INVARIANT MANIFOLDS, GLOBAL ATTRACTORS AND ALMOST PERIODIC SOLUTIONS OF NON-AUTONOMOUS DIFFERENCE EQUATIONS

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The talk is devoted to the study of quasi-linear non-autonomous difference equations: invariant manifolds, compact global attractors, almost periodic and recurrent solutions, invariant manifolds and chaotic sets. We prove that such equations admit an invariant continuous section (an invariant manifold). Then, we obtain the conditions of the existence of a compact global attractor and characterize its structure. We give a criterion for the existence of almost periodic and recurrent solutions of the quasi-linear non-autonomous difference equations. Finally, we prove that quasi-linear difference equations with chaotic base admits a chaotic compact invariant set. The obtained results are applied while studying triangular maps: invariant manifolds, compact global attractors, almost periodic and recurrent solutions and chaotic sets.

1. Introduction

In the qualitative theory of differential and difference equations non-local problems play the important role. It refers to questions of boundedness,

*Work partially supported by CNR (Consiglio Nazionale delle Ricerche, Italy. NATO-CNR autreach fellowships programme 2000) and award No.MM1-3016 of the Moldovan Research and Development Association (MRDA) and U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union.

periodicity, almost periodicity, stability by Poisson, asymptotic behaviour, dissipativity etc.

The present work belongs to this direction and is dedicated to the study of quasi-linear non-autonomous difference equations: invariant manifolds, compact global attractors, almost periodic and recurrent solutions, invariant manifolds and chaotic sets. The obtained results are applied while studying triangular maps. Triangular maps are close to one dimensional maps in the sense that some important dynamical features extend to triangular maps. On the other hand, they already display other important properties which are typical for higher dimensional maps and cannot be found in the one-dimensional maps.

Below we will give a new approach concerning the study of triangular maps. We study this problem in the framework of non-autonomous dynamical systems (cocyles) with discrete time. The main tool in the study triangular maps in our work are the continuous invariant sections (selectors) of cocyle.

2. Triangular maps and non-autonomous dynamical systems

Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ its Cartesian product. A continuous map $F : X \to X$ is called triangular, if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \tag{1}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all non-negative integer numbers.

Along with system (1) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega) \ (\omega \in \Omega), \tag{2}$$

which is equivalent to system (1). Let $\varphi(n, u, \omega)$ be a solution of equation (2) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

(1) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
(2) $\varphi(n + m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n\omega$;

(3) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ is continuous.

Denote by $(\Omega, \mathbb{Z}_+, \sigma)$ the semi-group dynamical system generated by positive powers of the map $g : \Omega \to \Omega$, i.e. $\sigma(n, \omega) := g^n \omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Definition 2.1. A triple $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ (or briefly φ) is called a cocycle (or non-autonomous dynamical system) [1, 2, 5] over the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W.

Taking into consideration this remark we can study triangular maps in the framework of non-autonomous dynamical systems (cocycles) with discrete time.

3. Linear non-autonomous dynamical systems

Let Ω be a compact metric space and $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system on Ω with discrete time.

Below we will suppose that the set Ω is invariant, i.e. $\sigma^n \Omega = \Omega$ for all $n \in \mathbb{Z}_+$. Let E be a finite-dimensional Banach space with the norm $|\cdot|$ and W be a complete metric space. Denote by $L(E)$ the space of all linear continuous operators on E and by $C(\Omega, W)$ the space of all the continuous functions $f : \Omega \to W$ endowed with the compact-open topology, i.e. the uniform convergence on compact subsets in Ω. The results of this section will be used in the next sections.

Consider a linear equation

$$u_{n+1} = A(\sigma^n \omega) u_n \quad (\omega \in \Omega,\ \sigma^n \omega := \sigma(n, \omega)) \tag{3}$$

and an in-homogeneous equation

$$u_{n+1} = A(\sigma^n \omega) u_n + f(\sigma^n \omega), \tag{4}$$

where $A \in C(\Omega, L(E))$ and $f \in C(\Omega, E)$.

Definition 3.1. Let $U(n, \omega)$ be the operator of Cauchy (a solution operator) of linear equation (3). Following [3] we will say that equation (3) has an exponential dichotomy on Ω, if there exists a continuous projection valued function $P : \Omega \to L(E)$ satisfying:

(1) $P(\sigma^n \omega) U(n, \omega) = U(n, \omega) P(\omega)$;
(2) $U_Q(n, \omega)$ is invertible as an operator from $ImQ(\omega)$ to $ImQ(\sigma^n \omega)$, where $U_Q(n, \omega) := U(n, \omega) Q(\omega)$;

(3) there exist constants $0 < q < 1$ and $N > 0$ such that

$$\|U_P(n,\omega)\| \leq Nq^n \text{ and } \|U_Q(n,\omega)^{-1}\| \leq Nq^n$$

for all $\omega \in \Omega$ and $n \in \mathbb{Z}_+$, where $U_P(n,\omega) := U(n,\omega)P(\omega)$.

Theorem 3.1. *If the equation (3) has an exponential dichotomy on Ω, then there exists a unique continuous function $\nu : \Omega \to E$ satisfying the following conditions:*

a. the equality

$$\nu(\sigma(n,\omega)) = \varphi(n,\nu(\omega),\omega)$$

holds for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$, where $\varphi(n,u,\omega)$ is the unique solution of equation (4) with the initial condition $\varphi(0,u,\omega) = u$.

b.

$$\|\nu\| \leq N\frac{1+q}{1-q}\|f\|.$$

Remark 3.1. In case the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is invertible, Theorem 3.1 is well known (see, for example, [3]).

