

# INVARIANT MANIFOLDS, GLOBAL ATTRACTORS, ALMOST AUTOMORPHIC AND ALMOST PERIODIC SOLUTIONS OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

DAVID CHEBAN AND BJOERN SCHMALFUSS

ABSTRACT. The paper is devoted to the study of non-autonomous evolution equations: invariant manifolds, compact global attractors, almost periodic and almost automorphic solutions. We study this problem in the framework of general non-autonomous (cocycle) dynamical systems. First, we prove that under some conditions such systems admit an invariant continuous section (an invariant manifold). Then, we obtain the conditions for the existence of a compact global attractor and characterize its structure. Third, we derive a criterion for the existence of almost periodic and almost automorphic solutions of different classes of non-autonomous differential equations (both ODEs (in finite and infinite spaces) and PDEs).

## 1. INTRODUCTION

The aim of this paper is the study of the problem of existence an invariant manifold (continuous invariant section), almost periodic and almost automorphic solutions of non-autonomous dynamical systems using the fixed point method.

In section 2 we give some known facts about compact global attractors of dynamical systems.

Section 3 is dedicated to Banach contraction principle and its generalizations. Our main result in this section is Theorem 3.7, which establishes the relation between the Krasnosel'kii generalized contraction principle [21, 22] and fixed point theorems of Boyd and Wong [4] and Browder [7].

Let  $(X, \rho)$  be a complete metric space.

**Theorem 1.1.** *(Classic) Let  $f : X \mapsto X$  be a contraction (there exists  $\alpha \in (0, 1)$  such that  $\rho(f(x_1), f(x_2)) \leq \alpha\rho(x_1, x_2)$  for all  $x_1, x_2 \in X$ ), then there exists a unique attractive fixed point  $p \in X$  of the map  $f$ , i.e. there exists a unique point  $p \in X$  such that  $f(p) = p$  and  $f^n(x) \rightarrow p$  as  $n \rightarrow \infty$  uniformly w.r.t.  $x$  on the bounded subsets from  $X$ .*

Let  $W$  and  $\Omega$  be two complete metric spaces and denote by  $X := W \times \Omega$  its Cartesian product. Recall that a continuous map  $F : X \rightarrow X$  is called triangular, if there are

---

*Date:* August 19, 2008.

*1991 Mathematics Subject Classification.* 37B25, 37B55, 39A11, 39C10, 39C55.

*Key words and phrases.* Global attractor; non-autonomous dynamical system; almost periodic solutions; invariant manifolds.

two continuous maps  $f : W \times \Omega \rightarrow W$  and  $g : \Omega \rightarrow \Omega$  such that  $F = (f, g)$ , i.e.  $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$  for all  $x =: (u, \omega) \in X$ .

**Theorem 1.2.** (*Hirsh M. W., Pugh C. C. and Shub M. - 1977, Non-autonomous version of Theorem 1.1*) Let  $\Omega$  be a compact metric space,  $X := W \times \Omega$  and  $F : X \mapsto X$  be a contraction in the first variable (i.e.  $k := \sup\{Lip(F(\cdot, \omega)) \mid \omega \in \Omega\} < 1$ ), then

- (i) there exists a unique continuous function  $\gamma : \Omega \mapsto W$  such that the graph of  $\gamma$  is  $F$  invariant;
- (ii)  $\rho(F^n(u, \omega), \gamma(g^n(\omega))) \leq k^n \rho(u, \gamma(\omega))$  for all  $u \in W, \omega \in \Omega$  and  $n \in \mathbb{Z}_+$ .

In section 4 we prove (Theorem 4.4 - the main result of paper) that under some conditions a non-autonomous dynamical system admits an invariant continuous section (an invariant manifold). This result generalizes the theorem 1.2 (first item) and the main result from [12].

Section 5 is dedicated to the study of compact global attractors of non-autonomous dynamical systems. The main result of this section is Theorem 5.3, which give the sufficient conditions of global asymptotically stability of continuous invariant section from Theorem 4.4. Thus the theorem 5.3 give the sufficient conditions of existence a compact global attractor for general non-autonomous dynamical systems.

Using the general results from Sections 3–5 we study the problem of existence the almost periodic and almost automorphic motions of non-autonomous dynamical systems in Section 6 and also we give the sufficient conditions of its global asymptotic stability.

In Section 6 are presented the application of results, obtained in Sections 3–6, to the study of the almost periodic and almost automorphis solutions of different classes of non-stationary evolution equations (Finite-dimensional ODEs, Carateadory's equations, ODEs with impulses, Infinite-dimensional systems (ODEs in Banach spaces, Evolution equations with monotone operators)).

## 2. COMPACT GLOBAL ATTRACTORS OF DYNAMICAL SYSTEMS

Let  $X$  be a topological space,  $\mathbb{R}$  ( $\mathbb{Z}$ ) be a group of real (integer) numbers,  $\mathbb{R}_+$  ( $\mathbb{Z}_+$ ) be a semi-group of the nonnegative real (integer) numbers,  $\mathbb{S}$  be one of the two sets  $\mathbb{R}$  or  $\mathbb{Z}$  and  $\mathbb{T} \subseteq \mathbb{S}$  ( $\mathbb{S}_+ \subseteq \mathbb{T}$ ) be a sub-semigroup of additive group  $\mathbb{S}$ .

Triplet  $(X, \mathbb{T}, \pi)$ , where  $\pi : \mathbb{T} \times X \rightarrow X$  is a continuous mapping satisfying the following conditions:

- (1)  $\pi(0, x) = x$ ;
- (2)  $\pi(s, \pi(t, x)) = \pi(s + t, x)$ ;

is called a dynamical system. If  $\mathbb{T} = \mathbb{R}$  ( $\mathbb{R}_+$ ) or  $\mathbb{Z}$  ( $\mathbb{Z}_+$ ), then the dynamical system  $(X, \mathbb{T}, \pi)$  is called a group (semi-group). In the case, when  $\mathbb{T} = \mathbb{R}_+$  or  $\mathbb{R}$  the dynamical system  $(X, \mathbb{T}, \pi)$  is called a flow, but if  $\mathbb{T} \subseteq \mathbb{Z}$ , then  $(X, \mathbb{T}, \pi)$  is called a cascade (discrete flow).

Sometimes, briefly, we will write  $xt$  instead of  $\pi(t, x)$ .

Below  $X$  will be a complete metric space with metric  $\rho$ .

A nonempty set  $M \subseteq X$  is called positively invariant (negatively invariant, invariant) with respect to dynamical system  $(X, \mathbb{T}, \pi)$  or, simple, positively invariant (negatively invariant, invariant), if  $\pi(t, M) \subseteq M$  ( $M \supseteq \pi(t, M)$ ,  $\pi(t, M) = M$ ) for every  $t \in \mathbb{T}$ .

A closed positively invariant set, which does not contain own closed positively invariant subset, is called minimal.

It easy to see that every positively invariant minimal set is invariant.

Let  $M \subseteq X$ . The set

$$\omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}$$

is called  $\omega$ -limit for  $M$ .

Let  $\mathfrak{M}$  be some family of subsets from  $X$ .

Dynamical system  $(X, \mathbb{T}, \pi)$  will be called  $\mathfrak{M}$ -dissipative if for every  $\varepsilon > 0$  and  $M \in \mathfrak{M}$  there exists  $L(\varepsilon, M) > 0$  such that  $\pi^t M \subseteq B(K, \varepsilon)$  for any  $t \geq L(\varepsilon, M)$ , where  $K$  is a certain fixed subset from  $X$  depending only on  $\mathfrak{M}$ . In this case  $K$  we will call the attractor for  $\mathfrak{M}$ .

For the applications the most important ones are the cases when  $K$  is bounded or compact and  $\mathfrak{M} = \{\{x\} \mid x \in X\}$  or  $\mathfrak{M} = C(X)$ , or  $\mathfrak{M} = \{B(x, \delta_x) \mid x \in X, \delta_x > 0\}$ , or  $\mathfrak{M} = B(X)$ .

The system  $(X, \mathbb{T}, \pi)$  is called:

- (i) point dissipative if there exist  $K \subseteq X$  such that for every  $x \in X$

$$(3) \quad \lim_{t \rightarrow +\infty} \rho(xt, K) = 0;$$

- (ii) compact dissipative if the equality (3) takes place uniformly w.r.t.  $x$  on the compacts from  $X$ ;  
 (iii) bounded dissipative if the equality (3) takes place uniformly w.r.t.  $x$  on every bounded subset from  $X$ .

Let  $(X, \mathbb{T}, \pi)$  be compactly dissipative and  $K$  be a compact set attracting every compact subset from  $X$ . Let us set

$$(4) \quad J := \omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, K)}.$$

It can be shown [9] that the set  $J$  defined by equality (4) doesn't depends on the choice of the attractor  $K$ , but is characterized only by the properties of the dynamical system  $(X, \mathbb{T}, \pi)$  itself. The set  $J$  is called a Levinson center of the compact dissipative dynamical system  $(X, \mathbb{T}, \pi)$ .

### 3. CONTRACTION PRINCIPLE AND ITS GENERALIZATIONS

This section is dedicated to Banach contraction principle and its generalizations. Our main result in this section is Theorem 3.7, which establishes the relation between the Krasnosel'kii generalized contraction principle [21, 22] and fixed point theorems of Boyd and Wong [4] and Browder [7].

**3.1. Classical Contraction Principle.** Let  $(X, \rho)$  be a complete metric space,  $\tau$  be a topology generated by  $\rho$ . Two metrics  $\rho_1$  and  $\rho_2$  are equivalent (respectively, topologically equivalent) if every fundamental sequence with respect to  $\rho_1$  is fundamental and with respect to  $\rho_2$  and vice versa (respectively, if the metrics  $\rho_1$  and  $\rho_2$  generate the same topology  $\tau$ ).

A mapping  $f : X \mapsto X$  is called  $\lambda$ -contracting ( $\lambda$ -contraction) if  $\rho(f(x_1), f(x_2)) \leq \lambda \rho(x_1, x_2)$  for all  $x_1, x_2 \in X$ , where  $\lambda \in (0, 1)$ .

A mapping  $f : X \mapsto X$  is called contracting (a contraction), if for every  $\lambda \in (0, 1)$  there exists a metric  $\rho_\lambda$  equivalent to initial metric  $\rho$  such that  $\rho_\lambda(f(x_1), f(x_2)) \leq \lambda \rho_\lambda(x_1, x_2)$  for all  $x_1, x_2 \in X$ .

It is well known the following result

**Theorem 3.1.** [27, 28] *The mapping  $f : X \mapsto X$  is a  $\lambda$ -contraction if and only the following conditions hold:*

- (F1) *the mapping  $f : X \mapsto X$  admits a unique fixed point, i.e. there exists a unique point  $p \in X$  such that  $f(p) = p$ ;*
- (F2) *the fixed point  $p \in X$  is globally attracting, i.e.*

$$\lim_{n \rightarrow +\infty} f^n(x) = p$$

*for every  $x \in X$ , where  $f^0 := Id_X$  and  $f^n := f^{n-1} \circ f$  for all  $n \in \mathbb{N}$ ;*

- (F3) *the fixed point  $p \in X$  is uniformly attracting, i.e. there exists  $\delta_0 > 0$  such that the equality (5) takes place uniformly with respect to  $x \in B(p, \delta_0) := \{x \in X \mid \rho(x, p) < \delta_0\}$ .*

**Remark 3.2.** *If the fixed point is uniformly attracting, then it is Lyapunov stable, i.e. for all  $\varepsilon > 0$  there exists a positive number  $\delta = \delta(\varepsilon)$  such that  $\rho(x, p) < \delta$  implies  $\rho(f^n(x), p) < \varepsilon$  for all  $n \in \mathbb{Z}_+$  (see, for example, [9, Ch.1]).*

**3.2. M. A. Krasnosel'skii's Generalized Contraction Principle.** The mapping  $f : X \mapsto X$  is called [28] a generalized contraction (in the sense of Krasnosel'skii M. A.), if for any  $\beta \geq \alpha > 0$  there exists a number  $\lambda(\alpha, \beta) \in (0, 1)$  such that  $\rho(f(x_1), f(x_2)) \leq \lambda(\alpha, \beta) \rho(x_1, x_2)$  for all  $(x_1, x_2) \in X^2(\alpha, \beta) := \{(x_1, x_2) \in X \times X \mid \alpha \leq \rho(x_1, x_2) \leq \beta\}$ .

