LEVITAN ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF V-MONOTONE DIFFERENTIAL EQUATIONS

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Abstract. In the present paper we consider a special class of equations

$$x' = f(t, x) \tag{1}$$

when the function $f : \mathbb{R} \times E \to E$ (E is a strictly convex Banach space) is V–monotone with respect to (w.r.t.) $x \in E$, i.e. there exists a continuous non-negative function $V : E \times E \to \mathbb{R}_+$, which equals to zero only on the diagonal, so that the numerical function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing w.r.t. $t \in \mathbb{R}_+$, where $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of (1) defined on$\mathbb{R}_+$.

The main result of this paper states that every V–monotone Levitan almost periodic (almost automorphic, Bohr almost periodic) equation (1) with bounded solutions admits at least one Levitan almost periodic (almost automorphic, Bohr almost periodic) solution. In particulary, we obtain some new criterions of existence of almost recurrent (Levitan almost periodic, almost automophic, recurrent in the sense of Birkgoff) solutions of forced vectorial Liénard equations.

1. Introduction

The problem of the almost periodicity of solutions of non-linear almost periodic ordinary differential equations

$$x' = f(t, x) \tag{2}$$

was studied by many authors (see, for example, [4, 9, 10, 12, 13, 14, 19, 25, 27, 28] and the bibliography therein).

In the present paper we consider a special class of equations (2), where the function $f : \mathbb{R} \times E \to E$ (E is a Banach space) is V–monotone with respect to (w.r.t.) $x \in E$, i.e. there exists a continuous non-negative function $V : E \times E \to \mathbb{R}_+$ which equals to zero only on the diagonal so that the numerical function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing w.r.t. $t \in \mathbb{R}_+$, where $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of (2) defined and bounded on $\mathbb{R}_+$. This class of non-linear differential equations (2) is interesting enough and well studied (see, for example, [10, 12, 16, 20, 18, 24, 25, 34] and the bibliography therein).

If the function $\alpha(t) = V(x_1(t), x_2(t))$ is strictly decreasing, then equation (2) admits a single almost periodic solution if there exists a bounded solution on $\mathbb{R}_+$.

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In general case (when the function $\alpha(t) = V(x_1(t), x_2(t))$ is non-increasing) the proof of the existence of an almost periodic solution (under the assumption that a bounded solution exists on $\mathbb{R}$) turns out to be difficult. For example, the difficulty consists in the fact that equation (2) might have an infinite number of bounded solutions on $\mathbb{R}$ (for instance, all solutions might be bounded on $\mathbb{R}$) and it is not clear how should we pick an almost periodic solution out of this set of bounded solutions.

Let $\varphi(t, u, f)$ be a unique solution of V-monotone equation (2) with the initial condition $\varphi(0, u, f) = u$ and let it be defined on $\mathbb{R}_+$. In virtue of the fundamental theory of ODEs with the V-monotone right hand side the mapping φ possesses the following properties:

1. $\varphi(0, u, f) = u$;
2. $\varphi(t + \tau, u, f) = \varphi(t, \varphi(\tau, f, z), f_\tau)$ for every $t, \tau \in \mathbb{R}^+$ and $u \in E$, where f_τ is a τ-translation of the function f;
3. φ is continuous;
4. $V(\varphi(t, u_1, f), \varphi(t, u_2, f)) \le V(u_1, u_2)$ for every $t \in \mathbb{R}_+$ and $u_1, u_2 \in E$.

Properties 1.-4. will make the basis of our research of the abstract V-monotone non-autonomous dynamical system (NDS).

The main result of this paper states that every V–monotone Levitan almost periodic (almost automorphic, Bohr almost periodic) equation (2) with bounded solutions admits at least one Levitan almost periodic (almost automorphic, Bohr almost periodic) solution. In particulary, we obtain some new criterions of existence of almost recurrent (Levitan almost periodic, almost automophic, recurrent in the sense of Birkgoff) solutions of forced vectorial Liénard equations. The problem of Bohr almost periodicity of solutions of forced vectorial Liénard equations was studied by P. Cieutat [15] (see also the bibliography therein).

This paper is organized as follows.

Section 2 contains the notions of cocycles, skew-product dynamical systems and non-autonomous dynamical systems. We give some examples of cocycles, generated by non-autonomous differential equations.

In Section 3 we introduce the notion of V-monotone non-autonomous dynamical systems and establish some properties of this class of NDS (Theorem 3.4 and Lemma 3.5).

In Section 4 we establish some general facts about compact motions of nonautonomous dynamical systems (Theorems 4.10 and 4.11).

Section 5 is devoted to the study of Levitan almost periodic (almost recurrent, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) motions of V-monoton NDS. This Section contains the main result of our paper (Theorems 5.30 and 5.33 and also Corollaries 5.31 and 5.34) where we prove that the V-montone NDS on the strictly convex metric space with compact motion admits at least one almost recurrent (Levitan almost periodic, almost automorphic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) motion.

In section 6 the description of the structure of the set of bounded motions of V-monotone NDS is given (Theorems 6.1 and 6.2).

Section 7 is devoted to the application of our general results obtained in sections 4-6 to the study of almost recurrent (Levitan almost periodic, almost automorphic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) solutions of certain classes of differential equations (forced vectorial Liénard equations, dissipative differential equations, the second order equation $w'' + B(\omega t, w') + Aw = f(\omega t)$).

2. Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems

Let X be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of the additive group $\mathbb{S}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system.

