Invariant Manifolds, Almost Periodic and Almost Automorphic Solutions of Second-Order Monotone Equations

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Abstract

We give sufficient conditions of the existence of a compact invariant manifold, almost periodic (quasi-periodic, almost automorphic, quasi-recurrent) solutions and chaotic sets of the second-order differential equation $x'' = f(t, x)$ on an arbitrary Hilbert space with the uniform monotone right hand side f.

Keywords: non-autonomous dynamical systems; skew-product systems; cocycles; continuous invariant sections of non-autonomous dynamical systems; almost periodic, almost automorphic, quasi-recurrent solutions; chaotic sets

1991 *Mathematics Subject Classification:* primary:34C35, 34D20, 34D40, 34D45, 58F10, 58F12, 58F39; secondary: 35B35, 35B40

1 Introduction

The problem of the almost periodicity of solutions of non-linear almost periodic second-order differential equations

$$x'' = f(t, x) \tag{1}$$

with the monotone (with respect to the spacial variable x) right hand side f was studied by many authors (see, for example, [3]-[15], [17, 18], [22, 23], [24], [29], [34] and the bibliography therein).

In the present paper we consider a special class of equations (1), where the function $f : \mathbb{R}\times H \to H$ (H is a Hilbert space) is uniformly monotone with respect to (w.r.t.) $x \in H$, i.e. $f'_x(t,x) \geq mI$, where $f_x(t,x)$ is a self-adjoint operator and I is a unit operator on H and $m > 0$. We also study a more general equation

$$x'' = f(\omega t, x) \quad (\omega \in \Omega), \tag{2}$$

with the uniform monotone (with respect to the spacial variable x) right hand side f, where Ω is a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ is a dynamical system on Ω and $\omega t := \sigma(t, \omega)$. We give sufficient conditions for the existence of a compact invariant manifold of equation (2). Almost periodic, quasi-periodic, almost automorphic, pseudo recurrent solutions and chaotic sets of equation (2) are studied too.

The problem of almost periodicity of solutions of equation (1) (with a monotone, but not strictly monotone, function f) was studied by Cieutat [22] and by Cheban [18] (with a uniform monotone function f).

A special class of such equations is the class of the Lagrangian system

$$x'' = \nabla_x V(t, x). \tag{3}$$

In [24] Corduneanu studied the existence of almost periodic solutions of (3) with a uniform monotone function $\nabla_x V$ (i.e. $\nabla_{xx} V(t,x) \geq mI$). Zakharin and Parasyuk [34] studied the problem of the existence of quasi-periodic solutions of the equation

$$x'' = \nabla_x V(\omega t, x) \ (\omega \in \Omega)$$

when Ω is an m-dimensional torus $\mathcal{T}^m$, $(\Omega, \mathbb{R}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$ and the right hand side $\nabla_x V$ is uniformly monotone.

For the equation

$$x'' = \nabla_x V(x) + f(t)$$

Carminati [17] gives sufficient conditions of the existence and uniqueness of a bounded solution and its almost periodicity (see also [3]-[15] and the bibliography therein).

This paper is organized as follows.

Section 2 contains the notions of different types of motions (almost periodic, almost automorphic, recurrent etc) and some properties of these classes of motions. We also give certain examples of shift dynamical systems which play a very important role in the study of the recurrence property (almost periodicity, almost automorphy, recurrence etc) of continuous functions and solutions of non-autonomous differential equations. Finally, in this section we present notions of cocycle, skew-product dynamical systems, non-autonomous dynamical systems and continuous invariant sections of non-autonomous dynamical systems which play a crucial role in our paper.

Section 3 is devoted to the study of invariant manifolds (invariant continuous sections) of the second order differential equation (2) with uniform monotone (with respect to spacial variable x) right hand side f. The main result of this paper is Theorem 3.4 which contains the sufficient conditions of the existence of compact invariant manifold of equation (2). Here, we study also the almost periodic, quasi-periodic, almost automorphic, pseudo recurrent solutions (Corollary 3.8 and Theorem 3.12) and chaotic sets (Theorem 3.15) of equation (2).

In section 4 we give sufficient conditions of the existence of at least one almost automorphic solution of differential equation (1) with almost authomorphic monotone right hand side (Theorem 4.5).

2 Almost Periodic and Almost Automorphic Motions of Dynamical Systems

2.1 Recurrent, Almost Periodic and Almost Automorphic Motions

Let X be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of the additive group $\mathbb{S}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system.

A number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ shift (respectively, almost period), if $\rho(x\tau, x) < \varepsilon$ (respectively, $\rho(x(\tau + t), xt) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon > 0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

Denote $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T}\ :\ \text{such that } \{\pi(t_n, x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called Levitan almost periodic [27], if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.1 *Let $x_i \in X_i$ $(i = 1, 2, \dots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then the point $x := (x_1, x_2, \dots, x_m)) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic in the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t, x) := (\pi_1(t, x_1), \pi_2(t, x_2), \dots, \pi_m(t, x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \dots, x_m) \in X$.*

A point $x \in X$ is called stable in the sense of Lagrange (st.L), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called almost automorphic [27, 31] in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;

(ii) there exists a dynamical system $(Y, \mathbb{T}, \sigma)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$ and an almost periodic in the sense of Bohr point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.2 *1. Every almost automorphic point $x \in X$ is also Levitan almost periodic.*

2. A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t, x) \ t \in \mathbb{T}\}$ is also almost automorphic (see [1, 2], [16], [27], [31] and also [25] and [28]). In other words, an Levitan almost periodic point x is almost periodic if and only if its trajectory $\{\pi(t, x) \ t \in \mathbb{T}\}$ is relatively compact.

3. Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems, $x \in X$ and the following conditions be fulfilled:

(i) a point $y \in Y$ is Levitan almost periodic;

(ii) $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Then the point x is Levitan almost periodic, too.

4. Let $x \in X$ be a st.L point, $y \in Y$ be an almost automorphic point and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then the point x is almost automorphic too.

Remark 2.3 *1. We note (see, for example, [27] and [33]) that if $y \in Y$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$, then the point $x = h(y)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{T}_1, \pi)$.*

2. If $y \in Y$ is an almost automorphic point of the dynamical system $(Y, \mathbb{T}, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{S}, \sigma)$ onto $(X, \mathbb{T}, \pi)$, then the point $x = h(y)$ is an almost automorphic point of the system $(X, \mathbb{T}, \pi)$.