**Theorem 3.3.** (Krasnosel'skii M. A. [22, 28]) *Let  $f$  be a generalized contracting mapping from the complete metric space  $X$  onto itself, then  $f$  possesses the properties (F1) – (F3).*

**Remark 3.4.** [28] *Let  $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function satisfying the following conditions:*

- (i)  $\varphi(t) > 0$  for all  $t > 0$ ;
- (ii)  $\varphi(t) \leq t$  for all  $t \in \mathbb{R}_+$ .

The mapping  $f$  is a generalized contraction if, for example,  $\rho(f(x_1), f(x_2)) \leq \rho(x_1, x_2) - \varphi(\rho(x_1, x_2))$  for all  $(x_1, x_2) \in X \times X$ .

Let  $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ . Denote by  $\varphi(t_0 + 0) := \lim_{t \rightarrow t_0, t > t_0} \varphi(t)$  (respectively,  $\varphi(t_0 - 0) := \lim_{t \rightarrow t_0, t < t_0} \varphi(t)$ ) if the last limit exists.

The mapping  $\varphi$  is called upper semi-continuous from the right at the point  $t_0 \in \mathbb{R}_+$ , if there exists  $\limsup_{t \rightarrow t_0, t > t_0} \varphi(t) \leq \varphi(t_0)$ .

**Lemma 3.5.** [14] *Let  $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$  and the following conditions be fulfilled:*

- (G1)  $\varphi(t) < t$  for all  $t > 0$ ;
- (G2)  $\varphi$  is monotone increasing, i.e.  $t_1 \leq t_2$  implies  $\varphi(t_1) \leq \varphi(t_2)$ ;
- (G3)  $\varphi$  is right continuous on  $\mathbb{R}_+$ , i.e.  $\varphi(t_0 + 0) = \varphi(t_0)$  for all  $t_0 \in \mathbb{R}_+$ .

Then for any  $\beta \geq \alpha > 0$  there exists  $\lambda(\alpha, \beta) := \max\{\varphi(t)/t \mid \alpha \leq t \leq \beta\}$  and  $\lambda(\alpha, \beta) \in [0, 1)$ .

The mapping  $f : X \mapsto X$  is called a  $\varphi$ -contraction, if  $\rho(f(x_1), f(x_2)) \leq \varphi(\rho(x_1, x_2))$  for all  $x_1, x_2 \in X$ , where  $\varphi$  is some mapping from  $\mathbb{R}_+$  to itself.

**Theorem 3.6.** [4, 7, 19] *Let  $f : X \mapsto X$  be a  $\varphi$ -contraction. Suppose that the mapping  $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$  satisfies the condition (G1)-(G3). Then  $f$  has a unique fixed point  $x_0$  and  $\lim_{n \rightarrow \infty} f^n(x) = x_0$  for all  $x \in X$ .*

**Theorem 3.7.** *Let  $f : X \mapsto X$  be a  $\varphi$ -contraction. Suppose that the mapping  $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$  satisfies the condition (G1)-(G3).*

*Then  $f$  possesses the properties (F1) – (F3).*

*Proof.* Let  $f$  be a  $\varphi$ -contraction. If the mapping  $\varphi$  satisfies the conditions (G1)-(G3), then by Lemma 3.5 for all  $\beta \geq \alpha > 0$  there exists  $\lambda(\alpha, \beta) \in (0, 1)$  such that  $\varphi(t) \leq \lambda(\alpha, \beta)t$  for all  $t \in [\alpha, \beta]$ . Let now  $(x_1, x_2) \in X \times X$  such that  $\alpha \leq \rho(x_1, x_2) \leq \beta$ . Then we have

$$\rho(f(x_1), f(x_2)) \leq \varphi(\rho(x_1, x_2)) \leq \lambda(\alpha, \beta)\rho(x_1, x_2).$$

To finish the proof it is sufficient to apply the Krasnosel'kii Theorem 3.3.  $\square$

#### 4. INVARIANT SECTIONS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

Below prove that under some conditions a non-autonomous dynamical system admits an invariant continuous section (an invariant manifold). This result generalizes the theorem 1.2 (first item) and the main result from [12].

Let  $(X, \mathbb{T}_1, \pi)$  and  $(Y, \mathbb{T}_2, \sigma)$  ( $\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ ) be two dynamical systems. A mapping  $h : X \rightarrow Y$  is called a homomorphism (isomorphism, respectively) of the dynamical system  $(X, \mathbb{T}_1, \pi)$  on  $(Y, \mathbb{T}_2, \sigma)$ , if the mapping  $h$  is continuous

(homeomorphic, respectively) and  $h(\pi(x, t)) = \sigma(h(x), t)$  ( $t \in \mathbb{T}_1$ ,  $x \in X$ ). In this case the dynamical system  $(X, \mathbb{T}_1, \pi)$  is an extension of the dynamical system  $(Y, \mathbb{T}_2, \sigma)$  by the homomorphism  $h$ , but the dynamical system  $(Y, \mathbb{T}_2, \sigma)$  is called a factor of the dynamical system  $(X, \mathbb{T}_1, \pi)$  by the homomorphism  $h$ . The dynamical system  $(Y, \mathbb{T}_2, \sigma)$  is called also a base of the extension  $(X, \mathbb{T}_1, \pi)$ .

A triplet  $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ , where  $h$  is a homomorphism from  $(X, \mathbb{T}_1, \pi)$  on  $(Y, \mathbb{T}_2, \sigma)$  and  $(X, h, Y)$  is a fiber space, is called a non-autonomous dynamical system.

A triplet  $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$  (or shortly  $\varphi$ ), where  $(Y, \mathbb{T}_2, \sigma)$  is a dynamical system on  $Y$ ,  $W$  is a complete metric space and  $\varphi$  is a continuous mapping from  $\mathbb{T}_1 \times W \times Y$  in  $W$ , possessing the following conditions:

- a.  $\varphi(0, u, y) = u$  ( $u \in W, y \in Y$ );
- b.  $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$  ( $t, \tau \in \mathbb{T}_1, u \in W, y \in Y$ ),

is called [30] a cocycle on  $(Y, \mathbb{T}_2, \sigma)$  with the fiber  $W$ .

Let  $X := W \times Y$  and define a mapping  $\pi : X \times \mathbb{T}_1 \rightarrow X$  as following:  $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$  (i.e.  $\pi = (\varphi, \sigma)$ ). Then it is easy to see that  $(X, \mathbb{T}_1, \pi)$  is a dynamical system on  $X$  which is called a skew-product dynamical system [30] and  $h = pr_2 : X \rightarrow Y$  is a homomorphism from  $(X, \mathbb{T}_1, \pi)$  on  $(Y, \mathbb{T}_2, \sigma)$  and, consequently,  $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$  is a non-autonomous dynamical system.

Thus, if we have a cocycle  $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$  on the dynamical system  $(Y, \mathbb{T}_2, \sigma)$  with the fiber  $W$ , then it generates a non-autonomous dynamical system  $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$  ( $X := W \times Y$ ) called a non-autonomous dynamical system generated by the cocycle  $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$  on  $(Y, \mathbb{T}_2, \sigma)$ .

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give an example of this.

**Example 4.1.** Let  $E$  be a finite-dimensional real or complex Euclidean space. Let us consider a differential equation

$$(5) \quad u' = f(t, u),$$

where  $f \in C(\mathbb{R} \times E, E)$ . Along with equation (5) we consider its  $H$ -class [6],[24], [30], [31], i.e. the family of equations

$$(6) \quad v' = g(t, v),$$

where  $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ ,  $f_\tau(t, u) = f(t + \tau, u)$  for all  $(t, u) \in \mathbb{R} \times E$  and by bar we denote the closure in  $C(\mathbb{R} \times E, E)$ . We will suppose also that the function  $f$  is regular, i.e. for every equation (6) the conditions of the existence, uniqueness and extendability on  $\mathbb{R}_+$  are fulfilled. Denote by  $\varphi(\cdot, v, g)$  the solution of equation (6) passing through the point  $v \in E$  at the initial moment  $t = 0$ . Then there is a correctly defined mapping  $\varphi : \mathbb{R}_+ \times E \times H(f) \rightarrow E$  satisfying the following conditions (see, for example, [6], [30]):

- 1)  $\varphi(0, v, g) = v$  for all  $v \in E$  and  $g \in H(f)$ ;

- 2)  $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$  for every  $v \in E$ ,  $g \in H(f)$  and  $t, \tau \in \mathbb{R}_+$ ;
- 3) the mapping  $\varphi : \mathbb{R}_+ \times E \times H(f) \rightarrow E$  is continuous.

Denote by  $Y := H(f)$  and  $(Y, \mathbb{R}, \sigma)$  a dynamical system of translations (a semigroup system) on  $Y$ , induced by the dynamical system of translations  $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ . The triplet  $\langle E, \varphi, (Y, \mathbb{R}, \sigma) \rangle$  is a cocycle on  $(Y, \mathbb{R}, \sigma)$  with the fiber  $E$ . Thus, equation (5) generates a cocycle  $\langle E, \varphi, (Y, \mathbb{R}, \sigma) \rangle$  and a non-autonomous dynamical system  $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ , where  $X := E^n \times Y$ ,  $\pi := (\varphi, \sigma)$  and  $h := pr_2 : X \rightarrow Y$ .

Let  $(Y, \mathbb{S}, \sigma)$  be a two-sided dynamical system,  $(X, \mathbb{S}_+, \pi)$  be a semi-group dynamical system and  $h : X \rightarrow Y$  be a homomorphism of  $(X, \mathbb{S}_+, \pi)$  onto  $(Y, \mathbb{S}, \sigma)$ .

Let  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  be a non-autonomous dynamical system. Recall that a mapping  $\gamma : Y \mapsto X$  is called a section (selector) of the homomorphism  $h$ , if  $h(\gamma(y)) = y$  for all  $y \in Y$ . The section  $\gamma$  of the homomorphism  $h$  is called invariant if  $\gamma(\sigma(t, y)) = \pi(t, \gamma(y))$  for all  $y \in Y$  and  $t \in \mathbb{S}_+$ .