Definition 2.1. *Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ ($\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$) be two dynamical systems. A mapping $h : X \to Y$ is called a homomorphism (isomorphism, respectively) of the dynamical system $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (homeomorphic, respectively) and $h(\pi(x, t)) = \sigma(h(x), t)$ ($t \in \mathbb{T}_1,\ x \in X$). In this case the dynamical system $(X, \mathbb{T}_1, \pi)$ is an extension of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ by the homomorphism h, but the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called a factor of the dynamical system $(X, \mathbb{T}_1, \pi)$ by the homomorphism h. The dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called also a base of the extension $(X, \mathbb{T}_1, \pi)$.*

Definition 2.2. *A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system (NDS).*

Definition 2.3. *A triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ (or shortly φ), where $(Y, \mathbb{T}_2, \sigma)$ is a dynamical system on Y, W is a complete metric space and φ is a continuous mapping from $\mathbb{T}_1 \times W \times Y$ to W, satisfying the following conditions:*

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$,

is called [28] *a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with the fiber W.*

Definition 2.4. *Let $X := W \times Y$ and define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as following: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X which is called a skew-product dynamical system* [28] *and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, consequently, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ $(X := W \times Y)$ called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of theses.

Example 2.5. Let E be a real or complex Banach space and Ω be a metric space. Denote by $C(\Omega \times E, E)$ the space of all continuous mappings $f : \Omega \times E \mapsto E$ endowed by compact-open topology. Consider the system of differential equations

$$\begin{cases} u' = & F(\omega, u) \\ \omega' = & G(\omega), \end{cases} \tag{3}$$

where $\Omega \subseteq E, G \in C(\Omega, E)$ and $F \in C(\Omega \times E, E)$. Suppose that for the system (3) the conditions of the existence, uniqueness, continuous dependence of initial data and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(\Omega, \mathbb{R}_+, \sigma)$ a dynamical system on Ω generated by the second equation of the system (3) and by $\varphi(t, u, \omega)$ – the solution of the equation

$$u' = F(\omega t, u) \ (\omega t := \sigma(t, \omega)) \tag{4}$$

passing through the point $u \in E$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \to E$ is continuous and satisfies the conditions: $\varphi(0, u, \omega) = u$ and $\varphi(t + \tau, u, \omega) = \varphi(t, \varphi(\tau, u, \omega), \omega t)$ for all $t, \tau \in \mathbb{R}_+$, $u \in E$ and $\omega \in \Omega$ and, consequently, the system (3) generates a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$ (where $X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to \Omega$).

We will give some generalization of the system (3). Namely, let $(\Omega, \mathbb{R}_+, \sigma)$ be a dynamical system on the metric space Ω. Consider the system

$$\begin{cases} u' = & F(\omega t, u) \\ \omega \in & \Omega, \end{cases} \tag{5}$$

where $F \in C(\Omega \times E, E)$. Suppose that for the equation (4) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The system $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}_+, \sigma), h \rangle$, where $X := E \times \Omega$, $\pi := (\varphi, \sigma)$, $\varphi(\cdot, u, \omega)$ is the solution of (4) and $h := pr_2 : X \to \Omega$ is a non-autonomous dynamical system generated by the equation (5).

Example 2.6. Let us consider a differential equation

$$u' = f(t, u), \tag{6}$$

where $f \in C(\mathbb{R} \times E, E)$. Along with equation (6) we consider its H-class [4],[25], [28], [30], i.e. the family of equations

$$v' = g(t, v), \tag{7}$$

where $g \in H(f) := \overline{\{f_\tau : \tau \in \mathbb{R}\}}$, $f_\tau(t, u) := f(t + \tau, u)$ for all $(t, u) \in \mathbb{R} \times E$ and by bar we denote the closure in $C(\mathbb{R} \times E, E)$. We will suppose also that the function f is regular, i.e. for every equation (7) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(\cdot, v, g)$ the solution of equation (7) passing through the point $v \in E$ at the initial moment $t = 0$. Then there is a correctly defined mapping $\varphi : \mathbb{R}_+ \times E \times H(f) \to E$ satisfying the following conditions (see, for example, [4], [28]):

1) $\varphi(0, v, g) = v$ for all $v \in E$ and $g \in H(f)$;
2) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for every $v \in E$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;

3) the mapping $\varphi : \mathbb{R}_+ \times E \times H(f) \to E$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}_+, \sigma)$ a dynamical system of translations (a semi-group system) on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$. The triplet $\langle E, \varphi, (Y, \mathbb{R}_+, \sigma)\rangle$ is a cocycle on $(Y, \mathbb{R}_+, \sigma)$ with the fiber E. Thus, equation (6) generates a cocycle $\langle E, \varphi, (Y, \mathbb{R}_+, \sigma)\rangle$ and a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$, where $X := E \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Remark 2.7. *Let $\Omega := H(f)$ and $(\Omega, \mathbb{R}, \pi)$ be the shift dynamical system on Ω. The equation (6) (the family of equation (7)) may be written in the form (4), where $F : \Omega \times E \mapsto E$ is defined by equality $F(g, u) := g(0, u)$ for all $g \in H(f) = \Omega$ and $u \in E$, then $F(g_t, u) = g(t, u)$ ($g_t(s, u) := \sigma(t, g)(s, u) = g(t + s, u)$ for all $t, s \in \mathbb{R}$ and $u \in E$).*

