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# INVARIANT MANIFOLDS AND ALMOST AUTOMORPHIC SOLUTIONS OF SECOND-ORDER MONOTONE EQUATIONS

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**Abstract.** We give sufficient conditions of the existence of a compact invariant manifold, almost periodic (almost automorphic) solutions of the second-order differential equation $x'' = f(t,x)$ on an arbitrary Hilbert space with the uniform monotone right hand side $f$.

**Key words.** Non-autonomous dynamical systems; skew-product systems; cocycles; continuous invariant sections of non-autonomous dynamical systems; almost periodic, almost automorphic, quasi-recurrent solutions; chaotic sets

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## 1. Introduction.

The problem of the almost periodicity of solutions of non-linear almost periodic second-order differential equations

$$x'' = f(t,x) \tag{1.1}$$

with the monotone (with respect to the spacial variable $x$) right hand side $f$ was studied by many authors (see, for example, [2, 3], [5, 6], [7], [12] and the bibliography therein).

In the present paper we consider a special class of equations (1.1), where the function $f : \mathbb{R} \times H \to H$ ($H$ is a Hilbert space) is uniformly monotone with respect to (w.r.t.) $x \in H$,

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i.e. $f'_x(t,x) \geq mI$, where $f_x(t,x)$ is a self-adjoint operator and $I$ is a unit operator on $H$ and $m > 0$. We also study a more general equation

$$x'' = f(\omega t, x) \quad (\omega \in \Omega), \tag{1.2}$$

with the uniform monotone (with respect to the spacial variable $x$) right hand side $f$, where $\Omega$ is a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ is a dynamical system on $\Omega$ and $\omega t := \sigma(t,\omega)$. We give sufficient conditions for the existence of a compact invariant manifold of equation (1.2). Almost periodic, quasi-periodic, almost automorphic, pseudo recurrent solutions and chaotic sets of equation (1.2) are studied too.

## 2. Almost Periodic and Almost Automorphic Motions of Dynamical Systems.

**2.1. Almost Periodic and Almost Automorphic Motions.** Let $X$ be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of the additive group $\mathbb{S}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system.

A number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ almost period, if $\rho(x(\tau + t), xt) < \varepsilon$ for all $t \in \mathbb{T}$.

A point $x \in X$ is called Bohr almost periodic, if for any $\varepsilon > 0$ there exists a positive number $l$ such that at any segment of length $l$ there is an $\varepsilon$ almost period of point $x \in X$.

Denote $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ :\ \text{such that } \{\pi(t_n, x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ is called

1. Levitan almost periodic [8], if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$;
2. stable in the sense of Lagrange ($st$.L), if its trajectory $\{\pi(t,x) : t \in \mathbb{T}\}$ is relatively compact;
3. almost automorphic [8, 10] in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(a) $x$ is st. $L$;
(b) there exists a dynamical system $(Y, \mathbb{T}, \sigma)$, a homomorphism $h$ from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$ and an almost periodic in the sense of Bohr point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

**2.2. Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** Below we indicate one general method of construction of dynamical systems on the space of continuous functions. In this way we will get many well known dynamical systems on the functional spaces (see, for example, [1, 11]).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on $X$, $Y$ be a complete pseudo metric space and $\mathcal{P}$ be a family of pseudo metrics on $Y$. We denote by $C(X, Y)$ the family of all continuous functions $f : X \to Y$ equipped with a compact-open topology. This topology is given by the following family of pseudo metrics $\{d^p_K\}$ $(p \in \mathcal{P},\ K \in C(X))$, where

$$d^p_K(f, g) := \sup_{x \in K} p(f(x), g(x))$$

and $C(X)$ a family of all compact subsets of $X$. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X, Y) \to C(X, Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau, x))\quad (x \in X)$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\forall \tau_1, \tau_2 \in \mathbb{T}$ $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}$;
c. $\forall \tau \in \mathbb{T}$ $\sigma_\tau$ is continuous.

