (24) holds.

Theorem 3.6. *(*[7],[9]*) Let* $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ *be a nonautonomous dynamical system and let* $(E, \mathbb{R}_+, \pi)$ *satisfy the condition of Ladyzhenskaya. Then the statements (i) and (ii) of Theorem 3.5 are equivalent.*

Applying the above general theorems about nonautonomous dissipative systems to nonautonomous system constructed in the example 3.1, we will obtain series of facts concerning the nonautonomous Lorenz system (4). In particular, from Theorems 2.2, 3.3 and 3.6, we have the following results.

Theorem 3.7. *(Compact global attractor) Let* Ω *be a compact metric space,* $(\Omega, \mathbb{R}, \sigma)$ *be a dynamical system on* Ω *and the conditions (5) and (6) are fulfilled. If the cocycle* φ *generated by nonautonomous Lorenz system (4) is asymptotic compact, then for every* $\omega \in \Omega$*, there exists a non-empty compact connected set* $I_\omega \subset H$ *such that the following conditions hold:*

(i) *The set* $I = \cup\{I_\omega \ : \ \omega \in \Omega\}$ *is compact and connected in* H*;*

(ii)
$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(U(t, \omega^{-t})M, I) = 0$$
for any bounded set $M \subset H$*, where* $U(t, \omega) = \varphi(t, \cdot, \omega)$ *and* β *is the semi-distance of Hausdorff;*

(iii) $U(t, \omega)I_\omega = I_{\omega t}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$*;*

(iv) I_ω *consists of those and only those points* $x \in H$*, such that a bounded solution (on* $\mathbb{R}$*) of the nonautonomous Lorenz system (4) passes through* x*.*

This theorem states that $I = \cup\{I_\omega \ : \ \omega \in \Omega\}$ is the compact global attractor of the nonautonomous Lorenz system (4) and also characterizes the structure of the sections I_ω of the attractor.

Theorem 3.8. *(Flow estimate on sections of global attractor) Under conditions of Theorem 3.7*

$$|\varphi(t, x, \omega)| \le \frac{\|f\|}{\alpha} \tag{27}$$

for all $t \in \mathbb{R}$*,* $\omega \in \Omega$ *and* $x \in I_\omega$*, where* φ *is the cocycle generated by Lorenz nonautonomous system (4). This establishes the flow estimate on each section of the compact global attractor.*

Proof. According to Theorem 3.3 the set $J = \bigcup\{I_\omega \times \{\omega\} : \omega \in \Omega\}$ is a Levinson's centre of dynamical system $(X, \mathbb{R}_+, \pi)$ and according to (25) for any point $(u_0, y_0) = z \in J$ there exists $t_n \to +\infty, u_n \in H$ and $\omega_n \in \Omega$ such that the sequence $\{u_n\}$ is bounded, $u_0 = \lim_{n\to+\infty} \varphi(t_n, u_n, \omega_n)$ and $\omega_0 = \lim_{n\to+\infty} \omega_n t_n$. From the inequality (15), it follows that $|u_0| \le \frac{\|f\|}{\alpha}$, i.e. $\varphi(t, x, \omega) \in I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, hence $|\varphi(t, x, \omega)| \le \frac{\|f\|}{\alpha}$ for any $t \in \mathbb{R}, x \in I_\omega$ and $\omega \in \Omega$. The theorem is proved. □

Theorem 3.9. *(Convergence Theorem) Let* φ *be the cocycle generated by the Lorenz nonautonomous system (4). Under conditions of Theorem 3.7 and further assume that* $\alpha^{-2}C_B\|f\| < 1$*. Then this cocycle* φ *is convergent, i.e. for any* $\omega \in \Omega$ *the set* I_ω *contains a single point* u_ω*.*

Proof. Let $\omega \in \Omega$ and $u_1, u_2 \in I_\omega$. We define $\psi(t) = \varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)$ and

$$w(t) = |\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|^2. \tag{28}$$

According to Theorem 3.8, the function $w(t)$ is bounded on $\mathbb{R}$. On the other hand, in view of (10) and (5), we have

$$w'(t) \leq -2\alpha w(t) + 2Re\langle B(\omega t)(\psi(t), \varphi(t, u_2, \omega)), \psi(t)\rangle. \tag{29}$$

From the inequalities (9), (29) and Theorem 3.8, it follows that

$$w'(t) \leq -2\alpha w(t) + 2C_B \frac{\|f\|}{\alpha} w(t).$$

Hence, $w(t) \leq w(0)e^{-2(\alpha - C_B \frac{\|f\|}{\alpha})t}$, i.e.

$$|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)| \leq |u_1 - u_2| e^{-(\alpha - \frac{\|f\|}{\alpha} C_B)t}$$

for all $t \geq 0$, $\omega \in \Omega$ and $u_1, u_2 \in I_\omega$. In particular,

$$|u_1 - u_2| \leq |\varphi(t, \varphi(-t, u_1, \sigma(-t, \omega)), \omega) - \varphi(t, \varphi(-t, u_2, \sigma(-t, \omega)), \omega)| e^{-(\alpha - \frac{\|f\|}{\alpha} C_B)t} \tag{30}$$

for all $t \geq 0, \omega \in \Omega$ and $u_1, u_2 \in J_\omega$. Note that $|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|$ is bounded on $\mathbb{R}$. Thus from (26) it follows that $u_1 = u_2$, where $\varphi(-t, x, \omega) := u_{\sigma(-t,\omega)}$ for all $x \in I_\omega$, $t \geq 0$ and $\omega \in \Omega$. The theorem is proved. □

4. Almost periodic motions and recurrent motions

In this section, we discuss almost periodic and recurrent solutions of nonautonomous Lorenz systems. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$ and $(X, \mathbb{T}, \pi)$ be a dynamical system. Let ρ be a metric on X. The point $x \in X$ is called a stationary (τ-periodic, $\tau > 0, \tau \in \mathbb{T}$) point, if $xt = x$ ($x\tau = x$ respectively) for all $t \in \mathbb{T}$, where $xt := \pi(t, x)$.

