RELATION BETWEEN DIFFERENT TYPES OF GLOBAL ATTRACTORS OF SET-VALUED NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Dedicated to our friend
Prof. ***Enrico Primo Tomasini***
on the occasion of his 55-th bithday.

Abstract: The article is devoted to the study of the relation between forward and pullback attractors of set-valued non-autonomous dynamical systems (cocycles). Here it is proved that every compact global forward attractor is also a pullback attractor of the set-valued non-autonomous dynamical system. The inverse statement, generally speaking, is not true, but we prove that every global pullback attractor of an α-condensing set-valued cocycle is always a local forward attractor. The obtained general results are applied while studying periodic and homogeneous systems. We give also a new criterion of the absolute asymptotic stability of non-stationary discrete linear inclusions.

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1. Introduction

Compact global attractors play a very important role in the study of dynamical systems (both autonomous – Babin and Vishik [1], Chueshov [15], Hale [22], Ladyzhenskaya [25], Sell and You [31] and non-autonomous – Cheban [11, 13], Chepyzhov and Vishik [14], Kloeden and Schmalfuss [24] and see also the bibliography therein). During the last ten years there were published many works, where the object of study is compact global attractors of set-valued dynamical systems (autonomous set-valued dynamical systems Ball [3, 4], Babin [2], Cheban and Fakeeh [6, 7],[19], Fakeeh [16, 17, 18] and Melnik [26], Melnik and Valero [27] and non-autonomous set-valued dynamical systems Cheban and Fakeeh [6, Ch.3-4], Cheban and Schmalfuss [9], Cheban and Mammana [12], Melnik and Valero [28] and Pilyugin [29] and see also the bibliography therein)

The present article is devoted to the study of the relation between forward and pullback attractors of set-valued non-autonomous (cocycles) dynamical systems. It is proved that every compact global forward attractor is also a pullback attractor of the set-valued non-autonomous dynamical system. The inverse statement, generally speaking, is not true, but we prove that every global pullback attractor of an α-condensing set-valued cocycle is always a local forward attractor. The main results of paper are Theorems 5.1 and 6.5 which establish the relation between forward and pullback attractors for set-valued non-autonomous dynamical system. The obtained general results are applied while studying periodic and homogeneous systems. We give also a new criterion of absolute asymptotic stability of non-stationary discrete linear inclusions.

This paper is organized as follows.

In Section 2 we give some notions and known facts from the theory of set-valued dynamical systems.

In Section 3 we introduce the notion of maximal compact invariant sets for set-valued dynamical systems (both autonomous and non-autonomous) and establish some of their properties. The main results in this section are Theorems 3.8 and 3.10.

Section 4 is devoted to the study of asymptotic stability in α-condensing set-valued dynamical systems. We give an infinite dimensional analogue of the well known theorem of Zubov on the asymptotical stability of compact invariant sets (see for example [36, Theorem 7] or [37, Theorem 8] and also [10]) for set-valued dynamical systems. Theorems 4.12, 4.13 and 4.17 contain the main results of this section.

In Section 5 we establish some properties of pullback attractors of set-valued non-autonomous dynamical systems. In particular, Theorem 5.1 states that every compact global pullback attractor is asymptotically stable (i.e. global pullback attractor is a local forward attractor) and Theorem 5.3 contains the conditions under which every compact global pullback attractor is an uniform global forward attractor. For dynamical systems with uniqueness these results were established in the paper [10].

Section 6 is devoted to the study of global attractors of set-valued cocycles. We prove that a compact global forward attractor is also a pullback attractor (Theorem 6.5). 5 and 6 sections contain the main results of paper.

In section 7 we give some applications of the general results obtained in sections 3.-6. In particular, it is shown that for periodical systems every global pullback attractor is also a global forward attractor (Theorem 7.2) and under some additional conditions we prove that a global pullback attractor will be a uniform forward attractor (Theorem 7.3). The applications of this results to periodical difference and differential inclusions are given. We prove that for a homogeneous set-valued cocycle the global pullback attractor $\{I_y \mid \ \in Y\}$ is trivial (i.e. $I_y = \{0\}$ for all $y \in Y$, where 0 is a null element of the euclidian space E^n) and it is a global forward attractor (Theorem 7.13). Finally, we obtain a criterion of the absolute asymptotic stability of non-stationary discrete linear inclusions (Theorem 7.17). This is a generalization of the well known result [21] for the non-autonomous case.

2. Global attractors of autonomous set-valued dynamical systems.

Let (X, ρ) be a complete metric space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a semigroup of the additive group $\mathbb{S}$. If $A \subseteq X$ and $x \in X$, then let us denote by $\rho(x, A)$ the distance from the point x to the set A, i.e. $\rho(x, A) = \inf\{\rho(x, a) : a \in A\}$. Denote by $B(A, \varepsilon)$ the ε-neighborhood of the set A, i.e. $B(A, \varepsilon) = \{x \in X : \rho(x, A) < \varepsilon\}$. And by $\mathcal{K}(X)$ we denote the family of all non-empty compact subsets of X. For every point $x \in X$ and number $t \in \mathbb{T}$ we associate with it closed compact subset $\pi(t, x) \in \mathcal{K}(X)$ and, hence, if $\pi(P, A) = \bigcup\{\pi(t, x) : t \in P, x \in A\} (P \subseteq \mathbb{T})$.

Definition 2.1. *The triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$, is called a set-valued semigroup dynamical system (disperse dynamical system or dynamical system without uniqueness)if the following conditions hold:*

1.1. $\pi(0, x) = x$ *for all* $x \in X$;
1.2. $\pi(t_2, \pi(t_1, x)) = \pi(t_1 + t_2, x)$ *for all* $x \in X$ *and* $t_1, t_2 \in \mathbb{T}$, *if* $t_2 \cdot t_2 > 0$;
1.3. *the mapping* π *is upper semi-continuous, i.e.* $\lim\limits_{x \to x_0, t \to t_0} \beta(\pi(t, x), \pi(t_0, x_0)) = 0$ *for all* $x_0 \in X$ *and* $t_0 \in \mathbb{T}$, *where* $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ *is the semi-deviation of the set* $A \subseteq X$ *from the set* $B \subseteq X$.

Let $\mathbb{T} = \mathbb{S}$ and the next condition be fulfilled:

1.4. If $p \in \pi(t, x)$, then $x \in \pi(-t, p)$ for all $x, p \in X$ and $t \in \mathbb{T}$.

Then it is said that there is defined a dynamical system $(X, \mathbb{T}, \pi)$ or dynamical system (bilateral or two-sided) without uniqueness.

Remark 2.2. *Later on under the set-valued dynamical system $(X, \mathbb{T}, \pi)$ we will understand a semi-group dynamical system unless otherwise stated, i.e. we will consider that $\mathbb{T} = \mathbb{S}_+$.*

Definition 2.3. *Let $\mathbb{T} \subset \mathbb{T}' \subset \mathbb{S}$. A continuous mapping $\gamma_x : \mathbb{T}' \to X$ is called a motion of the set-valued dynamical system $(X, \mathbb{T}, \pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}'$, if*

a. $\gamma_x(0) = x$;
b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ *for all* $t_1, t_2 \in \mathbb{T}'$ $(t_2 > t_1)$.

The set of all motions of $(X, \mathbb{T}, \pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\Phi_x(\pi)$ and $\Phi(\pi) := \bigcup\{\Phi_x(\pi) \mid x \in X\}$ (or simply Φ).

Definition 2.4. *The trajectory $\gamma \in \Phi(\pi)$ defined on $\mathbb{S}$ is called a full (entire) trajectory of the dynamical system $(X, \mathbb{T}, \pi)$.*

Denote by $\mathcal{F}(\pi)$ the set of all full trajectories of the dynamical system $(X, \mathbb{T}, \pi)$ and $\mathcal{F}_x(\pi) := \Phi_x(\pi) \bigcap \mathcal{F}(\pi)$.

Theorem 2.5. [35] *Let $(X, \mathbb{T}, \pi)$ be a set-valued dynamical system. The following affirmations hold:*

1. *for every point $x \in X$ there exists at least one $\gamma_x \in \Phi_x$, i.e. $\Phi_x \neq \emptyset$ for all $x \in X$;*
2. *if $\gamma^1, \gamma^2 \in \Phi$ are such that $\gamma^1(t_1) = \gamma^2(t_2)$ for certain $t_1, t_2 \in \tilde{\mathbb{T}} := D(\gamma^1) \cap D(\gamma^2)$ $(t_2 > t_1)$, where $D(\gamma^i)$ is the domain of definition of γ^i $(i = 1, 2)$, then the mapping $\gamma : \tilde{\mathbb{T}} \to X$ defined by the equality*
$$\gamma(t) := \gamma^1(t) \text{ if } t \in [0, t_1] \text{ and}$$
$$\gamma(t) := \gamma^2(t - t_1 + t_2), \text{ for } t > t_1,$$
belong to Φ
3. *if $\gamma \in \Phi$, $\tau \in \mathbb{T}$ and $\gamma^\tau(t) := \gamma(t + \tau)$ $(t \in \mathbb{T})$, then $\gamma^\tau \in \Phi$ and $D(\gamma^\tau) = -\tau + D(\gamma)$;*
4. *if $x_n \to x_0$, $\gamma^n \in \Phi_{x_n}$ and $D(\gamma_n) \subseteq D(\gamma_{n+1})$ for all $n \in \mathbb{N}$, then there exists $\gamma \in \Phi_{x_0}$ and a subsequence $\{\gamma^{n_k}\}$ such that γ^{n_k} uniformly converges to γ on every compact from $D(\gamma) = \cup_{n \in \mathbb{N}} D(\gamma_n)$.*

Denote by $C(\mathbb{T}, X)$ the space of all continuous functions $f : \mathbb{T} \to X$ equipped with the compact-open topology. This space can be metrized (see for example [33, 34] and [32]) for example by the distance

$$d(\gamma^1,\gamma^2) := \sup_{l>0} \min\{\max_{|t|\le l} \rho(\gamma^1(t),\gamma^2(t)), l^{-1}\}.$$

On the space $C(\mathbb{T},X)$ there is defined a dynamical system of translations (dynamical system of shifts or dynamical system of Bebutov [32] and [33, 34]) $(C(\mathbb{T},X),\mathbb{T},\sigma)$. The set Φ is a closed and invariant (with respect to translations) subset of $C(\mathbb{T},X)$ and, consequently, on Φ there is induced by $(C(\mathbb{T},X),\mathbb{T},\sigma)$ a dynamical system of translations $(\Phi,\mathbb{T},\sigma)$. This system (with uniqueness) plays a very important role in the study of the set-valued dynamical system $(X,\mathbb{T},\pi)$ (see, for example, [6]).