Definition 2.8. *The cocycle φ is called $V-$monotone (see* [10], [25], [34]*) if there exists a continuous function $\mathcal{V} : E \times E \times \Omega \to \mathbb{R}_+$ with the following properties:*

(i) *$\mathcal{V}(u_1, u_2, \omega) \geq 0$ for all $\omega \in \Omega$ and $u_1, u_2 \in E$;*
(ii) *$\mathcal{V}(u_1, u_2, \omega) = 0$ if and only if $u_1 = u_2$;*
(iii) *$\mathcal{V}(\varphi(t, \omega, u_1), \varphi(t, \omega, u_2), \theta_t\omega) \leq \mathcal{V}(u_1, u_2, \omega)$ for all $u_1, u_2 \in E, \omega \in \Omega$ and $t \in \mathbb{T}_+$.*

3. V-Monotone NDS

Definition 3.1. *A nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is said to be uniformly stable in the positive direction on compacts of X if, for arbitrary $\varepsilon > 0$ and compact subset $K \subseteq X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that inequality $\rho(x_1, x_2) < \delta$ ($x_1, x_2 \in K, h(x_1) = h(x_2)$) implies that $\rho(\pi^t x_1, \pi^t x_2) < \varepsilon$ for $t \in \mathbb{T}_1$, where $\pi^t := \pi(t, \cdot)$.*

Definition 3.2. *Denote by $X\dot{\times}X = \{(x_1, x_2) \in X \times X \mid h(x_1) = h(x_2)\ \}$. If there exists the function $V : X\dot{\times}X \to \mathbb{R}_+$ with the following properties:*

(i) *V is continuous;*
(ii) *V is positive defined, i.e. $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$;*
(iii) *$V(x_1 t, x_2 t) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X\dot{\times}X$ and $t \in \mathbb{T}_1^+ := \{t \in \mathbb{T}_1 \mid t \geq 0\}$,*

then the nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is called (see [10] *and* [34], [25]*) V - monotone.*

Definition 3.3. *Non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is called stable in the sense of Lagrange in positive direction (st.L^+), if for every compact subset $K \subseteq X$ the set $\bigcup\{\pi^t K \mid t \in \mathbb{T}_1^+\}$ is relatively compact.*

Theorem 3.4. *Every V - monotone st.L^+ nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is uniformly stable in the positive direction on compacts from X.*

Proof. Let $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ be a V - monotone nonautonomous dynamical system and it is not uniformly stable in the positive direction on compacts from X.

Then there is an $\varepsilon_0 > 0$, a sequence $\{t_n\} \subseteq \mathbb{T}_1$ ($t_n \to +\infty$ as $n \to +\infty$), a sequence $\delta_n \to 0$ ($\delta_n > 0$), a compact $K_0 \subseteq X$ and sequences $\{x_n^i\} \subseteq K_0$ $(i = 1, 2)$ such that

$$\rho(x_n^1, x_n^2) < \delta_n \quad \text{and} \quad \rho(x_n^1 t_n, x_n^2 t_n) \geq \varepsilon_0 \tag{8}$$

for all $n \in \mathbb{N}$. Since the dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is st.L^+, then without loss in generality we may suppose that the sequences $\{x_n^i\}$ $(i = 1, 2)$ and $\{x_n^i t_n\}$ $(i = 1, 2)$ are convergent. We denote by $x^i = \lim_{n\to+\infty} x_n^i$ $(i = 1, 2)$ and $\bar{x}^i = \lim_{n\to+\infty} x_n^i t_n$ $(i = 1, 2)$. According to inequality (8) we obtain $x^1 = x^2$ and $\bar{x}^1 \neq \bar{x}^2$. On the other hand in view of V - monotonicity of $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ we have

$$V(x_n^1 t_n, x_n^2 t_n) \leq V(x_n^1, x_n^2) \tag{9}$$

for all $n \in \mathbb{N}$. Passing to the limit in (9) as $n \to +\infty$ we obtain the equality $\bar{x}^1 = \bar{x}^2$ which contradicts to inequality (8). This contradiction proves Theorem 3.4. □

We denote by $\mathcal{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) \mid a(0) = 0,\ a \text{ is strict increasing}\}$.

Lemma 3.5. *Let $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ be a V-monotone non-autonomous dynamical system and there are two functions $a, b \in \mathcal{K}$ such that*

(i) *$Im(a) = Im(b)$, where $Im(a)$ is the set of the values of $a \in \mathcal{K}$;*
(ii) *$a(\rho(x_1, x_2)) \leq V(x_1, x_2) \leq b(\rho(x_1, x_2))$ for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$.*

Then $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ is uniformly stable in the positive direction on compacts from X.

Proof. Let $\varepsilon > 0$ be an arbitrary positive number and $\delta(\varepsilon) := b^{-1}(a(\varepsilon))$, then it easy to check that the inequality $\rho(x_1, x_2) \leq \delta(\varepsilon)$ implies $\rho(x_1 t, x_2 t) \leq \varepsilon$ for all $t \in \mathbb{T}_+$ and $x_1, x_2 \in X$ with condition $h(x_1) = h(x_2)$. □

4. Some General Properties of NDS

Let $(\Omega, \mathbb{T}, \sigma)$ be a group (two-sided) dynamical system.