A number $\tau \in \mathbb{T}$ is called $\varepsilon > 0$ shift (almost period) of point $x \in X$ if $\rho(x\tau, x) < \varepsilon$ ($\rho(x(\tau + t), xt) < \varepsilon$, for all $t \in \mathbb{T}$, respectively).

A point $x \in X$ is called almost recurrent (almost periodic) if for any $\varepsilon > 0$, there exists a positive number l such that on any segment of length l, there is a ε shift (almost period) of point $x \in X$.

If a point $x \in X$ is almost recurrent and the set $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

The solution $\varphi(t, x, \omega)$ of nonautonomous Lorenz system (4) is called recurrent (almost periodic, quasi-periodic, periodic), if the point $(x, \omega) \in H \times \Omega$ is a recurrent (almost periodic, quasi-periodic, periodic) point of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ ($X = H \times \Omega$ and $\pi = (\varphi, \sigma)$).

We note (see, for example, [21] and [19]) that if $\omega \in \Omega$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of dynamical system $(\Omega, \mathbb{R}, \sigma)$ and $h : \Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$, then the point $x = h(\omega)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{R}_+, \pi)$.

Let $X = H \times \Omega$ and $\pi = (\varphi, \sigma)$, then mapping $h : \Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$ if and only if $h(\omega) = (u(\omega), \omega)$ for all $\omega \in \Omega$, where $u : \Omega \to H$ is a continuous mapping with the condition that $u(\omega t) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

Theorem 4.1. *Let Ω be a compact metric space and suppose the cocycle φ, generated by the nonautonomous Lorenz system (4), is asymptotic compact. If the conditions (5), (7)-(8) are fulfilled with $\frac{\|f\|C_B}{\alpha^2} < 1$, then the set I_ω contains a unique point x_ω ($I_\omega = \{x_\omega\}$) for every $\omega \in \Omega$, the mapping $u : \Omega \to H$ defined by $u(\omega) := x_\omega$ is continuous and $u(\omega t) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.*

Proof. According to Theorems 3.3 and 3.9, it is sufficient to show that the mapping $u : \Omega \to H$ defined above is continuous. Let $\omega \in \Omega, \{\omega_n\} \subseteq \Omega$ and $\omega_n \to \omega$. Consider the sequence $\{x_n\} := \{x_{\omega_n}\} \subset I := \bigcup\{I_\omega \mid \omega \in \Omega\}$. Since the set I is compact, then the sequence $\{x_n\}$ is precompact. Let x' be a limit point of this sequence, then there is a subsequence $\{x_{k_n}\}$ such that $x_{k_n} \to x'$. Let J be a Levinson's centre of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$, generated by the cocycle φ. Note that the point $(x_{k_n}, \omega_{k_n}) \in J_{\omega_{k_n}} := I_{\omega_{k_n}} \times \{\omega_{k_n}\} \subseteq J$ and taking in the consideration that J is compact we obtain that $(x', \omega) \in J$. Thus $(x', \omega) \in J_\omega = I_\omega \times \{\omega\}$ and, consequently, $x' \in I_\omega = \{x_\omega\}$, i.e. the precompact sequence $\{x_n\}$ has a unique limit point x_ω. This means that the sequence $\{x_n\}$ converges to x_ω as $n \to +\infty$. The theorem is proved. □

Corollary 4.2. *Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal or periodic minimal) set of dynamical system $(\Omega, \mathbb{R}, \sigma)$. Then under the conditions of Theorem 4.1, the nonautonomous Lorenz system (4) admits a compact global attractor I, and for all $\omega \in \Omega$, the section I_ω of the attractor contains a unique point x_ω through which passes a recurrent (almost periodic, quasi-periodic, or periodic) solution of equation (4).*

Let H be a d-dimensional complex Euclidean space, i.e. $H = \mathbb{C}^d$. Denote by $HC(\mathbb{C}^d \times \Omega, \mathbb{C}^d)$ the space of all continuous functions $f : \mathbb{C}^d \times \Omega \to \mathbb{C}^d$ holomorphic in $z \in \mathbb{C}^d$ and equipped with compact-open topology. Consider the differential equation

$$\frac{dz}{dt} = f(z, \sigma_t\omega), \quad (\omega \in \Omega) \tag{31}$$

where $f \in HC(\mathbb{C}^d \times \Omega, \mathbb{C}^d)$. Let $\varphi(t, \omega, z)$ be the solution of equation (31) passing through point z at $t = 0$ and defined on $\mathbb{R}^+$. The mapping $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{C}^d \to \mathbb{C}^d$ has the following properties (see, for example, [12] and [15]):

a) $\varphi(0, z, \omega) = z$ for all $z \in \mathbb{C}^d$.

b) $\varphi(t + \tau, z, \omega) = \varphi(t, \varphi(\tau, z, \omega), \sigma_\tau\omega)$ for all $t, \tau \in \mathbb{R}^+, \omega \in \Omega$ and $z \in \mathbb{C}^d$.

c) Mapping φ is continuous.

d) Mapping $U(t, \omega) := \varphi(t, \cdot, \omega) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for any $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