**Remark 4.2.** *A continuous section  $\gamma \in \Gamma(Y, X)$  is invariant if and only if  $\gamma \in \Gamma(Y, X)$  is a stationary point of the semigroup  $\{S^t \mid t \in \mathbb{S}_+\}$ , where  $S^t : \Gamma(Y, X) \rightarrow \Gamma(Y, X)$  is defined by the equality  $(S^t \gamma)(y) := \pi(t, \gamma(\sigma(-t, y)))$  for all  $y \in Y$  and  $t \in \mathbb{S}_+$ .*

We consider a special case of the foregoing construction. Let  $\langle W, \varphi, (Y, \mathbb{S}, \sigma) \rangle$  be a cocycle over  $(Y, \mathbb{S}, \sigma)$  with the fiber  $W$  and  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  be the non-autonomous dynamical system generated by this cocycle. Then  $h \circ \gamma = Id_Y$  and since  $h = pr_2$ , then  $\gamma = (\psi, Id_Y)$ , where  $\gamma \in \Gamma(Y, X)$  and  $\psi : Y \rightarrow W$ . Hence, to each section  $\gamma$  there corresponds a mapping  $\psi : Y \rightarrow W$  and conversely. There being a one-to-one relation between  $\Gamma(Y, W \times Y)$  and  $C(Y, W)$ , where  $C(Y, W)$  is the space of continuous functions  $\psi : Y \rightarrow W$ , we identify these two objects from now on. The semi-group  $\{S^t \mid t \in \mathbb{S}_+\}$  naturally induces a semi-group  $\{Q^t \mid t \in \mathbb{S}_+\}$  of the mappings of  $C(Y, W)$ . Namely,

$$\begin{aligned} (S^t \gamma)(y) &= \pi^t \gamma(\sigma^{-t} y) = \pi^t(\psi, Id_Y)(\sigma^{-t} y) = \\ \pi^t(\psi(\sigma^{-t} y), \sigma^{-t} y) &= (U(t, \sigma^{-t} y) \psi(\sigma^{-t} y), y) = ((Q^t \psi)(y), y), \end{aligned}$$

where  $U(t, y) := \varphi(t, \cdot, y)$ .

Hence,  $S^t(\psi, Id_Y) = (Q^t \psi, Id_Y)$  with  $(Q^t \psi)(y) = U(t, \sigma^{-t} y) \psi(\sigma^{-t} y)$  ( $y \in Y$ ). We have the following properties:

- a.  $Q^0 = Id_{C(Y, W)}$ ;
- b.  $Q^t Q^\tau = Q^{t+\tau}$  ( $t, \tau \in \mathbb{S}_+$ ).

Let  $Y$  be a compact metric space. Consider a non-autonomous dynamical system  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  and denote by  $\Gamma(Y, X)$  the family of all continuous sections of the homomorphism  $h$ . By equality

$$(7) \quad d(\varphi_1, \varphi_2) := \max_{y \in Y} \rho(\varphi_1(y), \varphi_2(y))$$

there is defined a metric on  $\Gamma(Y, X)$ .

Let  $X \dot{\times} X := \{(x_1, x_2) : x_1, x_2 \in X, h(x_1) = h(x_2)\}$  and let  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  be a mapping satisfying the following conditions:

- (C1)  $a(\rho(x_1, x_2)) \leq V(x_1, x_2) \leq b(\rho(x_1, x_2))$  for all  $(x_1, x_2) \in X \dot{\times} X$ , where  $a, b$  are two functions from  $\mathfrak{A}$  ( $\mathfrak{A} := \{a \mid a : \mathbb{R}_+ \rightarrow \mathbb{R}_+, a \text{ is continuous, strongly increasing and } a(0) = 0\}$ ) and  $Im(a) = Im(b)$ ;
- (C2)  $V(x_1, x_2) = V(x_2, x_1)$  for all  $(x_1, x_2) \in X \dot{\times} X$ ;
- (C3)  $V(x_1, x_2) \leq V(x_1, x_3) + V(x_3, x_1)$  for all  $x_1, x_2, x_3 \in X$  such that  $h(x_1) = h(x_2) = h(x_3)$ .

From the conditions (C1)–(C3) follows that the function  $V$  on each fiber  $X_y = h^{-1}(y)$  defines some metric which is topologically equivalent to  $\rho$ .

**Lemma 4.3.** [9] *Suppose that the function  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  satisfies the conditions (C1)–(C3). Then by equality*

$$p(\gamma_1, \gamma_2) := \max\{V(\gamma_1(y), \gamma_2(y)) \mid y \in Y\}$$

on  $\Gamma(Y, X)$  there is defined a complete metric that is topologically equivalent to (7).

Denote by  $S^t : \Gamma(Y, X) \rightarrow \Gamma(Y, X)$  the mapping, defined by the equality  $(S^t \gamma)(y) = \pi^t \gamma(\sigma^{-1} y)$  for all  $t \in \mathbb{S}_+$ ,  $\gamma \in \Gamma(Y, X)$  and  $y \in Y$ . It is easy to check that the family of mappings  $\{S^t\}_{t \geq 0}$  is a commutative semigroup.

**Theorem 4.4.** *If there is a function  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  satisfying Conditions (C1)–(C3) and*

- (C4)  $V(x_1 t, x_2 t) \leq \omega(t, V(x_1, x_2))$  ( $\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0$ ), where  $\omega$  is a mapping from  $\mathbb{T}_+ \times \mathbb{R}_+$  into  $\mathbb{R}_+$  satisfying the following condition:
  - (a)  $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone increasing for all  $t > 0$ ;
  - (b) there exists a positive number  $t_0 \in \mathbb{S}_+$  such that
    - (i)  $\omega(t_0, r) < r$  for all  $r > 0$ ;
    - (ii) the mapping  $\omega(t_0, \cdot)$  is right continuous on  $\mathbb{R}_+$ .

Then the semigroup  $\{S^t\}_{t \geq 0}$  has a unique fixed point  $\gamma \in \Gamma(Y, X)$  which is an invariant section of  $h$ .

*Proof.* Note that

$$\begin{aligned} p(S^t \gamma_1, S^t \gamma_2) &= \sup\{V(\pi^t \gamma_1(\sigma^{-t} y), \pi^t \gamma_2(\sigma^{-t} y)) \mid y \in Y\} \\ &\leq \sup\{\omega(t, V(\gamma_1(\sigma^{-t} y), \gamma_2(\sigma^{-t} y))) \mid y \in Y\} \leq \omega(t, p(\gamma_1, \gamma_2)) \end{aligned}$$

for all  $t \in \mathbb{S}_+$  and  $\gamma_1, \gamma_2 \in \Gamma(Y, X)$ .

Thus for  $t_0 > 0$  we have  $p(S^{t_0} \gamma_1, S^{t_0} \gamma_2) \leq \omega(t_0, p(\gamma_1, \gamma_2))$  for all  $\gamma_1, \gamma_2 \in \Gamma(Y, X)$  and the mapping  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  possesses the properties (G1)–(G3). By Theorem 3.6 we have the unique fixed point  $\gamma$  for the mapping  $S^{t_0}$ . From this fact and taking into consideration that  $\{S^t\}_{t \geq 0}$  is commutative we obtain that  $\gamma$  is a unique fixed point of the semigroup  $\{S^t\}_{t \geq 0}$ . This means that  $\gamma$  is a unique invariant continuous section of  $h$ . Theorem is proved.  $\square$

**Corollary 4.5.** *If there is a function  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  satisfying Conditions (C1)–(C3) and*

- (C4.a)  $V(x_1 t, x_2 t) \leq N e^{-\nu t} V(x_1, x_2)$  ( $\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0$ ), where  $N$  and  $\nu$  are two positive numbers.



Then the semigroup  $\{S^t\}_{t \geq 0}$  has a unique fixed point  $\gamma \in \Gamma(Y, X)$  which is an invariant section of  $h$ .

*Proof.* This statement follows from Theorem 4.4. In fact, in this case it is sufficient to note that the mapping  $\omega(t, r) := \mathcal{N}e^{-\nu t}r$  possesses the properties (G1)-(G3) for all  $t > t_0 := \nu^{-1} \ln \mathcal{N}$ .  $\square$

**Corollary 4.6.** *If there is a function  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  satisfying Conditions (C1)-(C3) and*

$$(C4.b) \quad V(x_1 t, x_2 t) \leq \frac{V(x_1, x_2)}{(1 + (\alpha - 1)tV(x_1, x_2)^{\alpha - 1})^{1/(\alpha - 1)}} \quad (\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0),$$

where  $\alpha > 1$ .

Then the semigroup  $\{S^t\}_{t \geq 0}$  has a unique fixed point  $\gamma \in \Gamma(Y, X)$  which is an invariant section of  $h$ .

*Proof.* To prove this statement it is sufficient to note that the mapping  $\omega(t, r) := \frac{r}{(1 + (\alpha - 1)tr^{\alpha - 1})^{1/(\alpha - 1)}}$  possesses the properties (G1)-(G3) for all  $t > 0$ .  $\square$

**Lemma 4.7.** *Let  $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a function satisfying the following conditions:*

- (H1)  $f(0) = 0$ ;
- (H2)  $f(t) > 0$  for all  $t > 0$ ;
- (H3)  $f$  is locally Lipschitz;
- (H4)  $f$  satisfy the condition of Osgud, i.e. for all  $\varepsilon > 0 \int_0^\varepsilon \frac{dt}{f(s)} = +\infty$ .

Then the equation

$$(8) \quad u' = -f(u)$$

admits a unique solution  $\omega(t, r)$  with initial condition  $\omega(0, r) = r$  and the mapping  $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$  possesses the following properties:

- (i) the mapping  $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$  is continuous;
- (ii)  $\omega(t, r) < r$  for all  $r > 0$  and  $t > 0$ ;
- (iii) for all  $t \in \mathbb{R}_+$  the mapping  $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is increasing;
- (iv)  $\omega(0, t) = 0$  for all  $t \in \mathbb{R}_+$ ;
- (v)  $\lim_{t \rightarrow +\infty} \sup_{0 \leq r \leq r_0} \omega(t, r) = 0$  for all  $r_0 > 0$ .

*Proof.* This statement directly follows from the general properties of autonomous scalar differential equations (see, for example, [17, 20]).  $\square$

**Remark 4.8.** 1. *The local condition of Lipschitz in Lemma 4.7 guarantees the uniqueness of Cauchy problem for the equation (8). It may be replaced by more general condition (see, for example, [21]).*

2. *Lemma 4.7 gives us an algorithm of construction of function with properties (G1)-(G3). For example, the function  $\omega(t, r) := \ln \frac{e^{t+r}}{e^{t+r} + 1 - e^r}$  ( $t, r \in \mathbb{R}_+$ ) possesses the mentioned above properties, because it is a unique solution of differential equation (8) ( $f(u) = e^u - 1$  for all  $u \in \mathbb{R}_+$ ) with initial condition  $\omega(0, r) = r$ .*

3. The function  $\omega(t, r) := \frac{r}{(1+(\alpha-1)tr^{\alpha-1})^{1/(\alpha-1)}}$  is the solution of equation (8) ( $f(u) = u^\alpha$  for all  $u \in \mathbb{R}_+$  and  $\alpha > 1$ ) with initial condition  $\omega(0, r) = r$ .

4. If the function  $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$  satisfies the conditions (H1)-(H4), then the function  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ , defined by equality:  $g(t) := af(bt)$  for all  $t \in \mathbb{R}_+$  ( $a$  and  $b$  are some positive numbers), possesses the same property.