Definition 4.1. *The point $\omega \in \Omega$ is called (see, for example, [30] and [32]) positively (negatively) stable in the sense of Poisson if there exists a sequence $t_n \to +\infty$ ($t_n \to -\infty$ respectively) such that $\omega t_n \to \omega$ ($\omega t := \sigma(t, \omega)$). If the point ω is Poisson stable in both directions, in this case it is called Poisson stable.*

Denote by $\mathfrak{N}_\omega := \{\{t_n\} \mid \sigma_{t_n}\omega \to \omega\}$, $\mathfrak{N}_\omega^+ := \{\{t_n\} \in \mathfrak{N}_\omega \mid t_n \to +\infty\}$ and $\mathfrak{N}_\omega^- := \{\{t_n\} \in \mathfrak{N}_\omega | t_n \to -\infty\}$.

Definition 4.2. *(Conditional compactness). Let (X, h, Ω) be a fiber space, i.e. X and Ω be two metric spaces and $h : X \to \Omega$ be a homomorphism from X onto Ω. The subset $M \subseteq X$ is said to be conditionally relatively compact, if the pre-image $h^{-1}(\Omega') \bigcap M$ of every relatively compact subset $\Omega' \subseteq \Omega$ is a relatively compact subset of X, in particularly $M_\omega := h^{-1}(\omega) \bigcap M$ is relatively compact for every ω. The set M is called conditionally compact if it is closed and conditionally relatively compact.*

Example 4.3. *Let K be a compact space, $X := K \times \Omega$, $h = pr_2 : X \to \Omega$, then the triplet (X, h, Ω) be a fiber space, the space X is conditionally compact, but not compact.*

Let $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and $\omega \in \Omega$ be a positively Poisson stable point. Denote by

$$E^+_\omega := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}^+_\omega \quad \text{such that} \quad \pi^{t_n}|_{X_\omega} \to \xi\},$$

where $X_\omega := \{x \in X | \quad h(x) = \omega\}$ and $\to$ means the pointwise convergence.

Lemma 4.4. [8, 10] *Let $\omega \in \Omega$ be a positively Poisson stable point, $\langle (X, \mathbb{T}_1, \pi)$, $(\Omega, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and X be a conditionally compact space, then E^+_ω is a nonempty compact sub-semigroup of the semigroup $X_\omega^{X_\omega}$ (w.r.t. composition of mappings).*

Corollary 4.5. *Let $\omega \in \Omega$ be a negatively Poisson stable point, $\langle (X, \mathbb{T}, \pi)$, $(\Omega, \mathbb{T}, \sigma)$, $h\rangle$ be a two-sided non-autonomous dynamical system and X be a conditionally compact space, then $E^-_\omega := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}^-_\omega \quad$ such that $\quad \pi^{t_n}|_{X_\omega} \to \xi\}$ is a nonempty compact sub-semigroup of semigroup $X_\omega^{X_\omega}$.*

This assertion follows from Lemma 4.4.

Lemma 4.6. [8, 10] *Let $\omega \in \Omega$ be a two-sided Poisson stable point, $\langle (X, \mathbb{T}, \pi)$, $(\Omega, \mathbb{T}, \sigma), h\rangle$ be a two-sided non-autonomous dynamical system and X be a conditionally compact space, then $E_\omega = \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}_\omega \quad$ such that $\quad \pi^{t_n}|_{X_\omega} \to \xi\}$ is a nonempty compact sub-semigroup of the semigroup $X_\omega^{X_\omega}$.*

Corollary 4.7. *Under the conditions of Lemma 4.6 E^+_ω and E^-_ω are two nonempty sub-semigroups of the semigroup E_ω.*

Lemma 4.8. [8, 10] *Under the conditions of Lemma 4.6 the following assertions hold:*

- (i) *if $\xi_1 \in E^-_\omega$ and $\xi_2 \in E^+_\omega$, then $\xi_1 \cdot \xi_2 \in E^-_\omega \bigcap E^+_\omega$.*
- (ii) *$E^-_\omega \bigcap E^+_\omega$ is a sub-semigroup of the semigroup E^-_ω, E^+_ω and E_ω.*
- (iii) *$E^-_\omega \cdot E_\omega \subseteq E^-_\omega$ and $E^+_\omega \cdot E_\omega \subseteq E^+_\omega$, where $A_1 \cdot A_2 := \{\xi_1 \cdot \xi_2 | \xi_i \in A_i \quad (i = 1, 2)\}$ and $A_i \subseteq E_\omega$.*
- (iv) *if at least one of the sub-semigroups E^-_ω or E^+_ω is a group, then $E^-_\omega = E^+_\omega = E_\omega$.*

Lemma 4.9. [8, 10] *Let $\omega \in \Omega$ be a two-sided Poisson stable point, $\langle (X, \mathbb{T}, \pi)$, $(\Omega, \mathbb{T}, \sigma), h\rangle$ be a two-sided non-autonomous dynamical system and X be a conditionally compact space and*

$$\inf_{n \in \mathbb{N}} \rho(x_1 t_n, x_2 t_n) > 0$$

for all $\{t_n\} \in \mathfrak{N}^-_\omega$ and $x_1, x_2 \in X_\omega \quad (x_1 \neq x_2)$, then E^-_ω is a subgroup of the semigroup E_ω.

Let $x \in X$ denote by Φ_x the family of all entire trajectory of dynamical system $(X, \mathbb{T}_1, \pi)$ passing through point x for $t = 0$, i.e. $\gamma \in \Phi_x$ if and only if $\gamma : \mathbb{S} \to X$ is a continuous mapping with the properties: $\gamma(0) = x$ and $\pi^t\gamma(\tau) = \gamma(t + \tau)$ for all $t \in \mathbb{T}_1$ and $\tau \in \mathbb{S}$.