The cocycle $\langle \mathbb{C}^d, \varphi, (\Omega, \mathbb{T}, \theta)\rangle$ is called (see [8],[9]) $\mathbb{C}$-analytic if the mapping $U(t, \omega) : \mathbb{C}^d \to \mathbb{C}^d$ is holomorphic for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Example 4.3. *Let $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ be a dynamical system of translations on $HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ (Bebutov's dynamical system (see, for example, [21] and [9])). Denote by F the mapping from $\mathbb{C}^d \times HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$ to $\mathbb{C}^d$ defined by equality $F(z, f) := f(0, z)$ for all $z \in \mathbb{C}^d$ and $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$. Let Ω be the hull $H(f)$ of given function $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$, that is $\Omega = H(f) := \overline{\{f_\tau | \tau \in \mathbb{R}\}}$, where $f_\tau(t, z) := f(t + \tau, z)$ for all $t, \tau \in \mathbb{R}$ and $z \in \mathbb{C}^d$. Denote the restriction of $(HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d), \mathbb{R}, \sigma)$ on Ω by $(\Omega, \mathbb{R}, \sigma)$. Then, under appropriate restriction on the given function $f \in HC(\mathbb{R} \times \mathbb{C}^d, \mathbb{C}^d)$, the differential equation $\frac{dz}{dt} = f(z, t) = F(z, \sigma_t f)$ generates a $\mathbb{C}-$analytic cocycle.*

Theorem 4.4. *([8]) Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal, or periodic minimal) set of dynamical system $(\Omega, \mathbb{R}, \sigma)$, and let $\langle \mathbb{C}^d, \varphi, (\Omega, \mathbb{T}, \theta)\rangle$ be a $\mathbb{C}-$analytic cocycle admitting a compact global attractor $\{I_\omega | \quad \omega \in \Omega\}$. Then the following assertions hold:*

(i) *For every $\omega \in \Omega$, the set I_ω consists of a unique point $u(\omega)$.*
(ii) $u(\sigma_t\omega) = \varphi(t, u(\omega), \omega)$ *for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.*
(iii) *The mapping $\omega \to \gamma(\omega)$ is continuous, where $\gamma := (u, Id_\Omega)$.*
(iv) *Every point $\gamma(\omega)$ is recurrent (almost periodic, quasi-periodic or periodic).*
(v) *The continuous invariant section ν is global uniformly asymptotically stable, i.e.*
  a. *The fact that for arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\rho(z, \nu(\omega)) < \delta$, implies $\rho(\varphi(t, \omega, z), \nu(\sigma_t\omega)) < \varepsilon$ for all $t \geq 0$ and $\omega \in \Omega$.*
  b.
$$\lim_{t \to +\infty} \rho(\varphi(t, z, \omega), u(\sigma_t\omega)) = 0$$
  for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$.

Theorem 4.5. *Let $H = \mathbb{C}^d$, Ω be a compact minimal set and the conditions (5), (7)-(8) are fulfilled. Then the nonautonomous Lorenz system admits a compact global attractor $\{I_\omega \mid \omega \in \Omega\}$ and the set I_ω contains a unique point x_ω ($I_\omega = \{x_\omega\}$) for every $\omega \in \Omega$, the mapping $u : \Omega \to H$ defined by equality $u(\omega) := x_\omega$ is continuous and $u(\omega t) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where φ is a cocycle generated by the nonautonomous Lorenz system.*

Proof. We note that under the conditions of Theorem 4.5 the right-hand side $f(\omega, z) := A(\omega)z + B(\omega)(z, z) + f(\omega)$ is $\mathbb{C}$-analytic because $D_z f(\omega, z)h = A(\omega)h + B(\omega)(h, z) + B(\omega)(z, h)$ for all $\omega \in \Omega$ and $z \in \mathbb{C}^d$, where $D_z f(\omega, z)$ is a derivative of function $f(\omega, z)$ w.r.t. $z \in \mathbb{C}^d$. Now our statement directly results from Theorems 3.7 and 4.4. The proof is complete. □

Corollary 4.6. *(Almost periodic and recurrent motions) Let Ω be a compact minimal (almost periodic minimal, quasi-periodic minimal or periodic minimal) set of dynamical system $(\Omega, \mathbb{R}, \sigma)$. Then under the conditions of Theorem 4.5, the nonautonomous Lorenz system (4) admits a compact global attractor I and for all $\omega \in \Omega$, the set I_ω contains a unique point x_ω through which passes a recurrent (almost periodic, quasi-periodic or periodic) solution.*

5. Uniform averaging principle

Now we consider a uniform averaging principle for a general class of differential equations. In the next section, we apply this averaging principle to the nonautonomous Lorenz system (4).

Let $C(\mathbb{R} \times H, H)$ be the space of all continuous functions $f : \mathbb{R} \times H \to H$ equipped with compact open topology and let $\mathcal{F} \subseteq C(\mathbb{R} \times H, H)$. In Hilbert space H (with the norm $|\cdot|$ induced by the scalar product) we will consider the family of equations

$$x' = \varepsilon f(t, x), \quad f \in \mathcal{F}, \tag{32}$$

containing a small parameter $\varepsilon \in [0, \varepsilon_0]$ ($\varepsilon_o > 0$).

We assume that on the set $\mathbb{R}_+ \times B[0, r]$, where $B[0, r] := \{x \in H \mid |x| \leq r\}$ is a ball of radius $r > 0$ in H, the functions $f \in \mathcal{F}$ are uniformly bounded, i.e. there exists a positive constant M such that

$$|f(t, x)| \leq M \tag{33}$$

for every $f \in \mathcal{F}$, $t \in \mathbb{R}_+$ and $x \in B[0, r]$, and satisfies the condition of Lipschitz

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \ (x_1, x_2 \in B[0, r]) \tag{34}$$

with a constant $L > 0$ depending neither on $t \in \mathbb{R}_+$ nor on $f \in \mathcal{F}$.