**Theorem 4.9.** Let  $\langle W, \varphi, (Y, \mathbb{S}, \sigma) \rangle$  be a cocycle over dynamical system  $(Y, \mathbb{S}, \sigma)$  with fiber  $W$ . If there is a function  $\mathcal{V} : W \times W \times Y \mapsto \mathbb{R}_+$  satisfying the following conditions:

- (i)  $a(\rho(u_1, u_2)) \leq \mathcal{V}(u_1, u_2, y) \leq b(\rho(u_1, u_2))$  for all  $(u_1, u_2) \in W \times W$  and  $y \in Y$ , where  $a, b$  are two functions from  $\mathfrak{A}$ ;
- (ii)  $\mathcal{V}(u_1, u_2, y) = \mathcal{V}(u_2, u_1, y)$  for all  $(u_1, u_2) \in W \times W$  and  $y \in Y$ ;
- (iii)  $\mathcal{V}(u_1, u_2, y) \leq \mathcal{V}(u_1, u_3, y) + \mathcal{V}(u_3, u_2, y)$  for all  $u_1, u_2, u_3 \in W$  and  $y \in Y$ .
- (iv)  $\mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y), \sigma(t, y)) \leq \omega(t, \mathcal{V}(u_1, u_2, y))$  ( $\forall (u_1, u_2) \in W \times W, y \in Y$  and  $t \geq 0$ ), where  $\omega$  is a mapping from  $\mathbb{T}_+ \times \mathbb{R}_+$  into  $\mathbb{R}_+$  satisfying the following condition:
  - (a)  $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone increasing for all  $t > 0$ ;
  - (b) there exists a positive number  $t_0 \in \mathbb{S}_+$  such that
    - (i)  $\omega(t_0, r) < r$  for all  $r > 0$ ;
    - (ii) the mapping  $\omega(t_0, \cdot)$  is right continuous on  $\mathbb{R}_+$ .

Then the semigroup  $\{Q^t\}_{t \geq 0}$  ( $(Q^t \mu)(y) := U(t, \sigma^{-t}y)\mu(\sigma^{-t}y)$  ( $y \in Y$ ), where  $U(t, y) := \varphi(t, \cdot, y)$ ) has a unique fixed point  $\mu \in C(Y, W)$  which is an invariant section of cocycle  $\varphi$ .

*Proof.* Let  $X := W \times Y$ ,  $\pi := (\varphi, \sigma)$  and  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  ( $h := pr_2 : X \mapsto Y$ ) be a non-autonomous dynamical system associated by cocycle  $\varphi$ . We define the mapping  $V : X \times X \mapsto \mathbb{R}_+$  by equality:  $V(x_1, x_2) := \mathcal{V}(u_1, u_2, y)$ , where  $x_i := (u_i, y)$ ,  $u_i \in W$  ( $i = 1, 2$ ) and  $y \in Y$ . Now to finish the proof of Theorem it is sufficient to apply Theorem 4.4 for non-autonomous dynamical system  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  because under the conditions of Theorem 4.9 all conditions of Theorem 4.4 will hold.  $\square$

**Corollary 4.10.** Let  $\langle W, \varphi, (Y, \mathbb{S}, \sigma) \rangle$  be a cocycle over dynamical system  $(Y, \mathbb{S}, \sigma)$  with fiber  $W$ . If  $\rho(\varphi(t, u_1, y), \varphi(t, u_2, y)) \leq \omega(t, \rho(u_1, u_2))$  ( $\forall (u_1, u_2) \in W \times W$  and  $t \geq 0$ ), where  $\omega$  is a mapping from  $\mathbb{T}_+ \times \mathbb{R}_+$  into  $\mathbb{R}_+$  satisfying the following condition: there exists a positive number  $t_0 \in \mathbb{S}_+$  such that

- (i)  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone decreasing;
- (ii)  $\omega(t_0, r) < r$  for all  $r > 0$ ;
- (iii) the mapping  $\omega(t_0, \cdot)$  is right continuous on  $\mathbb{R}_+$ .

Then the semi-group  $\{Q^t\}_{t \geq 0}$  ( $(Q^t \mu)(y) := U(t, \sigma^{-t}y)\mu(\sigma^{-t}y)$  ( $y \in Y$ ), where  $U(t, y) := \varphi(t, \cdot, y)$ ) has a unique fixed point  $\mu \in C(Y, W)$  which is an invariant section of cocycle  $\varphi$ .

*Proof.* This statement follows from Theorem 4.9. In fact, it is easy to verify that the mapping  $\mathcal{V} : W \times W \times Y \mapsto \mathbb{R}_+$  defined by equality  $\mathcal{V}(u_1, u_2, y) := \rho(u_1, u_2)$  (for all  $u_1, u_2 \in W$  and  $y \in Y$ ) satisfies all conditions of Theorem 4.9.  $\square$

## 5. GLOBAL ATTRACTORS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

In this section we study the compact global attractors of non-autonomous dynamical systems. The main result is Theorem 5.3, which give the sufficient conditions of global asymptotically stability of continuous invariant section from Theorem 4.4. Thus the theorem 5.3 give the sufficient conditions of existence a compact global attractor for general non-autonomous dynamical systems.

**Lemma 5.1.** *Let  $\omega : \mathbb{S}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and there exists a positive number  $t_0 \in \mathbb{S}_+$  such that*

- a.  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone increasing;
- b.  $\omega(t_0, r) < r$  for all  $r > 0$ ;
- c. the mapping  $\omega(t_0, \cdot)$  is continuous on  $\mathbb{R}_+$ ;
- d.  $\omega(t + \tau, r) \leq \omega(t, \omega(\tau, r))$  for all  $t, \tau \in \mathbb{S}_+$  and  $r \in \mathbb{R}_+$ ;
- e. for every  $r \in \mathbb{R}_+$  the mapping  $\omega(\cdot, r) : \mathbb{S}_+ \rightarrow \mathbb{R}_+$  is continuous.

Then the following statements hold:

- (i) the equality

$$(9) \quad \lim_{t \rightarrow +\infty} \omega(t, r) = 0$$

takes place for all  $r \in \mathbb{R}_+$ ;

- (ii) if the mapping  $\omega : \mathbb{S}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous, then the equality (9) holds uniformly with respect to  $r$  on every compact subset from  $\mathbb{R}_+$ .

*Proof.* Consider the sequence  $\{c_n(r)\}$ , where  $c_n(r) := \omega(nt_0, r)$  for all  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_+$ . According to property d. the sequence  $\{c_n(r)\}$  is decreasing (i.e.  $c_{n+1}(r) \leq c_n(r)$  for all  $n \in \mathbb{Z}_+$  and  $r \in \mathbb{R}_+$ ) and, consequently, it is convergent. Let  $k(r) := \lim_{n \rightarrow \infty} c_n(r)$  for all  $r \in \mathbb{R}_+$ .

Now we will prove that  $k(r) = 0$  for all  $r \in \mathbb{R}_+$ . From the condition b. and c. it follows that  $k(0) = 0$ . If we suppose that there exists  $r_0 > 0$  such that  $k(r_0) > 0$ , then

$$(10) \quad c_{n+1}(r_0) = \omega((n+1)t_0, r_0) \leq \omega(t_0, \omega(nt_0, r_0)) = \omega(t_0, c_n(r_0))$$

for all  $n \in \mathbb{N}$ . Passing into limit in the inequality (10) we obtain  $k(r_0) \leq \omega(t_0, k(r_0))$ . Since  $k(r_0) > 0$ , the last inequality contradicts to the condition b. The obtained contradiction proves our statement.

Let now  $t \in \mathbb{S}_+$  ( $t \geq t_0$ ),  $n_t \in \mathbb{N}$  and  $\tau_t \in [0, t_0)$  such that  $t = t_0 n_t + \tau_t$ , then

$$(11) \quad \omega(t, r) = \omega(t_0 n_t + \tau_t, r) \leq \omega(t_0 n_t, \omega(\tau_t, r)) \leq \omega(t_0 n_t, m(r)),$$

where  $m(r) := \max_{0 \leq \tau \leq t_0} \omega(\tau, r)$ . From the inequality (11) we obtain (9).

If the mapping  $\omega : \mathbb{T}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous, then we put

$$m_0(l) := \max_{0 \leq \tau \leq t_0, 0 \leq r \leq l} \omega(\tau, r)$$

and from the inequality

$$\omega(t, r) = \omega(t_0 n_t + \tau_t, r) \leq \omega(t_0 n_t, \omega(\tau_t, r)) \leq \omega(t_0 n_t, m_0(l))$$

we conclude that the equality (9) takes place uniformly with respect to  $r$  on interval  $[0, l]$  for all  $l > 0$ . Lemma is proved.  $\square$

**Remark 5.2.** *It easy to see that Lemma 5.1 is not true without condition d.*

**Theorem 5.3.** *Let  $\omega : \mathbb{S}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous function satisfying the condition*

$$\rho(x_1 t, x_2 t) \leq \omega(t, \rho(x_1, x_2)) \quad (\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0).$$

*Suppose that there exists a positive number  $t_0 \in \mathbb{S}_+$  such that*

- a.  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone increasing;
- b.  $\omega(t_0, r) < r$  for all  $r > 0$ .

*Then the following statements hold:*

- (i) *the semigroup  $\{S^t\}_{t \geq 0}$  has a unique fixed point  $\gamma \in \Gamma(Y, X)$  which is an invariant section of  $h$ ;*
- (ii) *the dynamical system  $(X, \mathbb{T}, \pi)$  admits a compact global attractor  $J$ ;*
- (iii)  *$J = \gamma(Y)$ .*

*Proof.* The first statement follows from the theorem 4.4. Denote by  $J := \gamma(Y)$ , then  $J$  is a compact invariant set of dynamical system  $(X, \mathbb{T}, \pi)$ . We will show that the set  $J$  attracts every bounded subset  $B$  from  $X$ . Since  $B$  is bounded and  $J = \gamma(Y)$  is compact, then  $B \subseteq \mathcal{B}(J, R) := \cup_{y \in Y} \{x \in X_y \mid \rho(x, \gamma(y)) \leq R\}$ , where  $X_y := h^{-1}(y) = \{x \in X \mid h(x) = y\}$ ,  $R := \sup_{y \in Y} R_y$ ,  $R_y := \text{diam}(B) + \rho(\gamma(y), B_y)$  and  $B_y := B \cap X_y$ . Let  $x \in B$  and  $y := h(x)$ , then we have

$$(12) \quad \rho(\pi(t, x), \pi(t, \gamma(y))) \leq \omega(t, \rho(x, \gamma(y))) \leq \omega(t, R_y) \leq \omega(t, R)$$

for all  $x \in B$ . From the inequality (12) and Lemma 5.1 follows that

$$\sup_{x \in B} \rho(\pi(t, x), \pi(t, \gamma(h(x)))) \leq \omega(t, R) \rightarrow 0$$

as  $t \rightarrow +\infty$  and, consequently,

$$\lim_{t \rightarrow +\infty} \beta(\pi(t, B), J) = 0$$

for all bounded subset  $B$  from  $X$ . The Theorem is proved.  $\square$

**Corollary 5.4.** *If*

- (i)  $\rho(x_1 t, x_2 t) \leq \frac{\rho(x_1, x_2)}{(1+(\alpha-1)t\rho(x_1, x_2)^{\alpha-1})^{1/(\alpha-1)}} \quad (\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0)$ , where  $\alpha > 1$ .

*Then the following statements hold:*

- (i) *the semigroup  $\{S^t\}_{t \geq 0}$  has a unique fixed point  $\gamma \in \Gamma(Y, X)$  which is an invariant section of  $h$ ;*
- (ii) *the dynamical system  $(X, \mathbb{S}_+, \pi)$  admits a compact global attractor  $J$ ;*
- (iii)  *$J = \gamma(Y)$ .*

*Proof.* To prove this statement it is sufficient to note that the mapping  $\omega : \mathbb{S}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , where  $\omega(t, r) := \frac{r}{(1+(\alpha-1)tr^{\alpha-1})^{1/(\alpha-1)}}$ , is continuous and possesses the properties a. and b. of Theorem 5.3.  $\square$

A family of nonempty compact sets  $\{I_y \mid y \in Y\}$  is called a compact global attractor of cocycle  $\langle W, \varphi, (Y, \mathbb{S}, \sigma) \rangle$  (or shortly  $\varphi$ ) if the following conditions are fulfilled:

- (i)  $I = \cup\{I_y \mid y \in Y\}$  is relatively compact;
- (ii)  $\{I_y \mid y \in Y\}$  is invariant with respect to cocycle  $\varphi$ , i.e.  $\varphi(t, I_y, y) = I_{\sigma(t, y)}$  for all  $t \in \mathbb{S}_+$  and  $y \in Y$ ;
- (iii)  $\lim_{t \rightarrow +\infty} \sup_{y \in Y} \beta(\varphi(t, K, y), I) = 0$  for all  $K \in C(W)$ .