Lemma 2.1 ([4]). *The mapping $\sigma : \mathbb{T} \times C(X, Y) \to C(X, Y)$, defined by the equality $\sigma(\tau, f) := \sigma_\tau f\quad (f \in C(X, Y),\ \tau \in \mathbb{T})$, is continuous.*

Corollary 2.2. *The triple $(C(X, Y), \mathbb{T}, \sigma)$ is a dynamical system on $C(X, Y)$.*

Consider now some examples of dynamical systems of the form $(C(X, Y), \mathbb{T}, \sigma)$, useful in the applications.

Example 2.3. Let $X = \mathbb{T}$ and we denote by $(X, \mathbb{T}, \pi)$ a dynamical system on $\mathbb{T}$, where

$\pi(t, x) := x + t$. The dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ is called *Bebutov's dynamical system* [11] (a dynamical system of translations, or shifts dynamical system).

We will say that the function $\varphi \in C(\mathbb{T}, Y)$ possesses a property $(A)$, if the motion $\sigma(\cdot, \varphi) : \mathbb{T} \to C(\mathbb{T}, Y)$ possesses this property in the dynamical system of Bebutov $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$, generated by the function $\varphi$. As property $(A)$ we can take periodicity, almost periodicity, almost automorphy etc.

Example 2.4. Let $X := \mathbb{T} \times W$, where $W$ is some metric space and by $(X, \mathbb{T}, \pi)$ we denote a dynamical system on $X$ defined in the following way: $\pi(t, (s, w)) := (s + t, w)$. Using the general method proposed above we can define on $C(\mathbb{T} \times W, Y)$ a dynamical system of translations $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

The function $f \in C(\mathbb{T} \times W, Y)$ is called almost periodic (almost automorphic, etc) with respect to $t \in \mathbb{T}$ uniform on $w$ on every compact from $W$, if the motion $\sigma(\cdot, f)$ is almost periodic (almost automorphic, etc) in the dynamical system $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

**2.3. Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ $(\mathbb{S}_+ \subseteq \mathbb{T}_+)$.

A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where $h$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system.

Let $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on $Y$, $W$ be a complete metric space and $\varphi$ be a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in $W$, possessing the following properties:

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;

b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then the triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ (or shortly $\varphi$) is called [9] a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with the fiber $W$.

Let $X := W \times Y$ and let us define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as follows: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on $X$, which is called a skew-product dynamical system [9] and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, hence, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber $W$, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ $(X := W \times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system).

EXAMPLE 2.5. Consider the system of differential equations

$$\begin{cases} u' = F(y, u) \\ y' = G(y), \end{cases} \tag{2.1}$$

where $Y \subseteq E^m, G \in C(Y, E^n)$ and $F \in C(Y \times E^n, E^n)$. Suppose that for the system (2.1) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(Y, \mathbb{R}_+, \sigma)$ a dynamical system on $Y$ generated by the second equation of the system (2.1) and by $\varphi(t, u, y)$ we denote the solution of the equation

$$u' = F(\sigma(t, y), u)$$

passing through the point $u \in E^n$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times Y \to E^n$ satisfies the conditions a. and b. from the definition of cocycle and, consequently, system (2.1) generates a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ (where $X := E^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$).

EXAMPLE 2.6.Let $E$ be a Banach space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on the metric space $Y$. We consider the system

$$\begin{cases} u' = F(\sigma(y, t), u) \\ y \in Y, \end{cases} \tag{2.2}$$

where $F \in C(Y \times E, E)$. Suppose that for equation (2.2) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ (respectively, the cocycle $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ ), where $X := E \times Y$, $\pi := (\varphi, \sigma)$, $\varphi(\cdot, x, y)$ is the solution of (2.2) and $h := pr_2 : X \to Y$ is generated by equation (2.2).

**2.4. Invariant Sections of Non-Autonomous Dynamical Systems.** Let $\langle (X, \mathbb{S}_+, \pi),\ (Y, \mathbb{S}, \sigma), h)\rangle$ be a non-autonomous dynamical system.

