CONTINUOUS DEPENDENCE OF ATTRACTORS ON PARAMETERS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS AND INFINITE ITERATED FUNCTION SYSTEMS

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ABSTRACT. The paper is dedicated to the study of the problem of continuous dependence of compact global attractors on parameters of non-autonomous dynamical systems and infinite iterated function systems (IIFS). We prove that if a family of non-autonomous dynamical systems $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ depending on parameter $\lambda \in \Lambda$ is uniformly contracting (in the generalized sense), then each system of this family admits a compact global attractor J^λ and the mapping $\lambda \to J^\lambda$ is continuous with respect to the Hausdorff metric. As an application we give a generalization of well known Theorem of Barnsley concerning the continuous dependence of fractals on parameter.

1. INTRODUCTION

The aim of this paper is the study of the problem of existence of compact global attractors of non-autonomous dynamical systems and their continuous dependence on parameter. The problem of the upper semi-continuous dependence on parameter of global attractors of dynamical systems is well studied (both autonomous and non-autonomous, see for example Caraballo, Langa and Robinson [3], Caraballo and Langa [4], Cheban [6, 7], Hale and Raugel [15], Hale [16] and also see the bibliography therein). The problem of the lower semi-coninuous dependence on parameter of global attractors is less extensively studied. Note, for example, the works of Dupaix, Hilhorst and Kostin [11], Elliott and Kostin [13], Hale [16], Hale and Raugel [17], Kapitanskii and Kostin [20], Kostin [21], Li and Kloeden [22], Stuart and Humphries [28] and the bibliography therein).

The paper is dedicated to the study of the problem of continuous dependence of compact global attractors on parameter of non-autonomous dynamical systems and infinite iterated function systems (IIFS). We prove that if a family of non-autonomous dynamical systems $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ depending on parameter $\lambda \in \Lambda$ is uniformly contracting (in the generalized sense), then each system of this family admits a compact global attractor J^λ and the mapping $\lambda \to J^\lambda$ is continuous with respect to the Hausdorff metric. As an application we give a generalization of well known Theorem of Barnsley concerning the continuous dependence of fractals on parameters.

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This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of set-valued dynamical systems which we use in our paper.

Section 3 is dedicated to the study of upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems $\langle(X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$. They play a very important role in the study of non-autonomous dynamical systems. We give the sufficient conditions which guarantee the existence of a unique globally exponentially stable invariant section. The main result of this paper is Theorem 3.15. For the case of semi-group dynamical system $(Y,\mathbb{T}_2,\sigma)$ (i.e. $\sigma(t,\cdot): Y\mapsto Y$ is not invertible) Theorem 3.15 is formulated and proved for the first time in this paper.

We give in section 4 a new approach to the study of discrete inclusions (DI) which is based on non-autonomous dynamical systems (See also our previous works [8, 9, 10] , where we study the IFSs (both linear [8, 9] and nonlinear [10] cases) in the framework of non-autonomous dynamical systems (cocycles)). We show that every DI in a natural way generates some non-autonomous dynamical system (cocycle), which play an important role in its study (see Sections 6 and 7).

In section 5 we study some properties of Lipschitz maps. We introduce the notion of spectral radius for Lipschitz maps and we give the necessary and sufficient conditions that a Lipschitz mapping is contracting in the generalized sense in the term of its spectral radius (Lemma 5.6).

In Section 6 we study the relation between compact global attractor of cocycle and the skew-product dynamical system (respectively, set-valued dynamical system) associated by the given cocycle.

Section 7 is dedicated to the study of problem of continuous dependence of attractors of infinite iterated function systems. We give a generalization of well known Theorem of Barnsley concerning the continuous dependence of fractals on parameters (Theorem 7.2).

2. Set-Valued Dynamical Systems and Their Compact Global Attractors

Let (X,ρ) be a complete metric space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+\subseteq\mathbb{T}$) be a subsemi-group of $\mathbb{S}$. If $A\subseteq X$ and $x\in X$, then we denote by $\rho(x,A)$ the distance from the point x to the set A, i.e. $\rho(x,A)=\inf\{\rho(x,a): a\in A\}$. We denote by $B(A,\varepsilon)$ an ε-neighborhood of the set A, i.e. $B(A,\varepsilon)=\{x\in X:\rho(x,A)<\varepsilon\}$, by $K(X)$ we denote the family of all non-empty compact subsets of X. For every point $x\in X$ and number $t\in\mathbb{T}$ we put in correspondence a closed compact subset $\pi(t,x)\in K(X)$. So, if $\pi(P,A)=\bigcup\{\pi(t,x): t\in P, x\in A\}(P\subseteq\mathbb{T})$, then

(i) $\pi(0,x)=x$ for all $x\in X$;
(ii) $\pi(t_2,\pi(t_1,x))=\pi(t_1+t_2,x)$ for all $x\in X$ and $t_1,t_2\in\mathbb{T}$;

(iii) $\lim_{x\to x_0, t\to t_0} \beta(\pi(t,x),\pi(t_0,x_0)) = 0$ for all $x_0 \in X$ and $t_0 \in \mathbb{T}$, where $\beta(A,B) = \sup\{\rho(a,B) : a \in A\}$ is a semi-deviation of the set $A \subseteq X$ from the set $B \subseteq X$.

In this case it is said (see, for example, [27] and [23] and the bibligraphy therein) that there is defined a set-valued semi-group dynamical system.

Let $\mathbb{T} = \mathbb{S}$ and be fulfilled the next condition:

(i) if $p \in \pi(t,x)$, then $x \in \pi(-t,p)$ for all $x, p \in X$ and $t \in \mathbb{T}$.

Then it is said that there is defined a set-valued group dynamical system $(X,\mathbb{T},\pi)$ or a bilateral (two-sided) dynamical system.

Definition 2.1. *Let $\mathbb{T}' \subset \mathbb{S}$ $(\mathbb{T} \subset \mathbb{T}')$. A continuous mapping $\gamma_x : \mathbb{T} \to X$ is called a motion of the set-valued dynamical system $(X,\mathbb{T},\pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}'$, if*

a. $\gamma_x(0) = x$;
b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ *for all* $t_1, t_2 \in \mathbb{T}'$ $(t_2 > t_1)$.

The set of all motions of $(X,\mathbb{T},\pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\mathcal{F}_x(\pi)$ and we define $\mathcal{F}(\pi) := \bigcup\{\mathcal{F}_x(\pi) \mid x \in X\}$ (or simply $\mathcal{F}$).

Definition 2.2. *Any trajectory $\gamma \in \mathcal{F}(\pi)$ defined on $\mathbb{S}$ is called a full (entire) trajectory of the dynamical system $(X,\mathbb{T},\pi)$.*

Denote by $\Phi(\pi)$ the set of all full trajectories of the dynamical system $(X,\mathbb{T},\pi)$ and $\Phi_x(\pi) := \mathcal{F}_x(\pi) \bigcap \Phi(\pi)$.

Theorem 2.3. [27] *Let $(X,\mathbb{T},\pi)$ be a semi-group dynamical system and X be a compact and invariant set (i.e. $\pi(t,X) = X$ for all $t \in \mathbb{T}$. Then*

(i) *$\mathcal{F}(\pi) = \Phi(\pi)$, i.e. every motion $\gamma \in \mathcal{F}_x(\pi)$ can be extended on $\mathbb{S}$ (this means that there exists $\tilde{\gamma} \in \Phi_x(\pi)$ such that $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in \mathbb{T}$);*
(ii) *there exists a group (generally speaking set-valued) dynamical system $(X,\mathbb{S},\tilde{\pi})$ such that $\tilde{\pi}|_{\mathbb{T}\times X} = \pi$.*

Definition 2.4. *A system $(X,\mathbb{T},\pi)$ is called* [5, 7] *compactly dissipative, if there exists a nonempty compact $K \subseteq X$ such that*

$$\lim_{t\to+\infty} \beta(\pi(t,M),K) = 0;$$

for all $M \in K(X)$.

Let $(X,\mathbb{T},\pi)$ be compactly dissipative and K be a compact set attracting every compact subset of X. Let us set

$$J := \omega(K) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi(\tau,K)}. \tag{1}$$

It can be shown [5, 7] that the set J defined by equality (1) does not depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called a center of Levinson [29] (or global attractor) of the compact dissipative system $(X, \mathbb{T}, \pi)$.

Theorem 2.5. [5, 7] *If $(X, \mathbb{T}, \pi)$ is a compactly dissipative dynamical system and J is its global attractor, then :*

(i) *J is invariant, i.e. $\pi(t, J) = J$ for all $t \in \mathbb{T}$;*
(ii) *J is orbitally stable, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\beta(\pi(t, x), J) < \varepsilon$ for all $t \geq 0$;*
(iii) *J is an attractor of the family of all compact subsets of X;*
(iv) *J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$.*

3. Upper Semi-Continuous Invariant Sections of Non-Autonomous Dynamical Systems and their continuous dependence on parameters

In this section we study the upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems. They play a very important role in the study of non-autonomous dynamical systems. We give the sufficient conditions which guarantee the existence of a unique globally exponentially stable invariant section and their continuous dependence on parameters.