Theorem 4.10. *Let $(X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma)\rangle$ be a NDS with the following properties:*

(i) *It admits a conditionally relatively compact invariant set J (i.e. $\bigcup\{J_\omega \mid \omega\in\Omega'\}$ is relatively compact subset of X for any relatively compact subset Ω' of Ω).*
(ii) *The NDS $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ is positively uniformly stable on J;*
(iii) *evry point $\omega\in\Omega$ is two-sided Poisson stable.*

Then

(i) *all motions on J may be continued uniquely to the left and define on J a two-sided dynamical system $(J,\mathbb{S},\pi)$, i.e. the semi-group dynamical system $(X,\mathbb{T},\pi)$ generates on J a two-sided dynamical system $(J,\mathbb{S},\pi)$;*
(ii) *for every $\omega\in\Omega$ with $J_\omega\neq\emptyset$ there are two sequences $\{t_n^1\}\to+\infty$ and $\{t_n^2\}\to-\infty$ such that*
$$\pi(t_n^i,x)\to x\ (i=1,2)$$
as $n\to\infty$ for all $x\in J_\omega$.

Proof. First step: we will prove that the set $J\subset X$ is distal in the negative direction w.r.t. the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$, i.e. for all $\omega\in\Omega$ (with $J_\omega\neq\emptyset$) and $x_1,x_2\in J_\omega$ the following inequality holds

$$\inf_{t\le 0}\rho(\gamma_{x_1}(t),\gamma_{x_2}(t))>0\quad (x_1\neq x_2) \tag{10}$$

for all $\gamma_{x_i}\in\Phi_{x_i}(i=1,2)$, where by Φ_x it is denoted the family of all the entire trajectories of $(X,\mathbb{T},\pi)$ passing through point x and belonging to J. If it is not true, then there exist $\omega_0\in\Omega, x_i^0\in J_{\omega_0}\quad (x_1^0\neq x_2^0),\gamma_i^0\in\Phi_{x_i^0}\ (i=1,2)$ and $-t_n\to-\infty$ such that

$$\rho(\gamma_{x_1^0}(-t_n),\gamma_{x_2^0}(-t_n))\to 0 \tag{11}$$

as $n\to\infty$. Let $\varepsilon:=\rho(x_1^0,x_2^0)>0$ and $\delta=\delta(\varepsilon)>0$ be chosen from positively uniformly stability of NDS $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ on J. Then for sufficiently large n from (11) we have $\rho(\gamma_{x_1^0}(-t_n),\gamma_{x_2^0}(-t_n))<\delta$ and, consequently, $\varepsilon=\rho(x_1^0,x_2^0)=\rho(\pi^{t_n}\gamma_{x_1^0}(-t_n),\pi^{t_n}\gamma_{x_2^0}(-t_n))<\varepsilon$. The obtained contradiction proves our assertion.

Second step: we will prove that for any $\omega\in\Omega$ and $x\in J_\omega$ the set Φ_x contains only one entire trajectory of $(X,\mathbb{T},\pi)$ belonging to J. Let $\Phi:=\bigcup\{\Phi_x\mid x\in J\}\subset C(\mathbb{S},X)$, where $C(\mathbb{S},X)$ is a space of all the continuous functions $f:\mathbb{S}\to X$ equipped with compact-open topology and $(C(\mathbb{S},X),\mathbb{S},\lambda)$ is Bebutov's dynamical system (dynamical system of translations (see, for example, [10, 28, 29])). It is easy to verify that Φ is a closed and invariant subset of dynamical system $(C(\mathbb{S},X),\mathbb{S},\lambda)$ and, consequently, induces on the set Φ the dynamical system $(\Phi,\mathbb{S},\lambda)$. Let H be a mapping from Φ into Ω, defined by equality $H(\gamma):=h(\gamma(0))$, then it is possible to verify that the triplet $\langle(\Phi,\mathbb{S},\lambda),(\Omega,\mathbb{S},\sigma),H\rangle$ is a non-autonomous dynamical system. Now we will show that this non-autonomous dynamical system is distal on the negative direction, i.e.
$$\inf_{t\le 0}\rho(\gamma_{x_1}^t,\gamma_{x_2}^t)>0$$
for all $\gamma_{x_1},\gamma_{x_2}\in H^{-1}(\omega)\quad(\gamma_{x_1}\neq\gamma_{x_2})$ and $\omega\in\Omega$. Indeed, otherwise there exist $\omega_0,\gamma_{x_i}\in H^{-1}(\omega_0)$ ($i=1,2$ and $\gamma_{x_1}\neq\gamma_{x_2}$) and $t_n\to+\infty$ such that

$\rho(\gamma_{x_1}^{-t_n},\gamma_{x_2}^{-t_n}) \to 0$ (where $\gamma^{\tau} := \sigma(\gamma,\tau)$, i.e. $\gamma^{\tau}(s) := \gamma(\tau + t)$ for all $s \in \mathbb{S}$) as $n \to \infty$ and, consequently,

$$\rho(\gamma_{x_1}(-t_n),\gamma_{x_2}(-t_n)) \le \rho(\gamma_{x_1}^{-t_n},\gamma_{x_2}^{-t_n}) \to 0. \tag{12}$$