A mapping $\gamma : Y \to X$ is called a section (selector) of a homomorphism $h$, if $h(\gamma(y)) = y$ for all $y \in Y$. The section $\gamma$ of the homomorphism $h$ is called invariant, if $\gamma(\sigma(t, y)) = \pi(t, \gamma(y))$ for all $y \in Y$ and $t \in \mathbb{S}$.

Denote by $\Gamma = \Gamma(Y, X)$ the family of all continuous sections of $h$, i.e. $\Gamma(Y, X) = \{\gamma \in C(Y, X) :\ h \circ \gamma = Id_Y\}$. We will suppose that $\Gamma(Y, X) \neq \emptyset$. For applications this condition is fulfilled in many important cases.

## 3. Invariant Manifolds of Second Order Differential Equations.

**3.1. Invariant manifolds.** Let $\Omega$ be a compact metric space and $(\Omega, \mathbb{R}, \sigma)$ be an autonomous dynamical system on $\Omega$. Let $E$ be a Banach space. Denote by $[E]$ the space of all linear continuous operators acting on $E$ and endowed with an operator norm.

Denote by $H$ a Hilbert space with the scalar product $\langle \cdot, \cdot\rangle$ and the norm $|\cdot|^2 := \langle \cdot, \cdot\rangle$, by $C(\Omega, E)$ we denote the Banach space of all continuous function $\varphi : \Omega \to E$ equipped with the norm $\|\varphi\|_{C(\Omega,E)} := \max\limits_{\omega\in\Omega} |\varphi(\omega)|_E$.

A function $\varphi \in C(\Omega, E)$ is called:

– differentiable in the point $\omega_0$ along the flow $(\Omega, \mathbb{T}, \sigma)$, if there exists a limit

$$\dot{\varphi}_\sigma(\omega_0) := \lim_{s\to 0} \frac{\varphi(\sigma(s, \omega_0)) - \varphi(\omega_0)}{s};$$

In this case $\dot{\varphi}_\sigma(\omega_0)$ is called a derivative of the function $\varphi \in C(\Omega, E)$ at the point $\omega_0 \in \Omega$ along the flow $(\Omega, \mathbb{T}, \sigma)$ (shortly, $\sigma$).

– differentiable on $\Omega$ along the flow $\sigma$, if it is differentiable at every point $\omega \in \Omega$;
– continuously differentiable on $\Omega$ along the flow $\sigma$, if it is differentiable at $\Omega$ and $\dot{\varphi}_\sigma \in C(\Omega, E)$.

Denote by $\dot{C}^1(\Omega, E)$ a Banach space of all continuously differentiable (on $\Omega$ along the flow $\sigma$) functions $\varphi \in C(\Omega, E)$ endowed with the norm

$$\|\varphi\|_{\dot{C}^1(\Omega,E)} := \|\varphi\|_{C(\Omega,E)} + \|\dot{\varphi}\|_{C(\Omega,E)}.$$

Let us consider a differential equation of the second order

$$x'' = f(\omega t, x), \qquad (\omega \in \Omega) \tag{3.1}$$

where $f \in C(\Omega \times H, H)$, and give a criterion of the existence of an invariant manifold for this equation. Below we will suppose that the function $f$ is regular, i.e. for all $x \cdot y \in H$ the equation (3.1) admits a unique solution $\varphi(t, x, y, \omega)$ defined on $\mathbb{R}_+$ with the initial conditions $\varphi(0, x, y, \omega) = x$ and $\varphi'(0, x, y, \omega) = y$.

As we know, we can reduce the equation (3.1) to the equivalent system

$$\begin{cases} x' = y \\ y' = f(\omega t, x) \end{cases}$$

$(\omega \in \Omega)$ or to the equation

$$z' = F(\omega t, z) \tag{3.2}$$

on the product space $H^2 := H \times H$, where $z := (x, y)$ and $F \in C(\Omega \times H^2, H^2)$ is the function defined by the equality $F(\omega, z) := (y, f(\omega, x))$ for all $\omega \in \Omega$ and $z := (x, y) \in H^2$.