Lemma 3.1. *Let X and Λ be complete metric spaces. Let $(X, \mathbb{T}, \pi_\lambda)$ $(\lambda \in \Lambda)$ be a family of dynamical systems with uniqueness satisfying the following conditions:*

(i) *the family of dynamical systems $(X, \mathbb{T}, \pi_\lambda)$ $(\lambda \in \Lambda)$ is uniformly contracting, i.e. there exist two positive numbers $\mathcal{N}$ and ν such that $\rho(\pi_\lambda(t, x_1), \pi_\lambda(t, x_2)) \leq \mathcal{N} e^{-\nu t} \rho(x_1, x_2)$ for all $\lambda \in \Lambda, t \in \mathbb{T}$ and $x_1, x_2 \in X$;*
(ii) *for each $t \in \mathbb{T}$ the mapping $(\lambda, x) \mapsto \pi_\lambda(t, x)$ is continuous.*

Then for each $\lambda \in \Lambda$ the dynamical system $(X, \mathbb{T}, \pi_\lambda)$ admits a unique stationary point p_λ and the mapping $\lambda \mapsto p_\lambda$ is continuous.

Proof. Let Λ' be a compact subset of Λ. Denote by $C(\Lambda', X)$ the space of all continuous functions $\varphi : \Lambda' \mapsto X$ with distance $r(\varphi_1, \varphi_2) := \max\{\rho(\varphi_1(\lambda), \varphi_2(\lambda)) : \lambda \in \Lambda'\}$. $(C(\Lambda', X), r)$ is a complete metric space. Note that under the conditions of the lemma if $\varphi \in C(\Lambda', X)$ then also $\psi_t \in C(\Lambda', X)$, where $\psi_t(\lambda) := \pi_\lambda(t, \varphi(\lambda))$ for all $\lambda \in \Lambda'$, where $t \in \mathbb{T}$. Denote by $S^t_{\Lambda'}$ the mapping from $C(\Lambda', X)$ into itself defined by equality $(S^t_{\Lambda'}\varphi)(\lambda) := \pi_\lambda(t, \varphi(\lambda))$ for all $t \in \mathbb{T}$ and $\lambda \in \Lambda'$. It is easy to check that $\{S^t_{\Lambda'}\}_{t \in \mathbb{T}}$ is a commutative semi-group (with respect to composition) and $r(S^t_{\Lambda'}\varphi_1, S^t_{\Lambda'}\varphi_2) \leq \mathcal{N} e^{-\nu t} r(\varphi_1, \varphi_2)$ for all $t \in \mathbb{T}$ and $\varphi_1, \varphi_2 \in C(\Lambda', X)$. Hence there exists a unique common fix point $\varphi_{\Lambda'} \in C(\Lambda', X)$ of semi-group $\{S^t_{\Lambda'}\}_{t \in \mathbb{T}}$. In particularly $\pi_\lambda(t, \varphi_{\Lambda'}(\lambda)) = \varphi_{\Lambda'}(\lambda)$ for all $\lambda \in \Lambda'$, i.e. $p_\lambda := \varphi_{\Lambda'}(\lambda)$ is a unique stationary point of dynamical system $(X, \mathbb{T}, \pi_\lambda)$ and the mapping $\lambda \mapsto p_\lambda$ from Λ' into X is continuous.

Thus we have a family of commutative semi-groups $\{S^t_{\Lambda'}\}_{t \in \mathbb{T}}$ depending on parameter $\Lambda' \in K(\Lambda)$. It is easy to check that the following statements are true:

a. for each $\Lambda^{'} \in K(\Lambda)$ the commutative semi-group $\{S^t_{\Lambda^{'}}\}_{t\in\mathbb{T}}$ admits a unique stationary point $\varphi_{\Lambda^{'}} \in C(\Lambda^{'}, X)$;
b. if $\Lambda^{'} \subseteq \Lambda''$ then $\tilde{\varphi}_{\Lambda''} = \varphi_{\Lambda^{'}}$, where $\tilde{\varphi}_{\Lambda''}$ is the restriction on $\Lambda^{'}$ of function $\varphi_{\Lambda''}$;
c. $\varphi_{\Lambda^{'}}(\lambda) = \varphi_{\Lambda''}(\lambda)$ for all $\lambda \in \Lambda^{'} \cap \Lambda^{''}$ and $\Lambda^{'}, \Lambda^{''} \in K(\Lambda)$.

Denote by $C(\Lambda, X)$ the space of all continuous functions $\varphi : \Lambda \mapsto X$ equipped with compact-open topology (the topology of convergence uniform on every compact subset $\Lambda^{'} \subseteq \Lambda$). Let S^t be the mapping from $C(\Lambda, X)$ into itself defined by equality $(S^t\varphi)(\lambda) := \pi_\lambda(t, \varphi(\lambda))$ for all $t \in \mathbb{T}$ and $\lambda \in \Lambda$. It is easy to check that $\{S^t\}_{t\in\mathbb{T}}$ is a commutative semi-group (with respect to composition). We define now the mapping $\varphi : \Lambda \mapsto X$ as follow:

$$\varphi(\lambda) := \varphi_{\Lambda^{'}}(\lambda), \tag{2}$$

where $\Lambda^{'} \in K(\Lambda)$ is an arbitrary compact subset of Λ containing λ. According to properties a.-c. by equality (2) a function $\varphi \in C(\Lambda, X)$ is correctly defined and it is a unique stationary point of the semi-group $\{S^t\}_{t\in\mathbb{T}}$. This means that $S^t\varphi = \varphi$ for all $t \in \mathbb{T}$ or equivalently $\pi_\lambda(t, \varphi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in \Lambda$ and $t \in \mathbb{T}$, i.e. the point $p_\lambda := \varphi(\lambda)$ is a unique stationary point of dynamical system $(X, \mathbb{T}, \pi_\lambda)$ and the mapping $\lambda \mapsto p_\lambda$ is continuous. □

Remark 3.2. *Lemma 3.1 is also true if we consider in place of family of dynamical systems $(X, \mathbb{T}, \pi_\lambda)_{\lambda\in\Lambda}$ an arbitrary family of commutative semi-groups $\{\pi_\lambda(t, \cdot)\}_{t\in\mathbb{T}}$ ($\lambda \in \Lambda$) with conditions:*

(i) *for each $t \in \mathbb{T}$ the mapping $(\lambda, x) \mapsto \pi_\lambda(t, x)$ is continuous;*
(ii) *there are two positive numbers $\mathcal{N}$ and ν such that $\rho(\pi_\lambda(t, x_1), \pi_\lambda(t, x_2)) \leq \mathcal{N}e^{-\nu t}\rho(x_1, x_2)$ for all $\lambda \in \Lambda, t \in \mathbb{T}$ and $x_1, x_2 \in X$.*

Remark 3.3. *Lemma 3.1 and Remark 3.2 are also true if we replace the condition of uniform contraction by the following weaker condition: for each compact subset $\Lambda^{'} \subseteq \Lambda$ there are two positive numbers $\mathcal{N}_{\Lambda^{'}}$ and $\nu_{\Lambda^{'}}$ such that $\rho(\pi_\lambda(t, x_1), \pi_\lambda(t, x_2)) \leq \mathcal{N}_{\Lambda^{'}}e^{-\nu_{\Lambda^{'}}t}\rho(x_1, x_2)$ for all $\lambda \in \Lambda^{'}, t \in \mathbb{T}$ and $x_1, x_2 \in X$.*

Definition 3.4. *Let X be a metric space and Y be a topological space. The set-valued mapping $\gamma : Y \to K(X)$ is said to be upper semi-continuous (or β-continuous), if $\lim\limits_{y\to y_0} \beta(\gamma(y), \gamma(y_0)) = 0$ for all $y_0 \in Y$.*

Definition 3.5. *Let (X, h, Y) be a fiber space, i.e. $h : X \mapsto Y$ is a continuous mapping from X onto Y. The mapping $\gamma : Y \to K(X)$ is called a section (selector) of the fiber space (X, h, Y), if $h(\gamma(y)) = y$ for all $y \in Y$.*

Remark 3.6. *Let $X := W \times Y$. Then $\gamma : Y \to X$ is a section of the fiber space (X, h, Y) ($h := pr_2 : X \to Y$), if and only if $\gamma = (\psi, Id_Y)$ where $\psi : W \to K(W)$.*

Definition 3.7. *Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ ($\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$) be two dynamical systems. The mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (respectively homeomorphic) and $h(\pi(x, t)) = \sigma(h(x), t)$ ($t \in \mathbb{T}_1$, $x \in X$).*

Definition 3.8. *A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where h is a homomorphism of $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ and (X, h, Y) is a fiber space, is called a non-autonomous dynamical system.*

Definition 3.9. *The triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ (or shortly φ), where $(Y, \mathbb{T}_2, \sigma)$ is a dynamical system on Y, W is a complete metric space and φ is a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in W, possessing the following conditions:*

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$,

is called [7, 24] *a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with fiber W.*

Definition 3.10. *Let $X := W \times Y$ and we define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as following: $\pi(t, (u, y)) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X which is called a skew-product dynamical system* [1, 24] *and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ and, consequently, $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma), h\rangle$ is a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ $(X := W \times Y)$, which is called a non-autonomous dynamical system, generated by cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates some cocycle (non-autonomous dynamical system). Below we give some examples of this type.