Theorem 2.6. [35] *Let $\Phi \subseteq C(\mathbb{T},X)$ and the conditions 1.-4. of Theorem 2.5 be fulfilled. Then the triplet $(X,\mathbb{T},\pi)$, where $\pi(t,x) := \{\gamma(t) \mid \gamma\in\Phi$ and $\gamma(0)=x\}$ $(x\in X, t\in\mathbb{T})$, is a set-valued dynamical system and $\Phi(\pi)=\Phi$.*

Remark 2.7. *1. Theorems 2.5 and 2.6 show that there exist two possibilities of definition of set-valued dynamical systems: first - using semigroup of set-valued mappings and second - using family of motions.*

2. In works [3, 4] *and* [26, 27] *(see also the bibliography therein) there was considered and studied more general notion of set-valued dynamical systems.*

3. The relation between these two theories for multi-valued dynamical systems was studied in paper [5].

Definition 2.8. *Let $\mathfrak{M}$ be some family of subsets of X. We will call a dynamical system $(X,\mathbb{T},\pi)$ $\mathfrak{M}$-dissipative, if there exists a compact set $K\subseteq X$, such that for any $\varepsilon>0$ and $M\in\mathfrak{M}$ there exists $L=L(\varepsilon,M)>0$ such that $\pi^tM\subseteq B(K,\varepsilon)$ for every $t\ge L(\varepsilon,M)$, where $\pi^tM=\{\pi(t,x)\ :\ x\in M\}$. In addition, we will call the set K an attractor of the family $\mathfrak{M}$.*

The most interesting for applications are cases when $\mathfrak{M}=\{\{x\}:x\in X\},\mathfrak{M}=\mathcal{K}(X),\mathfrak{M}=\{B(x,\delta_x):x\in X,\delta_x>0 \text{ is fixed }\}$ or $\mathfrak{M}=B(X)$ (where $B(X)$ is the family of all bounded subsets of X).

Definition 2.9. *A system $(X,\mathbb{T},\pi)$ is called* [6]*:*

- *pointwise dissipative, if there exists $K\in B(X)$ such that for all $x\in X$*

(1)
$$\lim_{t\to+\infty}\beta(\pi^tx,K)=0;$$

- *compactly dissipative, if equality (1) holds uniformly w.r.t. x on compact subsets from X;*
- *locally dissipative, if for any point $p\in X$ there exists $\delta_p>0$ such that equality (1) holds uniformly w.r.t. $x\in B(p,\delta_p)$.*

Let $(X,\mathbb{T},\pi)$ be compactly dissipative and K be a compact set being the attractor of all compact subsets of X. Denote

(2)
$$J:=\omega(K)=\bigcap_{t\ge0}\overline{\bigcup_{\tau\ge t}\pi^\tau K}.$$

We can show [6] that the set J defined by equality (2) does not depend on the choice of the attractor K, but is characterized only by properties of the dynamical system $(X,\mathbb{T},\pi)$. The set J is called a center of Levinson of the compactly dissipative system $(X,\mathbb{T},\pi)$.

Let us state some known facts that we will need later.

Theorem 2.10. [6] *If $(X, \mathbb{T}, \pi)$ is a compactly dissipative dynamical system and J is its center of Levinson then :*

(1) *J is invariant, i.e. $\pi^t J = J$ for all $t \in \mathbb{T}$;*
(2) *J is orbitally stable, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\beta(\pi^t x, J) < \varepsilon$ for all $t \geq 0$;*
(3) *J is the attractor of the family of all compact subsets of X;*
(4) *J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$.*

Definition 2.11. *A dynamical system $(X, \mathbb{T}, \pi)$ is called :*

- *locally completely continuous, if for any point $p \in X$ there exists $\delta_p > 0$ and $l_p > 0$ such that $\pi^{l_p} B(p, \delta_p)$ is relatively compact ;*
- *trajectory dissipative, if there exists a non-empty compact K such that*
$$\lim_{t \to +\infty} \rho(\varphi_x(t), K) = 0$$
for all $x \in X$ and $\varphi_x \in \Phi_x$.

Remark 2.12. *Notice that every set-valued dynamical system $(X, \mathbb{T}, \pi)$ with locally-compact phase space X is locally completely continuous.*

Theorem 2.13. [6] *Let $(X, \mathbb{T}, \pi)$ be trajectory dissipative and $(X, \mathbb{T}, \pi)$ be locally completely continuous. Then $(X, \mathbb{T}, \pi)$ is locally dissipative.*

Lemma 2.14. [6] *Let $M \in B(X)$. Then the following conditions are equivalent :*

1. *for any $\{x_k\} \subseteq M$ and $t_k \to +\infty$ the sequence $\{y_k\}(y_k \in \pi(x_k, t_k))$ is relatively compact ;*
2. *$\omega(M) \neq \emptyset$, is compact, invariant and*
$$\lim_{t \to +\infty} \beta(\pi^t(M), \omega(M)) = 0;$$
3. *there exists a non-empty compact $K \subseteq X$ such that*
$$\lim_{t \to +\infty} \beta(\pi^t(M), K) = 0.$$

Theorem 2.15. [6] *A dynamical system $(X, \mathbb{T}, \pi)$ is pointwise (compactly) dissipative if and only if the dynamical system $(\Phi, \mathbb{T}, \sigma)$ also is.*

Theorem 2.16. [6] *If the dynamical system $(X, \mathbb{T}, \pi)$ is locally complete continuous, then the following conditions are equivalent:*

(1) *the dynamical system $(X, \mathbb{T}, \pi)$ is pointwise dissipative;*
(2) *the dynamical system $(X, \mathbb{T}, \pi)$ is locally dissipative.*

3. Maximal compact invariant sets.

Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two set-valued dynamical systems.

Definition 3.1. *A mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (respectively homeomorphic) and $h(\pi(t, x)) = \sigma(t, h(x))$ ($t \in \mathbb{T}_1$, $x \in X$). In this case the dynamical system $(X, \mathbb{T}_1, \pi)$ is an extension of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ w.r.t. the homomorphism h, but the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called a factor of the dynamical system $(X, \mathbb{T}_1, \pi)$ w.r.t. the homomorphism h. The dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called also a base of the extension $(X, \mathbb{T}_1, \pi)$.*

Definition 3.2. *The triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where h is a homomorphism of $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system.*

Let $W,\ Y$ be two complete metric spaces and $(Y,\mathbb{T}_2,\sigma)$ be a set-valued dynamical system on Y.

Definition 3.3. *The triplet $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ is said to be a set-valued cocycle over $(Y,\mathbb{T}_2,\sigma)$ with the fiber W, if φ is a mapping of $\mathbb{T}_1\times W\times Y$ onto $\mathcal{K}(W)$ and possesses the properties:*

1. *$\varphi(0,u,y)=u$ for all $u\in W$ and $y\in Y$;*
2. *$\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau\in\mathbb{T}_1$ and $(u,y)\in W\times Y$, where $\varphi(t,A,y):=\bigcup\{\varphi(t,u,y)\ :\ u\in A\}$;*
3. $$\lim_{t\to t_0,u\to u_0,y\to y_0}\beta(\varphi(t,u,y),\varphi(t_0,u_0,y_0))=0$$ *for all $(t_0,u_0,y_0)\in\mathbb{T}_1\times W\times Y$.*

We denote by $X:=W\times Y$; then $(X,\mathbb{T}_1,\pi)$ is a set-valued dynamical system on X defined by the equality $\pi:=(\varphi,\sigma)$, i.e. $\pi^t x:=\{(v,q)\ :\ v\in\varphi(t,u,y),\ q\in\sigma(t,y)\}$ for every $t\in\mathbb{T}_1$ and $x=(u,y)\in X=W\times Y$. Then the triplet $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is a set-valued non-autonomous dynamical system (a skew-product system), where $h:=pr_2:X\mapsto Y$.

Thus, if we have a set-valued cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ over the set-valued dynamical system $(Y,\mathbb{T}_2,\sigma)$ with the fiber W, then it generates a set-valued non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),h\rangle$ $(X:=W\times Y)$, which is called a non-autonomous dynamical system generated by the cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ over $(Y,\ \mathbb{T}_2,\ \sigma)$.

Example 3.4. Let E^n be an n–dimensional real or complex Euclidian space Y be a closed subset from E^m. Denote by $\mathcal{K}_V(E^n)$ the family of all convex compacts in E^n and by $C(Y\times E^n,\mathcal{K}_V(E^n))$ the set of all α–continuous mapping $F:Y\times E^n\to\mathcal{K}_V(E^n)$ endowed with the topology of uniform convergence on compacts. Let us consider the system of differential inclusions

$$\begin{cases}\dot u\in F(y,u),\\ \dot y\in G(y)\end{cases}\tag{3}$$

where $G\in C(Y,\mathcal{K}_V(E^m))$ and $F\in C(Y\times E^n,\mathcal{K}_V(E^n))$. We suppose that for system (3) the conditions of the existence and nonlocal continuability to the right are fulfilled. Denote by $(Y,\mathbb{R}_+,\sigma)$ the set-valued dynamical system on Y generated by the second inclusion of system (3). Along with system (3), we consider differential inclusions

$$\dot u\in F(\sigma(y,t),u)\quad(y\in Y)\tag{4}$$

and put $\varphi(t,u,y)=\{\varphi(t)\mid\varphi$ is a solution of inclusion (4) defined on $\mathbb{R}_+$ and $\varphi(0)=u\}$. Then the mapping $\varphi:\mathbb{R}_+\times Y\times E^n\to\mathcal{K}(E^n)$ is β–continuous and satisfies the following conditions:

1. $\varphi(0,u,y)=u$ for all $u\in W$ $(W:=E^n)$ and $y\in Y$;
2. $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau\in\mathbb{T}_+$ and $(u,y)\in W\times Y$, where $\varphi(t,A,y):=\bigcup\{\varphi(t,u,y)\ :\ u\in A\}$.

Thus the system (3) naturally generates a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R}_+,\ \sigma),\ h\rangle$ (where $X=E^m\times Y$, $\pi=(\varphi,\sigma)$ and $h=pr_2:X\to Y$).

Definition 3.5. *The family $\{I_y\mid y\in Y\}(I_y\subset W)$ of non-empty compact subsets of W is said to be the maximal compact invariant set of the set-valued cocycle φ, if the following conditions are fulfilled:*

1. *$\{I_y\mid y\in Y\}$ is invariant, i.e. $\varphi(t,I_y,y)=I_{yt}$ for every $y\in Y$ and $t\in\mathbb{T}_+$;*
2. *$\mathbb{I}=\bigcup\{I_y\mid y\in Y\}$ is relatively compact;*

(3) $\{I_y \mid y \in Y\}$ *is maximal, i.e. if the family* $\{I_y^{'} \mid y \in Y\}$ *is relatively compact and invariant, then* $I_y^{'} \subseteq I_y$ *for every* $y \in Y$.

Lemma 3.6. *The family* $\{I_y \mid y \in Y\}$ *is invariant w.r.t. set-valued the cocycle* φ *if and only if the set* $J = \bigcup\{J_y \mid y \in Y\}$ $(J_y = I_y \times \{y\})$ *is invariant w.r.t. the dynamical system* $(X, \mathbb{T}, \pi)$, *that is* $\pi^t J = J$ *for all* $t \in \mathbb{T}$ *where* $\pi^t := \pi(t, \cdot)$.

Proof. Let the family $\{I_y \mid y \in Y\}$ be invariant, $J = \bigcup\{J_y \mid y \in Y\}$ and $J_y = I_y \times \{y\}$. Then we have

$$\pi^t J = \bigcup\{\pi^t J_y \mid y \in Y\} = \bigcup\{(\varphi(t, I_y, y), yt) \mid y \in Y\}$$

$$= \bigcup\{I_{yt} \times \{yt\} \mid y \in Y\} = \bigcup\{J_{yt} \mid y \in Y\} = J \tag{5}$$

for all $t \in \mathbb{T}$.