2.2 Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions

Below we indicate one general method of construction of dynamical systems on the space of continuous functions. In this way we will get many well known dynamical systems on the functional spaces (see, for example, [16, 32]).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X, Y be a complete pseudo metric space and $\mathcal{P}$ be a family of pseudo metrics on Y. We denote by $C(X, Y)$ the family of all continuous functions $f : X \to Y$ equipped with a compact-open topology. This topology is given by the following family of pseudo metrics $\{d^p_K\}$ $(p \in \mathcal{P},\ K \in C(X))$, where

$$d^p_K(f, g) := \sup_{x \in K} p(f(x), g(x))$$

and $C(X)$ a family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X, Y) \to C(X, Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau, x))$ $(x \in X)$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;

b. $\forall \tau_1, \tau_2 \in \mathbb{T}$ $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}$;

c. $\forall \tau \in \mathbb{T}$ σ_τ is continuous.

Lemma 2.4 *[20] The mapping $\sigma : \mathbb{T} \times C(X, Y) \to C(X, Y)$, defined by the equality $\sigma(\tau, f) := \sigma_\tau f$ $(f \in C(X, Y),\ \tau \in \mathbb{T})$, is continuous.*

Corollary 2.5 *The triple $(C(X, Y), \mathbb{T}, \sigma)$ is a dynamical system on $C(X, Y)$.*

Consider now some examples of dynamical systems of the form $(C(X, Y), \mathbb{T}, \sigma)$, useful in the applications.

Example 2.6 Let $X = \mathbb{T}$ and we denote by $(X, \mathbb{T}, \pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t, x) := x + t$. The dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ is called *Bebutov's dynamical system* [32] (a dynamical system of translations, or shifts dynamical system).

We will say that the function $\varphi \in C(\mathbb{T}, Y)$ possesses a property (A), if the motion $\sigma(\cdot, \varphi) : \mathbb{T} \to C(\mathbb{T}, Y)$ possesses this property in the dynamical system of Bebutov $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$, generated by the function φ. As property (A) we can take periodicity, quasi-periodicity, almost periodicity, almost automorphy, recurrence etc.

Example 2.7 Let $X := \mathbb{T} \times W$, where W is some metric space and by $(X, \mathbb{T}, \pi)$ we denote a dynamical system on X defined in the following way: $\pi(t, (s, w)) := (s + t, w)$. Using the general method proposed above we can define on $C(\mathbb{T} \times W, Y)$ a dynamical system of translations $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

The function $f \in C(\mathbb{T} \times W, Y)$ is called almost periodic (quasi-periodic, recurrent, almost automorphic, etc) with respect to $t \in \mathbb{T}$ uniform on w on every compact from W, if the motion $\sigma(\cdot, f)$ is almost periodic (quasi-periodic, recurrent, almost automorphic, etc.) in the dynamical system $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

Remark 2.8 *Let W be a compact metric space, then the topology on $C(W, Y)$ is metrizable. For example by the equality*

$$d(f, g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f, g)}{1 + d_k(f, g)}$$

there is defined a complete metric on the space $C(W, X)$ which is compatible with the compact-open topology on $C(W, X)$, where $d_k(f, g) := \max_{|t| \le k,\ x \in W} \rho(f(t, x), g(t, x))$. The space $C(\mathbb{T} \times W, Y)$ is topologically isomorphic to $C(\mathbb{T}, C(W, Y))$ [32], and also the shifts dynamical systems $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$ and $(C(\mathbb{T}, C(W, Y)), \mathbb{T}, \sigma)$ are dynamically isomorphic.

2.3 Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ $(\mathbb{S}_+ \subseteq \mathbb{T}_+)$.

A triplet $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system.

Let $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on Y, W be a complete metric space and φ be a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in W, possessing the following properties:

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;

b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then the triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ (or shortly φ) is called [30] a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with the fiber W.

Let $X := W \times Y$ and let us define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as follows: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [30] and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, hence, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ $(X := W \times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of this type.

Example 2.9 Consider the system of differential equations

$$\begin{cases} u' = & F(y, u) \\ y' = & G(y), \end{cases} \tag{4}$$

where $Y \subseteq E^m, G \in C(Y, E^n)$ and $F \in C(Y \times E^n, E^n)$. Suppose that for the system (4) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(Y, \mathbb{R}_+, \sigma)$ a dynamical system on Y generated by the second equation of the system (4) and by $\varphi(t, u, y)$ we denote the solution of the equation

$$u' = F(\sigma(t, y), u)$$

passing through the point $u \in E^n$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times Y \to E^n$ satisfies the conditions a. and b. from definition of cocycle and, consequently, system (4) generates a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ (where $X := E^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$).

Example 2.10 Let E be a Banach space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on the metric space Y. We consider the system

$$\begin{cases} u' = & F(\sigma(y, t), u) \\ y \in & Y, \end{cases} \tag{5}$$

where $F \in C(Y \times E, E)$. Suppose that for equation (5) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ (respectively, the cocycle $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$), where $X := E \times Y$, $\pi := (\varphi, \sigma)$, $\varphi(\cdot, x, y)$ is the solution of (5) and $h := pr_2 : X \to Y$ is generated by equation (5).

2.4 Invariant Sections of Non-Autonomous Dynamical Systems

Let $\langle (X, \mathbb{S}_+, \pi),\ (Y, \mathbb{S}, \sigma),\ h)\rangle$ be a non-autonomous dynamical system.

A mapping $\gamma : Y \to X$ is called a section (selector) of a homomorphism h, if $h(\gamma(y)) = y$ for all $y \in Y$. The section γ of the homomorphism h is called invariant, if $\gamma(\sigma(t, y)) = \pi(t, \gamma(y))$ for all $y \in Y$ and $t \in \mathbb{S}$.

Remark 2.11 *Note that $(\gamma(Y), \mathbb{S}, \pi)$ is a group subsystem of the semigroup dynamical system $(X, \mathbb{S}_+, \pi)$, if γ is a continuous section of the homomorphism h from $(X, \mathbb{S}_+, \pi)$ onto $(Y, \mathbb{S}, \sigma)$.*

Denote by $\Gamma = \Gamma(Y, X)$ the family of all continuous sections of h, i.e. $\Gamma(Y, X) = \{\gamma \in C(Y, X) :\ h \circ \gamma = Id_Y\}$. We will suppose that $\Gamma(Y, X) \neq \emptyset$. For applications this condition is fulfilled in many important cases.