Example 3.11. Let E be a real or complex Banach space and Ω be a metric space. Denote by $C(\Omega \times E, E)$ the space of all continuous mappings $f : \Omega \times E \mapsto E$ endowed by compact-open topology. Consider the system of differential equations

$$\begin{cases} u' = & F(\omega, u) \\ \omega' = & G(\omega), \end{cases} \tag{3}$$

where $\Omega \subseteq E, G \in C(\Omega, E)$ and $F \in C(\Omega \times E, E)$. Suppose that for the system (3) the conditions of the existence, uniqueness, continuous dependence of initial data and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(\Omega, \mathbb{R}_+, \sigma)$ a dynamical system on Ω generated by the second equation of the system (3) and by $\varphi(t, u, \omega)$ – the solution of the equation

$$u' = F(\omega t, u) \ (\omega t := \sigma(t, \omega)) \tag{4}$$

passing through the point $u \in E$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \to E$ is continuous and satisfies the conditions: $\varphi(0, u, \omega) = u$ and $\varphi(t + \tau, u, \omega) = \varphi(t, \varphi(\tau, u, \omega), \omega t)$ for all $t, \tau \in \mathbb{R}_+$, $u \in E$ and $\omega \in \Omega$ and, consequently, the system (3) generates a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ (where $X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to \Omega$).

We will give some generalization of the system (3). Namely, let $(\Omega, \mathbb{R}_+, \sigma)$ be a dynamical system on the metric space Ω. Consider the system

$$\begin{cases} u' = & F(\omega t, u) \\ \omega \in & \Omega, \end{cases} \tag{5}$$

where $F \in C(\Omega \times E, E)$. Suppose that for the equation (4) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The system $\langle (X, \mathbb{R}_+, \pi), (\Omega,$

$\mathbb{R}_+,\sigma),h\rangle$, where $X := E\times\Omega$, $\pi := (\varphi,\sigma)$, $\varphi(\cdot,u,\omega)$ is the solution of (4) and $h := pr_2 : X\to\Omega$ is a non-autonomous dynamical system generated by the equation (5).

Example 3.12. Let us consider a differential equation

$$u' = f(t,u), \tag{6}$$

where $f\in C(\mathbb{R}\times E,E)$. Along with equation (6) we consider its H-class [24], i.e. the family of equations

$$v' = g(t,v), \tag{7}$$

where $g\in H(f) := \overline{\{f_\tau : \tau\in\mathbb{R}\}}$, $f_\tau(t,u) := f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times E$ and by bar we denote the closure in $C(\mathbb{R}\times E,E)$. We will suppose also that the function f is regular, i.e. for every equation (7) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(\cdot,v,g)$ the solution of equation (7) passing through the point $v\in E$ at the initial moment $t=0$. Then there is a correctly defined mapping $\varphi : \mathbb{R}_+\times E\times H(f)\to E$ satisfying the following conditions (see, for example, [24]):

1) $\varphi(0,v,g) = v$ for all $v\in E$ and $g\in H(f)$;
2) $\varphi(t,\varphi(\tau,v,g),g_\tau) = \varphi(t+\tau,v,g)$ for every $v\in E$, $g\in H(f)$ and $t,\tau\in\mathbb{R}_+$;
3) the mapping $\varphi : \mathbb{R}_+\times E\times H(f)\to E$ is continuous.

Denote by $Y := H(f)$ and $(Y,\mathbb{R}_+,\sigma)$ a dynamical system of translations (a semigroup system) on Y, induced by the dynamical system of translations $(C(\mathbb{R}\times E,E),\mathbb{R},\sigma)$. The triplet $\langle E,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ is a cocycle on $(Y,\mathbb{R}_+,\sigma)$ with the fiber E. Thus, equation (6) generates a cocycle $\langle E,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ and a non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$, where $X := E\times Y$, $\pi := (\varphi,\sigma)$ and $h := pr_2 : X\to Y$.

Remark 3.13. *1. Let $\Omega := H(f)$ and $(\Omega,\mathbb{R},\pi)$ be the shift dynamical system on Ω. The equation (6) (the family of equation (7)) may be written in the form (4), where $F:\Omega\times E\mapsto E$ is defined by equality $F(g,u) := g(0,u)$ for all $g\in H(f)=\Omega$ and $u\in E$, then $F(g_t,u) = g(t,u)$ ($g_t(s,u) := \sigma(t,g)(s,u) = g(t+s,u)$ for all $t,s\in\mathbb{R}$ and $u\in E$).*

2. In this work we show that every IFS generates some non-autonomous dynamical system (see Section 4 and also [10]*). Many examples of non-autonomous dynamical systems, generated by non-autonomous differential/difference equations (ODEs, PDEs and functional-differential equations) reader can find, for example, in the books* [7] *and* [24].

Definition 3.14. *A mapping $\gamma : Y\to X$ is called an invariant section of the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$, if it is a section of the fiber space (X,h,Y) and $\gamma(Y)$ is an invariant subset of the dynamical system $(X,\mathbb{T},\pi)$ (or, equivalently, $\pi^t\gamma(y) = \gamma(\sigma^t y)$ for all $t\in\mathbb{T}$ and $y\in Y$).*

Denote by $\alpha : K(X)\times K(X)\to\mathbb{R}_+$ the Hausdorff distance on $K(X)$, i.e.

$$\alpha(A,B) := \max(\beta(A,B),\beta(B,A)).$$

Theorem 3.15. *Let Λ be a metric space, $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ ($\lambda\in\Lambda$) be a family of non-autonomous dynamical system and suppose the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *Y is invariant, i.e. $\sigma^t Y = Y$ for all $t\in\mathbb{T}_2$;*
(iii) *the non-autonomous dynamical systems $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ are equicontracting in the extended sense, i.e. there exist positive numbers N and ν such that*

$$\rho(\pi_\lambda(t,x_1),\pi_\lambda(t,x_2))\le Ne^{-\nu t}\rho(x_1,x_2) \tag{8}$$

for all $\lambda\in\Lambda$, $x_1,x_2\in X$ ($h(x_1)=h(x_2)$) and $t\in\mathbb{T}_1$;
(iv) *for each $t\in\mathbb{T}_1$ the mapping $(\lambda,x)\to\pi_\lambda(t,x)$ from $\Lambda\times X$ into X is continuous;*
(v) *$\Gamma(Y,X)=\{\gamma\mid\gamma:Y\to K(X)$ is a set-valued $\beta-$continuous mapping and $h(\gamma(y))=y$ for all $y\in Y\}\neq\emptyset$.*

Then

(i) *for each $\lambda\in\Lambda$ there exists a unique invariant section $\gamma_\lambda\in\Gamma(Y,X)$ of the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$;*
(ii) *the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ is compactly dissipative (i.e. $(X,\mathbb{T}_1,\pi_\lambda)$ is compactly dissipative) and its global attractor center $J^\lambda=\gamma_\lambda(Y)$;*
(iii) *$\pi^t_\lambda J^\lambda_y = J^\lambda_{\sigma(t,y)}$ for all $t\in\mathbb{T}_1$ and $y\in Y$;*
(iv) *the mapping $\lambda\to\gamma_\lambda$ is continuous, i.e.*

$$\lim_{\lambda\to\lambda_0}\sup_{y\in Y}\alpha(\gamma_\lambda(y),\gamma_{\lambda_0}(y))=0;$$

(v) *if $(Y,\mathbb{T}_2,\sigma)$ is a group-dynamical system (i.e. $\mathbb{T}_2=\mathbb{S}$), then the unique invariant section γ_λ of the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ is one-valued (i.e. $\gamma_\lambda(y)$ consists a single point for any $y\in Y$) and*

$$\rho(\pi_\lambda(t,x),\pi_\lambda(t,\gamma_\lambda(h(x))))\le Ne^{-\nu t}\rho(x,\gamma_\lambda(h(x))) \tag{9}$$

for all $x\in X$ and $t\in\mathbb{T}$.

Proof. Since the space Y is compact and invariant, then according to Theorem 2.3 the semi-group dynamical system $(Y,\mathbb{T},\sigma)$ can be prolonged to a group set-valued dynamical system $(Y,\mathbb{S},\tilde\sigma)$ (this means that $\tilde\sigma(s,y)=\sigma(s,y)$ for all $(s,y)\in\mathbb{T}\times Y$).

Let $\alpha: K(X)\times K(X)\to\mathbb{R}_+$ be the Hausdorff's distance on $K(X)$ and $d:\Gamma(Y,X)\times\Gamma(Y,X)\to\mathbb{R}_+$ be the function defined by the equality

$$d(\gamma_1,\gamma_2):=\sup_{y\in Y}\alpha(\gamma_1(y),\gamma_2(y)). \tag{10}$$

Note that (10) defines a complete distance on $\Gamma(Y,X)$ (see [10]).