Since $\gamma_{x_1} \neq \gamma_{x_2}$, then there exists $t_0 \in \mathbb{S}$ such that $\gamma_{x_1}(t_0) \neq \gamma_{x_2}(t_0)$. Let $\tilde{\gamma}_{x_i}(t) := \gamma_{x_i}(t + t_0)$ for all $t \in \mathbb{S}$, then $\tilde{\gamma}_{x_i} \in \Phi_{\omega_0}$ and from inequality (12) we have

$$\rho(\tilde{\gamma}_{x_1}(-t_n),\tilde{\gamma}_{x_2}(-t_n)) \to 0. \tag{13}$$

as $n \to \infty, -t_n - t_0 \to -\infty$. Thus we found $\omega_0 := h(\gamma_{x_i}(t_0))$ and $\tilde{x}_i := pr_1\gamma_{x_i}(t_0)$ $(i = 1,2), \tilde{x}_1, \tilde{x}_2 \in J_{\omega_0}$ $(\tilde{x}_1 \neq \tilde{x}_2)$ and the entire trajectories $\gamma_{\tilde{x}_i} \in \Phi_{\tilde{x}_i} (i = 1,2)$ such that $\gamma_{\tilde{x}_1}$ and $\gamma_{\tilde{x}_2}$ are proximal (see (13)). But (13) and (10) are contradictory. Thus the negative distality of the non-autonomous dynamical system $\langle(\Phi,\mathbb{S},\sigma),(\Omega,\mathbb{S},\sigma),H\rangle$ is proved.

Now we can prove that for any $\omega \in \Omega$ and $x \in J_{\omega}$ the set Φ_x contains a unique entire trajectory. In fact, if it is not true, then there exists $\omega_0 \in \Omega$ and $x_0 \in J_{\omega_0}$ and two different trajectories $\gamma_1, \gamma_2 \in \Phi_{x_0} (\gamma_1 \neq \gamma_2)$. In virtue of above γ_1 and γ_2 are negatively distal with respect to $\langle(\Phi,\mathbb{S},\sigma),(\Omega,\mathbb{S},\sigma),H\rangle$, i.e.

$$\alpha(\gamma_1,\gamma_2) := \inf_{t \le 0} \rho(\gamma_1^t,\gamma_2^t) > 0.$$

It is easy to check that from the conditional compactness of the set J it follows that the set Φ of the NDS $\langle(\Phi,\mathbb{S},\sigma),(\Omega,\mathbb{S},\sigma),H\rangle$ is so. According to Lemmas 4.4-4.9 we have the sequence $\{t_n^1\} \to +\infty$ such that

$$\gamma_j^{t_n^1} \to \gamma_j \tag{14}$$

as $n \to \infty$ $(j = 1,2)$. In particularly from (14) we obtain

$$\gamma_j(s) = \lim_{n\to\infty} \gamma_j(s + t_n^1) \tag{15}$$

for all $s \in \mathbb{S}$. Since $\gamma_1(t) = \gamma_2(t)$ for all $t \ge 0$ then from (15) we have $\gamma_1 = \gamma_2$. The obtained contradiction proves our statement.

Third step: let now $\tilde{\pi}$ be a mapping from $\mathbb{S} \times J$ into J defined by equality

$$\tilde{\pi}(t,x) = \pi(t,x) \quad \text{if} \quad t \le 0 \quad \text{and} \quad \gamma_x(t) \quad \text{if} \quad t < 0$$

for all $x \in J$, where γ_x is a unique entire trajectory of the dynamical system $(X,\mathbb{T},\pi)$ passing through point x and belonging to J. To prove that $(J,\mathbb{S},\tilde{\pi})$ is a two-sided dynamical system on J it is sufficient to verify the continuity of the mapping $\tilde{\pi}$. Let $x \in J$, $\quad t \in \mathbb{S}_-, x_n \to x$ and $t_n \to t$, then there is a $l_0 > 0$ such that $t_n \in [-l_0, l_0]$ and, consequently,

$$\begin{aligned} \rho(\tilde{\pi}(t_n,x_n),\tilde{\pi}(t,x)) &= \rho(\pi^{t_n+l_0}\gamma_{x_n}(-l_0),\pi^{t+l_0}\gamma_x(-l_0)) \le \\ \rho(\pi^{t_n+l_0}\gamma_{x_n}(-l_0),\pi^{t_n+l_0}\gamma_x(-l_0)) &+ \rho(\pi^{t_n+l_0}\gamma_x(-l_0),\pi^{t+l_0}\gamma_x(-l_0)). \end{aligned} \tag{16}$$

Reasoning as in the proof of Lemma 4.6 it is possible to establish that the sequence $\{\gamma_{x_n}\}$ is relatively compact in $C(\mathbb{S},J)$ and that every limit point of this sequence $\gamma \in \Phi$ and $\gamma(0) = x$. Taking into account the result of the second step we claim that $\gamma_{x_n} \to \gamma_x$ uniformly on every segment $[-l,l] \subset \mathbb{S}$ $(l > 0)$. In particular, $\gamma_{x_n}(-l_0) \to \gamma_x(-l_0)$. Passing now to limit in inequality (16) when $n \to \infty$ we obtain the continuity of mapping $\tilde{\pi}$ in the point (t,x).