Remark 2.12 *A continuous section $\gamma \in \Gamma$ is invariant, if and only if $\gamma \in \Gamma$ is a stationary point of the semigroup $\{S^t \mid t \in \mathbb{S}_+\}$, where $S^t : \Gamma(Y, X) \to \Gamma(Y, X)$ is defined by the equality $(S^t\gamma)(y) := \pi(t, \gamma(\sigma(-t, y)))$ for all $y \in Y$ and $t \in \mathbb{S}_+$.*

We consider a special case of the foregoing construction. Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{S}, \sigma)$ with the fiber W and $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be the non-autonomous dynamical system generated by this cocycle. Then $h \circ \gamma = Id_Y$, and since $h = pr_2$, then $\gamma = (\psi, Id_Y)$, where $\gamma \in \Gamma(Y, X)$ and $\psi : Y \to W$. Hence, to each section γ a mapping $\psi : Y \to W$ corresponds, and vice versa. There is a one-on-one relation between $\Gamma(Y, W \times Y)$ and $C(Y, W)$, where $C(Y, W)$ is the space of continuous functions $\psi : Y \to W$, we identify these two objects from now on. The semigroup $\{S^t \mid t \in \mathbb{S}_+\}$ naturally induces a semigroup $\{Q^t \mid t \in \mathbb{S}_+\}$ of the mappings of $C(Y, W)$. Namely,

$$(S^t\gamma)(y) = \pi^t\gamma(\sigma^{-t}y) = \pi^t(\psi, Id_Y)(\sigma^{-t}y) =$$

$$\pi^t(\psi(\sigma^{-t}y), \sigma^{-t}y) = (U(t, \sigma^{-t}y)\psi(\sigma^{-t}y), y) = ((Q^t\psi)(y), y),$$

where $U(t, y) := \varphi(t, \cdot, y)$.

Hence, $S^t(\psi, Id_Y) = (Q^t\psi, Id_Y)$ with $(Q^t\psi)(y) = U(t, \sigma^{-t}y)\psi(\sigma^{-t}y)\ \ (y \in Y)$. We have the next properties:

a. $Q^0 = Id_{C(Y,W)}$;

b. $Q^tQ^\tau = Q^{t+\tau}\ (\ t, \tau \in \mathbb{S}_+)$.

A continuous function $\psi : Y \to W$ is called an invariant section of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ (or an invariant manifold of the cocycle φ), if $\psi(\sigma(t, y)) = \varphi(t, \psi(y), y)$ for all $t \in \mathbb{T}$ and $y \in Y$.

Remark 2.13 *Let $X := E \times Y$ and $\pi := (\varphi, \sigma)$. Then the mapping $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$, if and only if $h(y) = (\gamma(y), y)$ for all $y \in Y$, where $\gamma : Y \to E$ is a continuous mapping with the condition that $\gamma(yt) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{T}_2$.*

3 Invariant Manifolds of Second Order Differential Equations

3.1 Invariant manifolds

Let Ω be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on Ω. Let E be a Banach space. Denote by $[E]$ the space of all linear continuous operators acting on E and endowed with an operator norm.

Denote by H a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|^2 := \langle \cdot, \cdot \rangle$, by $C(\Omega, E)$ we denote the Banach space of all continuous function $\varphi : \Omega \to E$ equipped with the norm $\|\varphi\|_{C(\Omega,E)} := \max\limits_{\omega \in \Omega} |\varphi(\omega)|_E$.

A function $\varphi \in C(\Omega, E)$ is called:

- differentiable in the point ω_0 along the flow $(\Omega, \mathbb{T}, \sigma)$, if there exists a limit

$$\dot{\varphi}_\sigma(\omega_0) := \lim_{s \to 0} \frac{\varphi(\sigma(s, \omega_0)) - \varphi(\omega_0)}{s};$$

  In this case $\dot{\varphi}_\sigma(\omega_0)$ is called a derivative of the function $\varphi \in C(\Omega, E)$ at the point $\omega_0 \in \Omega$ along the flow $(\Omega, \mathbb{T}, \sigma)$ (shortly, σ).

- differentiable on Ω along the flow σ, if it is differentiable at every point $\omega \in \Omega$;

- continuously differentiable on Ω along the flow σ, if it is differentiable at Ω and $\dot{\varphi}_\sigma \in C(\Omega, E)$.

Denote by $\dot{C}^1(\Omega, E)$ a Banach space of all continuously differentiable (on Ω along the flow σ) functions $\varphi \in C(\Omega, E)$ endowed with the norm

$$\|\varphi\|_{\dot{C}^1(\Omega,E)} := \|\varphi\|_{C(\Omega,E)} + \|\dot{\varphi}\|_{C(\Omega,E)}.$$

Let us consider a differential equation of the second order

$$x'' = f(\omega t, x), \ (\omega \in \Omega) \tag{6}$$

where $f \in C(\Omega \times H, H)$, and give a criterion of the existence of an invariant manifold for this equation. Below we will suppose that the function f is regular, i.e. for all $x.y \in H$ the equation (6) admits a unique solution $\varphi(t,x,y,\omega)$ defined on $\mathbb{R}_+$ with the initial conditions $\varphi(0,x,y,\omega) = x$ and $\varphi'(0,x,y,\omega) = y$.

As we know, we can reduce the equation (6) to the equivalent system

$$\begin{cases} x' = y \\ y' = f(\omega t, x) \end{cases}$$

$(\omega \in \Omega)$ or to the equation

$$z' = F(\omega t, z) \tag{7}$$

on the product space $H^2 := H \times H$, where $z := (x,y)$ and $F \in C(\Omega \times H^2, H^2)$ is the function defined by the equality $F(\omega, z) := (y, f(\omega, x))$ for all $\omega \in \Omega$ and $z := (x,y) \in H^2$.

Remark 3.1 *1. Since $(\varphi(t,x,y,\omega), \varphi'(t,x,y,\omega))$ is a cocycle, generated by equation (7), then we have the following equality*

$$\varphi(t+\tau, x, y, \omega) = \varphi(t, \varphi(\tau,x,y,\omega), \varphi'(\tau,x,y,\omega), \omega\tau) \tag{8}$$

for all $t,\tau \in \mathbb{R}_+$, $x,y \in H$ and $\omega \in \Omega$.

2. The function $\mu := (\gamma,\delta) \in C(\Omega, H^2)$ $(\gamma,\delta \in C(\Omega,H))$ is a continuous invariant section of the cocycle $(\varphi(t,x,y,\omega), \varphi'(t,x,y,\omega))$, generated by equation (7), if and only if the following conditions are fulfilled:

(i) $\gamma \in \dot{C}^1(\Omega, H)$;

(ii) $\dot{\gamma}_\sigma = \delta$;

(iii) $\gamma(\omega t) = \varphi(t, \gamma(\omega), \dot{\gamma}_\sigma(\omega), \omega)$ *for all* $t \in \mathbb{R}$ *and* $\omega \in \Omega$.

Cheban [18] and Cieutat [22] have studied the existence of almost periodic and asymptotically almost periodic solutions of (6) (in the case, when $\Omega = H(f)$ and $(\Omega, \mathbb{R}, \sigma)$ is a shift dynamical system).

A special class of such systems is the class of the following Lagrangian system:

$$x'' = \nabla_x V(\omega t, x), \ (\omega \in \Omega), \tag{9}$$

where $V \in C(\Omega \times H, \mathbb{R})$ and $V(\omega, \cdot)$ is differentiable for each $\omega \in \Omega$ ($\nabla_x V(\omega, \cdot)$ denotes the gradient of the function $V(\omega, \cdot)$). Corduneanu [24] and Zakharin and Parasyuk [34] have studied the existence of almost periodic solutions of (9).