**Remark 5.5.** [9] 1. Let  $\{I_y \mid y \in Y\}$  be the compact global attractor of cocycle  $\varphi$  and  $(X, \mathbb{S}_+, \pi)$  be a skew product dynamical system generated by cocycle  $\varphi$ . Then the set  $J := \cup\{J_y \mid y \in Y\}$ , where  $J_y := I_y \times \{y\}$ , is the compact global attractor of  $(X, \mathbb{S}_+, \pi)$ .

2. If the skew-product dynamical system  $(X, \mathbb{S}_+, \pi)$  (generated by  $\varphi$ ) admits the compact global attractor  $J$ , then the cocycle  $\varphi$  has the compact global attractor  $\{I_y \mid y \in Y\}$ , where  $I_y := pr_1(pr_2^{-1}(y) \cap J)$ .

**Theorem 5.6.** Let  $\langle W, \varphi, (Y, \mathbb{S}, \sigma) \rangle$  be a cocycle over dynamical system  $(Y, \mathbb{S}, \sigma)$  with fiber  $W$ . Suppose that there exists a continuous mapping  $\omega : \mathbb{S}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that

$$\rho(\varphi(t, u_1, y), \varphi(t, u_2, y)) \leq \omega(t, \rho(u_1, u_2))$$

( $\forall u_1, u_2 \in W, y \in Y$  and  $t \geq 0$ ). If there exists a positive number  $t_0 \in \mathbb{S}_+$  such that

- (i)  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone increasing;
- (ii)  $\omega(t_0, r) < r$  for all  $r > 0$ .

Then the following statement hold:

- (i) the semigroup  $\{Q^t\}_{t \geq 0}$  ( $(Q^t \mu)(y) := U(t, \sigma^{-t}y)\mu(\sigma^{-t}y)$  ( $y \in Y$ ), where  $U(t, y) := \varphi(t, \cdot, y)$ ) has a unique fixed point  $\mu \in C(Y, W)$  which is an invariant section of cocycle  $\varphi$ ;
- (ii) the dynamical system  $(X, \mathbb{S}_+, \pi)$  admits a compact global attractor  $J$ ;
- (iii)  $J = \gamma(Y)$ .

*Proof.* Let  $X := W \times Y, \pi := (\varphi, \sigma)$  and  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  ( $h := pr_2 : X \mapsto Y$ ) be a non-autonomous dynamical system associated by cocycle  $\varphi$ . Now to finish the proof of Theorem it is sufficient to apply Theorem 5.3 for non-autonomous dynamical system  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  because under the conditions of Theorem 5.6 all conditions of Theorem 4.4 will hold.  $\square$

## 6. ALMOST PERIODIC AND ALMOST AUTOMORPHIC MOTIONS OF NON-AUTONOMOUS SYSTEMS

Using the general results from Sections 3–5 we study the problem of existence the almost periodic and almost automorphic motions of non-autonomous dynamical

systems in this section and also we give the sufficient conditions of its global asymptotic stability.

Let  $\mathbb{T} = \mathbb{R}$  or  $\mathbb{R}_+$ ,  $(X, \mathbb{T}, \pi)$  be a dynamical system,  $x \in X$ ,  $\tau, \varepsilon \in \mathbb{T}$ ,  $\tau > 0$ ,  $\varepsilon > 0$ . We denote  $\pi(x, t)$  by a short-hand notation  $xt$ .

The point  $x$  is called a stationary point if  $xt = x$  for all  $t \in \mathbb{T}$ . The point  $x$  is called  $\tau$ -periodic if  $x\tau = x$ .

The number  $\tau$  is called  $\varepsilon$ -shift ( $\varepsilon$ -almost period) of a point  $x$  if  $\rho(x\tau, x) < \varepsilon$  ( $\rho(x(t + \tau), xt) < \varepsilon$  for all  $t \in \mathbb{T}$ ).

The point  $x$  is called almost recurrent (almost periodic) if for any  $\varepsilon > 0$  there exists positive number  $l$  such that on every segment of length  $l$  can be found a  $\varepsilon$ -shift ( $\varepsilon$ -almost period) of the point  $x$ .

A point  $x$  is called recurrent if it is almost recurrent and the set  $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$  is compact.

Denote by  $\mathfrak{M}_x = \{\{t_n\} \mid \{xt_n\} \text{ is convergent}\}$ .

**Theorem 6.1.** ([31]) *Let  $(X, \mathbb{T}_1, \pi)$  and  $(Y, \mathbb{T}_2, \sigma)$  be dynamical systems with  $\mathbb{T}_1 \subset \mathbb{T}_2$ . Assume that  $h : X \rightarrow Y$  is a homomorphism from  $(X, \mathbb{T}_1, \pi)$  onto  $(Y, \mathbb{T}_2, \sigma)$ . If the point  $x \in X$  is stationary ( $\tau$ -periodic, quasi-periodic, almost periodic, recurrent), then the point  $y := h(x)$  is also stationary ( $\tau$ -periodic, quasi-periodic, almost periodic, recurrent) and  $\mathfrak{M}_x \subset \mathfrak{M}_y$ .*

An autonomous dynamical system  $(Y, \mathbb{T}, \sigma)$  is said to be pseudo recurrent if the following conditions are fulfilled:

- a)  $Y$  is compact;
- b)  $(Y, \mathbb{T}, \sigma)$  is transitive, i.e. there exists a point  $y_0 \in Y$  such that  $Y = \overline{\{y_0 t \mid t \in \mathbb{T}\}}$ ;
- c) every point  $y \in Y$  is stable in the sense of Poisson, i.e.

$$\mathfrak{N}_y = \{\{t_n\} \mid \sigma(t_n, y) \rightarrow y \text{ and } |t_n| \rightarrow +\infty\} \neq \emptyset.$$

A point  $y \in Y$  is said to be pseudo recurrent if the dynamical system  $(H(y), \mathbb{T}, \pi)$  is pseudo recurrent.

**Lemma 6.2.** ([10]) *Let  $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$  be a non-autonomous dynamical system and the following conditions are fulfilled:*

- 1)  $(Y, \mathbb{T}_2, \sigma)$  is pseudo recurrent;
- 2)  $\gamma \in \Gamma(Y, X)$  is an invariant section of the homomorphism  $h : X \rightarrow Y$ .

*Then the autonomous dynamical system  $(\gamma(Y), \mathbb{T}_2, \pi)$  is pseudo recurrent.*

Let  $\mathbb{T} = \mathbb{S}$  and  $(X, \mathbb{S}, \pi)$  be a two-sided dynamical system.

A recurrent point  $x \in X$  is called almost automorphic (see, for example, [32]) if whenever  $t_\alpha$  is a net with  $xt_\alpha \rightarrow x_*$ , then  $x_*(-t_\alpha) \rightarrow x$  too.

A motion  $\varphi(t, u_0, y_0)$  ( $u_0 \in E$  and  $y_0 \in Y$ ) of the cocycle  $\varphi$  is called recurrent (almost periodic, almost automorphic, quasi-periodic, periodic), if the point  $x_0 :=$

$(u_0, y_0) \in X := E \times Y$  is a recurrent (almost periodic, almost automorphic, quasi-periodic, periodic) point of the skew-product dynamical system  $(X, \mathbb{S}_+, \pi)$  ( $\pi := (\varphi, \sigma)$ ).

**Lemma 6.3.** [11] *If  $y \in Y$  is an almost automorphic point of the dynamical system  $(Y, \mathbb{S}, \sigma)$  and  $g : Y \rightarrow X$  is a homomorphism of the dynamical system  $(Y, \mathbb{S}, \sigma)$  onto  $(X, \mathbb{S}_+, \pi)$ , then the point  $x := g(y)$  is an almost automorphic point of the system  $(X, \mathbb{S}_+, \pi)$ .*

**Theorem 6.4.** *Let  $Y$  be a compact metric space,  $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$  be a non-autonomous dynamical system and  $y \in Y$  be a  $\tau$ -periodic (almost periodic, quasi-periodic, almost automorphic, recurrent, pseudo-recurrent) point. If there is a function  $V : X \dot{\times} X \rightarrow \mathbb{R}_+$  satisfying Conditions (C1)–(C3) and*

- (C4.c)  $V(x_1 t, x_2 t) \leq \omega(t, V(x_1, x_2))$  ( $\forall (x_1, x_2) \in X \dot{\times} X, t \geq 0$ ), where  $\omega$  is a mapping from  $\mathbb{R}_+^2$  into  $\mathbb{R}_+$  satisfying the following condition: there exists a positive number  $t_0 \in \mathbb{S}_+$  such that
- (a)  $\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is monotone decreasing;
  - (b)  $\omega(t_0, r) < r$  for all  $r > 0$ ;
  - (c) the mapping  $\omega(t_0, \cdot)$  is right continuous on  $\mathbb{R}_+$ .

Then there exists at least one  $\tau$ -periodic (almost periodic, quasi-periodic, almost automorphic, recurrent, pseudo-recurrent) point  $x \in X_y := h^{-1}(y) = \{x \in X \mid h(x) = y\}$  of dynamical system  $(X, \mathbb{S}_+, \pi)$ .

*Proof.* By Theorem 4.4 under the conditions of Theorem 6.4 the non-autonomous dynamical system  $(X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h$  admits a unique continuous invariant section  $\gamma \in G(Y, X)$ . By Theorem 6.1 and Lemmas 6.2, 6.3  $x := \gamma(y)$  is a  $\tau$ -periodic (almost periodic, quasi-periodic, almost automorphic, recurrent, pseudo-recurrent) point of dynamical system  $(X, \mathbb{S}_+, \pi)$ .  $\square$

## 7. APPLICATIONS

Below we give some application of results, obtained in Sections 3–6, to the study of the almost periodic and almost automorphic solutions of different classes of non-stationary evolution equations (Finite-dimensional ODEs, Carateadory's equations, ODEs with impulses, Infinite-dimensional systems (ODEs in Banach spaces, Evolution equations with monotone operators)).

**7.1. Finite-dimensional systems.** Denote by  $\mathbb{R}^n$  a real  $n$ -dimensional Euclidean space with the scalar product  $\langle \cdot, \cdot \rangle$  and the norm  $|\cdot|$  generated by the scalar product. Let  $[\mathbb{R}^n]$  be a space of all the linear mappings  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  equipped with the operator norm.

**Theorem 7.1.** *Let  $Y$  be a compact metric space,  $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ ,  $\mathcal{W} \in C(Y, [\mathbb{R}^n])$  and the following conditions be held:*

1. *the matrix-function  $\mathcal{W}$  is positively defined, i.e.  $\langle \mathcal{W}(y)u, u \rangle \in \mathbb{R}$  for all  $y \in Y, u \in \mathbb{R}^n$ , and there exists a positive constant  $a$  such that  $\langle \mathcal{W}(y)u, u \rangle \geq a|u|^2$  for all  $y \in Y$  and  $u \in \mathbb{R}^n$ .*

2. the function  $t \rightarrow \mathcal{W}(\sigma_t y)$  is differentiable for every  $y \in Y$  and  $\dot{\mathcal{W}}(y) \in C(Y, [\mathbb{R}^n])$ , where  $\dot{\mathcal{W}}(y) := \frac{d}{dt} \mathcal{W}(\sigma(t, y))|_{t=0}$ .
  3.  $\langle \dot{\mathcal{W}}(y)(u - v) + (\mathcal{W}(y) + \mathcal{W}^*(y))(F(y, u) - F(y, v)), u - v \rangle \leq -\theta(|u - v|^2)$  for all  $y \in Y$  and  $u, v \in \mathbb{R}^n$ , where  $\mathcal{W}^*(y)$  is a conjugate matrix and  $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a function possessing the properties (H1)-(H4) of Lemma 4.7;
- (H5). there exists  $r_0 > 0$  such that  $\theta(r)r^{-1/2} > M$  for all  $r > r_0$ , where  $M := \max_{y \in Y} |F(y, 0)|$ .