For $t\in\mathbb{T}_1$ and $\lambda\in\Lambda$, by S^t_λ we denote the mapping of $\Gamma(Y,X)$ into itself defined by the equality $(S^t_\lambda\gamma)(y)=\pi_\lambda(t,\gamma((\sigma^t)^{-1}y))$ for all $t\in\mathbb{T}_1$, $y\in Y$ and $\gamma\in\Gamma(Y,X)$.

It is easy to see that $S^t_\lambda\gamma \in \Gamma(Y,X)$, $S^t_\lambda S^\tau_\lambda = S^{t+\tau}_\lambda$ for all $t,\tau \in \mathbb{T}_1$ and $\gamma \in \Gamma(Y,X)$ and, hence, $\{S^t_\lambda\}_{t\in\mathbb{T}_1}$ forms a commutative semi-group. We will show that

$$d(S^t_\lambda\gamma_1, S^t_\lambda\gamma_2) \le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2) \tag{11}$$

for all $t\in\mathbb{T}_1$ and $\gamma_i \in \Gamma(Y,X)$ $(i=1,2)$. In fact. To prove the inequality (11) it is sufficient to show that

$$\alpha(\pi^t_\lambda\gamma_1(\sigma^{-t}y), \pi^t_\lambda\gamma_2(\sigma^{-t}y) \le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2) \tag{12}$$

for all $y\in Y$, where $\sigma^{-t}y := \{q\in Y \mid \sigma(t,q)=y\}$.

Let $v\in \pi^t_\lambda\gamma_2(\sigma^{-t}y)$ be an arbitrary element, then there is $q\in\sigma^{-t}y$ and $x_2(y)\in\gamma_2(q)$ so that $v=\pi(t,\cdot)_\lambda x_2(y)$. We choose $x_1(y)\in\gamma_1(q)$ such that

$$\rho(x_1(y),x_2(y)) \le \alpha(\gamma_1(q),\gamma_2(q)) \le d(\gamma_1,\gamma_2) \tag{13}$$

(by compactness of $\gamma_i(q)$ $(i=1,2)$ obviously an such $x_1(y)$ there exists and additionally $h(x_1(y))=h(x_2(y))=q$). Then we have

$$\rho(\pi^t_\lambda x_1(y), \pi^t_\lambda x_2(y)) \le \mathcal{N}e^{-\nu t}\rho(x_1(y),x_2(y)) \le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2),$$

i.e. for all $v\in\pi^t_\lambda\gamma_2(\sigma^{-t}y)$ there exists $u:=\pi^t x_1(y)\in\pi^t_\lambda\gamma_1(\sigma^{-t}y)$ so that $\rho(u,v)\le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2)$. This means that $\beta(\pi^t_\lambda\gamma_1(\sigma^{-t}y),\pi^t_\lambda\gamma_2(\sigma^{-t}y)) \le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2)$. Analogously, the inequality $\beta(\pi^t_\lambda\gamma_2(\sigma^{-t}y),\pi^t_\lambda\gamma_1(\sigma^{-t}y)) \le \mathcal{N}e^{-\nu t}\ d(\gamma_1,\gamma_2)$ can be established and, consequently, $\alpha(\pi^t_\lambda\gamma_1(\sigma^{-t}y),\pi^t_\lambda\gamma_2(\sigma^{-t}y)) \le \mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2)$ for all $y\in Y$ and $t\in\mathbb{T}_1$. Thus the inequality (12) is established.

We will show now that for each $t_0\in\mathbb{T}_1$ the mapping $(\lambda,\gamma)\to S^{t_0}_\lambda\gamma$ from $\Lambda\times\Gamma(Y,X)$ into $\Gamma(Y,X)$ is continuous. In fact. Let $\lambda_k\to\lambda_0$ and $\gamma_k\to\gamma_0$. We shall prove that $S^{t_0}_{\lambda_k}\gamma_k \to S^{t_0}_{\lambda_0}\gamma_0$ in the space Γ. Denote by

$$m(\lambda) := \sup_{x\in\gamma_0(Y)} \rho(\pi^{t_0}_\lambda x, \pi^{t_0}_{\lambda_0}x) \tag{14}$$

and note that $m(\lambda)\to 0$ as $\lambda\to\lambda_0$. If we suppose that it is not true, then there are $\varepsilon_0>0, \lambda_k\to\lambda_0$ and $x_k\to x_0$ $(x_k\in\gamma_0(Y))$ such that

$$\rho(\pi^{t_0}_{\lambda_k}x_k, \pi^{t_0}_{\lambda_0}x_k) \ge \varepsilon_0. \tag{15}$$

Passing into limit in (15) as $k\to+\infty$ we obtain $\varepsilon_0\le 0$. The obtained contradiction shows that $m(\lambda)\to 0$ as $\lambda\to\lambda_0$.

Let $y\in Y$ and $v\in\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y)$, then there are $q\in\sigma^{-t_0}y$ and $x\in\gamma_0(q)$ such that $v=\pi^{t_0}_{\lambda_0}x$. Denote by $u:=\pi^{t_0}_\lambda x$, then we have

$$\rho(u,v)=\rho(\pi^{t_0}_\lambda x,\pi^{t_0}_{\lambda_0}x) \le \sup_{x\in\gamma_0(Y)}\rho(\pi^{t_0}_\lambda x,\pi^{t_0}_{\lambda_0}x) = m(\lambda). \tag{16}$$

From the inequality (16) it follows $\beta(\pi^{t_0}_\lambda\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y)) \le m(\lambda)$. Analogously one can establish the inequality $\beta(\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y)) \le m(\lambda)$ and, consequently,

$$\alpha(\pi^{t_0}_\lambda\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y)) \le m(\lambda) \tag{17}$$

for all $y\in Y$ and $\lambda\in\Lambda$. From (17) it follows that

$$d(S^{t_0}_\lambda\gamma_0, S^{t_0}_{\lambda_0}\gamma_0) \le m(\lambda)\to 0 \tag{18}$$

as $\lambda\to\lambda_0$ and, consequently,

$$d(S^{t_0}_{\lambda_k}\gamma_k, S^{t_0}_{\lambda_0}\gamma_0) \le d(S^{t_0}_{\lambda_k}\gamma_k, S^{t_0}_{\lambda_k}\gamma_0) + d(S^{t_0}_{\lambda_k}\gamma_0, S^{t_0}_{\lambda_0}\gamma_0) \le$$

$$\mathcal{N}e^{-\nu t_0}d(\gamma_k,\gamma_0)+m(\lambda_k)\to 0$$

as $\lambda_k\to\lambda_0$. By Lemma 3.1 (see also Remark 3.2) for each $\lambda\in\Lambda$ the semi-group $\{S_\lambda^t\}_{t\in\mathbb{T}}$ admits a unique stationary point $\gamma_\lambda\in\Gamma(Y,X)$ and the mapping $\lambda\to\gamma_\lambda$ is continuous.

Let us write by $K_\lambda := \gamma_\lambda(Y)$, then K_λ is a nonempty compact and invariant set of the dynamical system $(X,\mathbb{T}_1,\pi_\lambda)$. From the inequality (8) it follows that

$$\lim_{t\to+\infty}\rho(\pi_\lambda^t M,K)=0$$

for all $M\in K(X)$ and, consequently, the dynamical system $(X,\mathbb{T}_1,\pi_\lambda)$ is compactly dissipative and its global attractor center $J_\lambda\subseteq K_\lambda$. On the other hand, $K_\lambda\subseteq J_\lambda$, because the set $K_\lambda=\gamma_\lambda(Y)$ is compact and invariant, but J_λ is the maximal compact invariant set of $(X,\mathbb{T}_1,\pi_\lambda)$. Thus we have $J_\lambda=\gamma_\lambda(Y)$.

Let now $\mathbb{T}_2=\mathbb{S}$. Then we will show that the set $\gamma_\lambda(y)$ contains a single point for any $y\in Y$. If we suppose that it is not true, then there are $y_0\in Y$ and $x_1,x_2\in\gamma_\lambda(y_0)$ $(x_1\neq x_2)$. Let $\phi_i\in\Phi_{x_i}$ $(i=1,2)$ be such that $\phi_i(\mathbb{S})\subseteq J_\lambda$. Then we have

$$\pi_\lambda^t(\phi_i(-t))=x_i\ \ (i=1,2) \tag{19}$$

for all $t\in\mathbb{T}_1$. Note that from inequality (8) and equality (19) it follows that

$$\begin{aligned}\rho(x_1,x_2)=\rho(\pi_\lambda^t(\phi_1(-t)),\pi_\lambda^t(\phi_2(-t)))\leq\\ Ne^{-\nu t}\rho(\phi_1(-t),\phi_2(-t))\leq Ne^{-\nu t}C\end{aligned} \tag{20}$$

for all $t\in\mathbb{T}$, where $C:=\sup\{\rho(\phi_1(s),\phi_2(s))\ :\ s\in\mathbb{S}\}$. Passing to the limit in (20) as $t\to+\infty$ we obtain $x_1=x_2$. The obtained contradiction proves our statement.