A particular case of equation (9) is the following equation:

$$x'' = \nabla_x V(x) + f(\omega t), \ (\omega \in \Omega), \tag{10}$$

where $f \in C(\Omega, H)$. Carminati [17] gives sufficient conditions for the existence and uniqueness of bounded or almost periodic solutions of (10).

Lemma 3.2 *Let $M > 0$ and $f \in C(\Omega, H)$. By the formula*

$$\gamma(\omega) = \frac{1}{2\sqrt{M}} \Big\{ \int\limits_0^{+\infty} e^{-\sqrt{M}\tau} f(\omega\tau) d\tau + \int\limits_{-\infty}^{0} e^{\sqrt{M}\tau} f(\omega\tau) d\tau \Big\} \tag{11}$$

there is defined a continuous function on Ω possessing the following properties:

1.
$$\gamma(\omega t) = \frac{1}{2\sqrt{M}} \Big\{ e^{\sqrt{M}t} \int\limits_t^{+\infty} e^{-\sqrt{M}\tau} f(\omega\tau) d\tau + e^{-\sqrt{M}\tau} \int\limits_{-\infty}^{t} e^{\sqrt{M}\tau} f(\omega\tau) d\tau \Big\}$$
for all $\omega \in \Omega$ and $t \in \mathbb{R}$;

2.
$$\dot{\gamma}_\sigma(\omega) = \frac{1}{2} \Big\{ \int\limits_0^{+\infty} e^{-\sqrt{M}\tau} f(\omega\tau) d\tau - \int\limits_{-\infty}^{0} e^{\sqrt{M}\tau} f(\omega\tau) d\tau \Big\}$$
for all $\omega \in \Omega$;

3. *$\gamma(\omega t) = \varphi(t, \gamma(\omega), \dot{\gamma}_\sigma(\omega), \omega)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$, where $\varphi(t, x, y, \omega)$ is a unique solution of the equation $x'' = Mx + f(\omega t)$ ($\omega \in \Omega$) with the initial conditions $\varphi(0, x, y, \omega) = x$ and $\varphi'(0, x, y, \omega) = y$.*

Proof. Since the integrals figuring in the equality (11) are convergent (uniformly in $\omega \in \Omega$), then by (11) there is correctly defined a continuous function γ on Ω. The fact that the function γ defined by equality (11) possess properties 1.–3. can be proved by a simple calculation. □

Corollary 3.3 *Let $\gamma : \Omega \to H$ be the function defined by (11), then the following statements hold:*

(i) $\gamma \in \dot{C}^1(\Omega, H)$;

(ii) $||\gamma||_{C(\Omega,H)} \le \frac{1}{M}||f||_{C(\Omega,H)}$;

(iii) $||\dot{\gamma}||_{C(\Omega,H)} \le \frac{1}{\sqrt{M}}||f||_{C(\Omega,H)}$;

(iv) $||\gamma||_{\dot{C}^1(\Omega,H)} \le (\frac{1}{M} + \frac{1}{\sqrt{M}})||f||_{C(\Omega,H)}$.

Theorem 3.4 *Let $f \in C(\Omega \times H, H)$ be continuously differentiable w.r.t. $x \in H$ and let exist $r_0 > 0$ such that*

(i) $|f(\omega, x)| \le A(r) < +\infty$ *for all* $(\omega, x) \in \Omega \times B[0, r]$ *and* $0 \le r \le r_0$*;*

(ii) there exists positive numbers m and $M(r)$ such that for all $(\omega, x) \in \Omega \times B[0, r]$, $0 \le r \le r_0$, $mI \le f'_x(\omega, x) \le M(r)I$ (I is a unit operator from $[H]$) and the operator $f'_x(\omega, x)$ is self-adjoint;

(iii) $A(0) \le mr_0$.

Then for an arbitrary $A(0)m^{-1} \le r \le r_0$ there exist a unique function $\gamma \in \dot{C}^1(\Omega, B[0,\ r])$ such that $\gamma(\omega t) = \varphi(t, \gamma(\omega), \dot{\gamma}(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, where $\varphi(t, u, v, \omega)$ is a unique solution of equation (6) with the initial conditions $\varphi(0, u, v) = u$ and $\varphi'(0, u, v) = v$.

Proof. Let $A(0)m^{-1} \le r \le r_0$. Assume $B_r(\Omega) := \{\varphi | \varphi \in C(\Omega, H), ||\varphi||_{C(\Omega,H)} \le r\}$. Further, define an operator Φ from $B_r(\Omega)$ to $B_r(\Omega)$ by the equality

$$(\Phi\varphi)(\omega) = \frac{1}{2\sqrt{M}}\Big\{\int_0^{+\infty} e^{-\sqrt{M}\tau}F(\omega\tau, \varphi(\omega\tau))d\tau + \int_{-\infty}^{0} e^{\sqrt{M}\tau}F(\omega\tau, \varphi(\omega\tau)d\tau\Big\},$$

where $F(\omega, x) = f(\omega, x) - Mx$. Let $\varphi \in B_r(\Omega)$. We consider a differential equation

$$\frac{d^2x}{dt^2} = Mx + f(\omega t, \varphi(\omega t)) - M\varphi(\omega t).$$

Note that $F'_x(\omega, x) = f'_x(\omega, x) - MI$, and since $f'_x(\omega, x)$ is self-adjoint, we have

$$\begin{aligned} ||F'_x(\omega, x)|| &= \sup_{|\xi|=1} |(F'_x(\omega, x)\xi, \xi)| = \sup_{|\xi|=1} |(f'_x(\omega, x)\xi, \xi) - M| = \\ &= \sup_{|\xi|=1} |M - (f'_x(\omega, x)\xi, \xi)| \le M(r) - m \end{aligned} \tag{12}$$

for all $\omega \in \Omega$ and $x \in B[0, r]$. From inequality (12) it follows that

$$|F(\omega, x_1) - F(\omega, x_2)| \le (M - m)|x_1 - x_2|$$

for all $\omega \in \Omega$ and $x_1, x_2 \in B[0, r]$.