Remark 1.

1. Since $(\varphi(t,x,y,\omega),\varphi'(t,x,y,\omega))$ is a cocycle, generated by equation (3.2), then we have the following equality

$$\varphi(t+\tau,x,y,\omega)=\varphi(t,\varphi(\tau,x,y,\omega),\varphi'(\tau,x,y,\omega),\omega\tau) \tag{3.3}$$

for all $t,\tau\in\mathbb{R}_+$, $x,y\in H$ and $\omega\in\Omega$.

2. The function $\mu:=(\gamma,\delta)\in C(\Omega,H^2)$ $(\gamma,\delta\in C(\Omega,H))$ is a continuous invariant section of the cocycle $(\varphi(t,x,y,\omega),\varphi'(t,x,y,\omega))$, generated by equation (3.2), if and only if the following conditions are fulfilled:

(i) $\gamma\in\dot{C}^1(\Omega,H)$;

(ii) $\dot{\gamma}_\sigma=\delta$;

(iii) $\gamma(\omega t)=\varphi(t,\gamma(\omega),\dot{\gamma}_\sigma(\omega),\omega)$ for all $t\in\mathbb{R}$ and $\omega\in\Omega$.

Theorem 3.1. *Let $f\in C(\Omega\times H,H)$ be continuously differentiable w.r.t. $x\in H$ and let exist $r_0>0$ such that*

1. *$|f(\omega,x)|\le A(r)<+\infty$ for all $(\omega,x)\in\Omega\times B[0,r]$ and $0\le r\le r_0$;*
2. *there exists positive numbers $m$ and $M(r)$ such that for all $(\omega,x)\in\Omega\times B[0,r]$, $0\le r\le r_0$, $mI\le f'_x(\omega,x)\le M(r)I$ ($I$ is a unit operator from $[H]$) and the operator $f'_x(\omega,x)$ is self-adjoint;*
3. *$A(0)\le mr_0$.*

*Then for an arbitrary $A(0)m^{-1}\le r\le r_0$ there exist a unique function $\gamma\in\dot{C}^1(\Omega,B[0,\ r])$ such that $\gamma(\omega t)=\varphi(t,\gamma(\omega),\dot{\gamma}(\omega),\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$, where $\varphi(t,u,v,\omega)$ is a unique solution of equation (3.1) with the initial conditions $\varphi(0,u,v)=u$ and $\varphi'(0,u,v)=v$.*

**4. Almost Automorphic Solutions of Monotone Second-Order Differential Equation.** In this section we suppose that the space $H$ is finite-dimensional. Let $W$ be a nonempty compact from $H$ and $(C(\mathbb{R}\times W,H),\mathbb{R},\sigma)$ be a shift dynamical system on $C(\mathbb{R}\times W,H)$. Recall, that $C(\mathbb{R}\times W,H)$ is topologically isomorphic to $C(\mathbb{R},$

$C(W,H))$ and the shift dynamical systems $(C(\mathbb{R}\times W,H),\mathbb{R},\sigma)$ and $(C(\mathbb{R},C(W,H)),\mathbb{R},\sigma)$ are dynamically isomorphic.

Let $K$ be a convex set of $H$.

The direction $n\in H$ is called normal to $K$ at the point $x\in K$, if $\langle n,u-x\rangle\le 0$ for all $u\in K$. The set of all normal directions is called normal cone to $K$ at $x$ and is denoted by $N(K,x)$.