Conversely. Let J be an invariant set w.r.t. $(X, \mathbb{T}, \pi)$. Then

$$\pi(t, J) = \bigcup_{y \in Y} \pi(t, J_y) = \bigcup_{y \in Y} J_{\sigma(t,y)} \tag{6}$$

and

$$\pi(t, J_y) = (\varphi(t, I_y, y), \sigma(t, y)),\ J_{\sigma(t,y)} = (I_{\sigma(t,y)}, \sigma(t, y)) \tag{7}$$

for all $t \in \mathbb{T}$ and $y \in Y$. From (6) and (7) we obtain the equality $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for all $t \in \mathbb{T}$ and $y \in Y$. □

Theorem 3.7. [35] *Let* $(X, \mathbb{T}, \pi)$ *be a set-valued dynamical system. If* $y \in \pi(t, x)$, *then there exists* $\varphi \in \Phi_x$ *such that* $\varphi(t) = y$.

Theorem 3.8. *Let a family of sets* $\{I_y \mid y \in Y\}$ *be maximal, relatively compact and invariant with respect to the set-valued cocycle* φ. *Then* $\mathbb{I} := \bigcup\{I_y \mid y \in Y\}$ *is closed.*

Proof. We note that the set $J = \bigcup\{J_y \mid y \in Y\}$ $(J_y = I_y \times \{y\})$ is relatively compact and according to Lemma 3.6 it is invariant. Let $K = \overline{J}$. Then K is compact. We have to show that K is invariant. In fact, if $x \in K$, then there exists $\{x_n\} \subset J$ such that $x = \lim\limits_{n\to+\infty} x_n$. Thus $x_n \in J = \pi^t J$ for all $t \in \mathbb{T}_+$, therefore by Theorem 3.7 for $t \in \mathbb{T}_+$ there exists $\bar{x}_n \in J$ and $\varphi_{\bar{x}_n} \in \Phi_{\bar{x}_n}$ such that $x_n = \varphi_{\bar{x}_n}(t)$. Since J is relatively compact, it is possible to suppose that the sequences $\{\bar{x}_n\}$ and $\{\varphi_{\bar{x}_n}\}$ are convergent, moreover the sequence $\{\varphi_{\bar{x}_n}\}$ uniformly converges on every compact from $\mathbb{T}$. We denote $\bar{x} = \lim\limits_{n\to+\infty} \bar{x}_n$ and $\varphi_x := \lim\limits_{n\to+\infty} \varphi_{x_n}$. Then $\varphi_{\bar{x}} \in \Phi_{\bar{x}}$, $\bar{x} \in \overline{J}, x = \varphi_{\bar{x}}(t) \in \pi^t \bar{x}$ and, consequently, $x \in \pi^t \overline{J}$ for all $t \in \mathbb{T}_+$, i.e. $\overline{J} = \pi^t \overline{J}$.

Let $\mathbb{I}' = pr_1 K$, where by pr_1 we denote the first projection of $X = W \times Y$ on W. Then we have $\mathbb{I}' = \bigcup\{I'_y \mid y \in Y\}$, where $I'_y = \{u \in W \mid (u, y) \in K\}$ and $K_y = I'_y \times \{y\}$. Since the set K is invariant, then according to Lemma 3.6 the set $\mathbb{I}'$ is also invariant w.r.t. the set-valued cocycle φ. The set $\mathbb{I}'$ is compact, because K is compact and $pr_1 : X \mapsto W$ is continuous. By the maximality of the family $\{I_y \mid y \in Y\}$ we have $I'_y \subseteq I_y$ for every $y \in Y$ and, consequently, $\mathbb{I}' \subseteq \mathbb{I}$.

On the other hand, $\mathbb{I} = pr_1 \overline{J} = \mathbb{I}'$ and, consequently, $\mathbb{I}' = \mathbb{I}$. Thus the set $\mathbb{I}$ is compact. The theorem is proved. □

Definition 3.9. *Let* $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *be a set-valued cocycle. The family* $\{I_y \mid y \in Y\}$ $(I_y \subset W)$ *of non-empty compact subsets of* W *is said to be a compact pullback attractor (uniform pullback attractor) of the set-valued cocycle* φ, *if the following conditions are fulfilled:*

a. $\mathbb{I} := \bigcup\{I_y \mid y \in Y\}$ *is relatively compact ;*

b. *I is invariant w.r.t. the set-valued cocycle* φ, *i.e.* $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ *for all* $t \in \mathbb{T}$ *and* $y \in Y$;

c. *for every* $y \in Y$ *and* $K \in C(W)$

$$\lim_{t\to+\infty} \beta(\varphi(t, K, \sigma(-t, y)), I_y) = 0, \tag{8}$$

where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ *is a semi-distance of Hausdorff (equality (8) holds uniformly with respect to* $y \in Y$*).*

Theorem 3.10. *The family* $\{I_y \mid y \in Y\}$ *of nonempty compact subsets of* W *will be maximal compact invariant set w.r.t. the set-valued cocycle* φ, *if* $\{I_y \mid y \in Y\}$ *is a compact pullback attractor w.r.t. the set-valued cocycle* φ.

Proof. Let $\{I_y \mid y \in Y\}$ be a compact pullback attractor of the set-valued cocycle φ. Since the family $\{I'_y \mid y \in Y\}$ is a compact and invariant set of the set-valued cocycle φ, we have

$$\beta(I'_y, I_y) = \beta(\varphi(t, I'_y, \sigma(-t, y), I_y) \leq \beta(\varphi(t, K, \sigma(-t, y), I_y) \to 0$$

as $t \to +\infty$, where $K = \overline{\bigcup\{I'_y \mid y \in Y\}}$. Hence $I'_y \subseteq I_y$ for every $y \in Y$, i.e. $\{I_y \mid y \in Y\}$ is maximal. □

Remark 3.11. *The family* $\{I_y \mid y \in Y\}$ $(I_y \subset W)$ *is a maximal compact invariant set w.r.t. the set-valued cocycle* φ *if and only if the set* $J = \bigcup\{J_y \mid y \in Y\}$, *where* $J_y = I_y \times \{y\}$, *is a maximal compact invariant set in the dynamical system* $(X, \mathbb{T}, \pi)$.

4. Asymptotic stability in α- condensing set-valued dynamical systems

Let $(X, \mathbb{T}, \pi)$ be a set-valued dynamical system. An ω-limit set of a set M is defined as follows:

$$\boldsymbol{\omega}(M) := \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \pi(t, M)}.$$

Definition 4.1. *A set* M *is called Lyapunov stable, if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\pi(t, B(M, \delta)) \subseteq B(U, \varepsilon)$ *for* $t \geq 0$.

Definition 4.2. *The set* $W^s(M) := \{x \in X \mid \lim_{t\to+\infty} \beta(\pi(t, x), M) = 0\}$ *is called a domain of attraction of the set* M.

Definition 4.3. *A set* M *is said to be a local attractor, if there exists a neighborhood* $B(M, \mu)$ $(\mu > 0)$ *of* M *such that* $B(M, \mu) \subseteq W^s(M)$.

Definition 4.4. *A Lyapunov stable set* M, *which is a local attractor, is called asymptotically stable.*

Theorem 4.5. [6] *Let* $(X, \mathbb{T}, \pi)$ *be a set-valued dynamical system and* $M \subseteq X$ *be a compact and positively invariant set. Then the next conditions are equivalents:*

(1) *the set* M *is asymptotically stable;*
(2) $W^s(M)$ *is open and for each* $\varepsilon > 0$ *and* $x \in W^s(M)$ *there are* $\delta = \delta(\varepsilon, x) > 0$ *and* $\tau = \tau(\varepsilon, x) > 0$ *such that*

$$\beta(\pi(t, B(x, \delta)), M) < \varepsilon$$

for all $t \geq \tau(\varepsilon, x)$;

(3) $W^s(M)$ *is open and the set* M *attracts every compact from* $W^s(M)$*, i.e.*

$$\lim_{t\to+\infty}\beta(\pi(t,K),M)=0$$

for every $K\in C(W^s(M))$*;*

(4) $W^s(M)$ *is open, every compact* $K\subset W^s(M)$ *is stable in the sense of Lagrange in positive direction (i.e. the set* $\Sigma^+(K):=\bigcup\{\pi(t,K)\mid t\in\mathbb{T}_+\}$ *is relatively compact) and* $\emptyset\neq J_x^+\subseteq M$ *for all* $x\in W^s(M)$*.*

Definition 4.6. *An invariant compact set* M *is said to be locally maximal, if there exists a number* $\delta>0$ *such that any invariant compact set contained in the open* δ*-neighborhood* $B(M,\delta)$ *of* M *in fact is contained in* M*.*

Definition 4.7. *The mapping* $\lambda : B(X)\to\mathbb{R}_+$*, satisfying the following conditions:*

(1) $\lambda(A)=0$ *if and only if* $A\in B(X)$ *is relatively compact;*
(2) $\lambda(A\cup B)=\max(\lambda(A),\lambda(B))$ *for every* A*,* $B\in B(X)$*.*

is called [22, 30] *a measure of non-compactness on* X*.*

Definition 4.8. *The measure of non-compactness of Kuratowsky* $\alpha : B(X)\to\mathbb{R}_+$ *is defined by the equality* $\alpha(A):=\inf\{\varepsilon>0\mid A$ *admits a finite* ε*-covering* $\}$*.*

Definition 4.9. *A set-valued dynamical system* $(X,\mathbb{T},\pi)$ *is called* α*–condensing, if there exists* $t_0\in\mathbb{T}$ $(t_0>0)$ *such that* $\pi(t_0,B)$ *is bounded and*

$$\alpha(\pi(t_0,B))<\alpha(B)$$

for any bounded set B *of* X *with* $\alpha(B)>0$*.*

Remark 4.10. *In the book* [22] *there are a some class of dynamical systems (with uniqueness) which possesses with property of* α*-condensingness. For example: every dynamical system on finite-dimensional space, every dynamical system with compact* $\pi^t(t>0)$ *or* $\pi^t=m(t)+r(t)$*, where* $m(t):X\to X$ *is compact for every* $t>0$ *and* $r(t)x\to 0$ *as* $t\to+\infty$ *uniformly w.r.t.* x *on every bounded subset from* X*.*

Definition 4.11. *An entire trajectory of the semi-group dynamical system* $(X,\mathbb{T},\pi)$ *(respectively, of the cocycle* $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ *over* $(Y,\mathbb{S},\sigma)$ *with the fiber* W *), passing through the point* $x\in X$ *(respectively,* $(u,y)\in W\times Y$ *) is called a continuous mapping* $\gamma:\mathbb{S}\to X$ *(respectively,* $\nu:\mathbb{S}\to W$*) satisfying the conditions :* $\gamma(0)=x$ *(respectively,* $\nu(0)=w$*) and* $\gamma(t+\tau)\in\pi^t\gamma(\tau)$ *(respectively,* $\nu(t+\tau)\in\varphi(t,\nu(\tau),y\tau)$*) for all* $t\in\mathbb{T}$ *and* $\tau\in\mathbb{S}$*.*

Theorem 4.12. *Let* M *be a compact positively invariant set for an* α*-condensing semi-dynamical system* $(X,\mathbb{T},\pi)$*. Then* M *is Lyapunov stable if and only if*

$$\alpha_x\cap M=\emptyset$$

for all $x\notin M$*, where* $\alpha_x:=\{y\in X$ *: there exist* $\{\gamma_n\}\subset\Phi_x$ *and* $\{t_n\}\to+\infty$ *such that* $[-t_n,+\infty)\subset D(\gamma_n)$ *and* $\{\gamma_n(-t_n)\}\to y\}$*, where* $[-\tau,+\infty):=\{t\in\mathbb{S}\ :\ t+\tau\geq 0\}$*.*