Furthermore, we assume that the mean value of f is uniform with respect to (w.r.t.) $f \in \mathcal{F}$ and $x \in B[0,r]$

$$f_0(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t,x)dt, \tag{35}$$

i.e. for every $\varepsilon > 0$ there exists a $l = l(\varepsilon) > 0$ such that

$$|\frac{1}{T} \int_0^T f(t,x)dt - f_0(x)| < \varepsilon \tag{36}$$

for all $T \geq l(\varepsilon),\ x \in B[0,r]$ and $f \in \mathcal{F}$, and the function f_0 does not depend on $f \in \mathcal{F}$.

Lemma 5.1. *The condition (35) holds if and only if there exists a decreasing continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying the condition $m(t) \to 0$ as $t \to 0$, such that*

$$|\frac{1}{T} \int_0^T f(t,x)dt - f_0(x)| \leq m(T) \tag{37}$$

for all $T > 0,\ x \in B[0,r]$ and $f \in \mathcal{F}$. The function m depends neither on $x \in B[0,r]$ nor on $f \in \mathcal{F}$.

Proof. Denote by

$$k(T) := \sup_{f \in \mathcal{F}, x \in B[0,r]} |\frac{1}{T} \int_0^T f(t,x)dt - f_0(x)|. \tag{38}$$

The mapping k possesses the following properties:

(i) $0 \leq k(T) \leq 2M$, where $M := \sup\{|f(t,x)| \ :\ f \in \mathcal{F}, |x| \leq r\}$;
(ii) $k(T) \to 0$ as $T \to +\infty$.

Let

$$c_n := \sup_{T \geq n} k(T),$$

then $c_o \geq c_1 \geq ... \geq c_n \geq ...$ and $c_n \to 0$ as $n \to +\infty$. Define now the function $m : \mathbb{R}_+ \to \mathbb{R}_+$ by the equality

$$m(t) := c_{n-1} + (t-n)(c_n - c_{n-1}) \ \ (n \leq t \leq n+1, n = 0,1,...),$$

where $c_{-1} := c_0 + 1$. The lemma is proved. □

Lemma 5.2. *Let $\mathcal{F} \subseteq C(\mathbb{R} \times E, E)$ be a family of functions satisfying the condition (35), then for every $L > 0$*

$$l(\varepsilon) := \sup\{|\int_0^\tau f(\frac{t}{\varepsilon}, x)dt - f_0(x)| \ :\ 0 \leq \tau \leq L,\ f \in \mathcal{F},\ |x| \leq r\} \to 0$$

as $\varepsilon \to 0$.

Proof. According to Lemma 5.1 there exists a decreasing continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$ with the condition $m(t) \to 0$ as $t \to 0$ and such that the inequality (37) holds. Let $\nu \in (0,1)$, then

$$
\begin{aligned}
& l(\varepsilon) \le \sup\{|\int_0^\tau f(\frac{t}{\varepsilon}, x)dt| \ : \ 0 \le \tau \le \varepsilon^\nu, \ f \in \mathcal{F}, \ |x| \le r\} + \\
& \sup\{|\int_0^\tau f(\frac{t}{\varepsilon}, x)dt| \ : \ \varepsilon^\nu \le \tau \le L, \ f \in \mathcal{F}, \ |x| \le r\} = \\
& \sup\{\tau|\frac{\varepsilon}{\tau}\int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt| \ : \ 0 \le \tau \le \varepsilon^\nu, \ f \in \mathcal{F}, \ |x| \le r\} + \\
& \sup\{\tau|\frac{\varepsilon}{\tau}\int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt| \ : \ \varepsilon^\nu \le \tau \le L, \ f \in \mathcal{F}, \ |x| \le r\} \le \\
& m(0)\varepsilon^\nu + Lm(\varepsilon^{\nu-1}) \to 0
\end{aligned} \tag{39}
$$

as $\varepsilon \to 0$. The lemma is proved. □

Under the assumptions above, it is expedient to consider along with equation (32) the *averaged* equation

$$x' = \varepsilon f_o(x). \tag{40}$$

From (35) we see that the function f_0 also satisfies the conditions (33) and (34). Let $\varphi(t,x)$ $(0 \le t \le T_0)$ be a solution of equation

$$y' = f_0(y). \tag{41}$$

taking values in $B[0,r]$ and passing through the point x at the initial moment $t = 0$. Then, as can easily be seen, the function $\varphi(t,x,\varepsilon) := \varphi(\varepsilon t, x)$ is the solution of equation (41) on the interval $0 \le t \le \frac{T_0}{\varepsilon}$. We will establish a connection between $\varphi(t,x,\varepsilon)$ and the solution $\psi(t,x,f,\varepsilon)$ of the unaveraged equation (32) with the initial condition $\psi(0,x,f,\varepsilon) = x$.

More precisely, we will prove the following assertion.

Theorem 5.3. *(Uniform averaging principle) Suppose that on $\mathbb{R}_+ \times B[0,r]$ the function $f \in \mathcal{F}$ satisfy the conditions (33)-(35). Then for any $\eta > 0$ there exists an $\varepsilon > 0$ $(0 < \varepsilon < \varepsilon_0)$ such that the estimate*

$$|\psi(t,x,f,\varepsilon) - \varphi(t,x,\varepsilon)| \le \eta, \ \ 0 \le t \le \frac{T_0}{\varepsilon}$$

holds uniformly w.r.t. $f \in \mathcal{F}$ and $x \in B[0,r]$.

We need an auxiliary result in order to

prove this theorem. Denote by $\mathcal{K}$ the family of all solutions (bounded by r) $x : [0,T_0] \to B[0,r]$ of the equation (41).

Lemma 5.4. *Let $\mathcal{F} \subseteq C(\mathbb{R} \times H, H)$ be a family of functions satisfying the conditions (33)-(35). Then the equality*

$$\lim_{\varepsilon \to 0} \int_0^\tau f(\frac{s}{\varepsilon}, x(s))ds = \int_0^\tau f_0(x(s))ds, \quad 0 < \tau \le T_0$$

holds uniformly w.r.t. $x \in \mathcal{K}$, $\tau \in [0,T_0]$ and $f \in \mathcal{F}$.