Note that $g \in C(\Omega, H)$, where $g(\omega) := F(\omega, \varphi(\omega))$. According to Lemma 3.2 $\psi := \Phi(\varphi) \in C(\Omega, H)$ and

$$\begin{aligned} & ||\psi|| \le \frac{1}{M}||g|| = \frac{1}{M} \max_{\omega \in \Omega} |F(\omega, \varphi(\omega))| \le \\ & \frac{1}{M} \max_{\omega \in \Omega} |F(\omega, \varphi(\omega)) - F(\omega, 0)| + \frac{1}{M} \max_{\omega \in \Omega} |F(\omega, 0)| \le \\ & \frac{1}{M}(M-m)||\varphi||_{C(\Omega,H)} + \frac{A(0)}{M} \le \frac{M-m}{M} r + \frac{A(0)}{M}. \end{aligned} \tag{13}$$

From inequality (13) it follows that $\psi \in B_r(\Omega)$, because $r \ge A(0)m^{-1}$. From the above said it follows that $\Phi B_r(\Omega) \subseteq B_r(\Omega)$. In addition, $B_r(\Omega)$ is a closed subspace of the full metric space $C(\Omega, H)$. Let us show that $\Phi : B_r(\Omega) \to B_r(\Omega)$ is a contracting mapping. Let $\varphi_1, \varphi_2 \in B_r(\Omega)$, $\psi_i := \Phi\varphi_i$ $(i = 1, 2)$ and $\psi := \psi_1 - \psi_2$. Then the function $\psi(\omega t)$ $(\omega \in \Omega)$ satisfies the equation

$$\frac{d^2x}{dt^2} = Mx + F(\omega t, \varphi_1(\omega t)) - F(\omega t, \varphi_2(\omega t))$$

and can be estimated like this:

$$||\psi||_{C(\Omega,H)} = ||\psi_1 - \psi_2||_{C(\Omega,H)} \le M^{-1} \max_{\omega \in \Omega} |F(\omega, \varphi_1(\omega)) -$$

$$F(\omega, \varphi_2(\omega))| \le \frac{M-m}{M}||\varphi_1 - \varphi_2||_{C(\Omega,H)},$$

i.e.

$$||\Phi\varphi_1 - \Phi\varphi_2||_{C(\Omega,H)} \le \alpha||\varphi_1 - \varphi_2||_{C(\Omega,H)}$$

for all $\varphi_1, \varphi_2 \in B_r(\Omega)$, where $\alpha = M^{-1}(M - m) < 1$. Consequently, there exists a unique fixed point of the operator Φ that, obviously, is the desired function. The theorem is proved. □

Corollary 3.5 *Let $(\Omega, \mathbb{R}, \sigma)$ be a compact minimal dynamical system. If the point $\omega \in \Omega$ is almost periodic (respectivelly, almost automorphic), then under the conditions of Theorem 3.4 the equation (6) admits at least one almost periodic (respectively, almost automorphic) solution.*

Remark 3.6 *1. For almost periodic system Corollary 3.5 was proved before by first author [18] (see, also, [22]).*

2. For almost periodic finite-dimensional Lagrangian systems (9) Corollary 3.5 was established by Corduneanu [24].

3.2 Linear case

Corollary 3.7 *Let $A \in C(\Omega, [H])$ be a self-adjoint operator-function. If there exist positive numbers m and M such that for all $\omega \in \Omega$*

$$mI \leq A(\omega) \leq MI, \tag{14}$$

then for any function $f \in C(\Omega, H)$ there exists a unique function $\gamma \in \dot{C}^1(\Omega, H)$ such that

(i) $\gamma(\omega t) = \varphi(t, \gamma(\omega), \dot{\gamma}(\omega), \omega)$ *for all $\omega \in \Omega$ and $t \in \mathbb{R}$, where $\varphi(t, u, v, \omega)$ is a unique solution of the equation*

$$x'' = A(\omega t)x + f(\omega t) \tag{15}$$

with the initial conditions $\varphi(0, u, v, \omega) = u$ and $\varphi'(0, u, v, \omega) = v$;

(ii)

$$\|\gamma\|_{C(\Omega,H)} \leq \frac{\|f\|_{C(\Omega,H)}}{m}.$$

Proof. Let $F(\omega, x) := A(\omega)x + f(\omega)$, $A \in C(\Omega, [H])$ be a self-adjoint operator-function, $f \in C(\Omega, H)$ and the condition (14) be held. Note that $|F(\omega, x)| \leq A(r) := r\|A\|_{C(\Omega,[H])} + \|f\|_{C(\Omega,H)}$ for all $\omega \in \Omega$ and $x \in B[0, r]$, where $r \in [m^{-1}\|f\|_{C(\Omega,H)}, r_0]$ and $r_0 > \|f\|_{C(\Omega,H)}m^{-1}$. Now to finish the proof it is sufficient to apply Theorem 3.4, because all its conditions are fulfilled. □

3.3 Quasi-Periodic Solutions

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$.

A function $\varphi : \mathbb{T} \to H$ is called quasi-periodic with the frequency $\omega := (\omega_1, \omega_2, \ldots, \omega_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to H$ such that $\varphi(t) := \Phi(\omega t)$ for all $t \in \mathbb{T}$, where $\omega t := \sigma(t, \omega)$ and $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$.

Corollary 3.8 *Let the conditions of Theorem 3.4 be fulfilled and the point $\omega \in \Omega$ be stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent). Then equation (6) has a unique stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent) solution.*

Remark 3.9 *1. For almost periodic equations (6) Corollary 3.8 was proved by Cheban [18] and for finite-dimensional Lagrangian equation it was established by Corduneanu [24].*

2. For finite-dimensional quasi-periodic Lagrangian equations (9) Corollary 3.8 improves Theorem 4.3 of Zakharin and Parasyuk [34].

3.4 Invariant Manifold of The Equation $x'' = \nabla V(x) + f(\omega t)$

Let us consider now equation (10).

Corollary 3.10 *Assume that the following conditions are held:*

(i) the function $V \in C(H, \mathbb{R})$ has a local minimum at $x_0 \in H$, and let r_0 be a positive number such that V is bounded on $B[x_0, r_0]$;

(ii) for all $x \in B[x_0, r_0] := \{x \in H \ : \ |x - x_0| \leq r_0\}$ the function V is of the form

$$V(x) := \frac{1}{2}\langle Ax, x\rangle + v(x),$$

where $A \in [H]$ is a self-adjoint operator and

$$\langle Ax, x\rangle \geq \alpha |x|^2$$

for all $x \in H$, and $v \in C^1(H, \mathbb{R})$ is a convex function on $B[x_0, r_0]$, i.e $\nabla_{xx} v(x) \geq 0$ for $x \in B[x_0, r_0]$;

(iii) the function $f \in C(\Omega, H)$ satisfies the inequality

$$\|f\|_{C(\Omega,H)} \leq \alpha r_0.$$

Then

(i) for an arbitrary $r \in [\|f\|_{C(\Omega,H)}\alpha^{-1}, r_0]$ there exist a unique function $\gamma \in \dot{C}^1(\Omega, B[0, r_0])$ such that

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \dot{\gamma}(\omega), \omega)$$

for all $\omega \in \Omega$ and $t \in \mathbb{R}$, where $\varphi(t, u, v, \omega)$ is a unique solution of equation (10) with the initial conditions $\varphi(0, u, v) = u$ and $\varphi'(0, u, v) = v$;