Proof. The proof of necessity was given by Zubov in [36, Theorem 7] for dynamical systems with uniqueness on the locally compact space X. This proof remains also true for set-valued dynamical systems on non locally compact space under consideration here. Indeed, let M be a compact positively invariant set of $(X,\mathbb{T},\pi)$, stable in positive direction. If we suppose that this assertion is not true, then there exist $x\notin M$ such that $\alpha_x\bigcap M\neq\emptyset$, i.e., there exist $\gamma_x\in\Phi_x$ and $\tau_n\to-\infty$ such that $\rho(\gamma_x(\tau_n),M)\to 0$ as $n\to\infty$. Let $0<\varepsilon<\rho(x,M)$ and $\delta(\varepsilon)>0$ be the corresponding positive number from the definition of the stability of the set M. Then for a sufficiently large n we have $\rho(\gamma_x(\tau_n),M)<\delta(\varepsilon)$ and, consequently, $\beta(\pi^t\gamma_x(\tau_n),M)<\varepsilon$ for all $t\geq 0$. In particular, for $t=-\tau_n$ we have $x\in\pi^{-\tau_n}\gamma_x(\tau_n)$ and $\rho(x,M)\leq\beta(\pi^{-\tau_n}\gamma_x(\tau_n),M)<\varepsilon$. The obtained contradiction proves our assertion. To prove sufficiency let us consider first the case when $\mathbb{T}=\mathbb{Z}_+$. Suppose that M is not

Lyapunov stable, but that the other condition of the theorem holds. Then there exist $\varepsilon_0 > 0$ and sequences $\delta_n \to 0$, $x_n \in B(M, \delta_n)$, $k_n \to \infty$ such that $\pi(kt_0, x_n) \subseteq B(M, \varepsilon_0)$ for $0 \leq k \leq k_n - 1$ and $\pi(k_n t_0, x_n) \nsubseteq B(M, \varepsilon_0)$ (t_0 is the positive number from the α-condensing property of the set-valued dynamical system $(X, \mathbb{T}, \pi)$). Since the set M is compact and positively invariant, then the number ε_0 has to be chosen sufficiently small so that

$$\beta(\pi(t, B(M, \varepsilon_0)), M) < \frac{\delta_0}{2}.$$

for all $t \in [0, 2t_0]$. Define $A = \{x_n\}$ and $B = \cup_{n \in \mathbb{N}} \{\pi(kt_0, x_n) \mid 0 \leq k \leq k_n - 1\}$. Then $\alpha(A)$ $= 0$ (α is the measure of non-compactness of Kuratowsky), since A is relatively compact. In addition, $\pi(t_0, B) \subseteq B(M, \delta_0)$, so $\pi(t_0, B)$ is bounded. Suppose that B is not relatively compact, so $\alpha(B) > 0$. The properties of the measure of non-compactness for the non relatively compact set B imply that

$$\alpha(B) = \alpha(A \cup \pi(t_0, B) \cap B) \leq \max(\alpha(A), \alpha(\pi(t_0, B))) = \alpha(\pi(t_0, B)) < \alpha(B)$$

and this is a contradiction. It shows that B is relatively compact. Let $y_n \in \pi(k_n t_0, x_n) \bigcap (X \setminus B(M, \varepsilon_0))$ and $\tilde{\gamma_n} \in \Phi_{x_n}$ such that $\tilde{\gamma_n}(k_n t_0) = y_n$. We may suppose that the sequence $\{y_n\}$ is convergent. We define γ_n by equality $\gamma_n(s) := \tilde{\gamma}(k_n t_0 + s)$ for all $-k_n t_0 \leq s < +\infty$. Without loss of generality we may suppose that the sequence $\{\gamma_n\}$ is convergent. Let $\gamma := \lim\limits_{n \to +\infty} \gamma_n$, i.e., $\gamma(k) = \lim\limits_{n \to +\infty} \gamma_n(k)$ for every $k \in \mathbb{Z}$. Then $\gamma \in \Phi_y$ ($y := \lim\limits_{n \to +\infty} y_n$). It is clear that $\alpha_\gamma \bigcap M \neq \emptyset$, where

$$\alpha_\gamma := \bigcap_{t \leq 0} \overline{\bigcup_{\tau \leq t} \gamma(\tau)}.$$

On the other hand $\gamma(0) = y$, $\alpha_\gamma \subseteq \alpha_y$ and $y \in B(M, \varepsilon_0) \setminus M$, so $\alpha_y \bigcap M = \emptyset$ holds by our assumptions. This contradiction proves the sufficiency of the condition in the discrete–time case.

Now let $\mathbb{T} = \mathbb{R}_+$ and suppose that $\alpha_x \cap M = \emptyset$ where $x \notin M$ holds for the continuous–time system. Then this also holds for the restricted discrete–time system generated by $\pi_1 := \pi(t_0, \cdot)$, because $\tilde{\alpha}_x \subseteq \alpha_x$, where $\tilde{\alpha_x}$ (respectively α_x) is the α-limit set of point x with respect to discrete-time dynamical system (X, π_1) (respectively, dynamical system $(X, \mathbb{R}_+, \pi)$). Hence, the set M is Lyapunov stable with respect to the restricted discrete–time dynamical system generated by π_1. Since M is compact, for every $\varepsilon > 0$ there exists μ > 0 such that

$$\rho(\pi(t, x), M) < \varepsilon \quad \text{for all } t \in [0, t_0], \ x \in B(M, \mu).$$

In view of the first part of the proof above, there is $\delta > 0$ such that

$$\rho(\pi(nt_0, x), M) < \min(\mu, \varepsilon) \quad \text{for } \ x \in B(M, \delta) \quad \text{for } n \in \mathbb{Z}_+.$$

The Lyapunov stability of M for the continuous dynamical system $(X, \mathbb{R}_+, \pi)$ then follows from the semi–group property of π. □

Theorem 4.13. *Let M be a compact and positively invariant subset of the set-valued dynamical system $(X, \mathbb{T}, \pi)$. Then M is asymptotically stable if and only if $\boldsymbol{\omega}(M)$ is locally maximal and asymptotically stable.*

Proof. Suppose that M is asymptotically stable. Then there exists a closed positively invariant closed neighborhood C of M contained in its domain of attraction $W^s(M)$. The mapping π can be restricted to the complete metric space C to form a semi–dynamical system $(C, \mathbb{T}, \pi)$. Since M is a locally attracting set it attracts compact subsets of C. The assertion then follows from Theorem 2.10 because $\boldsymbol{\omega}(M) = \bigcap_{t \in \mathbb{T}} \pi(t, M)$.

Suppose instead that $\boldsymbol{\omega}(M)$ is asymptotically stable and locally maximal. Since M is compact, $\boldsymbol{\omega}(M)=\bigcap_{t\geq 0}\pi(t,M)$. Hence there exist $\eta>0$ and $\tau\in\mathbb{T}$ such that

$$\pi(\tau,M)\subset B(\boldsymbol{\omega}(M),\eta)\subset W^s(\boldsymbol{\omega}(M)).$$

Since M is compact and the mapping $\pi(\tau,\cdot):X\to\mathcal{K}(X)$ is upper semi-continuous, the set $\pi(\tau,M)$ is compact and, consequently, there is a number $\nu>0$ such that

$$B(\pi(\tau,M),\nu)\subset B(\omega(M),\eta)\subseteq W^s(\omega(M)).$$

In view of the integral continuity for the numbers τ and ν there exists a number $\delta>0$ such that

$$\pi(\tau,B(M,\delta))\subset B(\pi(\tau,M),\nu)\subset B(\omega(M),\eta)\subseteq W^s(\omega(M)). \tag{9}$$

From the inclusion (9) it follows that

$$B(M,\delta)\subseteq W^s(\omega(M))\subseteq W^s(M).$$

Then, if M was not Lyapunov stable, there would exist $\varepsilon_0>0$, $\delta_n\to 0$, $x_n\in B(M,\delta_n)$, $\varphi_{x_n}\in\Phi_{x_n}$ and $t_n\to\infty$ such that

$$\rho(\varphi_{x_n}(t_n),M)\geq\varepsilon_0. \tag{10}$$

For sufficiently large n_0, the set $\overline{\{x_n\}_{n\geq n_0}}$ would then be contained in $B(M,\delta)\subseteq W^s(\omega(M))$. Since the set M is compact, the set $\overline{\{x_n\}_{n\geq n_0}}$ also is. According to Theorem 4.5 this set would thus be attracted by $\boldsymbol{\omega}(M)\subset M$, which contradicts (10) . □

Definition 4.14. *A set-valued dynamical system $(X,\mathbb{T},\pi)$ is called asymptotically compact, if for every bounded and positively invariant set $M\subseteq X$ there exists a nonempty compact set $K\subseteq X$ such that $\lim\limits_{t\to+\infty}\beta(\pi(t,M),K)=0$.*

Lemma 4.15. *Let M be a compact and positively invariant set for the asymptotically compact semi–dynamical system $(X,\mathbb{T},\pi)$. Then the set M is asymptotically stable if and only if $\boldsymbol{\omega}(M)$ is locally maximal and Lyapunov stable.*

Proof. The necessity follows from Theorem 4.13. Suppose instead that $\boldsymbol{\omega}(M)$ is locally maximal and Lyapunov stable. Then for any $\varepsilon>0$ there exists $\delta>0$ such that

$$\pi(t,B(\boldsymbol{\omega}(M),\delta))\subset B(\boldsymbol{\omega}(M),\varepsilon)\qquad\text{for all } t\geq 0.$$

In virtue of the assumption of the asymptotic compactness of $(X,\mathbb{T}_+,\pi)$ and Lemma 2.14 the set $\boldsymbol{\omega}(B(\boldsymbol{\omega}(M),\delta))$ is nonempty and compact with

$$\lim_{t\to\infty}\beta(\pi(t,B(\boldsymbol{\omega}(M),\delta)),\boldsymbol{\omega}(B(\boldsymbol{\omega}(M),\delta)))=0.$$

Since $\boldsymbol{\omega}(M)$ is locally maximal, $\boldsymbol{\omega}(B(\boldsymbol{\omega}(M),\delta))\subset\boldsymbol{\omega}(M)$ for sufficiently small $\delta>0$, which means that $\boldsymbol{\omega}(M)$ is asymptotically stable. So, the conclusion follows from Theorem 4.13. □

Corollary 4.16. *Let $(X,\mathbb{T},\pi)$ be asymptotically compact and let M be a compact invariant set. Then M is asymptotically stable if and only if M is locally maximal and Lyapunov stable.*

Proof. Indeed, $M=\boldsymbol{\omega}(M)$ here, so we just apply Lemma 4.15. □

The next theorem is a generalization for infinite dimensional spaces and α-condensing set-valued dynamical systems of Theorem 8 of Zubov [36] characterizing the asymptotic stability of a compact set.

Theorem 4.17. *Let $(X, \mathbb{T}, \pi)$ be an α-condensing semi–dynamical system and let $M \subset X$ be a compact invariant set. Then the set M is asymptotically stable if and only if*

(i) *M is locally maximal, and*
(ii) *$\alpha_x \cap M = \emptyset$ for any $x \notin M$.*

Proof. By Lemmas 2.3.1 and 2.3.2 in [6] any α-condensing set-valued dynamical system is asymptotically compact, so the assertion follows easily from Theorem 4.12 and Corollary 4.16. □

Definition 4.18. *A set-valued cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is called α-condensing if there exists $t_0 > 0$ ($t_0 \in \mathbb{T}$) such that the set $\varphi(t_0, B, Y)$ is bounded and*

$$\alpha(\varphi(t_0, B, Y)) < \alpha(B)$$

for any bounded subset B of W with $\alpha(B) > 0$.