Proof. Observe that

$$\lim_{\varepsilon \to 0} \int_0^\tau f(\frac{\tau}{\varepsilon}, x)d\tau = \tau f_0(x) \tag{42}$$

or, equivalently,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\tau}\int_0^{\frac{\tau}{\varepsilon}} f(t,x)dt = f_0(x) \tag{43}$$

uniformly w.r.t. $x\in\mathcal{K},\ \tau\in[0,T_0]$ and $f\in\mathcal{F}$. In fact, according to Lemma 5.2

$$|\frac{\varepsilon}{\tau}\int_0^{\frac{\varepsilon}{\tau}} f(t,x)dt - f_0(x)|\to 0$$

as $\varepsilon\to 0$ uniformly w.r.t. $x\in B[0,r],\ f\in\mathcal{F}$ and $\tau\in[0,T_0]$. Let us note that the equality (43) is equivalent to (35). From (42) it follows that for any $\tau_1,\tau_2\in[0,T_0]$ we have

$$\lim_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2} f(\frac{\tau}{\varepsilon},x)d\tau = \int_{\tau_1}^{\tau_2} f_0(x)d\tau$$

uniformly w.r.t. $x\in B[0,r],\ \tau\in[0,T_0]$ and $f\in\mathcal{F}$. Hence for any $0\le\tau_1<\tau_2<...\tau_{n-1}<\tau_n=T_0,\ x_k\in B[0,r]\ (k=1,2,...,n)$, we conclude that

$$\lim_{\varepsilon\to 0}\sum_1^n\int_{\tau_{k-1}}^{\tau_k} f(\tau,x_k,\varepsilon)d\tau = \sum_1^n\int_{\tau_{k-1}}^{\tau_k} f_0(x_k)d\tau \tag{44}$$

uniformly w.r.t. $x_1,x_2,...,x_n\in B[0,r]$ and $f\in\mathcal{F}$.

If we introduce the step functions $\tilde{x}_n(\tau):=x(\tau_k)\ \ (\tau_{k-1}\le\tau\le\tau_k;\ \tau_k-\tau_{k-1}=\frac{1}{n};\ k=1,2,...,n$ and $x\in\mathcal{K})$, then from the equality (44), we have the following relation

$$\lim_{\varepsilon\to 0}\int_0^\tau f(\frac{s}{\varepsilon},\tilde{x}_n(s))ds = \int_0^\tau f_0(\tilde{x}_n(s))ds. \tag{45}$$

Under our assumption the family of functions $\mathcal{K}$ is equicontinuous on $[0,T_0]$ and, consequently,

$$\sup_{x\in\mathcal{K}}\sup_{0\le\tau\le T_0}\|\tilde{x}_n(\tau)-x(\tau)\|\to 0 \tag{46}$$

as $n\to+\infty$. Using the condition of Lipschitz (34) for the family of functions $\mathcal{F}$ we obtain the estimate

$$\begin{aligned}
|\int_0^\tau f(\frac{s}{\varepsilon},x(s))ds-\int_0^\tau f_0(x(s))ds| &\le \int_0^\tau|f(\frac{s}{\varepsilon},x(s))-f(\frac{s}{\varepsilon},\tilde{x_n}(s))|ds+ \\
|\int_0^\tau[f(\frac{s}{\varepsilon},\tilde{x_n}(s))-f_0(\tilde{x_n}(s))]ds| &+ \int_0^\tau|f_0(x(s))-f_0(\tilde{x_n}(s))|ds\le \\
2LT_0\sup_{x\in\mathcal{K}}\sup_{0\le\tau\le T_0}|\tilde{x}_n(\tau)-x(\tau)| &+ |\int_0^\tau[f(\frac{s}{\varepsilon},\tilde{x_n}(s))-f_0(\tilde{x_n}(s))]ds|.
\end{aligned} \tag{47}$$

From (44) - (47) immediately we obtain the results in the lemma. □

Now we prove Theorem 5.3.

Proof. of Theorem 5.3. Denote by $\psi(\tau,x,f,\varepsilon)$ (respectively $\bar{\psi}(\tau,x)$) a unique solution of equation

$$x'=f(\frac{\tau}{\varepsilon},x) \tag{48}$$

(respectively (41)) passing through point $x\in B[0,r]$ at the moment $\tau=0$ and defined on $[0,\frac{T_0}{\varepsilon}]$. The functions $\psi(\tau,x,f,\varepsilon)$ and $\bar{\psi}(\tau,x)$ satisfy the integral equations

$$\psi(\tau,x,f,\varepsilon)=x+\int_0^\tau f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon))ds$$

and

$$\bar{\psi}(\tau,x) = x + \int_0^\tau f_0(\bar{\psi}(s,x))ds,$$

respectively. Using the condition of Lipschitz (34), we obtain the estimate

$$|\psi(\tau,x,f,\varepsilon) - \bar{\psi}(\tau,x)| = |\int_0^\tau [f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon)) - f_0(\bar{\psi}(s,x))]ds| \leq$$

$$\int_0^\tau |f(\frac{s}{\varepsilon},\psi(s,x,f,\varepsilon)) - f(\frac{s}{\varepsilon},\bar{\psi}(s,x))|ds + |\int_0^\tau [f(\frac{s}{\varepsilon},\bar{\psi}(s,x)) - f_0(\bar{\psi}(s,x))]ds| \leq$$

$$L\int_0^\tau |\psi(s,x,f,\varepsilon) - \bar{\psi}(s,x)|ds + c(\varepsilon),$$

where

$$c(\varepsilon) := \sup_{0\leq\tau\leq T_0, x\in\mathcal{K}} |\int_0^\tau [f_0\frac{s}{\varepsilon}, x(s)) - f_0(x(s))]ds|.$$