Then

- (i) the equation

$$(13) \quad u' = F(\sigma(t, y), u)$$

generates a cocycle  $\varphi$  on  $\mathbb{R}^n$  which admits a unique invariant section  $\mu \in C(Y, \mathbb{R}^n)$ ;

- (ii) the equation (13) admits at least one stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution, if the point  $y \in Y$  is stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent);
- (iii) the cocycle  $\varphi$ , generated by equation (13), admits a compact global attractor  $\{I_y : y \in Y\}$ , where  $I_y := \mu(y)$  for all  $y \in Y$ .

*Proof.* Since the function  $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ , then, according to the Peano theorem (see, for example, [17]), the equation (13) admits at least one solution  $u(t)$  ( $t \in [0, t_u), t_u > 0$ ) with the condition  $u(0) = u_0$  for every  $u_0 \in \mathbb{R}^n$ . We will show that under the conditions of Theorem 7.1 this solution is unique. In fact, let  $u_i(t)$  ( $i = 1, 2$ ) be two solutions of the equation (13) defined on  $[0, t_0)$  with the condition  $u_i(0) = x$  ( $i = 1, 2$ ). We consider the function  $\langle \mathcal{W}(\sigma_t y)(u_1(t) - u_2(t)), u_1(t) - u_2(t) \rangle$ . According to the conditions 1. and 3. of Theorem 7.1 we have

$$a|u_1(t) - u_2(t)|^2 \leq 0$$

for all  $t \in [0, t_0)$  and, consequently,  $u_1(t) = u_2(t)$  for all  $t \in [0, t_0)$ .

Now we will prove that every solution of the equation (13) is defined on  $\mathbb{R}_+$ . Let  $u \in \mathbb{R}^n$  and  $\varphi(t, u, y)$  be the unique solution of the equation (13) defined on  $[0, t_{(u, y)})$ . To prove that  $t_{(u, y)} = +\infty$  it is sufficient to show that the solution  $\varphi(t, u, y)$  is bounded on  $[0, t_{(u, y)})$ . We denote by  $b := \max_{y \in Y} \|\mathcal{W}(y)\|$ ,  $T_1(u, y) := \{t \in [0, t_{(u, y)}) \mid |\varphi(t, u, y)| \leq r_0\}$  and  $T_2(u, y) := [0, t_{(u, y)}) \setminus T_1(u, y)$ . It is clear that the set  $T_2(u, y)$  is open and, consequently,  $T_2(u, y) = \bigcup_{\alpha} \{(t_{\alpha}, t_{\beta}) \mid \beta = \beta(\alpha)\}$ . For all  $t \in T_2(u, y)$  there exists  $\alpha$  such that  $t \in (t_{\alpha}, t_{\beta})$ ,  $|\varphi(t_{\alpha}, u, y)| = |\varphi(t_{\beta}, u, y)|$  and  $|\varphi(t, u, y)| > r_0$ .

Denote by  $\mathcal{V} : \mathbb{R}^n \mapsto \mathbb{R}^+$  the function defined by the equality  $\mathcal{V}(u, y) := \langle W(y)u, u \rangle$  for all  $(u, y) \in X := \mathbb{R}^n \times Y$ . If  $|\varphi(t, u, y)| > r$  for all  $t \in (t_1, t_2) \subset \mathbb{R}_+$ , then

$$\frac{d}{dt} \mathcal{V}(\sigma(t, y), \varphi(t, u, y)) = \langle \dot{W}(\sigma(t, y))\varphi(t, u, y), \varphi(t, u, y) \rangle +$$



$\langle (W(\sigma(t, y)) + W_*(\sigma(t, y)))F(\sigma(t, y), \varphi(t, u, y)), \varphi(t, u, y) \rangle \leq -c(|\varphi(t, u, y)|) < 0$ ,  
 where  $c(r) := -\theta(r^2) + Mr$  and  $M := \max_{y \in Y} |F(y, 0)|$ .

Consequently,

$$(14) \quad a|\varphi(t, u, y)|^2 \leq \langle \mathcal{W}(\sigma(t, y))\varphi(t, u, y), \varphi(t, u, y) \rangle \leq \langle \mathcal{W}(yt_\alpha)\varphi(t_\alpha, u, y), \varphi(t_\alpha, u, y) \rangle \leq br_0^2.$$

From the inequality (14) follows that  $|\varphi(t, u, y)| \leq \sqrt{\frac{b}{a}}r_0$  and, consequently, we obtain  $\sup\{|\varphi(t, u, y)| \mid t \in [0, t_{(u, y)})\} \leq R_0 := \max\{r_0, \sqrt{\frac{b}{a}}r_0\}$ . Thus, the equation (13) defines a cocycle  $\varphi$  on  $\mathbb{R}^n$ .

Let  $X = \mathbb{R}^n \times Y$ ,  $(X, \mathbb{R}_+, \pi)$  be a skew-product dynamical system and  $((X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h)$  be the non-autonomous dynamical system generated by the equation (13). Denote by  $\mathcal{V} : \mathbb{R}^n \times \mathbb{R}^n \times Y \rightarrow \mathbb{R}^+$  the function defined by the equality  $\mathcal{V}(u_1, u_2, y) := \langle \mathcal{W}(y)(u_1 - u_2), u_1 - u_2 \rangle$  for all  $(u_1, u_2, y) \in \mathbb{R}^n \times \mathbb{R}^n \times Y$ . Then

$$\begin{aligned} \frac{d}{dt} \mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y), \sigma(t, y)) &= \langle \dot{\mathcal{W}}(\sigma(t, y))(\varphi(t, u_1, y) - \varphi(t, u_2, y)), \\ &\varphi(t, u_1, y) - \varphi(t, u_2, y) \rangle + \langle (\mathcal{W}(\sigma(t, y)) + \mathcal{W}^*(\sigma(t, y)))(F(\sigma(t, y), \varphi(t, u_1, y)) - \\ &F(\sigma(t, y), \varphi(t, u_2, y))), \varphi(t, u_1, y) - \varphi(t, u_2, y) \rangle \leq -\theta(|\varphi(t, u_1, y) - \varphi(t, u_2, y)|^2) \end{aligned}$$

and, consequently,  $\mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y), \sigma(t, y)) \leq \omega(t, \mathcal{V}(u_1, u_2, y))$  for all  $y \in Y$ ,  $u_1, u_2 \in \mathbb{R}^n$  and  $t \in \mathbb{R}_+$ , where  $\omega(t, r)$  is a unique solution of equation  $z' = -\theta(b^{-1}z)$  with initial condition  $\omega(0, r) = r$ . By Lemma 4.7 for all  $t > 0$  the mapping  $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  possesses the following properties:

- (i)  $\omega(t, \cdot)$  is monotone increasing;
- (ii)  $\omega(t, r) < r$  for all  $r > 0$ ;
- (iii) the mapping  $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$  is continuous;
- (iv)  $\omega(t + \tau, r) = \omega(t, \omega(\tau, r))$  for all  $t, \tau, r \in \mathbb{R}_+$ .

By Corollary 5.4 the cocycle  $\varphi$ , generated by equation (13), admits a unique invariant continuous section  $\mu \in C(Y, \mathbb{R}^n)$ . By Theorem 6.1  $\varphi(t, \mu(y), y) = \mu(\sigma(t, y))$  is a stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution of equation (13).

The last statement of Theorem follows from Theorem 5.6. □

**Remark 7.2.** Note that the condition 4. (there exists  $r_0 > 0$  such that  $\theta(r)r^{-1/2} > M$  for all  $r > r_0$ ) it was used only to prove the extendability on  $\mathbb{R}_+$  the solutions of equation (13).

Let  $E$  be a Banach space. The function  $F \in C(Y \times E, E)$ , is called regular, if for all  $u \in E$  and  $y \in Y$  there exists a unique solution  $\varphi(t, u, y)$  of equation (13) defined on  $\mathbb{R}_+$  with initial condition  $\varphi(0, u, y) = u$ .

**Theorem 7.3.** Let  $Y$  be a compact metric space,  $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$  be a regular function,  $\mathcal{W} \in C(Y, [\mathbb{R}^n])$  and the following conditions be held:

1. the matrix-function  $W$  is positively defined, i.e.  $\langle \mathcal{W}(y)u, u \rangle \in \mathbb{R}$  for all  $y \in Y$ ,  $u \in \mathbb{R}^n$  and there exists a positive constant  $a$  such that  $\langle \mathcal{W}(y)u, u \rangle \geq a|u|^2$  for all  $y \in Y$  and  $u \in \mathbb{R}^n$ .
2. the function  $t \rightarrow \mathcal{W}(\sigma(t, y))$  is differentiable for every  $y \in Y$  and  $\dot{\mathcal{W}}(y) \in C(Y, [\mathbb{R}^n])$ , where  $\dot{\mathcal{W}}(y) := \frac{d}{dt} \mathcal{W}(\sigma(t, y))|_{t=0}$ .
3.  $\langle \dot{\mathcal{W}}(y)(u - v) + (\mathcal{W}(y) + \mathcal{W}^*(y))(F(y, u) - F(y, v)), u - v \rangle \leq -\theta(|u - v|^2)$  for all  $y \in Y$  and  $u, v \in \mathbb{R}^n$ , where  $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a function possessing the properties (H1)-(H4) of Lemma 4.7.

Then

- (i) the equation

$$(15) \quad u' = F(\sigma(t, y), u)$$

generates a cocycle  $\varphi$  on  $\mathbb{R}^n$  which admits a unique invariant section  $\mu \in C(Y, \mathbb{R}^n)$ ;

- (ii) the equation (15) admits at least one stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution, if the point  $y \in Y$  is stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent);
- (iii) the cocycle  $\varphi$ , generated by equation (13), admits a compact global attractor  $\{I_y : y \in Y\}$ , where  $I_y := \{\mu(y)\}$  for all  $y \in Y$ .

*Proof.* Taking into account the Remark 7.2 we can prove this statement by slight modification the proof of Theorem 7.1.  $\square$

**Example 7.4.** As an example that illustrates these theorems we can consider the following equation

$$u' = g(u) + f(\sigma(t, \omega)),$$

where  $f \in C(\Omega, \mathbb{R}^n)$  and  $\langle A(g(u_1) - g(u_2)), u_1 - u_2 \rangle \leq -\alpha|u_1 - u_2|^p$  ( $p \geq 2$ ) for all  $u_1, u_2 \in \mathbb{R}^n$ , where  $A \in [\mathbb{R}^n]$  is a self-adjoint positive definite matrix and  $\alpha > 0$ .

**7.2. Caratheodory's differential equations.** Let us consider now the equation (13) with the right hand side  $f$  satisfying the conditions of Caratheodory (see, for example, [30]). The space of all Caratheodory's functions we denote by  $\mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ . Topology on this space is defined by the family of semi-norms [30]

$$d_{k,m}(f) := \int_{-k}^k \max_{|x| \leq m} |f(t, x)| dt.$$

This space is metrizable, and on  $\mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  there can be defined a dynamical system of translations  $(\mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$ .