Lemma 4.19. *Suppose that the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is α-condensing. Then the associated skew–product flow $(X, \mathbb{T}, \pi)$ is also α-condensing.*

Proof. Let $M := \bigcup\{M_y \times \{y\} \mid y \in Y\}$ be a bounded set in X. Then M can be covered by finite number of balls $M_i \subset X$, $i = 1, \cdots, n$, of largest radius $\alpha(M) + \varepsilon$ for an arbitrary $\varepsilon > 0$. The sets $\mathrm{pr}_1 M_i \subset W$, $i = 1, \cdots, n$, cover $\mathrm{pr}_1 M$. The sets M_i are balls, so $\alpha(\mathrm{pr}_1 M_i) = \alpha(M_i) < \alpha(M) + \varepsilon$ for $i = 1, \cdots, n$. It is easily seen that

$$\pi(t, M) = \bigcup\{\pi(t, (M_y, y)) \mid y \in Y\} =$$

$$\bigcup\{(\varphi(t, M_y, y), \sigma_t y) \mid y \in Y\} \subset \varphi(t, \mathrm{pr}_1 M, Y) \times Y.$$

Since φ is α-condensing, there exists $t_0 \in \mathbb{T}$ ($t_0 > 0$) such that the set $\varphi(t_0, \mathrm{pr}_1 M, Y)$ is bounded. Hence

$$\begin{aligned} \alpha(\pi(t_0, M)) &\leq \alpha(\varphi(t_0, \mathrm{pr}_1 M, Y) \times Y) \\ &\leq \alpha(\varphi(t_0, \mathrm{pr}_1 M, Y)) < \alpha(\mathrm{pr}_1 M) \leq \alpha(M). \end{aligned} \tag{11}$$

The second inequality above is true by the compactness of Y. Indeed, Y can be covered by finite number of open balls Y_i of arbitrarily small radius. Hence

$$\alpha(\varphi(t_0, \mathrm{pr}_1 M, Y) \times Y) \leq \max_i \alpha(\varphi(t_0, \mathrm{pr}_1 M, Y) \times Y_i) \leq \alpha(\varphi(t_0, \mathrm{pr}_1 M, Y)) + \varepsilon$$

for an arbitrarily small $\varepsilon > 0$. The conclusion of the lemma follows in virtue of (11). □

5. Uniform pullback attractors and global attractors

It was seen earlier that the set $\bigcup\{I_y \times \{y\} \mid y \in Y\} \subset X$ which was defined in terms of the pullback attractor $I = \{I_y \mid y \in Y\}$ of the set-valued cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is the maximal π-invariant compact subset of the associated skew–product set-valued system $(X, \mathbb{T}, \pi)$, but not necessarily is a global attractor [10]. However, this set is always a local attractor under the additional assumption that the set-valued cocycle φ is α-condensing.

Theorem 5.1. *Let Y be a compact space, $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be an α-condensing set-valued cocycle with the pullback attractor $I = \{I_y \mid y \in Y\}$ and let us define $J = \bigcup\{I_y \times \{y\} \mid y \in Y\}$. Then*

(i) *the α-limit set α_x of any point $x \in X \setminus J$ is empty;*
(ii) *J is asymptotically stable with respect to π.*

Proof. Suppose that there exists a point $x = (u, y) \in X \setminus J$ such that $\alpha_x \neq \emptyset$. Then there exist sequences $\{\gamma_n\} \subset \Phi_x$ and $-\tau_n \to \infty$ such that $\gamma_n(\tau_n)$ converges to a point in α_x. The set $K = \mathrm{pr}_1\overline{\bigcup_{n\in\mathbb{N}} \gamma_n(\tau_n)}$ is compact, since $\overline{\bigcup_{n\in\mathbb{N}} \gamma_n(\tau_n)}$ is compact too. Also $I = \{I_y \mid y \in Y\}$ is a pullback attractor, so

$$\lim_{n\to+\infty} \beta(\varphi(-\tau_n, K, y\tau_n), I_y) = 0$$

from which it follows that $u \in I_y$. Hence $(u, y) \in J$, which is a contradiction. This proves the first assertion.

By Lemma 4.19 $(X, \mathbb{T}, \pi)$ is α-condensing. According to Theorem 3.10 and Remark 3.11 the set J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$, as I is the pullback attractor of the cocycle φ. The second assertion then follows from Theorem 5.1 and from the first assertion of this theorem. □

Remark 5.2. *If in addition to the assumptions of Theorem 5.1 the stable set $W^s(J)$ of J satisfies $W^s(J) = X$, then the skew-product dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and J is its Levinson center (see Lemma 1.4.10 and Theorem 1.4.11 from* [6]*).*

Theorem 5.3. *Suppose that Y compact, $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is a set-valued cocycle with the pullback attractor $I = \{I_y \mid y \in Y\}$ and suppose that $W^s(J) = X$, where $J = \bigcup\{I_y \times \{y\} \mid y \in Y\}$.*

If the mapping $y \to I_y$ is lower semi–continuous, then I is a uniform pullback attractor and hence a uniform forward attractor.

Proof. Suppose that the uniform convergence

$$\lim_{t\to\infty} \sup_{y\in Y} \beta(\varphi(t, D, y), I_{\sigma(t,y)}) = 0$$

is not held for some $D \in C(W)$. Then there exist $\varepsilon_0 > 0$, a set $D_0 \in C(W)$ and sequences $t_n \to \infty$, $y_n \in Y$, $p_n \in \sigma(t_n, y_n)$, $u_n \in D_0$ and $\tilde{u_n} \in \varphi(t_n, u_n, y_n)$ such that:

$$\rho(\tilde{u}_n, I_{p_n}) \geq \varepsilon_0. \tag{12}$$

Now Y is compact and J is a global attractor by Remark 5.2, so we can suppose that the sequences $\{\tilde{u}_n\}$ and $\{p_n\}$ are convergent. Let $\bar{u} = \lim_{n\to\infty} \tilde{u}_n$ and $\bar{y} = \lim_{n\to\infty} p_n$. Then $\bar{u} \in I_{\bar{y}}$ because $\bar{x} = (\bar{u}, \bar{y}) \in J$. On the other hand, by (12),

$$\varepsilon_0 \leq \rho(\tilde{u}_n, I_{p_n}) \leq \rho(\tilde{u}_n, I_{\bar{y}}) + \beta(I_{\bar{y}}, I_{p_n}).$$

By the lower semi–continuity of $y \to I_y$ it follows then that $\bar{u} \notin I_{\bar{y}}$, which is a contradiction. □

Remark 5.4. *Theorem 5.3 is in general not true without the assumption that $W^s(J) = X$ (see* [10] *and also* [13, Ch.8]*).*

6. Global attractors of set-valued cocycles

Let $(Y, \mathbb{T}, \sigma)$ be a bilateral set-valued dynamical system on Y, W be a complete metric space and $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a set-valued cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber W.

If $M \subseteq W$, then we will set

$$\omega_y(M) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} U(\tau, \sigma(-\tau, y))M} \tag{13}$$

for every $y \in Y$, where $U(t, y) := \varphi(t, \cdot, y)$.

Lemma 6.1. *The following affirmations take place:*

1. *a point* $w \in \omega_y(M)$ *if and only if, there exist* $t_n \to +\infty, \{x_n\} \subseteq M$ *and* $w_n \in U(t_n, \sigma(-t_n, y))x_n$ *such that* $w = \lim_{n\to+\infty} w_n$;
2. *if the mapping* $U(t, y)$ *is lower semi-continuous, then* $U(t, y)\omega_y(M) \subseteq \omega_{yt}(M)$ *for all* $y \in Y$ *and* $t \in \mathbb{T}$;
3. *if there exists a non-empty compact* $K \subset W$ *such that*

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \sigma(-t, y)), K) = 0, \tag{14}$$

then $\omega_y(M) \neq \emptyset$, *is compact,*

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \sigma(-t, y)), \omega_y(M)) = 0, \tag{15}$$

and

$$\omega_{yt}(M) \subseteq U(t, y)\omega_y(M) \tag{16}$$

for all $t \in \mathbb{T}$.

Proof. The first affirmation of the lemma follows directly from equality (13). The second affirmation of the lemma follows from the definition of the sets $U(t, y)\omega_y(M)$ and $\omega_{yt}(M)$, from the equality $U(t, y)U(\tau, \sigma(-\tau, y)) = U(t + \tau, \sigma(-t - \tau, \sigma(t, y)))$ for all $t, \tau \geq 0, y \in Y$ and the α-continuity of the mapping $\varphi : \mathbb{T}_+ \times W \times Y \to C(W)$. Indeed,

$$U(t, y)\omega_y(M) = U(t, y)(\bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} U(\tau, \sigma(-\tau, y))M})$$
$$\subseteq \bigcap_{s\geq 0} U(t, y)(\overline{\bigcup_{\tau\geq s} U(\tau, \sigma(-\tau, y))M}) \subseteq \bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} U(t, y)U(\tau, \sigma(-\tau, y))M} =$$
$$\bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} U(t + \tau, \sigma(-t - \tau, \sigma(t, y)))M} \subseteq \bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} U(\tau, \sigma(-\tau, \sigma(t, y)))M} = \omega_{\sigma(t,y)}(M).$$

Equality (15) follows directly from the first affirmation of lemma and equality (14).

We will show that inequality(16) takes place. For this aim it is enough to show that $\omega_{\sigma(t,y)}(M) \subseteq U(t, y)\omega_y(M)$ for all $y \in Y$ and $t \in \mathbb{T}$. Let $y \in Y, t \in \mathbb{T}$ and $w \in \omega_{\sigma(t,y)}(M)$. Then according to the first affirmation of the lemma there exists $x_n \in M, t_n \to +\infty$ and $w_n \in U(t_n, \sigma(t - t_n, y))x_n$ such that $w = \lim_{n\to+\infty} w_n$. Since $U(t_n, \sigma(t - t_n, y))x_n = U(t, y)U(t_n - t, \sigma(t - t_n, y))x_n$ for a sufficiently big n ($t_n \geq t$), then there exists $\overline{w}_n \in U(t_n - t, \sigma(t - t_n, y))x_n$ such that $w_n \in U(t, y)\overline{w}_n$. Under the conditions of Lemma 6.1 we can suppose that the sequence $\{\overline{w}_n\}$ is convergent. Let $\overline{w} = \lim_{n\to+\infty} \overline{w}_n$. Then according to the first affirmation of the lemma $\overline{w} \in \omega_y(M)$ and, consequently, $w \in U(t, y)\omega_y(M)$, i.e. $\omega_{\sigma(t,y)}(M) \subseteq U(t, y)\omega_y(M)$. The lemma is proved. □

Definition 6.2. *The cocycle* φ *over* $(Y, \mathbb{T}, \sigma)$ *with the fiber* W *is said to be compactly dissipative, if there exists a non-empty compact* $K \subseteq W$ *such that*

$$\lim_{t\to+\infty} \sup\{\beta(U(t, y)M, K) : y \in Y\} = 0 \tag{17}$$

for all $M \in \mathcal{K}(W)$.