According to the Gronwall-Bellman inequality [15], we can now conclude that

$$|\psi(\tau,x,f,\varepsilon) - \bar{\psi}(\tau,x)| \leq \exp(2L\tau)c(\varepsilon)$$

and it remains only to note that in virtue of Lemma 5.4, $c(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$|x(t,\varepsilon) - y(\varepsilon t)| = |\psi(\tau,x,f,\varepsilon) - \bar{\psi}(\tau,x)| \leq \exp(2L\tau)c(\varepsilon) = \exp(2L\varepsilon t))c(\varepsilon)$$

for all $t \in [0, \frac{T_0}{\varepsilon}]$. The theorem is thus proved. □

In the next section, we will also need the following lemma.

Lemma 5.5. *Let $\mathcal{F}$ be a transitive subset of $C(\mathbb{R}\times H, H)$, i.e. there exists a function $g \in \mathcal{F}$ such that $\mathcal{F} = H(g)$, the hull of g. Then the following two assertions are equivalent:*

(i) *There exists $f_0 \in C(H,H)$ such that*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(t,x)dt = f_0(x)$$

uniformly w.r.t. $f \in \mathcal{F}$ and $x \in B[0,r]$;

(ii) *There exists $f_0 \in C(H,H)$ such that*

$$\lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} g(\tau,x)d\tau = f_0(x)$$

uniformly w.r.t $t \in \mathbb{R}$ and $x \in B[0,r]$.

Proof. It is evident that (i) implies (ii) because $g_t \in \mathcal{F}$ for all $t \in \mathbb{R}$ and, consequently,

$$\frac{1}{T}\int_t^{t+T} g(\tau,x)d\tau = \frac{1}{T}\int_0^T g(t+\tau,x)d\tau \to f_0(x)$$

as $T \to +\infty$ uniformly w.r.t $t \in \mathbb{R}$ and $x \in B[0,r]$.

Let now $\varepsilon > 0$ and $f \in \mathcal{F} = H(g)$, then there exists a sequence $\{t_n\} \subset \mathbb{R}$ and $L(\varepsilon) > 0$ such that $g_{t_n} \to f$ and

$$|\frac{1}{T}\int_0^T g(\tau+t_n,x)d\tau - f_0(x)| < \varepsilon \tag{49}$$

for all $T > L(\varepsilon)$. Passing to limit as $n \to +\infty$ in the inequality (49) we obtain

$$|\frac{1}{T}\int_0^T f(\tau,x)d\tau - f_0(x)| \leq \varepsilon$$

for all $T > L(\varepsilon)$. From the latter inequality, the required statement immediately follows. This proves the lemma. □

Remark 5.6. *All the results of this section are also true in an arbitrary Banach space, not only for Hilbert spaces.*

6. Global averaging principle

Now we consider a global averaging principle for the nonautonomous Lorenz systems. Let Ω be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω. We consider the "perturbed" nonautonomous Lorenz equation

$$\frac{dx}{dt} = \varepsilon A(\omega t)x + \varepsilon B(\omega t)(x,x) + \varepsilon f(\omega t), \tag{50}$$

where $\varepsilon \in [0, \varepsilon_0]$ ($\varepsilon_0 > 0$) is a small parameter. Suppose that the conditions (5)–(8) are fulfilled and the following averaging values exist uniformly w.r.t. $\omega \in \Omega$:

$$\overline{A} = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A(\omega t)dt, \tag{51}$$

$$\overline{B} = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} B(\omega t)dt, \tag{52}$$

and

$$\overline{f} = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(\omega t)dt. \tag{53}$$

Remark 6.1. *The conditions* (51) − (53) *are fulfilled if a dynamical system* $(\Omega, \mathbb{R}, \sigma)$ *is strictly ergodic, i.e. there exists on* Ω *a unique invariant measure* μ *w.r.t.* $(\Omega, \mathbb{R}, \sigma)$.

Along with equation (50), we will also consider the averaged equation

$$\frac{dx}{dt} = \varepsilon\overline{A}x + \varepsilon\overline{B}(x,x) + \varepsilon\overline{f}. \tag{54}$$

If we introduce the "slow time" $\tau := \varepsilon t$ ($\varepsilon > 0$), then the equations (50) and 54) can be written as

$$\frac{dx}{d\tau} = A(\omega\frac{\tau}{\varepsilon})x + B(\omega\frac{\tau}{\varepsilon})(x,x) + f(\omega\frac{\tau}{\varepsilon}) \tag{55}$$

and

$$\frac{dx}{d\tau} = \overline{A}x + \overline{B}(x,x) + \overline{f}. \tag{56}$$

Remark 6.2. *a. From the conditions (7) and (52) it follows that*

$$Re\langle \overline{B}(u,v),v\rangle = 0 \tag{57}$$

for all $u, v \in H$*;*

b. From the inequality (5) it follows that

$$Re\langle \overline{A}x,x)\rangle \le -\alpha|x|^2 \tag{58}$$

for all $x \in H$*.*

Theorem 6.3. *Assume the conditions enumerated above are all satisfied. Then for all* $T > 0$ *and* $\rho \ge r_0 := \frac{\|f\|}{\alpha} > 0$*, the solution for the "perturbed" nonautonomous Lorenz equation (50) approaches the solution of the averaged Lorenz equation (54) in the following sense:*

$$\max\{|\varphi(t,x,\omega,\varepsilon) - \overline{\varphi}(t,x,\varepsilon)| : \ 0 \le t \le T/\varepsilon, \ |x| \le \rho, \quad \omega \in \Omega\} \to 0 \tag{59}$$

as $\varepsilon \to 0$*, where* $\varphi(t,x,\omega,\varepsilon)$ *(respectively* $\overline{\varphi}(t,x,\varepsilon)$*) is a solution of equation (50) (respectively (54)), passing through point* x *at the initial moment* $t = 0$*.*