Thus, if $\mathbb{T}_2=\mathbb{S}$, the unique fix point $\gamma_\lambda\in\Gamma(Y,X)$ of the semi-group of operators $\{S_\lambda^t\}_{t\in\mathbb{T}_1}$ is a single-valued function and, consequently, it is continuous. Finally, inequality (9) follows from (8), because $h(\gamma_\lambda(h(x)))=(h\circ\gamma_\lambda)(h(x))=h(x)$ for all $x\in X$. □

Remark 3.16. *If $(Y,\mathbb{T}_2,\sigma)$ is a semi-group dynamical system (i.e. $\mathbb{T}_2=\mathbb{R}_+$ or $\mathbb{Z}_+$), then the unique invariant section γ_λ of the non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi_\lambda),\ (Y,\mathbb{T}_2,\ \sigma),h\rangle$ is multi-valued (i.e. $\gamma_\lambda(y)$ contains, generally speaking, more than one point). This fact is confirmed by the below example, which is a slight modification of example from* [25, Ch1,p.42-43].

Example 3.17. Let $Y:=[-1,1]$ and $(Y,\mathbb{Z}_+,\sigma)$ be a cascade generated by positive powers of the odd function g, defined on $[0,1]$ in the following way:

$$g(y)=\begin{cases}-2y & ,\ \ 0\leq y\leq\frac{1}{2}\\ 2(y-1) & ,\ \ \frac{1}{2}<y\leq 1.\end{cases}$$

It is easy to check that $g(Y)=Y$. Let us put $X:=\mathbb{R}\times Y$ and denote by $(X,\mathbb{Z}_+,\pi)$ a cascade generated by the positive powers of the mapping $P: X\to X$

$$P\begin{pmatrix}u\\ y\end{pmatrix}=\begin{pmatrix}f(u,y)\\ g(y)\end{pmatrix}, \tag{21}$$

where $f(u,y):=\frac{1}{10}u+\frac{1}{2}y$. Finally, let $h=pr_2: X\to Y$. From (21), it follows that h is a homomorphism of $(X,\mathbb{Z}_+,\pi)$ onto $(Y,\mathbb{Z}_+,\sigma)$ and, consequently,

$\langle(X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z}_+,\sigma),h\rangle$ is a non-autonomous dynamical system. Note that

$$|(u_1,y)-(u_2,y)| = |u_1-u_2| = 10|P(u_1,y)-P(u_2,y)|. \tag{22}$$

From (22), it follows that

$$|P^n(u_1,y)-P^n(u_2,y)| \le \mathcal{N}e^{-\nu n}|\langle u_1,y\rangle - \langle u_2,y\rangle| \tag{23}$$

for all $n\in\mathbb{Z}_+$, where $\mathcal{N}=1$ and $\nu=\ln 10$. By Theorem 3.15 there exists a unique β-continuous invariant section $\gamma\in\Gamma(Y,X)$ of non-autonomous dynamical system $\langle(X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z}_+,\sigma),h\rangle$. According to [25, p.43] $\gamma(y)$ is homeomorphic to the Cantor set for all $y\in[-1,1]$.

4. Iterated Function Systems, Discrete Inclusions and Cocycles

Definition 4.1. *A iterated function system (IFS) cosists of a complete metric space (X,ρ) together with a finite set of mappings $f_i : X\mapsto X$ $(i=1,\dots,m)$ (the notation $\{X;\ f_i,\ i=1,\dots,m\}$). The IFS $\{X;\ f_i,\ i=1,\dots,m\}$ is called hyperbolic if every function f_i $(i=1,\dots,m)$ is a contraction.*

Let W be a topological space. Denote by $C(W)$ the space of all continuous operators $f:W\to W$ equipped with the compact-open topology. Consider a set of operators $\mathcal{M}\subseteq C(W)$ and, respectively, an ensemble (collage) of discrete dynamical systems $(W,f)_{f\in\mathcal{M}}$ ((W,f) is a discrete dynamical system generated by positive powers of map f).

Definition 4.2. *A discrete inclusion $DI(\mathcal{M})$ is (see, for example, [14]) a set of all sequences $\{\{x_j\}\mid j\ge 0\}\subset W$ such that*

$$x_j = f_{i_j}x_{j-1}$$

for some $f_{i_j}\in\mathcal{M}$ (trajectory of $DI(\mathcal{M})$), i.e.

$$x_j = f_{i_j}f_{i_{j-1}}...f_{i_1}x_0 \ \textit{all}\ f_{i_k}\in\mathcal{M}.$$

Definition 4.3. *A bilateral sequence $\{\{x_j\}\mid j\in\mathbb{Z}\}\subset W$ is called a full trajectory of $DI(\mathcal{M})$ (entire trajectory or trajectory on $\mathbb{Z}$), if $x_{n+j}=f_{i_j}x_{n+j-1}$ for all $n\in\mathbb{Z}$ and $j\in\mathbb{Z}_+$.*

Let us consider the set-valued function $F:W\to K(W)$ defined by the equality $F(x):=\{f(x)\,|f\in\mathcal{M}\}$. Note that the set $F(x)$ is compact because $\mathcal{M}$ is so. Then the discrete inclusion $DI(\mathcal{M})$ is equivalent to the difference inclusion

$$x_j\in F(x_{j-1}). \tag{24}$$

Denote by $\mathcal{F}_{x_0}$ the set of all trajectories of discrete inclusion (24) (or $DI(\mathcal{M})$) issuing from the point $x_0\in W$ and $\mathcal{F}:=\bigcup\{\mathcal{F}_{x_0}\mid x_0\in W\}$.

Below we will give a new approach concerning the study of discrete inclusions $DI(\mathcal{M})$ (or difference inclusion (24)). Denote by $C(\mathbb{Z}_+,W)$ the space of all continuous mappings $f:\mathbb{Z}_+\to W$ equipped with the compact-open topology. Let $(C(\mathbb{Z}_+,W),\mathbb{Z}_+,\sigma)$ be the dynamical system of translations (shift dynamical system or dynamical system of Bebutov [24, 26]) on $C(\mathbb{Z}_+,W)$, i.e. $\sigma(k,f):=f_k$ and f_k is a $k\in\mathbb{Z}_+$ shift of f (i.e. $f_k(n):=f(n+k)$ for all $n\in\mathbb{Z}_+$).

We may now rewrite equation (24) in the following way:

$$x_{j+1} = \omega(j)x_j, \ \ (\omega \in \Omega := C(\mathbb{Z}_+, \mathcal{M})) \tag{25}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(j) := f_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$. We denote by $\varphi(n, x_0, \omega)$ the solution of equation (25) issuing from the point $x_0 \in W$ at the initial moment $n = 0$. Note that $\mathcal{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathcal{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in W, \omega \in \Omega\}$, i.e. $DI(\mathcal{M})$ (or inclusion (24)) is equivalent to the family of non-autonomous equations (25) ($\omega \in \Omega$).

From the general properties of difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in W \times \Omega$;
(ii) $\varphi(n + \tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in W \times \Omega$;
(iii) the mapping φ is continuous;
(iv) for any $n, \tau \in \mathbb{Z}_+$ and $\omega_1, \omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that

$$U(n, \omega_2)U(\tau, \omega_1) = U(n + \tau, \omega_3), \tag{26}$$

where $\omega \in \Omega$, $U(n, \omega) := \varphi(n, \cdot, \omega) = \prod_{k=0}^{n} \omega(k)$, $\omega(k) := f_{i_k}$ $(k = 0, 1, \dots, n)$ and $f_{i_0} := Id_W$.

Let W, Ω be two topological spaces and $(\Omega, \mathbb{T}, \sigma)$ be a semi-group dynamical system on Ω.

Definition 4.4. *Recall* [24] *that a triplet* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ *(or briefly* φ*) is called a cocycle over* $(\Omega, \mathbb{T}, \sigma)$ *with the fiber* W*, if* φ *is a mapping from* $\mathbb{T} \times W \times \Omega$ *to* W *satisfying the following conditions:*

1. $\varphi(0, x, \omega) = x$ *for all* $(x, \omega) \in W \times \Omega$*;*
2. $\varphi(n+\tau, x, \omega) = \varphi(n, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ *for all* $n, \tau \in \mathbb{T}$ *and* $(x, \omega) \in W \times \Omega$*;*
3. *the mapping* φ *is continuous.*

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T} \to X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to check that $(X, \mathbb{T}, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [1], [24]; but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}, \pi)$ onto $(\Omega, \mathbb{T}, \sigma)$ and hence $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ over the dynamical system $(\Omega, \mathbb{T}, \sigma)$ with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ $(X := W \times \Omega)$, which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ over $(\Omega, \mathbb{T}, \sigma)$.