Let $K\subset H$ be nonempty, compact, convex subset of $H$ and $f\in C(\mathbb{R}\times K,H)$. We formulate the following assumptions:

(C1) $f$ is almost automorphic in $t$ uniformly for $x\in K$, i.e. the motion $\sigma(t,f)$ is almost automorphic in the shift dynamical system $(C(\mathbb{R}\times K,H),\mathbb{R},\sigma)$;

(C2) the function $f$ is monotone in $x\in K$ uniformly for $t\in\mathbb{R}$, i.e. $\langle f(t,x_1)-f(t,x_2),x_1-x_2\rangle\ge 0$ for all $x_1,x_2\in K$ and $t\in\mathbb{R}$;

(C3) there exists $t_0\in\mathbb{R}$ such that $\langle f(t_0,x_1)-f(t_0,x_2),x_1-x_2\rangle>0$ for all $x_1,x_2\in K$, such that $x_1\ne x_2$;

(C4) $\langle f(t,x),n\rangle\ge 0$ for each $x\in\partial K$, $n\in N(K,x)$ and $t\in\mathbb{R}$.

LEMMA 4.1. *Let $W\subset H$ be a nonempty compact. The function $f\in C(\mathbb{R}\times W,H)$ is almost automorphic in $t\in\mathbb{R}$ uniformly for $x\in W$, if and only if the following conditions hold:*

(i) *the function $f$ is bounded, i.e. there exists a constant $C\ge 0$ such that $|f(t,x)|\le C$ for all $(t,x)\in\mathbb{R}\times W$;*

(ii) *the function $f$ is uniformly continuous on $\mathbb{R}\times W$;*

(iii) *the function $f$ is Levitan almost periodic in $t\in\mathbb{R}$ uniformly for $x\in W$.*

THEOREM 4.2 ([5]). *Let $f\in C(\mathbb{R}\times K,H)$ be a bounded on $\mathbb{R}\times K$ function. Then the following statements hold:*

(i) *if the assumption (C4) is fulfilled, then the equation*

$$x''=f(t,x) \tag{4.1}$$

*has at lest one bounded on $\mathbb{R}$ solution;*

(ii) *if the assumptions* (C2) *and* (C4) *are fulfilled and equation* (4.1) *has two solutions* $\varphi_1$ *and* $\varphi_2$ *defined on* $\mathbb{R}$ *with their values in* $K$, *then* $\varphi_1(t) - \varphi_2(t) = costant$ *for all* $t \in \mathbb{R}$;

(iii) *if, in addition, the condition* (C3) *is fulfilled, then* (4.1) *has a unique solution defined and bounded on* $\mathbb{R}$.

Denote $X_0 := \{(\varphi, f) \mid \varphi \in C(\mathbb{R}, H),\ f \in C(\mathbb{R} \times H, H)$, and let $\varphi$ be a solution of equation (4.1)$\}$.

Lemma 4.3. *The set* $X_0$ *is invariant and closed in the product dynamical system* $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H), \mathbb{R}, \sigma)$.

Corollary 4.4.

1. $X_0$ *is a complete metric subspace of the product space* $C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H)$.
2. *On the space* $X_0$ *there is defined a shift dynamical system, induced by the product dynamical system* $(C(\mathbb{R}, H) \times C(\mathbb{R} \times H, H), \mathbb{R}, \sigma)$.

Theorem 4.5. *Let the assumptions* (C1), (C2) *and* (C4) *be fulfilled. Then the following statements hold:*

(i) *equation* (4.1) *admits at least one almost automorphic solution;*

(ii) *if the equation* (4.1) *has two solutions* $\varphi_1$ *and* $\varphi$ *defined on* $\mathbb{R}$ *with their values in* $K$, *then* $\varphi_1(t) - \varphi_2(t) = costant$ *for all* $t \in \mathbb{R}$;

(iii) *if, in addition, we assume that* (C3) *is fulfilled, then equation* (4.1) *has a unique almost automorphic solution.*