Lemma 6.3. *Let* Y *be a compact and* $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *be a set-valued cocycle over* $(Y, \mathbb{T}, \sigma)$ *with the fiber* W. *For* $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *to be compactly dissipative it is necessary and sufficient that the skew-product dynamical system* $(X, \mathbb{T}_+, \pi)(X := W \times Y, \pi := (\varphi, \sigma))$ *would be compactly dissipative.*

Proof. The formulated affirmation directly follows from the respective definitions. □

Definition 6.4. *We will say that the space X has property (S), if for any compact $K \subseteq X$ there exists a compact connected set $M \subseteq X$ such that $K \subseteq M$.*

Theorem 6.5. *Let Y be a compact, $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a compactly dissipative cocycle, K be a non-empty compact appearing in equality (17), $(X, \mathbb{T}, \pi)$ be a skew-product dynamical system associated by cocycle φ and J be the Levinson center of $(X, \mathbb{T}, \pi)$. Then:*

1. *$I_y := pr_1(J_y) \neq \emptyset$, is a compact subset from K and*

(18) $$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))K, I_y) = 0$$

for every $y \in Y$, where $J_y := J \bigcap X_y$ and $X_y := h^{-1}(y)$ ($h := pr_2 : X \to Y$, and $X := W \times Y$);

2. *$U(t, y)I_y = I_{\sigma(t,y)}$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3.

(19) $$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))M, I_y) = 0$$

for all $M \in \mathcal{K}(W)$ and $y \in Y$;

4.

(20) $$\lim_{t\to+\infty} \sup\{\beta(U(t, y)M, I) : y \in Y\} = 0$$

for any $M \in \mathcal{K}(W)$, where $\mathbb{I} := pr_1 J = \bigcup\{I_y : y \in Y\}$;

5. *the set $\mathbb{I}$ is compact and connected if one of the following two conditions is fulfilled:*
   - a. *$\mathbb{T} = \mathbb{R}_+$ and spaces W and Y are connected;*
   - b. *$\mathbb{T} = \mathbb{Z}_+$ and the space $W \times Y$ has the property (S) or it is connected and locally connected.*

Proof. Let $\tilde{K} := K \times Y$ and $\omega(\tilde{K})$ its ω-limit set with respect to $(X, \mathbb{T}, \pi)$. Under the conditions of Theorem the set $\tilde{K}$ attracts every compact subset from $(X, \mathbb{T}, \pi)$ and, consequently, $J = \omega(\tilde{K})$. It easy to see that $\omega_y(K) \times \{y\} \subseteq \omega_y(\tilde{K})$ for all $y \in Y$ and, consequently, $\omega_y(K) \subseteq I_y$ for all $y \in Y$, where $\omega_y(\tilde{K}) := \omega(\tilde{K}) \bigcap X_y$. Since $\omega_y(K) \subseteq I_y$ then by Lemma 6.1 we have the equality (18).

To prove the seconde statement we note that $\pi(t, J_y) = J_{\sigma(t,y)}$ for all $y \in Y$ and $t \in \mathbb{T}$. Since $J_y = I_y \times \{y\}$ and $\pi(t, J_y) = (U(t, y)I_y, \sigma(t, y))$, then we obtain $U(t, y)I_y = I_{\sigma(t,y)}$.

If we will suppose that equality (19) does not take place, then there are $\varepsilon_0 > 0, y_0 \in Y, M_0 \in C(W), t_n \to +\infty$ and $w_n \in U(t_n, y_0^{-t_n})M_0$ such that

(21) $$\rho(w_n, I_{y_0}) \geq \varepsilon_0.$$

According to (18) for ε_0 and $y_0 \in Y$ there will be found $t_0 = t_0(\varepsilon_0, y_0) > 0$ such that

(22) $$\beta(U(t, y_0^{-t})K, I_{y_0}) < \frac{\varepsilon_0}{2}$$

for all $t \geq t_0$. Notice that

(23) $$U(t_n, y_0^{-t_n}) = U(t_0, y_0^{-t_0})U(t_n - t_0, y_0^{-t_n}).$$

Let $\overline{w}_n \in U(t_n - t_0, y_0^{-t_n})M_0$ so that $w_n \in U(t_0, y_0^{-t_0})\overline{w}_n$. By the compact dissipativity of the set-valued cocyle $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ the sequences $\{w_n\}$ and $\{\overline{w}_n\}$ can be considered convergent. Let the set $\overline{w} = \lim_{n\to+\infty} \overline{w}_n$ and $w = \lim_{n\to+\infty} w_n$. Then $w \in U(t_0, y_0^{-t_0})\overline{w}$ and according to (17) $\overline{w} \in K$. Passing to the limit in (21) as $n \to +\infty$ and taking into account (23) we will get

(24) $$\rho(w, I_{y_0}) \geq \varepsilon_0.$$

On the other hand, since $\overline{w} \in K$, then from (22) we have $U(t_0, y_0^{-t_0})\overline{w} \subseteq B(I_{y_0}, \frac{\varepsilon_0}{2})$ and, consequently,

$$w \in U(t_0, y_0^{-t_0})\overline{w} \subseteq B(I_{y_0}, \frac{\varepsilon_0}{2}),$$

which contradicts to (24). The obtained contradiction proves the required affirmation.

Now we will prove equality (20). If it does not take place, then there exist $\varepsilon_0 > 0, M_0 \in C(W), y_n \in Y, \{x_n\} \subseteq M_0, t_n \to +\infty$ and $w_n \in U(t_n, y_n^{-t_n})x_n$ such that

(25) $$\rho(w_n, I) \geq \varepsilon_0.$$

In virtue of the compactness of Y we can suppose that the sequence $\{y_n\}$ is convergent. Let $y_0 = \lim_{n \to +\infty} y_n$ and according to (19) for the number $\varepsilon_0 > 0$ and $y_0 \in Y$ there is $t_0 = t_0(\varepsilon_0, y_0) > 0$ such that equality (22) takes place for all $t \geq t_0(\varepsilon_0, y_0)$. By the compactness of M_0 and compact dissipativity of the cocycle $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ the sequences $\{\overline{w}_n\}$ and w_n can be considered convergent, where $\{\overline{w}_n\} \in U(t_n - t_0, y_n^{-t_n})x_n$ and $w_n \in U(t_0, y_n^{-t_0})\{\overline{w}_n\}$. Let us notice that according to (17) $\overline{w} \in K$. From equality (23) it follows that $w \in U(t_0, y_0^{-t_0})\overline{w}$, hence from (25) we have

(26) $$w \notin B(I_{y_0}, \frac{\varepsilon_0}{2}).$$

Relation (26) contradicts to (22), which completes the proof of the fourth affirmation of theorem.

The compactness and connectedness of the set $\mathbb{I}$ follows from that fact that under the conditions of Theorem 6.5 the center of Levinson J of the dynamical system $(X, \mathbb{T}, \pi)$ is compact and connected according to Consequence 1.8.7 and Theorem 1.8.15 from [6], and consequently, I is also connected as a continuous image of a connected set. The theorem is proved. □

7. Applications

Several examples illustrating the application of the above results are now presented.

7.1. Periodic systems.

Let $(Y, \mathbb{S}, \sigma)$ be a bilateral set-valued dynamical system and Ψ_y be the set of all the trajectories of this system passing through the point $y \in Y$ at the initial moment $t = 0$ and $\Psi(\sigma) := \bigcup\{\Psi_y \mid y \in Y\}$ (or simply Ψ). Note that $\Psi \subseteq C(\mathbb{S}, Y)$ and it is invariant (with respect to translations) and closed in $C(\mathbb{S}, Y)$ and, consequently, on the set Ψ by the dynamical system of Bebutov $(C(\mathbb{S}, Y), \mathbb{S}, \sigma)$ (dynamical system of translations or dynamical system of shifts) there is induced the dynamical system of translations $(\Psi, \mathbb{S}, \sigma)$.

Definition 7.1. *A bilateral set-valued dynamical system $(Y, \mathbb{S}, \sigma)$ is called periodic, if every trajectory $\psi \in \Psi$ is periodic in the dynamical system $(\Psi, \mathbb{S}, \sigma)$, i.e. there exists a positive number $\tau \in \mathbb{S}$ such that $\psi(t + \tau) = \psi(t)$ for all $t \in \mathbb{S}$.*

Let us consider a periodical dynamical system $(Y, \mathbb{S}, \sigma)$.

Theorem 7.2. *Suppose that a set-valued α-condensing cocycle $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ over the periodical dynamical system $(Y, \mathbb{S}, \sigma)$ has a pullback attractor $I = \{I_y \mid y \in Y\}$. Then I is a forward attractor for $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$, i.e.*

$$\lim_{t \to +\infty} \sup\{\beta(U(t, y)M, \mathbb{I}) : y \in Y\} = 0$$

whatever is $M \in \mathcal{K}(W)$, where $\mathbb{I} = \bigcup\{I_y : y \in Y\}$.

Proof. Note that the set $J := \bigcup\{J_y \mid y \in Y\}$, where $J_y := I_y \times \{y\}$, is the maximal compact invariant set of the skew-product dynamical system $(X, \mathbb{T}, \pi)$ ($X := W \times Y$ and $\pi := (\varphi, \sigma)$). By Theorem 5.1 the set J is asymptotically stable.

Let now $\Phi \subseteq C(\mathbb{T}, X)$ be the set of all motions of the set-valued dynamical system $(X, \mathbb{T}, \pi)$ and $(\Phi, \mathbb{T}, \sigma)$ be the dynamical system (with uniqueness) of shifts on Φ. $(\Phi, \mathbb{T}, \sigma)$ is a subsystem of Bebutov's dynamical system $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$. Denote $\mathcal{F} := \{\gamma \in \Phi \mid \gamma$ is an entire trajectory of $(X, \mathbb{T}, \pi)$ and $\gamma(\mathbb{S}) \subseteq J\}$. Then $\mathcal{F}$ is a nonempty maximal compact invariant set of the dynamical system $(\Phi, \mathbb{T}, \pi)$.

We will prove that under the conditions of Theorem 7.2 the set $\mathcal{F}$ is orbitally stable with respect to $(\Phi, \mathbb{T}, \pi)$. In fact, if we suppose that it is not so, then there are $\varepsilon_0 > 0$, $\delta_n \searrow 0$, $\gamma_n \in \Phi$ and $t_n \in \mathbb{T}$ such that

$$d(\gamma_n, \mathcal{F}) < \delta_n \text{ and } d(\gamma_n^{t_n}, \mathcal{F}) \geq \varepsilon_0, \tag{27}$$

where γ^s is an s-shift of γ (i.e. $\gamma^s(t) := \gamma(t+s)$ for all $t \in \mathbb{T}$) and by d there is denoted the distance on Φ. Since the set $\mathcal{F}$ is invariant, then $t_n \to +\infty$. From inequality (27) it follows the existence of a sequence $\{t'_n\}$ ($t'_n = t_n + \tau_n$ and $|\tau_n| \leq \frac{1}{\varepsilon_0}$) such that

$$\rho(\gamma_n(t_n), J) \geq \varepsilon_0 \text{ and } \rho(\gamma_n(0), J) < \delta_n \tag{28}$$

for all $n \in \mathbb{N}$ because $\rho(\gamma(0), J) \leq d(\gamma, \mathcal{F})$ for all $\gamma \in \Phi$. But the inequality (28) contradicts to the asymptotic stability of the set J. The obtained contradiction proves the orbital stability of the set $\mathcal{F}$ w.r.t. $(\Phi, \mathbb{T}, \sigma)$.