We consider the equation

$$(16) \quad \frac{dx}{dt} = f(t, x),$$

where  $f \in \mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ , and the family of equations

$$(17) \quad \frac{dx}{dt} = g(t, x),$$

where  $g \in H(f) := \overline{\{f_\tau \mid \tau \in \mathbb{R}\}}$ , and  $f_\tau$  is a  $\tau$ -translation of the function  $f$  w.r.t. the variable  $t$ , i.e.  $f_\tau(t, x) := f(t + \tau, x)$  for all  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^n$  and by bar is denoted the closure in the space  $\mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ .

If the function  $f \in \mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  is regular, then  $\varphi$  is a cocycle on  $\mathbb{R}^n$  (see, for example, [30]) with the base  $H(f)$ . Hence, we may apply the general results from Section 4–6 to the cocycle  $\varphi$  generated by the equation (16) with a Caratheodory's right hand side and obtain some results for this type of equations.

For instance, the following assertion holds.

**Theorem 7.5.** *Let  $f \in \mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  be a regular and almost periodic function in  $t \in \mathbb{R}$  (in the sense of Stepanoff [24]) uniformly w.r.t.  $x$  on compacts from  $\mathbb{R}^n$ , i.e. for every  $\varepsilon > 0$  and compact  $K \subset \mathbb{R}^n$  the set*

$$\mathfrak{Z}(\varepsilon, f, K) := \left\{ \tau \in \mathbb{R} \mid \int_0^1 \max_{x \in K} |f(t + \tau + s, x) - f(t + s, x)| ds < \varepsilon \right\}$$

is relatively dense on  $\mathbb{R}$ . Suppose that

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq -\theta(|x_1 - x_2|^2)$$

for all  $t \in \mathbb{R}$  and  $x_1, x_2 \in \mathbb{R}^n$ , where  $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a function possessing the properties (H1)-(H4) of Lemma 4.7.

Then

- (i) the equation (16) generates a cocycle  $\varphi$  on  $\mathbb{R}^n$  which admits a unique invariant section  $\mu \in C(H(f), \mathbb{R}^n)$ ;
- (ii) every equation (17) admits a unique Bohr almost periodic solution  $\mu(g_t) = \varphi(t, \mu(g), g)$  (for all  $t \in \mathbb{R}$ );
- (iii) the cocycle  $\varphi$ , generated by equation (16), admits a compact global attractor  $\{I_g : g \in H(f)\}$ , where  $I_g := \{\mu(g)\}$  for all  $g \in H(f)$ .

**7.3. ODEs with impulse.** Let  $\{t_k\}_{k \in \mathbb{Z}}$  be a two-sided sequence of real numbers,  $p : \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuously differentiable on every interval  $(t_k, t_{k+1})$  function, continuous to the right in every point  $t = t_k$ , bounded on  $\mathbb{R}$ , almost periodic in the sense of Stepanoff [15, 24] and

$$p'(t) = \sum_{k \in \mathbb{Z}} s_k \delta_{t_k},$$

where  $s_k := p(t_k + 0) - p(t_k - 0)$ .

Consider the equation with impulse

$$\frac{dx}{dt} = f(t, x) + \sum_{k \in \mathbb{Z}} s_k \delta_{t_k}$$

or, what is equivalent,

$$(18) \quad \frac{dx}{dt} = f(t, x) + p'(t)$$

and parallelly consider the family of equations

$$(19) \quad \frac{dx}{dt} = g(t, x) + q'(t),$$

where  $(g, q) \in H(f, p) := \overline{\{(f_\tau, p_\tau) \mid \tau \in \mathbb{R}\}}$  and by bar we denote the closure in the product-space  $\mathfrak{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \times \mathfrak{C}(\mathbb{R}, \mathbb{R}^n)$ .

Suppose that the function  $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  is regular, then for all  $(g, q) \in H(f, p)$  the equation (19) admits a unique solution  $\varphi(t, x, g, q)$  (see [15] and [29]) satisfying the initial condition  $\varphi(0, x, g, q) = x$ . This solution is continuous on every interval  $(t_k, t_{k+1})$  and continuous to the right in every point  $t = t_k$  (see [15] and [29]).

By the transformation

$$(20) \quad x := y + q(t)$$

we can bring the equation (19) to the equation

$$(21) \quad \frac{dy}{dt} = g(t, y + q(t)).$$

**Theorem 7.6.** *Let  $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  be a regular and Bohr almost periodic function in  $t \in \mathbb{R}$  uniformly with respect to  $x$  on every compact subset from  $\mathbb{R}^n$  and  $p \in \mathfrak{C}(\mathbb{R}, \mathbb{R}^n)$  be a Stepanoff's almost periodic function. Suppose that  $\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq -\theta(|x_1 - x_2|^2)$  for all  $t \in \mathbb{R}$  and  $x_1, x_2 \in \mathbb{R}^n$ , where  $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a function possessing the properties (H1)-(H4) of Lemma 4.7.*

*Then every equation (18) admits a unique Stepanovs almost periodic solution which is globally asymptotically stable.*

*Proof.* Let  $\varphi(t, x, g, q)$  be the cocycle generated by the family of equations (19) and  $\tilde{\varphi}(t, y, g, q)$  be the cocycle generated by the family of equations (21). Then we have the following equality

$$(22) \quad \varphi(t, x, g, q) = q(t) + \tilde{\varphi}(t, x - q(0), g, q).$$

We will show that it is possible to apply Theorem 7.5 to the equation

$$\frac{dy}{dt} = f(t, y + p(t)).$$

In fact,

$$\langle f(t, y_1 + p(t)) - f(t, y_2 + p(t)), y_1 - y_2 \rangle \leq -\theta(|y_1 - y_2|^2)$$

for all  $t \in \mathbb{R}$  and  $y_1, y_2 \in \mathbb{R}^n$ . To finish the proof of the theorem it is sufficient to apply Theorem 7.5 and take into consideration the relations (20) and (22). The theorem is proved.  $\square$

## 7.4. Infinite-dimensional systems.

7.4.1. *ODEs in Banach space.* Let  $(E, |\cdot|)$  be a Banach space. The upper semi-inner product is defined [13, 26] by

$$\langle x, y \rangle_+ := |y| \lim_{t \rightarrow 0^+} \frac{1}{t} (|y + tx| - |y|)$$

and the lower semi-inner product by

$$\langle x, y \rangle_- := |y| \lim_{t \rightarrow 0^+} \frac{1}{t} (|y| - |y - tx|).$$

Both limits exist for every norm, and they coincide with the inner product, if  $E$  is a Hilbert space. In the case when  $E$  is a uniformly convex space we have  $\langle x, y \rangle_+ = \langle x, y \rangle_-$ .

**Theorem 7.7.** *Let  $Y$  be a compact metric space,  $F \in C(Y \times E, E)$  be a regular function,  $\mathcal{W} \in C(Y, [E])$  and the following conditions be held:*

1. *the operator-function  $W$  is positively defined, i.e.  $\langle W(y)u, u \rangle \in \mathbb{R}$  for all  $y \in Y$ ,  $u \in E$  and there exists a positive constant  $a$  such that  $\langle W(y)u, u \rangle \geq a|u|^2$  for all  $y \in Y$  and  $u \in E$ .*
2. *the function  $t \rightarrow \mathcal{W}(\sigma(t, y))$  is differentiable for every  $y \in Y$  and  $\dot{\mathcal{W}}(y) \in C(Y, [E])$ , where  $\dot{\mathcal{W}}(y) := \frac{d}{dt}\mathcal{W}(\sigma(t, y))|_{t=0}$ .*
3.  *$\text{Re}\langle \dot{\mathcal{W}}(y)(u-v) + (\mathcal{W}(y) + \mathcal{W}^*(y))(F(y, u) - F(y, v)), u-v \rangle_- \leq -\theta(|u-v|^2)$  for all  $y \in Y$  and  $u, v \in E$ , where  $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a function possessing the properties (H1)-(H4) of Lemma 4.7.*

Then

- (i) *the equation*

$$(23) \quad u' = F(\sigma(t, y), u)$$

*generates a cocycle  $\varphi$  on  $E$  which admits a unique invariant section  $\mu \in C(Y, E)$ ;*

- (ii) *the equation (23) admits at least one stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution, if the point  $y \in Y$  is stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent);*
- (iii) *the cocycle  $\varphi$ , generated by equation (23), admits a compact global attractor  $\{I_y : y \in Y\}$ , where  $I_y := \{\mu(y)\}$  for all  $y \in Y$ .*

*Proof.* Let  $X := E \times Y$ ,  $(X, \mathbb{R}_+, \pi)$  be a skew-product dynamical system and  $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$  be the non-autonomous dynamical system generated by equation (23). Denote by  $\mathcal{V} : E \times E \times Y \rightarrow \mathbb{R}^+$  the function defined by the equality  $\mathcal{V}(u_1, u_2, y) := \langle \mathcal{W}(y)(u_1 - u_2), u_1 - u_2 \rangle$  for all  $(u_1, u_2, y) \in E \times E \times Y$ . Let us note that the Dini derivative

$$D^-u(t) := \limsup_{h \rightarrow 0^-} \frac{u(t+h) - u(t)}{h},$$

then, one has

$$\begin{aligned} D^-(\mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y)), \sigma(t, y)) &= \langle \dot{\mathcal{W}}(\sigma(t, y))(\varphi(t, u_1, y) - \varphi(t, u_2, y)), \\ &\varphi(t, u_1, y) - \varphi(t, u_2, y) \rangle_- + \langle (\mathcal{W}(\sigma(t, y)) + \mathcal{W}^*(\sigma(t, y)))(F(\sigma(t, y), \varphi(t, u_1, y)) - \\ &F(\sigma(t, y), \varphi(t, u_2, y))), \varphi(t, u_1, y) - \varphi(t, u_2, y) \rangle_- \leq -\theta(|\varphi(t, u_1, y) - \varphi(t, u_2, y)|^2) \end{aligned}$$

and, consequently,  $\mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y), \sigma(t, y)) \leq \omega(t, \mathcal{V}(u_1, u_2, y))$  for all  $y \in Y$ ,  $u_1, u_2 \in E$  and  $t \in \mathbb{R}_+$ , where  $\omega(t, r)$  is a unique solution of equation  $z' = -\theta(b^{-1}z)$  with initial condition  $\omega(0, r) = r$ , where  $b := \max_{y \in Y} \|\mathcal{W}(y)\|$ . By Lemma 4.7 for all  $t > 0$  the mapping  $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  possesses the following properties:

- (i)  $\omega(t, \cdot)$  is monotone decreasing;

- (ii)  $\omega(t, r) < r$  for all  $r > 0$ ;
- (iii) the mapping  $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$  is continuous.

By Corollary 4.6 the cocycle  $\varphi$ , generated by equation (23), admits a unique invariant continuous section  $\mu \in C(Y, E)$ . By Theorem 6.1  $\varphi(t, \mu(y), y) = \mu(\sigma(t, y))$  is a stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution of equation (23). The last statement of Theorem follows from Theorem 5.6.  $\square$

**Remark 7.8.** *In the almost periodic case Theorem 7.7 improves the result of Arino O. and Hanebaly E. [1] and Trubnikov Yu. V. and Perov A. I. [33].*

**Example 7.9.** *As an example that illustrates Theorem 7.7 we can consider the following equation*

$$u' = -|u|^\alpha u + f(\sigma(t, y)),$$

where  $f \in C(Y, E)$  and  $\alpha \geq 0$  if  $E$  is a Hilbert space and  $0 \leq \alpha \leq 1$  if  $E$  is a Banach space [1, 3].