(ii)

$$\|\gamma - x_0\|_{C(\Omega,H)} \leq \alpha^{-1}\|f\|_{C(\Omega,H)}.$$

Proof. Making the change of variable $x = x_0 + y$ in equation (10) we obtain

$$y'' = \nabla_y V(x_0 + y) + f(\omega t).$$

We denote by $F(\omega, y) := \nabla_y V(x_0 + y) + f(\omega)$ $((\omega, y) \in \Omega \times B[0, r_0])$. Then

$$F'_y(\omega, y) = \nabla_{yy} V(x_0 + y) = A + \nabla_{yy} v(x_0 + y),$$

because

$$V(x_0 + y) := \frac{1}{2}\langle A(x_0 + y), x_0 + y\rangle + v(x_0 + y).$$

Since v is convex on $B[x_0, r]$ (this means that $\nabla_{xx} v(x) \geq 0$ on $B[x_0, r]$), we obtain

$$\alpha \cdot I \leq F'_y(\omega, y) \leq M(r) \cdot I$$

for all $(\omega, y) \in \Omega \times B[0, r]$ and $r \in [0, r_0]$. As the function v is bounded on $B[x_0, r_0]$, then there exists a function $A : [0, r_0] \to \mathbb{R}_+$ such that $|v(x_0 + y)| \leq A(r)$ for all $y \in B[0, r]$ (in our case, for example, we can take

$$A(r) := \sup_{y \in B[0,r]} |\nabla_y V(x_0 + y)| + \|f\|_{C(\Omega, H)}$$

and $A(0) = \|f\|_{C(\Omega,H)}$). To finish the proof of Corollary 3.10 it is sufficient to apply Theorem 3.4. □

3.5 Pseudo Recurrent Motions

A dynamical system $(\Omega, \mathbb{T}, \sigma)$ is said to be pseudo recurrent (see [19]), if the following conditions are fulfilled:

a) Ω is compact;

b) $(\Omega, \mathbb{T}, \sigma)$ is transitive, i.e. there exists a point $\omega_0 \in \Omega$ such that $\Omega = \overline{\{\sigma(t, \omega_0) \mid t \in \mathbb{T}\}}$;

c) every point $\omega \in \Omega$ is stable in the sense of Poisson, i.e. $\mathfrak{N}_\omega =\neq \emptyset$.

Lemma 3.11 *[21] Let $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h \rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

1) $(\Omega, \mathbb{T}, \sigma)$ *is pseudo recurrent;*

2) $\gamma \in C(\Omega, X)$ *is an invariant section of the homomorphism* $h : X \to \Omega$.

Then the autonomous dynamical system $(\gamma(\Omega), \mathbb{T}, \pi)$ is pseudo recurrent too.

Lemma 3.11 implies that under the conditions of Theorem 3.4 (respectively, Corollaries 3.7 and 3.10) equation (6) (respectively, equation (15) or equation (10)) admits a pseudo recurrent invariant manifold.

Therefore, we have the following result.

Theorem 3.12 *Assuming that the driving dynamical system $(\Omega, \mathbb{T}, \sigma)$ is pseudo recurrent, and assuming the conditions of Theorem 3.4 (respectively of Corollaries 3.7 or 3.10) are satisfied, we get that equation (6) (respectively, equation (15) or equation (10)) admits a pseudo-recurrent invariant manifold.*

3.6 Chaotic Motions

Let (X, ρ) be a metric space and $(X, \mathbb{T}, \pi)$ be a dynamical system.

A subset $M \subseteq X$ is called transitive, if there exists a point $x_0 \in X$ such that $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{T}\}} = M$.

$\{p, q\} \subseteq X$ is called a Li-Yorke pair, if simultaneously

$$\liminf_{t\to+\infty} \rho(\pi(t,p), \pi(t,q)) = 0 \text{ and } \limsup_{t\to+\infty} \rho(\pi(t,p), \pi(t,q)) > 0.$$

A set $M \subseteq X$ is called scrambled, if any pair of distinct points $\{p, q\} \subseteq M$ is a Li-Yorke pair.

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be chaotic, if X contains an uncountable subset M satisfying the next conditions:

(i) the set M is transitive;

(ii) M is scrambled;

(iii) $\overline{P(M)} = M$, where $P(M) := \{x \in M \mid \mathfrak{N}_x \neq \emptyset\}$ (i.e. $x \in P(M)$, if and only if x is contained in its omega limit set) and by bar we denote the closure in X.

Theorem 3.13 *[21] Let $(X, \mathbb{T}, \pi)$ and $(\Omega, \mathbb{T}, \sigma)$ be two dynamical systems and $\nu : X \to \Omega$ be a homeomorphism of $(\Omega, \mathbb{T}, \sigma)$ onto $(X, \mathbb{T}, \pi)$. Assume that $(\Omega, \mathbb{T}, \sigma)$ is chaotic. Then the dynamical system $(X, \mathbb{T}, \pi)$ is chaotic too.*

Remark 3.14 *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \pi)\rangle$ be a cocycle over $(\Omega, \mathbb{T}, \pi)$ with the fiber W and $w : \Omega \to W$ be a continuous function satisfying the condition $w(\sigma(t, \omega)) = \varphi(t, w(\omega), \omega)$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$. Then if the dynamical system $(\Omega, \mathbb{T}, \sigma)$ is* *chaotic ,the skew-product dynamical system* $(X, \mathbb{T}, \pi)$ *(*$X := W$ *and* $\pi(t,(u,\omega)) := (\varphi(t,u,\omega),\sigma(t,\omega))$ *for all* $(u,\omega) \in X$ *and* $t \in \mathbb{T}$*) is chaotic too. In this case we say that the cocycle* φ *is chaotic.*

Using Theorem 3.13, Remark 3.14 and the results from sections 2-3 we obtain some criteria of the existence of chaotic sets for the second-order differential equations. For instance, the following statement holds.

Theorem 3.15 *Let* $(\Omega, \mathbb{T}, \pi)$ *be a chaotic dynamical system. Then under the conditions of Theorem 3.4 the cocycle* φ *defined by equation (6) admits a compact invariant chaotic set.*

4 Almost Automorphic Solutions of Monotone Second-Order Differential Equation

In this section we suppose that the space H is finite-dimensional. Let W be a nonempty compact from H and $(C(\mathbb{R} \times W, H), \mathbb{R}, \sigma)$ be a shift dynamical system on $C(\mathbb{R} \times W, H)$. Recall, that $C(\mathbb{R} \times W, H)$ is topologically isomorphic to $C(\mathbb{R}, C(W, H))$ and the shift dynamical systems $(C(\mathbb{R} \times W, H), \mathbb{R}, \sigma)$ and $(C(\mathbb{R}, C(W, H)), \mathbb{R}, \sigma)$ are dynamically isomorphic.

Let K be a convex set of H.

The direction $n \in H$ is called normal to K at the point $x \in K$, if $\langle n, u - x\rangle \leq 0$ for all $u \in K$. The set of all normal directions is called normal cone to K at x and is denoted by $N(K, x)$.