Consider the two-sided NDS $\langle (J,\mathbb{S},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$. Let $\omega\in\Omega$. Under the conditions of Theorem we may apply Lemmas 4.4-4.9 and, consequently, $E^{-}_{\omega}=E^{+}_{\omega}=E_{\omega}$. From the last equality we have two sequences $\{t^i_n\}$ $(i=1,2)$ such that $t^1_n\to+\infty$, $t^2_n\to-\infty$ and $\lim\limits_{n\to+\infty}\pi(t^i_n,x)=x$ for all $x\in J_{\omega}$. The theorem is completely proved. □

Theorem 4.11. *Let $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ be a NDS with the following properties:*

(i) *It admits a conditionally relatively compact invariant set J;*
(ii) *The point $\omega_0\in\Omega$ is two-sided Poisson stable and $J_{\omega_0}\neq\emptyset$;*
(iii) *The set J_{ω_0} of NDS $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ is positively uniformly stable, i.e., $\forall\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\rho(x_1,x_2)<\delta$ $(x_1,x_2\in J_{\omega_0})$ implies $\rho(x_1t,x_2t)<\varepsilon\ \forall\ t\geq 0$;*
(iv) *Every point $\omega\in\Omega$ is two-sided Poisson stable.*

Then

(i) *all motions on J_{ω_0} may be continued uniquely to the left;*
(ii) *there are two sequences $\{t^1_n\}\to+\infty$ and $\{t^2_n\}\to-\infty$ such that*
$$\pi(t^i_n,x)\to x$$
as $n\to\infty$ for all $x\in J_{\omega_0}$.

Proof. This statement my be proved with slight modification the proof of Theorem 4.10. □

5. Almost Automorphic Motios of V-Monotone NDS.

5.1. **Recurrent, Almost Periodic and Almost Automorphic Motions.** Let $(X,\mathbb{T},\pi)$ be a dynamical system.

Definition 5.1. *A number $\tau\in\mathbb{T}$ is called an $\varepsilon>0$ shift of x (respectively, almost period of x), if $\rho(x\tau,x)<\varepsilon$ (respectively, $\rho(x(\tau+t),xt)<\varepsilon$ for all $t\in\mathbb{T}$).*

Definition 5.2. *A point $x\in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon>0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x\in X$.*

Definition 5.3. *If the point $x\in X$ is almost recurrent and the set $H(x):=\overline{\{xt\mid t\in\mathbb{T}\}}$ is compact, then x is called recurrent.*

Definition 5.4. *A point $x\in X$ of the dynamical system $(X,\mathbb{T},\pi)$ is called Levitan almost periodic* [25]*, if there exists a dynamical system $(Y,\mathbb{T},\sigma)$ and a Bohr almost periodic point $y\in Y$ such that $\mathfrak{N}_y\subseteq\mathfrak{N}_x$.*

Remark 5.5. *Let $x_i\in X_i$ $(i=1,2,\dots,m)$ be a Levitan almost periodic point of the dynamical system $(X_i,\mathbb{T},\pi_i)$. Then the point $x:=(x_1,x_2,\dots,x_m))\in X:=X_1\times X_2\times\ldots\times X_m$ is also Levitan almost periodic in the product dynamical system $(X,\mathbb{T},\pi)$, where $\pi:\mathbb{T}\times X\to X$ is defined by the equality $\pi(t,x):=(\pi_1(t,x_1),\pi_2(t,x_2),\ \dots,\ \pi_m(t,x_m))$ for all $t\in\mathbb{T}$ and $x:=(x_1,x_2,\dots,x_m)\in X$.*

Definition 5.6. *A point $x \in X$ is called stable in the sense of Lagrange (st.L), if its trajectory $\{\pi(t,x)\ :\ t \in \mathbb{T}\}$ is relatively compact.*

Definition 5.7. *A point $x \in X$ is called almost automorphic in the dynamical system $(X,\mathbb{T},\pi)$, if the following conditions hold:*

(i) *x is st.L;*
(ii) *there exists a dynamical system $(Y,\mathbb{T},\sigma)$, a homomorphism h from $(X,\mathbb{T},\pi)$ onto $(Y,\mathbb{T},\sigma)$ and an almost periodic in the sense of Bohr point $y \in Y$ such that $h^{-1}(y) = \{x\}$.*

Remark 5.8. *1. Every almost automorphic point $x \in X$ is also Levitan almost periodic.*

2. A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is also almost automorphic (see [2, 3], [4], [25], [31] *and also* [17] *and* [26]*). In other words, an Levitan almost periodic point x is almost automorphic if and only if its trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is relatively compact.*

Lemma 5.9. [11] *Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$ be two dynamical systems, $x \in X$ and the following conditions be fulfilled:*

(i) *a point $y \in Y$ is Levitan almost periodic;*
(ii) *$\mathfrak{N}_y \subseteq \mathfrak{N}_x$.*

Then the point x is Levitan almost periodic, too.

Corollary 5.10. *Let $x \in X$ be a st.L point, $y \in Y$ be an almost automorphic point and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then the point x is almost automorphic too.*

Proof. Let y be an almost automorphic point, then by Lemma 5.9 the point $x \in X$ is Levitan almost periodic. Since x is st.L, then by Remark 5.8 it is almost automorphic. □

Remark 5.11. *We note (see, for example,* [25] *and* [30]*) that if $y \in Y$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the dynamical system $(Y,\mathbb{T}_2,\sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y,\mathbb{T}_2,\sigma)$ onto $(X,\mathbb{T}_1,\pi)$, then the point $x = h(y)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X,\mathbb{T}_1,\pi)$.*