From that has been presented above it follows that every $DI(\mathcal{M})$ (respectively, inclusion (24)) in a natural way generates a cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is the solution of equation (25) issuing from the point $x \in W$ at the initial moment $n = 0$. Thus, we can study inclusion (24) (respectively, $DI(\mathcal{M})$) in the framework of the theory of cocycles with discrete time.

Theorem 4.5. [10] *The following statements hold:*

(i) $\Omega = \overline{Per(\sigma)}$, *where* $Per(\sigma)$ *is the set of all periodic points of* $(\Omega, \mathbb{Z}_+, \sigma)$ *(i.e.* $\omega \in Per(\sigma)$, *if there exists* $\tau \in \mathbb{N}$ *such that* $\sigma(\tau, \omega) = \omega$*);*
(ii) *the set* Ω *is compact;*
(iii) Ω *is invariant, i.e.* $\sigma^t \Omega = \Omega$ *for all* $t \in \mathbb{Z}_+$;
(iv) *if* $\mathcal{M}$ *is a compact subset of* $C(W)$ *and* $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ *is a cocycle generated by* $DI(\mathcal{M})$, *then* φ *satisfies the condition (26).*

5. Some properties of Lipschitz mappings

In this section we study some properties of Lipschitz maps, because they play the important role in the study of generalized contraction Iterated Function Systems. We introduce the notion of spectral radius for Lipschitz maps and we give the necessary and sufficient conditions that a Lipschitz mapping is contracting in the generalized sense in the term of its spectral radius.

Let (W, ρ) be a metric space.

Definition 5.1. *A mapping* $f : W \to W$ *satisfies the Lipschitz condition, if there exists a constant* $L > 0$ *such that* $\rho(f(x_1), f(x_2)) \leq L\rho(x_1, x_2)$ *for all* $x_1, x_2 \in W$. *The smallest constant with above mentioned property is called the Lipschitz constant* $Lip(f)$ *of the mapping* f.

Denote by $Lip(W) := \{f : W \mapsto W \mid Lip(f) < \infty\}$.

Lemma 5.2. *Let* $f \in Lip(W)$, *then the following statement hold:*

(i) $f^n \in Lip(W)$ *for all* $n \in \mathbb{N}$, *where* $f^n := f^{n-1} \circ f$ *(*$\forall\ n \geq 2$*);*
(ii) $Lip(f^n) \leq Lip(f)^n$ *(*$\forall\ n \in \mathbb{N}$*);*
(iii) *there exists the limit*

$$r_f := \lim_{n\to\infty} (Lip(f^n))^{\frac{1}{n}};$$

(iv) $r_f \leq Lip(f)$.

Proof. The first, second and fourth statements are obvious. To prove the third statement we note that the sequence $\{b_n\}$ ($b_n := \ln(Lip(f^n))$) is sub-additive, i.e. $b_{n_1+n_2} \leq b_{n_1} + b_{n_2}$ for all $n_1, n_2 \in \mathbb{N}$. Thus there exists the limit $\lim\limits_{n\to\infty} \frac{b_n}{n}$ (see, for example, [19, p.27]) and, consequently, there exists also the limit

$$\lim_{n\to\infty} (Lip(f^n))^{\frac{1}{n}} = e^{\lim\limits_{n\to\infty} \frac{b_n}{n}}.$$

□

Definition 5.3. *The spectral radius of function* $f \in Lip(W)$ *is said to be the number* $r_f := \lim\limits_{n\to\infty} (Lip(f^n))^{\frac{1}{n}}$.

Definition 5.4. *The function* $f \in Lip(W)$ *is said to be generalized contraction (contracting in the extended sense) if* $r_f < 1$.

Remark 5.5. *1. If* $f \in Lip(W)$ *is a contraction (i.e.,* $Lip(f) < 1$*), then* $r_f < 1$ *because* $r_f \leq Lip(f)$.

2. If $f \in Lip(W)$ and $r_f < 1$ then, generally speaking, f is not a contraction. This fact is confirmed by the below example. In fact, let $W := C[0,1]$ and $f \in Lip(W)$ is defined by equality

$$(f\varphi)(t) := \frac{3}{2}\int_0^t \varphi(s)ds$$

($t \in [0,1]$ and $\varphi \in C[0,1]$). It is easy to verify that $Lip(f^n) = (\frac{3}{2})^n \frac{1}{n!}$. In particular, $Lip(f) = \frac{3}{2}, Lip(f^2) = \frac{9}{8}$ and $Lip(f^3) = \frac{27}{32}$. In addition $Lip(f^n) \le 2(\frac{3}{4})^n$ for all $n \in \mathbb{N}$. Thus the mapping f is a generalized contraction, but $Lip(f) \ge 1$.

Lemma 5.6. *The function $f \in Lip(W)$ is a generalized contraction if and only if there exist positive numbers $\mathcal{N}$ and ν ($0 < \nu < 1$) such that*

$$Lip(f^n) \le \mathcal{N}\nu^n \tag{27}$$

for all $n \in \mathcal{N}$.

Proof. It is easy to see that from (27) we have $r_f \le \nu < 1$.

Let now $r_f < 1$ and $\varepsilon \in (0, 1 - r_f)$. Then there is a number $n_0 = n_0(\varepsilon) \in \mathcal{N}$ such that $(Lip(f^n))^{\frac{1}{n}} < r_f + \varepsilon$ for all $n \in \mathbb{N}$ with $n \ge n_0$. We put $\nu := r_f + \varepsilon$ and $\mathcal{N} := \max\{1, \nu Lip(f), \nu^2 Lip(f^2), \dots, \nu^{n_0} Lip(f^{n_0})\}$, then $Lip(f^n) \le \mathcal{N}\nu^n$ for all $n \in \mathbb{N}$. □

Corollary 5.7. *The mapping f is a generalized contraction if and only if one of its iterates is contracting.*

Definition 5.8. *A subset of operators $\mathcal{M} \subseteq C(W)$ is said to be generally contracting (contracting in the extended sense), if there are positive numbers $\mathcal{N}$ and $\nu < 1$ such that*

$$L(f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1}) \le \mathcal{N}\nu^n$$

for all $f_{i_1}, f_{i_2}, \dots, f_{i_n} \in \mathcal{M}$ and $n \in \mathbb{N}$.

Remark 5.9. *1. If the subset of operators $\mathcal{M} \subseteq C(W)$ is generally contracting, then*

- (i) *every function $f \in \mathcal{M}$ is generally contracting;*
- (ii) *every function $f := f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1}$ ($f_{i_k} \in \mathcal{M}$ for all $k = 1, \dots, n$) is a generalized contraction.*

2. If $r_f < 1$ for every function $f \in \mathcal{M}$,then the subset of operators $\mathcal{M} \subseteq C(W)$, generally speaking, is not a generalized contraction. In fact, let $W := \mathbb{R}^2$ and $\mathcal{M} \subseteq C(W)$ consists from two functions $\{f_1, f_2\}$, where $f_1(x_1, x_2) := (2x_2, \frac{x_1}{4})$ and $f_2(x_1, x_2) := (5x_2, \frac{x_1}{6})$. Then $r_{f_1} = \frac{\sqrt{2}}{2}$, $r_{f_2} = \sqrt{\frac{5}{6}}$ and $r_{f_1 f_2} = \frac{5}{4}$ (see [12]) and, consequently, $\mathcal{M} := \{f_1, f_2\}$ is not generally contracting.

Lemma 5.10. *Let $\mathcal{M} = \{f_1, f_2, \dots, f_m\}$, then the following statements hold:*

- (i) *If $Lip(f_i) < 1$ for all $1 \le i \le m$, then the subset of operators $\mathcal{M} \subseteq C(W)$ is generally contracting;*
- (ii) *Let $r_{f_i} < 1$ for all $1 \le i \le m$ and the mappings $f_1, \dots, f_m$ are permutable (i.e. $f_i \circ f_j = f_j \circ f_i$ for all $1 \le i, j \le m$), then the set of operators $\mathcal{M} = \{f_1, \dots, f_m\}$ is generally contracting.*

Proof. Let $Lip(f_i) < 1$ for all $i = 1, \dots, m$. Then $Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) \leq Lip(f_{i_n}) \dots Lip(f_{i_1}) \leq \nu^n$ for all $n \in \mathbb{N}$, where $\nu := \max\{Lip(f_k) \mid 1 \leq k \leq m\}$.

Let $n \in \mathbb{N}$ and $f_{i_k} \in \mathcal{M} := \{f_1, \dots, f_m\}$ $(1 \leq i_k \leq m$ for all $1 \leq k \leq n)$. Then $f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1} = f_1^{k_1} \dots f_m^{k_m}$, where $k_i \in \mathbb{Z}_+$ $(1 \leq i \leq m)$ with condition $k_1 + \dots + k_m = n$. Thus we have

$$Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) = Lip(f_1^{k_1}) \dots Lip(f_m^{k_m}). \tag{28}$$

Since $r_{f_i} < 1$, then by Lemma 5.6 there are positive numbers $\mathcal{N}_i$ and $\nu_i < 1$ such that

$$Lip(f_i^n) \leq \mathcal{N}_i \nu_i^n \tag{29}$$

for all $n \in \mathbb{N}$.