Recall [26, p.137] that $N(K, x) \neq \emptyset$ for each $x \in \partial K$ and $N(K, x) = \{0\}$ for each $x \in Int(K)$, where ∂K is the boundary of K and $Int(K)$ is its interior.

Let $K \subset H$ be nonempty, compact, convex subset of H and $f \in C(\mathbb{R} \times K, H)$. We formulate the following assumptions:

(C1) f is almost automorphic in t uniformly for $x \in K$, i.e. the motion $\sigma(t, f)$ is almost automorphic in the shift dynamical system $(C(\mathbb{R} \times K, H), \mathbb{R}, \sigma)$;

(C2) the function f is monotone in $x \in K$ uniformly for $t \in \mathbb{R}$, i.e. $\langle f(t, x_1) - f(t, x_2), x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in K$ and $t \in \mathbb{R}$;

(C3) there exists $t_0 \in \mathbb{R}$ such that $\langle f(t_0, x_1) - f(t_0, x_2), x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in K$, such that $x_1 \neq x_2$;

(C4) $\langle f(t, x), n\rangle \geq 0$ for each $x \in \partial K$, $n \in N(K, x)$ and $t \in \mathbb{R}$.

Lemma 4.1 *Let $W \subset H$ be a nonempty compact. The function $f \in C(\mathbb{R} \times W, H)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for $x \in W$, if and only if the following conditions hold:*

(i) the function f is bounded, i.e. there exists a constant $C \geq 0$ such that $|f(t,x)| \leq C$ for all $(t,x) \in \mathbb{R} \times W$;

(ii) the function f is uniformly continuous on $\mathbb{R} \times W$;

(iii) the function f is Levitan almost periodic in $t \in \mathbb{R}$ uniformly for $x \in W$.

Proof. According to Remark 2.2, the motion $\sigma(t,f)$ is almost automorphic, if and only if it is Levitan almost periodic and stable in the sense of Lagrange. Now to finish the proof of the lemma it is sufficient to note that, by Theorem 7 [30, p.37], the motion $\sigma(t,f)$ is stable in the sense of Lagrange in the shift dynamical system $(C(\mathbb{R} \times W, H), \mathbb{R}, \sigma)$, if and only if the function f is bounded and uniformly continuous on the set $\mathbb{R} \times W$. $\square$

Theorem 4.2 *[22] Let $f \in C(\times K, H)$ be a bounded on $\mathbb{R} \times K$ function. Then the following statements hold:*

(i) if the assumption $(C4)$ is fulfilled, then the equation

$$x'' = f(t,x) \tag{16}$$

has at lest one bounded on $\mathbb{R}$ solution;

(ii) if the assumptions $(C2)$ and $(C4)$ are fulfilled and equation (16) has two solutions φ_1 and φ_2 defined on $\mathbb{R}$ with their values in K, then $\varphi_1(t) - \varphi_2(t) =$ costant for all $t \in \mathbb{R}$;

(iii) if, in addition, the condition $(C3)$ is fulfilled, then (16) has a unique solution defined and bounded on $\mathbb{R}$.

Denote $X_0 := \{(\varphi, f) \mid \varphi \in C(\mathbb{R}, H),\ f \in C(\mathbb{R} \times H, H)$, and let φ be a solution of equation (16)$\}$.

Lemma 4.3 *The set X_0 is invariant and closed in the product dynamical system $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H), \mathbb{R}, \sigma)$.*

Proof. Let $\varphi \in C(\mathbb{R}, H)$ be a solution of equation (16), then it is twice continuously differentiable and

$$\varphi''(t) = f(t, \varphi(t)) \tag{17}$$

for all $t \in \mathbb{R}$. From (17) it follows that $\varphi''(t+\tau) = f(t+\tau, \varphi(t+\tau))$ for all $t, \tau \in \mathbb{R}$, i.e. $\sigma(\tau, (\varphi, f)) := (\sigma(\tau, \varphi), \sigma(\tau, f)) \in X_0$. We will show that the set X_0 is closed in the product space $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H)$. Let $(\psi, g) \in \overline{X}_0$ ($\overline{X}_0$ is the closure of X_0 in $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H))$. Then there exists a sequence $\{(\varphi_n, f_n)\} \subseteq X_0$ such that:

(i) $\{\varphi_n\} \to \psi$ in $C(\mathbb{R}, H)$;

(ii) $\{f_n\} \to g$ in $C(\mathbb{R} \times H, H)$;

(iii) $\varphi_n \in C(\mathbb{R}, H)$ is twice differentiable and

$$\varphi_n''(t) = f_n(t, \varphi_n(t)) \tag{18}$$

for all $t \in \mathbb{R}$.

Let $l > 0$ be an arbitrary number. Since $\{\varphi_n\} \to \psi$ in $C(\mathbb{R}, H)$, then the set $Q(l) := \overline{\cup_{n=1}^{\infty} \varphi_n([-l, l])}$ is a compact subset of H. Note that the sequence $\{\theta_n\} \to \theta$ in $C(\mathbb{R}, H)$, where $\theta_n(t) := \varphi_n''(t) = f_n(t, \varphi_n(t))$ and $\theta(t) := g(t, \psi(t))$ for all $t \in \mathbb{R}$. Really,

$$\begin{aligned} &|\theta(t) - \theta_n(t)| \le |g(t, \psi(t)) - g(t, \varphi_n(t))| + \\ &|g(t, \varphi_n(t)) - f_n(t, \varphi_n(t))| \le \alpha_n + \beta_n \end{aligned} \tag{19}$$

for all $t \in [-l, l]$, where

$$\alpha_n := \max_{|t| \le l} |g(t, \psi(t)) - g(t, \varphi_n(t))| \text{ and } \beta_n := \max_{|t| \le l, x \in Q(l)} |g(t, x) - f_n(t, x)|.$$

Since $\{f_n\} \to g$ in $C(\mathbb{R} \times H, H)$, then $\{\beta_n\} \to 0$ as $n \to +\infty$. Note that $\{\alpha_n\}$ also converges to 0 as $n \to +\infty$. If we suppose that it is not true, then there are $\varepsilon_0 > 0$ and a sequence $\{t_n\} \subset [-l, l]$ such that

$$|g(t_n, \psi(t_n)) - g(t_n, \varphi_n(t_n))| \ge \varepsilon_0 \tag{20}$$

for all $n \in \mathbb{N}$. Without loss of generality we can suppose that the sequence $\{t_n\}$ is convergent. Denote by t_0 its limit. Then $\lim_{n \to +\infty} \varphi_n(t_n) = \varphi(t_0)$, since

$$|\varphi_n(t_n) - \varphi(t_0)| \le |\varphi_n(t_n) - \varphi(t_n)| + |\varphi(t_n) - \varphi(t_0)| \le$$

$$\max_{|t| \le l} |\varphi_n(t) - \varphi(t)| + |\varphi(t_n) - \varphi(t_0)|.$$

Passing into limit in inequality (20) as $n \to +\infty$, we get $0 \ge \varepsilon_0$. The obtained contradiction proves our statement. Now, passing into limit in inequality (19) as $n \to +\infty$ we obtain that $\theta = \lim_{n \to +\infty} \theta_n$ in the space $C(\mathbb{R}, H)$.