Proof. According to Theorem 2.2, we have $|\varphi(t,x,\omega,\varepsilon)| \le \rho$ and $\bar{\varphi}(t,x,\varepsilon)| \le \rho$ for all $t \ge 0$, $|x| \le \rho, \omega \in \Omega$ and $\varepsilon \in (0,\varepsilon_0]$. If we take $\mathcal{F} := \{F_\omega \mid \omega \in \Omega\} \subset C(\mathbb{R} \times H, H)$, where $f_\omega(t,x) := A(\omega t)x + B(\omega t)(x,x) + f(\omega t)$ for all $t \in \mathbb{R}$ and $x \in H$, then the relation (59) follows from Theorem 5.3. This completes the proof. □

Theorem 6.4. *(Global averaging principle for nonautonomous Lorenz systems) Let* φ_ε *be the cocycle generated by the "perturbed" nonautonomous Lorenz system (50). Assume the conditions enumerated above are all satisfied. If the cocycle* φ_ε*(* $\varepsilon \in [0,\varepsilon_0]$*) is asymptotic compact, then the following assertions hold:*

(i) *The averaged equation (56) admits a compact global attractor* $\overline{I} \subset H$*;*
(ii) *For every* $\varepsilon \in (0,\varepsilon_0]$ *the equation (50) has a compact global attractor* $\{I^\varepsilon_\omega \mid \omega \in \Omega\}$*;*
(iii) *The set* $I = \cup\{I^\varepsilon \mid \varepsilon \in [0,\varepsilon_0]\}$ *is bounded, where* $I^0 = \overline{I}$ *and* $I^\varepsilon = \cup\{I^\varepsilon_\omega \mid \omega \in \Omega\}$*;*
(iv)

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega} \beta(I^\varepsilon_\omega, \overline{I}) = 0 \tag{60}$$

and, in particular,

$$\lim_{\varepsilon\to 0} \beta(I^\varepsilon, \overline{I}) = 0.$$

Proof. The first three statements of the theorem follow from Theorems 2.2, 3.7 and Remark 6.2. Now we will prove the fourth statement of the theorem. To this end, we will use the same arguments as in [16, 6]. Let $\lambda > 0$ and $B(\overline{I},\lambda) = \{x \in H \mid \rho(x,\overline{I}) < \lambda\}$. According to orbital stability of the set $\overline{I}$ (see, for example, [14, Ch.I] or Theorem 1.2.4 from [9]), for given λ there exists $\delta = \delta(\lambda) > 0$ (we may consider $\delta(\lambda) < \lambda/2$) such that

$$\overline{\varphi}(t, B(\overline{I},\delta)) \subset B(\overline{I},\lambda/2) \tag{61}$$

for all $t \ge 0$. In virtue of boundedness of the set $I = \cup\{I^\varepsilon \mid 0 \le \varepsilon \le \varepsilon_0\}$ we may choose $\rho \le r_0$ such that $I \subset B(0,\rho) = \{x \in H \mid |x| < \rho\}$. Since $\overline{I}$ is a compact global attractor of the system (56), then for the closed ball $B[0,\rho] := \{x \in H | \, |x| \le \rho\}$ and the number $\delta > 0$ there exists $T = T(\rho,\delta) > 0$ such that

$$\overline{\varphi}(t, B[0,\rho]) \subset B(\overline{I},\delta/2), \quad t \ge T. \tag{62}$$

Let $x \in B[0, \rho]$. Then in virtue of Theorem 6.3 for the numbers $\rho \geq r_0$ and $T(\rho, \delta) > 0$ there exists $\mu = \mu(\rho, \delta) > 0$ such that $0 < \varepsilon \leq \mu,\ m(\varepsilon) < \lambda/2$ (see (59)), i.e.

$$|\varphi(t, x, \omega, \varepsilon) - \overline{\varphi}(t, x)| < \delta/2 \tag{63}$$

for all $x \in B[0, \rho],\ \omega \in \Omega,\ t \in [0, T/\varepsilon]$ and $0 < \varepsilon \leq \mu$. According to (62) we have $\overline{\varphi}(T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \delta/2)$. Thus, taking into account (63), we obtain $\varphi(T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \delta)$. Let us take the initial point $x_1 := \varphi(T/\varepsilon, x, \omega, \varepsilon)$ and we will repeat for this point the same reasoning as above. Taking into consideration the equality $\varphi(t, x, \sigma(T/\varepsilon, \omega), \varepsilon) = \varphi(t + T/\varepsilon, x, \omega, \varepsilon)$, we will have

$$|\varphi(t + T/\varepsilon, x, \omega, \varepsilon)| = |\overline{\varphi}(t, x_1)| < \delta/2 \tag{64}$$

for all $t \in [0, T/\varepsilon]$, $x \in B[0, \rho]$ and $\omega \in \Omega$, where $x_1 = \varphi(T/\varepsilon, x, \omega, \varepsilon)$.