From the relations (28) and (29) follows that

$$Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) \leq \mathcal{N} \nu^n$$

for all $n \in \mathbb{N}$, where $\mathcal{N} := \max\{\mathcal{N}_k \mid 1 \leq k \leq m\}$ and $\nu := \max\{\nu_k \mid 1 \leq k \leq m\}$. □

6. Relation Between Compact Global Attractors of Skew-Product Systems, Collages and Cocycles

Theorem 6.1. [10] *Suppose the following conditions are fulfilled:*

(i) $\mathcal{M} := \{f_i \ : i \in I\}$ *is a compact subset from* $C(W)$*;*
(ii) *the set* $\mathcal{M}$ *of operators is contracting in the extended sense.*

Then the set-valued cascade (W, F) *(discrete dynamical system generated by positive powers of mapping* F*) is compactly dissipative, , where* $F(x) := \{f(x) \mid f \in \mathcal{M}\}$ $(\forall x \in W)$*.*

Theorem 6.2. [10] *Let* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma \rangle$ *be a cocycle,* Ω *be a compact space and* $f : \mathbb{T} \times W :\to K(W)$ *be a mapping defined by the equality*

$$f(t, u) = \varphi(t, u, \Omega) \tag{30}$$

for all $u \in W$ *and* $t \in \mathbb{T}$*.*

Then the mapping f *possesses the following properties:*

a. $f(0, u) = u$ *for all* $u \in W$*;*
b. $f(t, f(\tau, u)) \subseteq f(t + \tau, u)$ *for all* $t, \tau \in \mathbb{T}$ *and* $u \in W$*;*
c. $f : \mathbb{T} \times W \to K(W)$ *is upper semi-continuous, i.e.*

$$\lim_{t \to t_0, u \to u_0} \beta(f(t, u), f(t_0, u_0)) = 0 \quad \forall (t_0, u_0) \in \mathbb{T} \times W;$$

d. *if the cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma) \rangle$ *satisfies the following condition:*

$$\forall t, \tau \in \mathbb{T}, u_1, u_2 \in W \ \exists u_3 \text{ such that } \varphi(t, \varphi(\tau, x, u_1), u_2) = \varphi(t + \tau, x, u_3), \tag{31}$$

then

$$f(t, f(\tau, u)) = f(t + \tau, u)$$

for all $t, \tau \in \mathbb{T}$ *and* $u \in W$*.*

Corollary 6.3. *Every cocycle $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma\rangle$ with the compact Ω and satisfying the condition (31) generates a set-valued dynamical system $(W, \mathbb{T}, f)$, where $f : \mathbb{T} \times W \to K(W)$ is defined by equality (30).*

Definition 6.4. *A cocycle φ over $(\Omega, \mathbb{T}, \sigma)$ with the fiber W is said to be a compactly dissipative one, if there is a nonempty compact $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(U(t,\omega)M, K) \mid \omega \in \Omega\} = 0 \tag{32}$$

for any $M \in K(W)$, where $U(t, \omega) := \varphi(t, \cdot, \omega)$.

Definition 6.5. [7, Ch.II] *A metric space X possesses the property (S), if for every compact subset $K \subseteq X$ there exists a connected compact subset $L \subseteq X$ such that $K \subseteq L$.*

Theorem 6.6. [7, Ch.II] *Let Y be compact, $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (32). Then the following statements hold:*

(i) *$w \in I_y$ $(y \in Y)$ if and only if there exits a complete trajectory $\nu : \mathbb{S} \to W$ of the cocycle φ, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S})$ is relatively compact;*
(ii) *I_y $(y \in Y)$ is connected, if the space W possesses the property (S).*

Definition 6.7. *The smallest compact set $I \subseteq W$ with property (32) is said to be a Levinson center (global attractor) of cocycle φ.*

Theorem 6.8. [10]

(i) *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a cocycle with the compact Ω and satisfying the condition (31). Then the following statements are equivalent:*
  (a) *the cocycle φ is compactly dissipative;*
  (b) *the skew-product dynamical system $(X, \mathbb{T}, \pi)$ generated by the cocycle φ is compactly dissipative;*
  (c) *the set-valued dynamical system $(W, \mathbb{T}, f)$ generated by the cocycle φ is compactly dissipative.*
(ii) *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a compact dissipative cocycle and the following conditions be fulfilled:*
  (a) *Ω is compact and invariant ($\sigma^t\Omega = \Omega$ for all $t \in \mathbb{T}$);*
  (b) *the cocycle φ satisfies condition (31).*
  Then $I = pr_1(J)$, where J is the global attractor center of the skew-product dynamical system $(X, \mathbb{T}, \pi)$ (generated by the cocycle φ) and I is the global attractor of the set-valued dynamical system $(W, \mathbb{T}, f)$ (generated by the cocycle φ).

Denote by $\Phi(\varphi)$ the set of all full trajectories of the cocycle φ.

Corollary 6.9. *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a compactly dissipative cocycle and the following conditions be fulfilled:*

(i) *Ω is compact and invariant;*
(ii) *the cocycle φ satisfies condition (31).*

Then $I = \{u \in W \ : \ \exists \eta \in \Phi(\varphi),\ \eta(0) = u$ and $\eta(\mathbb{S})$ is relatively compact$\}$.

7. Continuous dependence of Attractors of IFS

Theorem 7.1. [10] *Suppose that the following conditions are fulfilled:*

(i) $\mathcal{M}$ *is a compact subset of* $C(W)$;
(ii) $\mathcal{M}$ *is contracting in the extended sense.*

Then

(i) $I_\omega := \{u \in W \ : \ $ *a solution* $\varphi(n,u,\omega)$ *of equation (25) is defined on* $\mathbb{Z}$ *and* $\varphi(\mathbb{Z},u,\omega)$ *is relatively compact*$\} \neq \emptyset$ *for all* $\omega \in \Omega$, *i.e. every equation (25) admits at least one solution defined on* $\mathbb{Z}$ *with relatively compact range of values;*
(ii) *the sets* I_ω $(\omega \in \Omega)$ *and* $I := \bigcup\{I_\omega \ : \ \omega \in \Omega\}$ *are compact;*
(iii) *the set-valued map* $\omega \to I_\omega$ *is upper semi-continuous;*
(iv) *the family of compact sets* $\{I_\omega \ : \ \omega \in \Omega\}$ *is invariant with respect to the cocycle* φ, *i.e.* $\varphi(n, I_\omega, \omega) = I_{\sigma^n\omega}$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$;
(v) $\rho(\varphi(n,u_1,\omega),\varphi(n,u_2,\omega)) \le \mathcal{N}e^{-\nu n}\rho(u_1,u_2)$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$ *and* $u_1,u_2 \in W$, *where* $\mathcal{N}$ *and* ν *are positive numbers from the definition of the contractivity of* $\mathcal{M}$ *in the extended sense;*
(vi) *if every map* $f \in \mathcal{M}$ *is invertible, then*
  (a) I_ω *consists of a single point* u_ω;
  (b) *the map* $\omega \to u_\omega$ *is continuous;*
  (c) $\varphi(n,u_\omega,\omega) = u_{\sigma(n,\omega)}$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$;
  (d) $\rho(\varphi(n,u,\omega),\varphi(n,u_\omega,\omega)) \le \mathcal{N}e^{-\nu n}\rho(u,u_\omega)$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$.

Let Λ be a compact metric space. Denote by $C(\Lambda\times W, W)$ the space of all continuous functions $f : \Lambda \times W \mapsto W$ equipped with compact-open topology. If $f \in C(\Lambda \times W, W)$ then we denote by $f^\lambda := f(\lambda,\cdot) \in C(W)$ and $\mathcal{M}^\lambda := \{f^\lambda \mid f \in \mathcal{M}\}$.

Consider a set of operators $\mathcal{M} \subseteq C(\Lambda \times W, W)$ and, respectively, an ensemble (collage) of discrete dynamical systems $(W, f_\lambda)_{f_\lambda \in \mathcal{M}_\lambda}$ ((W, f_λ) is a discrete dynamical system generated by positive powers of map f_λ).

We consider the equation

$$x_{j+1} = \omega(\cdot, j)x_j, \ (\omega \in \Omega := C(\Lambda \times \mathbb{Z}_+, \mathcal{M})) \tag{33}$$

or

$$x_{j+1} = \omega(\lambda, j)x_j, \ (\lambda \in \Lambda, \ \omega(\lambda,\cdot) \in \Omega_\lambda := C(\mathbb{Z}_+, \mathcal{M})), \tag{34}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(\cdot, j) := f_{i_{j+1}} \in C(\Lambda \times W, W)$ (or $\omega(\lambda, j) := f^\lambda_{i_{j+1}} \in C(W,W)$ for all $\lambda \in \Lambda$) for all $j \in \mathbb{Z}_+$, i.e. $\omega(j)$ is a continuous function depending on two variables $\lambda \in \Lambda$ and $x \in W$. We denote by $\varphi(\cdot, n, x_0, \omega)$ the solution of equation (33) (respectively, by $\varphi(\lambda, n, x_0, \omega)$ the solution of equation (34)) issuing from the point $x_0 \in W$ at the initial moment $n = 0$.