*Proof.* According to Lemma 4.1 and Theorem 4.2, to prove this theorem it is sufficient to show that equation (4.1), under the conditions of the theorem, admits at least one almost automorphic solution. Let $\varphi$ be a bounded on $\mathbb{R}$ solution of equation (4.1). By Landau's inequality, we have

$$\sup_{t \in \mathbb{R}} |\varphi'(t)| \le 2\sqrt{\sup_{t \in \mathbb{R}} |\varphi''(t)|}\sqrt{\sup_{t \in \mathbb{R}} |\varphi(t)|}$$

and, consequently, $|\varphi'(t)| \le 2ab$ for all $t \in \mathbb{R}$, where

$$a := \sup_{t\in\mathbb{R}} |f(t,\varphi(t))| \le \sup_{t\in\mathbb{R},\ x\in W} |f(t,x)| \text{ and } b := \sup_{t\in\mathbb{R}} |\varphi(t)|.$$

Thus, the function $\varphi \in C(\mathbb{R}, H)$ is bounded and uniformly continuous on $\mathbb{R}$ and by Theorem 7 [9, p.37] the motion $\sigma(t,\varphi)$ is stable in the sense of Lagrange in the shift dynamical system $(C(\mathbb{R},H),\mathbb{R},\sigma)$. Let us consider a non-autonomous dynamical system $\langle (X,\mathbb{R},\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $Y := H(\tilde{f})$ ($\tilde{f}$ is the restriction on $\mathbb{R}\times W$ of $f$, where $W := \overline{\varphi(\mathbb{R})}$ ) and $(Y,\mathbb{R},\sigma)$ is the shift dynamical system on $H(\tilde{f})$ induced by $(C(\mathbb{R}\times H,H),\mathbb{R},\sigma)$, $X := H(\varphi,\tilde{f}) \subset X_0$ and $(X,\mathbb{R},\pi)$ is the shift dynamical system induced by $(X_0,\mathbb{R},\sigma)$ and $h := pr_2 : X \to Y$ is the second projection. Now we will prove that $\mathfrak{N}_{\tilde{f}} \subset \mathfrak{N}_{(\varphi,\tilde{f})}$. In fact, let $\{t_n\} \in \mathfrak{N}_{\tilde{f}}$. Then $\{\tilde{f}_{t_n}\} \to \tilde{f}$ in the space $C(\mathbb{R}\times W,H)$ $(\tilde{f}_\tau := \sigma(\tau,\tilde{f}))$. Since $\varphi \in C(\mathbb{R},H)$ is stable in the sense of Lagrange, then $H(\varphi) := \overline{\{\varphi_\tau \mid \tau\in\mathbb{R}\}}$ is a compact invariant set and the sequence $\{\varphi_{t_n}\}$ is relatively compact. Let $\{t'_n\}$ be a subsequence of the sequence $\{t_n\}$, such that $\{\varphi_{t'_n}\}$ converges and denote by $P(\varphi) := \lim\limits_{n\to+\infty} \varphi_{t'_n} \in H(\varphi)$. By Lemma 4.3, the function $P(\varphi)$ is a solution of equation (4.1) defined on $\mathbb{R}$. Since $P(\varphi)(\mathbb{R}) \subseteq W$, then by Theorem 4.2 there exists $c \in H$ such that

$$P(\varphi)(t) = \varphi(t) + c \tag{4.2}$$

for all $t \in \mathbb{R}$. From equality (4.2) we have $P^2(\varphi) = P(\varphi)+c = \varphi+2c,\ \ldots,\ P^k(\varphi) = \varphi+kc$ for all $k \in \mathbb{N}$. On the other hand, $\{P^k(\varphi)\} \subseteq H(\varphi)$ and taking into account the compactness of the set $H(\varphi)$ we obtain $c = 0$, i.e. $P(\varphi) = \varphi$. Thus the sequence $\{\varphi_{t_n}\}$ is relatively compact and it has a unique limit point $\varphi$. This means that the sequence $\{\varphi_{t_n}\}$ is convergent, and consequently, $\{t_n\} \in \mathfrak{N}_{(\varphi,\tilde{f})}$. But $\tilde{f}$, under the conditions of Theorem, is almost automorphic in $t \in \mathbb{R}$ uniformly for $x \in W$, and, hence, the function $(\varphi,\tilde{f})$ is also almost automorphic (and, in particular, the function $\varphi$ is too). □

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