We will show that the sequence $\{\varphi_n'(0)\}$ is convergent. Indeed, since

$$\varphi_n(t) = \varphi_n(0) + \varphi_n'(0)t + \int_0^t \int_0^s \theta_n(\tau) d\tau ds$$

and the sequences $\{\varphi_n\}, \{\theta_n\} \subset C(\mathbb{R}, H)$ are convergent, we obtain that the sequence $\{\varphi_n'(0)\} \subset H$ is convergent (in the space H) too. From this fact and the equality

$$\varphi_n'(t) = \varphi_n'(0) + \int_0^t \theta_n(s) ds$$

we receive the convergence of the sequence $\{\varphi_n'\}$ in the space $C(\mathbb{R}, H)$. Thus, the sequences $\{\varphi_n\}$, $\{\varphi_n'\}$ and $\{\varphi_n''\}$ are convergent in the space $C(\mathbb{R}, H)$ and, consequently, the function ψ is twice continuously differentiable, $\psi'(t) = \lim_{n\to+\infty} \varphi_n'(t)$ and $\psi''(t) := \lim_{n\to+\infty} \varphi_n''(t)$ for all $t \in \mathbb{R}$. Finally, passing into limit in equality (18) as $n \to +\infty$, we obtain $\psi''(t) = g(t, \psi(t))$ for all $t \in \mathbb{R}$, i.e. $(\psi, g) \in X_0$. $\square$

Corollary 4.4 *1. X_0 is a complete metric subspace of the product space $C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H)$.*

2. On the space X_0 there is defined a shift dynamical system, induced by the product dynamical system $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H), \mathbb{R}, \sigma)$.

Theorem 4.5 *Let the assumptions $(C1)$, $(C2)$ and $(C4)$ be fulfilled. Then the following statements hold:*

(i) equation (16) admits at least one almost automorphic solution;

(ii) if the equation (16) has two solutions φ_1 and φ defined on $\mathbb{R}$ with their values in K, then $\varphi_1(t) - \varphi_2(t) = costant$ for all $t \in \mathbb{R}$;

(iii) if, in addition, we assume that $(C3)$ is fulfilled, then equation (16) has a unique almost automorphic solution.

Proof. According to Lemma 4.1 and Theorem 4.2, to prove this theorem it is sufficient to show that equation (16), under the conditions of the theorem, admits at least one almost automorphic solution. Let φ be a bounded on $\mathbb{R}$ solution of equation (16). By Landau's inequality, we have

$$\sup_{t\in\mathbb{R}} |\varphi'(t)| \le 2\sqrt{\sup_{t\in\mathbb{R}} |\varphi''(t)|}\sqrt{\sup_{t\in\mathbb{R}} |\varphi(t)|}$$

and, consequently, $|\varphi'(t)| \le 2ab$ for all $t \in \mathbb{R}$, where

$$a := \sup_{t\in\mathbb{R}} |f(t, \varphi(t))| \le \sup_{t\in\mathbb{R},\ x\in W} |f(t, x)| \text{ and } b := \sup_{t\in\mathbb{R}} |\varphi(t)|.$$

Thus, the function $\varphi \in C(\mathbb{R}, H)$ is bounded and uniformly continuous on $\mathbb{R}$ and by Theorem 7 [30, p.37] the motion $\sigma(t, \varphi)$ is stable in the sense of Lagrange in the shift dynamical system $(C(\mathbb{R}, H), \mathbb{R}, \sigma)$. Let us consider a non-autonomous dynamical system $\langle (X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $Y := H(\tilde{f})$ ($\tilde{f}$ is the restriction on $\mathbb{R} \times W$ of f, where $W := \overline{\varphi(\mathbb{R})}$) and $(Y, \mathbb{R}, \sigma)$ is the shift dynamical system on $H(\tilde{f})$ induced by $(C(\mathbb{R} \times H, H), \mathbb{R}, \sigma)$, $X := H(\varphi, \tilde{f}) \subset X_0$ and $(X, \mathbb{R}, \pi)$ is the shift dynamical system induced by $(X_0, \mathbb{R}, \sigma)$ and $h := pr_2 : X \to Y$ is the second projection. Now we will prove that $\mathfrak{N}_{\tilde{f}} \subset \mathfrak{N}_{(\varphi, \tilde{f})}$. In fact, let $\{t_n\} \in \mathfrak{N}_{\tilde{f}}$. Then $\{\tilde{f}_{t_n}\} \to \tilde{f}$ in the space $C(\mathbb{R} \times W, H)$ ($\tilde{f}_\tau := \sigma(\tau, \tilde{f})$). Since $\varphi \in C(\mathbb{R}, H)$ is stable in the sense of Lagrange, then $H(\varphi) := \overline{\{\varphi_\tau \mid \tau \in \mathbb{R}\}}$ is a compact invariant set and the sequence $\{\varphi_{t_n}\}$ is relatively compact. Let $\{t'_n\}$ be a subsequence of the sequence $\{t_n\}$, such that $\{\varphi_{t'_n}\}$ converges and denote by $P(\varphi) := \lim_{n \to +\infty} \varphi_{t'_n} \in H(\varphi)$. By Lemma 4.3, the function $P(\varphi)$ is a solution of equation (16) defined on $\mathbb{R}$. Since $P(\varphi)(\mathbb{R}) \subseteq W$, then by Theorem 4.2 there exists $c \in H$ such that

$$P(\varphi)(t) = \varphi(t) + c \tag{21}$$

for all $t \in \mathbb{R}$. From equality (21) we have $P^2(\varphi) = P(\varphi) + c = \varphi + 2c, \ \dots, \ P^k(\varphi) = \varphi + kc$ for all $k \in \mathbb{N}$. On the other hand, $\{P^k(\varphi)\} \subseteq H(\varphi)$ and taking into account the compactness of the set $H(\varphi)$ we obtain $c = 0$, i.e. $P(\varphi) = \varphi$. Thus the sequence $\{\varphi_{t_n}\}$ is relatively compact and it has a unique limit point φ. This means that the sequence $\{\varphi_{t_n}\}$ is convergent, and consequently, $\{t_n\} \in \mathfrak{N}_{(\varphi, \tilde{f})}$. But $\tilde{f}$, under the conditions of Theorem, is almost automorphic in $t \in \mathbb{R}$ uniformly for $x \in W$, and, hence, the function $(\varphi, \tilde{f})$ is also almost automorphic (and, in particular, the function φ is too). $\square$

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