Now we will establish the equality $W^s(X) = X$. Let $x = (u, y) \in X = W \times Y$ and $\gamma_{(u,y)} \in \Phi_x \subseteq \Phi$. Then $\nu_y \in \Psi_y$, where $\nu_y(t) := pr_2\gamma_{(u,y)}(t)$ (for all $t \in \mathbb{T}$), Ψ_y is the set of all the motions of the set-valued dynamical system $(Y, \mathbb{T}, \sigma)$ passing through the point $y \in Y$ at the initial moment $t = 0$ and $\Psi := \bigcup\{\Psi_y \mid y \in Y\}$. Since the dynamical system $(Y, \mathbb{S}, \sigma)$ is periodic, then there exists a positive number $\tau \in \mathbb{S}$ such that $\nu_y(t+\tau) = \nu_y(t)$ for all $t \in \mathbb{S}$. Under the conditions of Theorem 7.2 we have

$$\lim_{n \to +\infty} \rho(\gamma_{(u,y)}(n\tau), J_y) = \lim_{n \to +\infty} \rho(\gamma_{(u,y)}(n\tau), J_{\nu_y(-n\tau)}) = 0$$

because

$$\rho(\gamma_{(u,y)}(n\tau), J_{\nu_y(-n\tau)}) \leq \mathrm{dist}_W(U(n\tau, y)u, I_{\sigma(-n\tau, y)}) \to 0$$

as $n \to +\infty$. Thus the sequence $\{\gamma_{(u,y)}(n\tau)\}$ is relatively compact and, consequently, the functional sequence $\{\gamma_{(u,y)}(t + n\tau)\}$ also is (in the compact-open topology in $C(\mathbb{T}_+, X)$). This fact implies that the positive semi-trajectory of the point $\gamma_{(u,y)} \in \Psi$ is relatively compact and, consequently, $\gamma_{(u,y)} \in W^s(\mathcal{F})$, because $\mathcal{F}$ is the maximal compact invariant set of $(\Psi, \mathbb{T}, \sigma)$ and the ω-limit set of the point $\gamma_{(u,y)}$ is nonempty, compact and invariant. This means that $W^s(\mathcal{F}) = \Phi$ and, consequently (see, for example, [13, Ch.I]), $(\Phi, \mathbb{T}, \sigma)$ is compactly dissipative. By Theorem 2.15 the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative too. Now to finish the proof of the theorem it is enough to apply Lemmas 6.1 and 6.3. □

Theorem 7.3. *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a set-valued α-condensing cocycle and the following conditions be held:*

1. *the dynamical system $(Y, \mathbb{S}, \sigma)$ is periodic and minimal (i.e. $(Y, \mathbb{S}, \sigma)$ is a dynamical system with uniqueness and there exists a τ-periodic point $y_0 \in Y$ such that $Y = \{\sigma(t, y_0) \ |t \in [0, \tau)\}$);*
2. *the set-valued cocycle φ has a pullback attractor $I = \{I_y \mid y \in Y\}$;*

(3) *the mapping* $\varphi : \mathbb{T} \times W \times Y \to \mathcal{K}(W)$ *is continuous with respect to the Hausdorff distance* α $(\alpha(A,B) := max\{\beta(A,B), \beta(B,A)\}$ *for all* $A, B \in \mathcal{K}(W))$.

Then I *is a uniform forward attractor for* $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$, *i.e.*

$$\lim_{t\to+\infty} \sup\{\beta(U(t,y)M, I_{\sigma(t,y)}) : y \in Y\} = 0 \tag{29}$$

whatever is $M \in \mathcal{K}(W)$.

Proof. This statement directly follows from Theorems 5.3 and 7.2. For this is sufficient to note that under the conditions of the theorem the set-valued mapping $y \to I_y$ is lower semi-continuous. Really, consider a sequence $y_n \to y$. By the periodicity and minimality of the system $(Y, \mathbb{T}, \sigma)$ there exists a sequence $\tau_n \in [0, \tau]$ such that $y_n = \sigma(\tau_n, y)$. By its compactness, there is a convergent subsequence (indexed here for our convenience like a full one) $\tau_n \to \tau_0 \in [0, \tau]$. Hence

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sigma(\tau_n, y) = \sigma(\tau_0, y)$$

which means that $\tau_0 = 0$ or τ. Suppose that $\tau_0 = \tau$. Then

$$\lim_{n\to\infty} \beta(I_y, I_{y_n}) = \lim_{n\to\infty} \beta(I_y, \varphi(\tau_n, I_y, y))$$
$$= \beta(I_y, \varphi(\tau, I_y, y)) = \beta(I_y, I_{\sigma(\tau,y)}) = 0,$$

since φ is continuous with respect to the Hausdorff distance H and $I_{y_n} = I_{\sigma(\tau_n,y)} = \varphi(\tau_n, I_y, y)$ by the φ–invariance of $\{I_y \mid y \in Y\}$. Hence the set valued mapping $y \to I_y$ is lower semi–continuous. □

Example 7.4. (Differential inclusions) Let E^n be an n-dimensional Euclidean space. We denote by $\mathcal{K}_V(E^n)$ the family of all convex compacts from E^n, and by $C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$ we denote the set of all continuous in Hausdorff's metric mappings $F : \mathbb{R} \times E^n \to \mathcal{K}_V(E^n)$ allotted by the uniform convergence topology on compacts. Let us consider the differential inclusion

$$u' \in F(t,u), \tag{30}$$

where $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$. Along with inclusions (30) we will also consider the family of differential inclusions

$$v' \in G(t,v), \tag{31}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}, F_\tau$ is a translation by variable t of the function F on τ and by bar there is denoted the closure in $C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$.

Remark 7.5. *1. Let* $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$. *The set* $H(F)$ *is compact if and only if the function* F *is bounded on* $\mathbb{R}$ *w.r.t.* t *uniformly w.r.t.* x *on every compact subset from* E^n.

2. If the function $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$ *is almost periodic w.r.t.* t *uniformly w.r.t.* x *on every compact subset from* E^n, *then* $H(F)$ *is compact.*

Definition 7.6. *A function* $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$ *is said to be regular, if for every inclusion (31) there is fulfilled the condition of existence and non-local extendability to the right, i.e. for any* $G \in H(F)$ *and* $v \in E^n$ *there exists at least one solution* $\varphi_{(v,G)}(t)$ *of the inclusion (31) passing through the point* v *when* $t = 0$ *and defined on* $\mathbb{R}_+$.

Let $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$ be regular. We put $\varphi(t, v, g) = \{\varphi_{(v,G)}(t) : \varphi_{(v,G)} \in \Phi_{(v,G)}\}$, where $\Phi_{(v,G)}$ is the set of all solutions of inclusion (31) defined on $\mathbb{R}_+$ and passing through the point v at the initial moment $t = 0$. From the general properties of differential inclusions [20] it follows that the following properties take place :

a. $\varphi(0, v, G) = v$ for all $v \in E^n, G \in H(F)$;
b. $\varphi(t + \tau, v, G) = \varphi(t, \varphi(\tau, v, G), G_\tau)$ for all $v \in E^n, G \in H(F)$ and $t, \tau \in \mathbb{R}_+$;
c. the mapping $\varphi : \mathbb{R}_+ \times E^n \times H(F) \to C(E^n)$ is β-continuous.

Denote $Y := H(F)$ and by $(Y, \mathbb{R}, \sigma)$ the dynamical system of translations on Y. Then the triplet $\langle E^n, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is a set-valued cocycle over $(Y, \mathbb{R}, \sigma)$ with the fiber E^n. Thus differential inclusion (30) with the regular right hand side $F \in C(\mathbb{R}\times E^n, \mathcal{K}_V(E^n))$ generates a non-autonomous set-valued dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = E^n \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$.

Applying to the constructed dynamical system the general results from sections 3.-6. we will obtain, for example, the following results.

Theorem 7.7. *Let $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$,$H(F)$ be τ-periodic ($F(t + \tau, x) = F(t, x)$ for all $(t, x) \in \times E^n$, where $\tau > 0$) and let F be regular. Then the next statements are equivalent:*

(1) *equation (30) admits a compact global pullback attractor, i.e. there exists a family of nonempty compacts $\{I_s \mid s \in [0, \tau]\}$ such that*
  (a) *the set $I := \bigcup\{I_s \mid s \in [0, \tau]\}$ is relatively compact in E^n;*
  (b) *$I_{s+\tau} = I_s$ for all $s \in [0, \tau]$ and $\varphi(s, I_0, F) = I_s$ for all $s \in [0, \tau]$;*
  (c) *the equality*
$$\lim_{t\to+\infty} \beta(\varphi(t, M, F_{s-t}), I_s) = 0$$
  holds for each $M \in \mathcal{K}(E^n)$ and $s \in [0, \tau]$.

(2) *equation (30) admits a compact global forward attractor, i.e. there exists a family of nonempty compacts $\{I_s \mid s \in [0, \tau]\}$ such that*
  (a) *the set $I := \bigcup\{I_s \mid s \in [0, \tau]\}$ is relatively compact in E^n;*
  (b) *$I_{s+\tau} = I_s$ for all $s \in [0, \tau]$ and $\varphi(s, I_0, F) = I_s$ for all $s \in [0, \tau]$;*
  (c) *the equality*
$$\lim_{t\to+\infty} \sup_{s\in[0,\tau]} \beta(\varphi(t, M, F_s), I) = 0$$
  holds for every $M \in C(E^n)$, where $I := \bigcup\{I_s \mid s \in [0, \tau]\}$.

Theorem 7.8. *Let $F \in C(\mathbb{R} \times E^n, \mathcal{K}_V(E^n))$,$H(F)$ be τ-periodic ($F(t + \tau, x) = F(t, x)$ for all $(t, x) \in \times E^n$, where $\tau > 0$), F be regular and the mapping $\varphi : \mathbb{R}_+ \times E^n \times H(F) \to \mathcal{K}(E^n)$ be continuous w.r.t. the Hausdorff's distance in $\mathcal{K}(E^n)$. Then the following statements are equivalent:*

(1) *equation (30) admits a compact global pullback attractor;*
(2) *equation (30) admits a compact global uniform forward attractor, i.e. there exists a family of nonempty compacts $\{I_s \mid s \in [0, \tau]\}$ such that*
  (a) *the set $I := \bigcup\{I_s \mid s \in [0, \tau]\}$ is relatively compact in E^n;*
  (b) *$I_{s+\tau} = I_s$ for all $s \in [0, \tau]$ and $\varphi(s, I_0, F) = I_s$ for all $s \in [0, \tau]$;*
  (c) *the equality*
$$\lim_{t\to+\infty} \sup_{s\in[0,\tau]} \beta(\varphi(t, M, F_s), I_s) = 0$$
  holds for every $M \in \mathcal{K}(E^n)$, where $I := \bigcup\{I_s \mid s \in [0, \tau]\}$.

Example 7.9. (Difference inclusions). Let us consider the difference inclusion

$$u(t + 1) \in F(t, u(t)), \tag{32}$$

where $F \in C(\mathbb{Z} \times E^n, \mathcal{K}(E^n))$. Along with difference inclusion (32) we will consider the family of difference inclusions

$$v(t + 1) \in G(t, v(t)), \tag{33}$$

where $G \in H(F) = \overline{\{F_\tau : \tau \in \mathbb{Z}\}}, F_\tau(t,u) = F(t+\tau,u)$ and by bar there is denoted the closure in $C(\mathbb{Z} \times E^n, \mathcal{K}(E^n))$.