By the inequality (64) we obtain again $x_2 := \varphi(2T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \delta)$ and, consequently,

$$\varphi(t + T/\varepsilon, x, \omega, \varepsilon) \in B(\overline{I}, \lambda/2 + \delta/2) \subset B(\overline{I}, \lambda).$$

If we continue this process and later (in virtue of uniformity w.r.t. $|x| \leq \rho$ and $\omega \in \Omega$ of the estimation (63) it is possible), we will obtain

$$\varphi(t, x, \omega, \varepsilon) \in B(\overline{I}, \lambda) \tag{65}$$

for all $t \geq T/\varepsilon$, $x \in B[0, \rho]$, $\omega \in \Omega$ and $o \leq \varepsilon \leq \mu$ and, consequently,

$$\varphi(t, x, \sigma(-t, \omega), \varepsilon) \in B(\overline{I}, \lambda)$$

for all $t \geq T/\varepsilon$ and $|x| \leq \rho$. Since $I = \cup\{I^\varepsilon \mid 0 \leq \varepsilon \leq \varepsilon_0\} \subseteq B(0, \rho)$, then according to Theorem 3.3

$$I^\varepsilon_\omega = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, B[0, \rho], \sigma(-\tau, \omega), \varepsilon)}.$$

Therefore, from (65) we have $I^\varepsilon_\omega \subset B(\overline{I}, \lambda)$ for all $\omega \in \Omega$ and $0 < \varepsilon < \mu$. Note that λ is arbitrarily chosen. Hence from the last inclusion we obtain the equality (60). The theorem is proved. □

Acknowledgment: The research described in this publication was made possible in part by the Award MM1-3016 of the Moldovan Research and Development Association (MRDA) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF). This paper was written while the first author was visiting Illinois Institute of Technology (Department of Applied Mathematics) in spring of 2002. He would like to thank people in that institution for their very kind hospitality.

References

[1] Arnold L., Random Dynamical Systems. Springer-Verlag, 1998.
[2] Bronshteyn I. U., Extensions of Minimal Transformation Group. Noordhoff, 1979.
[3] Bronshteyn I. U., Nonautonomous Dynamical Systems. Kishinev, Shtiintsa, 1984.(in Russian)
[4] Caraballo T., Langa J. A. and Robinson J., Upper semicontinuity of attractors for small random perturbations of dynamical systems. Comm. in Partial Differential Equations, 23:1557-1581, 1998.
[5] Caraballo T. and Langa J. A., On the upper semicontinuity of cocycle attractors for nonautonomous and random dynamical systems (to appear).
[6] Cheban D. N., Principle of Averaging on the Semi-axis for the Dissipative Systems. Dynamical System and Equations of Mathematical Physics. Kishinev, Shtiintsa, 1988, p.149-161.
[7] Cheban D.N., Global Attractors of Infinite-Dimensional Nonautonomous Dynamical Systems: Part I. Bulletin of Academy of Sciences of Republic of Moldova. Mathematics. 1997, N3 (25), p. 42-55
[8] Cheban D. N., Global Pullback Attractors of ℂ-Analytic Nonautonomous Dynamical Systems. Stochastics and Dynamics, Vol.1, No.4, 2001, pp.211-536.

[9] Cheban D. N., Global Attractors of Nonautonomous Dynamical Systems. Kishinev, State University of Moldova, 2002 (in Russian).

[10] Cheban D. N., Kloeden P. E. and Schmalfuß B., Relation Between pullback and Global Attractors of Nonautonomous Dynamical Systems. Nonlinear Dynamics and Systems Theory, v.2, No.2(2002), pp.9-28.

[11] Chepyzhov V. V. and Vishik M. I., Attractors for Equations of Mathematical Physics. Amer. Math. Soc., Providence, RI, 2002.

[12] Coddington E.A. and Levinson N., Theory of Ordinary Differential Equations. McGraw Hill, New York, 1955.

[13] Crauel, H., Debussche, A., and Flandoli, F., Random attractors, J. Dyn. Diff. Eqns, **9** (1997), 307–341.

[14] Hale J. K., Asymptotic Behaviour of Dissipative Systems. Amer. Math. Soc., Providence, RI, 1988.

[15] Hartman P., Ordinar Differential Equations. Birkhäuser, 1982.

[16] Ilyin A. A., Averaging Principle for Dissipative Dynamical System with Rapidly Oscillating Right-Hand Sides. Mathematicheskii Sbornik, 187(1996), No.5, 15-58: English Transl. in Sbornik: Mathematics, 187(1996).

[17] Kloeden P. E. and Schmalfuß B., Cocycle Attractors of Variable Time-step Discretizations of Lorenz Systems. Technical reports. Mathematics series.(Deakin University, School of Computing & Mathematics). TR M 94/08 2nd November, 1994.

[18] Ladyzhenskaya O. A., Attractors for Semigroups and Evolution Equations. Lizioni Lincei, Cambridge Univ. Press, Cambridge, New-York, 1991.

[19] Levitan B. M. and Zhikov V. V., Almost Periodic Functions and Differential Equations. Cambridge Univ. Press, Cambridge, 1982.

[20] Lorenz E.N., Deterministic Nonperiodic Flow. Journal of the Atmospheric Sciences, 1962, 20, p.130-141.

[21] Scherbakov B. A., Topological Dynamic and Poisson's Stability of Solutions of Differential Equations. Kishinev, Shtiintsa, 1972. (Russian)

[22] Schmalfuß, B., Backward cocycles and attractors of stochastic differential equations, in International Seminar on Applied Mathematics–Nonlinear Dynamics: Attractor Approximation and Global Behaviour, V. Reitmann, T. Riedrich, and N. Koksch (eds.), TU Dresden, 1992, pp 185–192.

[23] Sell G. R., Topological Dynamics and Ordinary Differential Equations. Van Nostrand-Reinhold, 1971.

[24] Sell G. R. and You Y., Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.

[25] Sinai Yu. G. and Shilnikov L. P. (eds.), Strange Attractors (Collection of articles). Moscow, "Mir", 1981.

[26] Temam, R., Navier–Stokes Equation–Theory and Numerical Analysis, North-Holland, Amsterdam, 1979.

(D. Cheban) State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova

E-mail address, D. Cheban: `cheban@usm.md`

(J. Duan) Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA

E-mail address, J. Duan: `duan@iit.edu`