4. Quasi-linear non-autonomous dynamical systems

Let us consider the following quasi-linear equation

$$u_{n+1} = A(\sigma^n\omega)u_n + f(\sigma^n\omega) + F(u_n, \sigma^n\omega), \tag{5}$$

where $A \in C(\Omega, L(E))$, $f \in C(\Omega, E)$ and $F \in C(E \times \Omega, E)$.

Theorem 4.1. *If the equation (3) has an exponential dichotomy on Ω and there exist positive numbers $L < L_0 := \frac{1-q}{N(1+q)}$ and $r < r_0 := \varepsilon_0 \frac{N(1+q)}{1-q}(1 - NL_0\frac{1+q}{1-q})^{-1}$ such that*

$$\|F(x_1,\omega) - F(x_2,\omega)\| \leq L\|x_1 - x_2\|$$

for all $\omega \in \Omega$ and $x_1, x_2 \in B[Q,r] = \{x \in E \mid \rho(x,Q) \leq r\}$, where $Q := \nu(\Omega)$, $\nu \in C(\Omega, E)$ is the unique function from $C(\Omega, E)$ figuring in Theorem 3.1 and $\varepsilon_0 = \max_{\omega\in\Omega} \|F(\nu(\omega),\omega)\|$. Then there exists a function $w \in C(\Omega, B[Q,r])$ such that

$$w(\sigma^n\omega) = \psi(n, w(\omega), \omega)$$

for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$, where $\psi(\cdot, u, \omega)$ is the unique solution of quasi-linear equation (5) with the initial condition $\psi(0,u,\omega) = u$.

5. Global attractors of quasi-linear triangular systems

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k + F(u_k, \sigma^n \omega), \tag{6}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies to "the condition of smallness".

Denote by $U(k, \omega)$ the Cauchy's matrix for the linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k.$$

Definition 5.1. A family $\{I_\omega \mid \omega \in \Omega\}(I_\omega \subset E)$ of nonempty compact subsets is called [1, 2] a compact global attractor of the cocycle φ, if the following conditions are fulfilled:

(1) the set $I := \bigcup\{I_\omega \mid \omega \in \Omega\}$ is relatively compact;
(2) the family $\mathbb{I} := \{I_\omega \mid \omega \in \Omega\}$ is invariant with respect to the cocycle φ, i.e. $\varphi(n, I_\omega, \omega) = I_{\sigma^n \omega}$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$;
(3) the equality

$$\lim_{n \to +\infty} \sup_{\omega \in \Omega} \beta(\varphi(n, K, \omega), I) = 0$$

takes place for every $K \in C(E)$, where $C(E)$ is a family of all compacts from E.

Theorem 5.1. *Suppose that the following conditions hold:*

(1) there are positive numbers N and $q < 1$ such that

$$\|U(n, \omega)\| \le Nq^n \qquad (n \in \mathbb{Z}_+); \tag{7}$$

(2) $|F(u, \omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1-q)N^{-1})$ for all $u \in E$ and $\omega \in \Omega$.

Then the equation (6) (i.e. the cocycle φ generated by equation (6)) admits a compact global attractor $\{I_\omega \mid \omega \in \Omega\}$ with the following properties:

(1) the component I_ω $(\omega \in \Omega)$ is connected;
(2) the set $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is connected, if the space Ω also is.

6. Almost periodic, almost automorphic and recurrent solutions

Theorem 6.1. *Let $(\Omega, \mathbb{Z}_+, \pi)$ be a dynamical system and Ω consist of m-periodic (almost periodic, recurrent) points. Then under the conditions of*

Theorem 4.1 equation (5) admits an invariant manifold consisting of m-periodic (almost periodic, almost automorphic, recurrent) solutions.

7. Chaos in triangular maps

Let (X, ρ) be a metric space and $(X, \mathbb{Z}_+, \pi)$ be a discrete dynamical system generated by positive powers of the map $f : X \to X$, i.e. $\pi(n, x) = f^n x$ for all $x \in X$ and $n \in \mathbb{Z}_+$, where $f^n := f^{n-1} \circ f$.

Definition 7.1. $\{p_1, p_2\} \subseteq X$ is called a Li-Yorke pair, if simultaneously

$$\liminf_{n\to+\infty} \rho(\pi(n, p_1), \pi(n, p_2)) = 0 \text{ and } \limsup_{n\to+\infty} \rho(\pi(n, p_2), \pi(n, p_2)) > 0.$$

Definition 7.2. A set $M \subseteq X$ is called scrambled, if any pair of distinct points $\{p_1, p_2\} \subseteq M$ is a Li-Yorke pair.

Denote by $\mathfrak{N}_x := \{\{t_n\} \mid \{xt_n\}$ *converges to* $x\}$, where $xt := \pi(t, x)$ $(x \in X, t \in \mathbb{Z}_+)$.

Definition 7.3. A dynamical system $(X, \mathbb{Z}_+, \pi)$ is said to be chaotic, if X contains an uncountable subset M satisfying the following conditions:

(1) the set M is transitive;
(2) M is scrambled;
(3) $\overline{P(M)} = M$, where $P(M) := \{x \in M \mid \mathfrak{N}_x \neq \emptyset\}$ (i.e. $x \in P(M)$, if and only if x is stable in the sense of Poisson [4]) and by bar we denote the closure in X.

Theorem 7.1. *Let* $(\Omega, \mathbb{Z}_+, \pi)$ *be a chaotic dynamical system. Then under the conditions of Theorem 4.1 equation (5) (respectively triangular map) admits a compact invariant chaotic set.*

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