We denote by $\varphi_{(v,G)}(n)$ a solution of inclusion (33) passing through the point v for $t = 0$ end defined for all $t \geq 0$. We set $\varphi(t,v,G) = \{\varphi_{(v,G)}(t) : \varphi_{(v,G)} \in \Phi_{(v,G)}\}$, where $\Phi_{(v,G)}$ is the set of all solutions of inclusion (33) , passing through the point v for $t = 0$. From the general properties of difference inclusions it follows that the mapping $\varphi : \mathbb{Z}_+ \times E^n \times H(F) \to \mathcal{K}(E^n)$ possesses the next properties :

1. $\varphi(0,v,G) = v$ for all $v \in E^n, G \in H(F)$;
2. $\varphi(t+\tau,v,G) = \varphi(t,\varphi(\tau,v,G),G_\tau)$ for all $v \in E^n, G \in H(F)$ and $t,\tau \in \mathbb{Z}_+$;
3. the mapping $\varphi : \mathbb{Z}_+ \times E^n \times H(F) \to \mathcal{K}(E^n)$ is $\beta-$ continuous.

Assume $Y = H(F)$ and denote by $(Y,\mathbb{Z},\sigma)$ the dynamical system of translations on Y. Then the triplet $\langle E^n, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ is a set-valued cocycle over $(Y,\mathbb{Z},\sigma)$ with the fiber E^n. Thus, non-autonomous difference inclusion (32) in a natural way generates a non-autonomous set-valued dynamical system $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z},\sigma),h\rangle$, where $X = E^n \times Y, \pi = (\varphi,\sigma)$ and $h = pr_2 : X \to Y$. Applying the results of paragraphs 3.-6. to the constructed above non-autonomous dynamical system we will obtain the analogues of Theorems 7.7 and 7.8 for difference inclusions.

7.2. **Homogeneous set-valued dynamical systems.** Let Y be a compact metric space and (X,h,Y) be a fiber bundle [23] with the fiber E, (X,ρ) be a complete metric space, $\mathbb{T} = \mathbb{S}_+ := \{s \in \mathbb{S} \mid s \geq 0\}$, where $\mathbb{S} := \mathbb{R}$ or $\mathbb{Z}$.

Definition 7.10. *$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ ($\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$) is said to be homogeneous, if for any $x \in X$ and any $\gamma_x \in \Phi_x$ the function $\tilde{\gamma} : D(\gamma_x) \to X$ ($\mathbb{T}_1 \subseteq D(\gamma_x) := [r_x, +\infty)$ is the domain of the definition of γ_x, where $r_x \in \mathbb{S}$) defined by the relation $\tilde{\gamma}(t) := \lambda\gamma_x(t)$ is the motion of $(X,\mathbb{T}_1,\pi)$ issuing from the point $\lambda x \in X$, i.e. $\tilde{\gamma} \in \Phi_{\lambda x}$.*

Note that non-autonomous dynamical systems from Examples 7.4 and 7.9 are homogeneous, if the set-valued mapping F which figures in these examples is homogeneous, i.e. $F(t,\lambda x) = \lambda F(t,x)$ for all $(t,x) \in \mathbb{T} \times E^n$.

If $x \in X$, then we put $|x| := \rho(x,\theta_{h(x)})$, where θ_y $(y \in Y)$ is the null (trivial) element of the linear space X_y and $\Theta := \{\theta_y \mid y \in Y\}$ is the null (trivial) section of the vectorial bundle (X,h,Y). Let $A \in \mathcal{K}(X)$, then we denote $|A| := \max\{|a| \ : \ a \in A\}$. Denote by X^s a stable manifold of $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma)\rangle$, i.e. $X^s := \{x \mid x \in X, \ \lim\limits_{t\to+\infty} |\pi(t,x)| = 0\}$.

Lemma 7.11. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ ($\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$) be a homogeneous non-autonomous set-valued dynamical system and L be a nonempty maximal compact invariant set from X. Then $J_y = \theta_y$ for all $y \in Y$, where $J_y := \{x \in J \mid h(x) = y\}$.*

Proof. If we suppose that the statement of the lemma is not true, then there exists a point $y_0 \in Y$ and $x_0 \in J_{y_0}$ such that $x_0 \neq \theta_{y_0}$. By the homogeneity of the system $(X,\mathbb{T}_1,\pi)$ we have $\lambda\mathcal{F}_{\lambda x_0} = \mathcal{F}_{\lambda x_0}$ for all $\lambda \geq 0$ $(\lambda \in \mathbb{R})$ and, consequently, $\lambda x_0 \in J$. But the last inclusion contradicts to the compactness of the set J. The obtained contradiction proves our statement. □

Definition 7.12. *Let W be a Banach space. The cocycle $\langle W,\varphi,(Y,\mathbb{S},\sigma),h\rangle$ is said to be homogeneous, if the skew-product set-valued dynamical system $(X,\mathbb{T},\pi)$ also is ($X := W \times Y, \pi := (\varphi,\sigma)$).*

Theorem 7.13. *Let W be a Banach space, $\langle W, \varphi, (Y, \mathbb{S}, \sigma), h\rangle$ be a homogeneous set-valued α-condensing cocycle admitting a compact global pullback attractor $\{I_y \mid y \in Y\}$. Then the following statements hold:*

(1) $I_y = 0$ *(0 is the null element of Banach space W) for all $y \in Y$;*
(2)
$$\lim_{t\to+\infty} \sup_{y\in Y} |\varphi(t, M, \sigma(-t, y))| = 0$$
for all $M \in \mathcal{K}(W)$.

Proof. Let $\{I_y \mid y \in Y\}$ be a compact global pullback attractor of the set-valued α-condensing cocycle φ. Then by Theorem 3.10 $J := \bigcup\{J_y \mid y \in Y\}$, where $J_y := I_y \times \{y\}$ is the maximal compact invariant set of the skew-product set-valued dynamical system $(X, \mathbb{T}, \pi)$ $(X := W \times Y$ and $\pi := (\varphi, \sigma))$. According to Lemma 7.11 $J_y = \theta_y = \{0\} \times \{y\}$ and, consequently, $I_y = \{0\}$ for all $y \in Y$.

Since the set J is compact, the set-valued mapping $y \to J_y$ is upper semi-continuous and, consequently, the mapping $y \to I_y$ is too because $I_y = pr_1(J_y)$. Taking into account that I_y contains a single point we obtain its continuity. Now to finish the proof of the theorem it is sufficient to apply Theorem 5.3. □

Example 7.14. *(Non-autonomous discrete linear inclusions). Let us consider a finite set of non-stationary matrices $\mathcal{M} := \{A_i(t) \mid i \in \{1, ..., m\}\}$, with each $A_i(t) : E^n \to E^n$ $(t \in \mathbb{Z}_+)$. Let $F : \mathbb{Z}_+ \times E^n \to \mathcal{K}(E^n)$ be the set-valued mapping defined by the equality $F(t, x) := \{A_1(t)x, A_2(t)x, ..., A_m(t)x\}$ for all $t \in \mathbb{Z}_+$ and $x \in E^n$. Consider a difference inclusion*

$$x(t + 1) \in F(t, x(t)). \tag{34}$$

Note that the solution of this inclusion is a sequence $\{x(t)\}_{t\in\mathbb{Z}_+}$ of vectors in E^n such that $x(t + 1) = A_{i_t}(t)x(t)$ for some $A_{i_t}(t) \in \mathcal{M}$, i.e.

$$x(t) = A_{i_t}(t)A_{i_{t-1}}(t - 1)...A_{i_1}(1)x(0) \ (A_{i_t}(t) \in \mathcal{M}).$$

Along with equation (34) we consider its H-class (see Example 7.9), i.e. the family of inclusions

$$x(t + 1) \in G(t, x(t)), \tag{35}$$

where $G \in H(F) := \overline{\{F_s \mid s \in \mathbb{Z}_+\}}$ and $F_s(t, x) := F(t + s, x)$ for all $(t, x) \in \mathbb{Z}_+ \times E^n$.

Definition 7.15. *Following* [21]*, inclusion (34) is said to be absolutely asymptotically stable (AAS) if for any trajectory $\{x(t)\}_{t\in\mathbb{Z}_+}$ of any inclusion (35)*

$$\lim_{t\to+\infty} x(t) = 0.$$

Remark 7.16. *We note that in work* [21] *only stationary case is considered, i.e. the matrices $A_i(t)$ $(i = 1, 2, ..., m)$ are not dependant on time $t \in \mathbb{Z}_+$. In this case the H-class of inclusion (34) contains only the inclusion (34).*

Theorem 7.17. *Suppose that the matrices $A_1(t), A_2(t), ..., A_m(t)$ are bounded on $\mathbb{Z}_+$, i.e. there is a positive number C such that $\|A_i(t)\| \le C$ for all $i \in \{1, 2, ..., m\}$ and $t \in \mathbb{Z}_+$. Then the following two affirmations are equivalent:*

1. *inclusion (34) is absolutely asymptotically stable;*

2. *inclusion (34) is uniformly exponentially stable, i.e. there are positive numbers* N *and* a *(*$a \in (0,1)$*) such that*

$$|x(t)| \le Na^t|x(0)|$$

for all $t \in \mathbb{Z}_+$*, where* $\{x(t) \ |t \in \mathbb{Z}_+\}$ *is an arbitrary solution of arbitrary inclusion (35).*

Proof. The implication 2. $\to$ 1. is obvious, therefore to prove Theorem it is sufficient to show that 1. implies 2. Let $\langle E^n, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ be the set-valued cocycle generated by inclusion (34). From condition 1. it follows that the skew-product set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := E^n \times Y$ and $\pi := (\varphi, \sigma)$) is trajectory dissipative and by Theorem 2.15 the dynamical system $(\Phi, \mathbb{Z}_+, \sigma)$ is pointwise dissipative, where by Φ there is denoted the set of all motions of the set-valued dynamical system $(X, \mathbb{Z}_+, \pi)$ and by $(\Phi, \mathbb{Z}_+, \sigma)$ we denote the dynamical system of shifts on Φ. Under the conditions of the theorem the set $Y = H(A)$ is a compact subset of $C(Z_+, \mathcal{K}(E^n))$ (the space $C(Z_+, \mathcal{K}(E^n))$ is equipped with the compact-open topology and $\mathcal{K}(E^n)$ is a metric space with the distance of Hausdorff). Since the phase space $X = E^n \times H(A)$ is locally compact, then (see [6, Ch.I]) the space Φ (Φ is a subspace of $C(\mathbb{Z}_+, X)$ which is equipped with the compact-open topology) is also locally compact. Thus, the dynamical system $(\Phi, \mathbb{Z}_+, \sigma)$ is pointwise dissipative and its phase space Φ is locally compact. According to Theorem 2.16 the dynamical system system $(\Phi, \mathbb{Z}_+, \sigma)$ is locally dissipative and by Theorem 1.6.7 [6] the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is also locally dissipative and, in particular, we have

$$\lim_{t \to +\infty} \sup_{|x| \le 1} |\pi(t, x)| = 0.$$

Note that the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ ($h = pr_2 : X \to Y$) generated by the set-valued cocycle φ obviously is homogeneous. Now to finish the proof of the theorem it is sufficient to apply Theorem 1.1 from [8], which states that for a homogeneous set-valued non-autonomous dynamical system the local dissipativity implies its uniform exponential stability. □

Remark 7.18. *Theorem 7.17 is a generalization of the well know result (see* [21]*) on absolute asymptotic stability for non-autonomous discrete linear inclusions .*

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