**7.4.2. Evolution equations with monotone operators.** Let  $H$  be a real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$ ,  $|\cdot| := \sqrt{\langle \cdot, \cdot \rangle}$  and  $E$  be a reflexive Banach space contained in  $H$  algebraically and topologically. Furthermore, let  $E$  be dense in  $H$ , and here  $H$  can be identified with a subspace of the dual  $E'$  of  $E$  and  $\langle \cdot, \cdot \rangle$  can be extended by continuity to  $E' \times E$ . Finally, let  $(Y, \mathbb{R}, \sigma)$  be a dynamical system on the compact metric space  $Y$ .

We consider the initial value problem

$$(24) \quad u'(t) + Au(t) = f(\sigma(t, y)) \quad (y \in Y)$$

$$(25) \quad u(0) = u,$$

where  $A : E \rightarrow E'$  is bounded (generally nonlinear),

$$|Au|_{E'} \leq C|u|_E^{p-1} + K, u \in E, p > 1,$$

coercive,

$$\langle Au, u \rangle \geq a|u|_E^p, u \in E, a > 0,$$

uniformly monotone,

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \geq \alpha|u_1 - u_2|^\beta \quad (\forall u_1, u_2 \in E, \text{ where } \beta \geq 2),$$

and semi-continuous (see [25]).

A nonlinear "elliptic" operator

$$Au = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \phi \left( \frac{\partial u}{\partial x_i} \right) \quad \text{in } D \subset \mathbb{R}^n$$

$$u = 0 \text{ on } \partial D,$$

where  $D$  is a bounded domain in  $\mathbb{R}^n$ ,  $\phi(\cdot)$  is an increasing function satisfying,  $\phi(0) = 0$ ,  $c|\xi - \eta|^p \leq \sum_{i=1}^n (\xi_i - \eta_i)(\phi(\xi_i) - \phi(\eta_i)) \leq C|\xi - \eta|^p$  (for all  $\xi, \eta \in \mathbb{R}^n$ ), and provides an example with  $H = L^2(D)$ ,  $E = W_0^{1,p}(D)$ ,  $E' = W^{-1,p'}(D)$ ,  $p' = \frac{p}{p-1}$ .

The following result is established in [25] (Ch.2 and Ch.4). If  $x \in H$  and  $f \in C(\Omega, E')$ ,  $p' = \frac{p}{p-1}$ , then there exists a unique solution  $\varphi \in C(\mathbb{R}_+, H)$  of (24) – (25).

We denote by  $\varphi(\cdot, u, \omega)$  the unique solution of (24) and (25). According to [18],  $\varphi(\cdot, u, \omega)$  is a continuous cocycle on  $H$ .

**Theorem 7.10.** *Suppose that the operator  $A$  satisfies the conditions above. Then the following statements hold:*

- (i) *the cocycle  $\varphi$  admits a unique invariant section  $\mu \in C(Y, E)$ ;*
- (ii) *the equation (24) admits at least one stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent) solution, if the point  $y \in Y$  is stationary (respectively,  $\tau$ -periodic, quasi-periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent);*
- (iii) *the cocycle  $\varphi$ , generated by equation (23), admits a compact global attractor  $\{I_y : y \in Y\}$ , where  $I_y := \{\mu(y)\}$  for all  $y \in Y$ .*

*Proof.* Let  $X := H \times Y$ ,  $(X, \mathbb{R}_+, \pi)$  be a skew-product dynamical system and  $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$  be a non-autonomous dynamical system generated by the equation (24). Denote by  $\mathcal{V} : X \dot{\times} X \mapsto \mathbb{R}^+$  the function defined by the equality  $\mathcal{V}(u_1, u_2, y) := \langle u_1 - u_2, u_1 - u_2 \rangle$  for all  $(u_i, y) \in X := H \times Y$  ( $i = 1, 2$ ). Then

$$\begin{aligned} \frac{d}{dt} V(\varphi(t, u_1, y), \varphi(t, u_2, y)) &= 2 \left\langle \frac{d}{dt} (\varphi(t, u_1, y) - \varphi(t, u_2, y)), \right. \\ &\quad \left. \varphi(t, u_1, y) - \varphi(t, u_2, y) \right\rangle = 2 \langle -A(\varphi(t, u_1, y)) + A(\varphi(t, u_2, y)), \\ &\quad \varphi(t, u_1, y) - \varphi(t, u_2, y) \rangle \leq -\theta (|\varphi(t, u_1, y) - \varphi(t, u_2, y)|^2) \end{aligned}$$

and, consequently,  $V(\varphi(t, u_1, y), \varphi(t, u_2, y)) \leq \omega(t, V(u_1, u_2, y))$  for all  $y \in Y$ ,  $u_1, u_2 \in H$  and  $t \in \mathbb{R}_+$ , where  $\theta(x) := 2\alpha x^{\beta/2}$  (for all  $x \in \mathbb{R}_+$ ) and  $\omega(\cdot, \cdot)$  is the unique solution of equation  $x' = -2\alpha x^{\beta/2}$  with initial condition  $\omega(0, x) = x$ . It is easy to verify that the function  $V$  satisfies all the conditions of Theorems 4.4 and 5.3. To finish the proof of Theorem it is to refer to Theorems 4.4 and 5.3.  $\square$

### Acknowledgment

This paper was written while the first author was visiting the Institut of Mathematics, University of Paderborn, in the autumn of 2006. He would like to thank people in that institution for their very kind hospitality. He also gratefully acknowledge the financial support of the Deutscher Akademischer Austauschdienst.

### REFERENCES

- [1] Arino O. and Hanebaly E., Solutions presque periodiques de  $x' + |x|^\alpha x = h(t)$  ( $\alpha > 0$ ) sur les espaces de Banach. C.R. Acad. Sci. Paris, t.306, serie 1, 1988, pp.707-710.
- [2] Babin A. V. and Vishik M. I., *Attractors of Evolutionary Equations*. Nauka, Moscow, 1989; English translation, North-Holland, Amsterdam, 1992.
- [3] Bayliss A., Forced oscillations in quadratically damped systems. *Com. Pure and Apl. Math.*, Vol. 31, 1978, pp.69-88.
- [4] Boyd D. W. and Wong J. S. W., On nonlinear contractions. *Proc. Amer. Math. Soc.* 20 (1969), pp.458-464.

- [5] Brezis H., *Operateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert. Math.Studies*, v.5. North Holland, 1973.
- [6] Bronsteyn I. U., *Extensions of Minimal Transformation Group*. Noordhoff, 1979.
- [7] Browder F. E., On the convergence of successive approximations for nonlinear functional equations. *Nederl. Akad. Wetensch. Proc.*, Ser. A 71, Indag. Math. 30, 1968, pp.27–35.
- [8] Cheban D. N., Locally Dissipative Dynamical Systems and Some Their Applications. *Bulletin of Academy of Sciences of Republic of Moldova. Mathematics*, No.1, 1992, pp.7–14.
- [9] Cheban D. N., *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*. Interdisciplinary Mathematical Sciences 1. River Edge, NJ: World Scientific, 2004, 528pp.
- [10] Cheban D. N., Duan J. and Gherco A. I., Generalization of the Second Bogolyubov's Theorem for Non-Almost Periodic Systems. *Nonlinear Analysis: Real World Applications*, v.4, No.4, 2003, pp.599-613.
- [11] Cheban D.N. and Duan J., Almost Periodic Solutions and Global Attractors of Non-Autonomous Navier-Stokes Equations. *Journal of Dynamics and Differential Equations*, v.16, No.1, 2004, pp.1-34.
- [12] Cheban D. N. and Mammana C., Global Compact Attractors of Discrete Inclusions. *Nonlinear Analyses*, v.65 , No.8 , 2006, pp.1669-1687.
- [13] Deimling K., *Nonlinear Functional Analysis*. Springer, New-York, 1985.
- [14] Glavan V. and Guțu V., Attractors and fixed points of weakly contracting relations. *Fixed Point Theory*, v.5, No.2, 2004, pp.265-284.
- [15] Halanay A. and Wexler D, Teoria calitativă a sistemelor cu impulsuri. București, 1968.
- [16] Hale J. K., *Asymptotic Behaviour of Dissipative Systems*. Amer. Math. Soc., Providence, RI, 1988.
- [17] Hartman P., *Ordinary Differential Equations*. Birkhauser, Boston–Basel–Stuttgart, 1982.
- [18] Hassani N., *Systems Dynamiques Nonautonomes Contractants et leur Applications*. Thése de magister. Algerie, USTHB, 1983.
- [19] W. A. Kirk and B. Sims, Handbook of Metric Fixed Point Theory. *Kluwer Academic Publishers*, Dodrecht/Boston/London/2001, 703 pp.
- [20] Kopanskii A. Ya. and Sibirskii K. S., Conditions for the representation of dynamic systems on the line by means of differential equations. *Differential equations and operator theory. Mat. Issled.* No. 67 (1982), pp.61–68 (in Russian).
- [21] Krasnosel'skii M. A., *Shift Operator Along of Trajectories of Differential Equations*. Moscow, Nauka, 1966 (in Russian).
- [22] Krasnosel'skii M. A., Burd V. Sh. and Kolesov Yu. S., *Nonlinear almost periodic oscillations*. Translated from the Russian by A. Libin. Translation edited by D. Louvish. Halsted Press [A division of John Wiley& Sons], New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973.
- [23] Ladyzhenskaya O. A., On the Determination of Minimal Global Attractors for Navier–Stokes' Equations and Other Equations with Partial Derivatives. *Uspekhi Mat. Nauk*, 42(6(258)):25–60, 1987.
- [24] Levitan B. M. and Zhikov V. V., *Almost Periodic Functions and Differential Equations*. Cambridge Univ. Press, Cambridge, 1982.
- [25] Loins J. L., *Quelques Methodes de Résolution des Problèmes aux Limites non Linéaires*. Dunod, Paris, 1969.
- [26] Lumer G., Semi inner product spaces. *Trans. Math. Math. Soc.*, 100, 1961, pp.29-43.
- [27] Meyers Ph. R., A convers to Banach's contraction theorem. *J. Res. Nat. Bur. Stand. (US)*, 71B:1,2 (1967), pp.73-76.
- [28] Opoitsev V. I., A converse of the contraction mapping principle. *Uspehi Mat. Nauk*, 31 (1976), no. 4 (190), pp.169–198.
- [29] Samoilenko A.M. and Perestiuk N.A., *Differential equations with impulse*. Kiev, High School,1987 (in Russian).
- [30] Sell G. R., *Topological Dynamics and Ordinary Differential Equations*. Van Nostrand-Reinhold, London, 1971.
- [31] Shcherbakov B. A., *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*. Știința, Chișinău, 1985.
- [32] Shen Wenxian and Yi Yingfei, *Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows*. Mem. Amer. Math. Soc., 136 (1998), no. 647, x+93 pp.



- [33] Trubnikov Yu. V. and Perov A. I., *The Differential Equations with Monotone Nonlinearity*. Nauka i Tehnika. Minsk, 1986 (in Russian).

(D. Cheban) STATE UNIVERSITY OF MOLDOVA, DEPARTMENT OF MATHEMATICS AND INFORMATICS,  
A. MATEEVICH STREET 60, MD-2009 CHIȘINĂU, MOLDOVA

*E-mail address*, D. Cheban: [cheban@usm.md](mailto:cheban@usm.md)

(B. Schmalfluss) UNIVERSITY OF PADERBORN, MATHEMATICAL INSTITUTE, STREET, 33098 PADER-  
BORN, GERMANY

*E-mail address*, B. Schmalfluss: [schmalfu@math.uni-paderborn.de](mailto:schmalfu@math.uni-paderborn.de)