From the general properties of difference equations it follows that the mapping $\varphi : \Lambda \times \mathbb{Z}_+ \times W \times \Omega \to W$ satisfies the following conditions:

(i) $\varphi(\lambda, 0, x_0, \omega) = x_0$ for all $(\lambda, x_0, \omega) \in \Lambda \times W \times \Omega$;

(ii) $\varphi(\lambda, n + \tau, x_0, \omega) = \varphi(\lambda, n, \varphi(\lambda, \tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(\lambda, x_0, \omega) \in \Lambda \times W \times \Omega$;

(iii) the mapping φ is continuous;

(iv) for any $n, \tau \in \mathbb{Z}_+$ and $\omega_1, \omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that

$$U(\lambda, n, \omega_2)U(\lambda, \tau, \omega_1) = U(\lambda, n + \tau, \omega_3),$$

where $\omega \in \Omega$, $U(\lambda, n, \omega) := \varphi(\lambda, n, \cdot, \omega) = \prod_{k=0}^{n} \omega(\lambda, k)$, $\omega(\lambda, k) := f^{\lambda}_{i_k}$ $(k = 0, 1, \dots, n)$ and $f^{\lambda}_{i_0} := Id_W$.

Let $X := W \times \Omega$, and define the mapping $\pi_\lambda : X \times \mathbb{T} \to X$ by the equality: $\pi_\lambda((u, \omega), t) := (\varphi(\lambda, t, u, \omega), \sigma(t, \omega))$ (i.e. $\pi_\lambda = (\varphi_\lambda, \sigma)$). Then it is easy to check that for each $\lambda \in \Lambda$ the triplet $(X, \mathbb{T}, \pi_\lambda)$ is a dynamical system on X, but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}, \pi_\lambda)$ onto $(\Omega, \mathbb{T}, \sigma)$ and hence $\langle (X, \mathbb{T}, \pi_\lambda), (\Omega, \mathbb{T}, \sigma), h \rangle$ is a family of non-autonomous dynamical systems depending on parameter $\lambda \in \Lambda$. Applying Theorem 3.15 to the family of dynamical systems $\langle (X, \mathbb{T}, \pi_\lambda), (\Omega, \mathbb{T}, \sigma), h \rangle$ we will receive the following result.

Theorem 7.2. *Suppose that the following conditions hold:*

(i) Λ *be a compact metric space;*

(ii) $\mathcal{M}$ *be a nonempty compact subset of* $C(\Lambda \times W, W)$, *where* W *is a complete metric space;*

(iii) *the subset* $\mathcal{M} \subseteq C(\Lambda \times W, W)$ *is generalized contracting, i.e. there are two positive numbers* $\mathcal{N}$ *and* $\nu < 1$ *such that* $Lip(f^{\lambda}_{i_n} \circ \dots \circ f^{\lambda}_{i_1}) \le \mathcal{N}\nu^n$ *for all* $\lambda \in \Lambda$, $n \in \mathbb{N}$ *and* $i_1, \dots, i_n \in \mathbb{N}$ *where* $f^{\lambda}_k := f_k(\lambda, \cdot)$ *and* $f_k \in \mathcal{M}$.

Then the following statements hold:

(i) *for each* $\lambda \in \Lambda$ *the non-autonomous dynamical system* $\langle (X, \mathbb{Z}_+, \pi_\lambda)$, $(\Omega, \mathbb{Z}_+, \sigma)$, $h\rangle$ *is compactly dissipative;*

(ii)

$$\rho(\pi_\lambda(n, x_1), \pi_\lambda(n, x_2)) \le \mathcal{N}\nu^n \rho(x_1, x_2) \tag{35}$$

for all $n \in \mathbb{Z}_+$ *and* $x_, x_2 \in X$ $(h(x_1) = h(x_2))$, *i.e. the family of non-autonomous dynamical systems* $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h \rangle$ *is generalized contracting;*

(iii) *for each* $(\lambda, \omega) \in \Lambda \times \Omega$ *the set* $I^{\lambda}_{\omega} := \{u \in W \mid$ *the solution* $\varphi(\lambda, n, u, \omega)$ *of equation (34) defined on* $\mathbb{Z}$ *with relatively compact range of values* $\varphi(\lambda, \mathbb{Z}, u, \omega)\}$ *is nonempty and compact;*

(iv) *for each* $\lambda \in \Lambda$ *the family of subsets* $\mathcal{I}^{\lambda} := \{I^{\lambda}_{\omega} \mid \omega \in \Omega\}$ *is invariant with respect to cocycle* $\varphi_\lambda := \varphi(\lambda, \cdot, \cdot, \cdot)$, *i.e.* $\varphi_\lambda(t, I^{\lambda}_{\omega}, \omega) = I^{\lambda}_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{Z}_+$ *and* $\omega \in \Omega$*;*

(v) $I^{\lambda}_{\omega} = pr_1(J^{\lambda}_{\omega})$ *for all* $\lambda \in \Lambda$ *and* $\omega \in \Omega$, *where* J^{λ} *is the global attractor of dynamical system* $(X, \mathbb{Z}_+, \pi_\lambda)$;

(vi) *for each* $\lambda \in \Lambda$ *the set* $\mathbb{I}^{\lambda} := \cup\{I^{\lambda}_{\omega} \mid \omega \in \Omega\} = pr_1(J^{\lambda})$ *and, consequently, it is compact;*

(vii)

$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \alpha(I^{\lambda}_{\omega}, I^{\lambda_0}_{\omega}) = 0 \tag{36}$$

and, consequently, we have also

$$\lim_{\lambda \to \lambda_0} \alpha(\mathbb{I}^{\lambda}, \mathbb{I}^{\lambda_0}) = 0. \tag{37}$$

Proof. Let φ_λ be the cocycle generated by equation (34). Denote by $(X, \mathbb{Z}_+, \pi_\lambda)$ the skew-product dynamical system generated by cocycle φ_λ (i.e. $X := W \times \Omega$ and $\pi_\lambda := (\varphi_\lambda, \sigma)$). Let $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ be the non-autonomous dynamical system associated by cocycle φ_λ, where $h := pr_2 : X \mapsto \Omega$. Under the conditions of Theorem the family of non-autonomous dynamical systems $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ satisfies the inequality (35) because $\pi_\lambda(n, x) = (\varphi_\lambda(n, u, \omega), \sigma(n, \omega))$ $(x := (u, \omega))$ and $\varphi_\lambda(n, u, \omega) = \omega(\lambda, n) \circ \ldots \circ \omega(\lambda, 1)u$. By Theorem 3.15 for each $\lambda \in \Lambda$ dynamical system $(X, \mathbb{Z}_+, \pi_\lambda)$ admits a compact global attractor J^λ and there exists the unique invariant section $\gamma_\lambda \in \Gamma(\Omega, X)$ such that:

(i) the mapping $\lambda \mapsto \gamma_\lambda$ is continuous, i.e.

$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \alpha(\gamma_\lambda(\omega), \gamma_{\lambda_0}(\omega)) = 0; \tag{38}$$

(ii) $J^\lambda_\omega = \gamma_\lambda(\omega)$ for all $\omega \in \Omega$ and, consequently, $J^\lambda = \gamma_\lambda(\Omega)$,where $J^\lambda_\omega := X_\omega \cap J^\lambda$ and $X_\omega := h^{-1}(\omega)$.

Since $(X, \mathbb{Z}_+, \pi_\lambda)$ is a skew-product dynamical system and $X = W \times \Omega$, then γ_λ has the form $(\phi_\lambda, Id_\Omega)$, where $\phi_\lambda \in C(\Omega, W)$. Note that $I^\lambda_\omega = pr_1(J^\lambda_\omega)$ and, consequently, it is non-empty and compact. On the other hand $\pi_\lambda(n, J^\lambda_\omega) = J^\lambda_{\sigma(n,\omega)}$ for all $\lambda \in \Lambda$, $n \in \mathbb{Z}_+$ and $\omega \in \Omega$ because Ω is invariant (i.e. $\sigma(n, \Omega) = \Omega$ for all $n \in \mathbb{Z}_+$) and, consequently, $\varphi_\lambda(n, I^\lambda_\omega, \omega) = \phi_\lambda(\pi_\lambda(n, J^\lambda) = \phi_\lambda(J^\lambda_{\sigma(n,\omega)}) = I^\lambda_{\sigma(n,\omega)}$.

From the equalities (38) and $\gamma_\lambda = (\phi_\lambda, Id_\Omega)$ follow the equalities (36) and (37). □

Remark 7.3. *If $\mathcal{M} \subseteq C(\Lambda \times W, W)$ is a finite set, i.e. $\mathcal{M} = \{f_1, \ldots, f_m\}$, then the equality (37) coincides with Barnsley's theorem of continuous dependence of fractals on parameters* [2, Th.1,Ch.III] *(see also* [18]*).*

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