Lemma 5.12. [11] *If $y \in Y$ is an almost automorphic point of the dynamical system $(Y,\mathbb{T},\sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y,\mathbb{T},\sigma)$ into $(X,\mathbb{T},\pi)$, then the point $x = h(y)$ is an almost automorphic point of the system $(X,\mathbb{T},\pi)$.*

5.2. **The Principle of Invariance for V-Monotone NDS.**

Theorem 5.13. *Suppose that the following conditions hold:*

(i) *$\omega \in \Omega$ is a two-sided Poisson stable point;*
(ii) *the NDS $\langle (X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\Theta),h\rangle$ admits a conditionally precompact invariant set J;*
(iii) *$\langle (X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ be a V-monotone non-autonomous dynamical system and there are two functions $a,b \in \mathcal{K}$ such that*

(a) *$Im(a) = Im(b)$, where $Im(a)$ is the set of the values of $a \in \mathcal{K}$;*
(b) *$a(\rho(x_1, x_2)) \le V(x_1, x_2) \le b(\rho(x_1, x_2))$ for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$.*

Then $V(x_1t, x_2t) = V(x_1, x_2)$ for all $t \in \mathbb{S}$ and $x_1, x_2 \in J_\omega$.

Proof. Let $x_1, x_2 \in J_\omega$. Under the conditions of Theorem 5.13 by Lemma 3.5 and Theorem 4.10 there are two sequences $\{t_n^1\} \to +\infty$ and $\{t_n^2\} \to -\infty$ such that

$$\pi(t_n^i, x) \to x \ (i = 1, 2) \tag{17}$$

as $n \to \infty$ for all $x \in J_\omega$. Define the function $\psi : \mathbb{S} \mapsto \mathbb{R}_+$ by equality $\psi(s) := V(x_1s, x_2s)$ for all $s \in \mathbb{S}$. It easy to check that $0 \le \psi(s) \le \psi(0)$ for all $s \in \mathbb{S}_+$ and $\psi(s_1) \le \psi(s_2)$ for all $s_1 \le s_2$ $(s_1, s_2 \in \mathbb{S}_+)$. Thus there exists $\lim\limits_{s \to +\infty} \psi(s) = \psi_0 \in [0, \psi(0)]$. According to relation (17) we have $\psi_0 = \lim\limits_{n \to +\infty} V(x_1(s+t_n^1), x_2(s+t_n^1)) = V(x_1s, x_2s)$ for every $s \in \mathbb{S}$ and, consequently, $\psi_0 = \psi(0)$. □

Corollary 5.14. *Suppose that the following conditions hold:*

(i) *every point $\omega \in \Omega$ is two-sided Poisson stable;*
(ii) *the NDS $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{S}, \sigma), h \rangle$ admits a conditionally precompact invariant set J;*
(iii) *$\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{S}, \sigma), h \rangle$ be a V-monotone non-autonomous dynamical system and there are two functions $a, b \in \mathcal{K}$ such that*
    (a) *$Im(a) = Im(b)$, where $Im(a)$ is the set of the values of $a \in \mathcal{K}$;*
    (b) *$a(\rho(x_1, x_2)) \le V(x_1, x_2) \le b(\rho(x_1, x_2))$ for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$.*

Then $V(x_1t, x_2t) = V(x_1, x_2)$ for all $t \in \mathbb{S}$, $x_1, x_2 \in J_\omega$ and $\omega \in \Omega$.

5.3. **Comparability of Motions by the Character of Recurrence.** In this subsection following B. A. Shcherbakov we introduce the notion of comparability of motions of dynamical system by the character of their recurrence. While studying stable in the sense of Poisson motions this notion plays the very important role (see, for example, [29, 30]).

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be dynamical systems, $x \in X$ and $y \in Y$. Denote by $\mathfrak{M}_{x,p} := \{\{t_n\} \subset \mathbb{R} \mid \{xt_n\} \to p\}$, $\mathfrak{L}_{x,p} := \{\{t_n\} \subseteq \mathfrak{M}_{x,p} \mid \{t_n\} \to \infty\}$, $\mathfrak{L}_{x,p}^{+\infty}$ the set of the sequences $\{t_n\} \in \mathfrak{M}_{x,p}$ such that $t_n \to +\infty$. Assume $\mathfrak{L}_x^{+\infty}(M) := \cup\{\mathfrak{L}_{x,p} : p \in M\}$ and $\mathfrak{L}_x^{+\infty} = \mathfrak{L}_x^{+\infty}(X)$.

Definition 5.15. *A point $x \in X$ is called comparable by the character of recurrence with $y \in Y$ w.r.t. $M \subset Y$ or, in short, comparable with y w.r.t. the set M if $\mathfrak{L}_y^{+\infty}(M) \subseteq \mathfrak{L}_x^{+\infty}$.*

Denote by $H(M) := \overline{\{\pi(t, x) : x \in M, t \in \mathbb{T}\}}$. Let $(Y, \mathbb{S}, \sigma)$ be a group dynamical system.

Lemma 5.16. [6, 9] *If $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$, then $\mathfrak{L}_{y,\sigma(t,q)}^{+\infty} \subseteq \mathfrak{L}_{x,\pi(t,p)}^{+\infty}$ for all $t \in \mathbb{T} \subseteq \mathbb{S}$.*

Corollary 5.17. *Under the conditions of Lemma 5.16 if $\mathfrak{L}_y^{+\infty}(M) \subseteq \mathfrak{L}_x^{+\infty}$, then $\mathfrak{L}_y^{+\infty}(\Sigma_M) \subset \mathfrak{L}_x^{+\infty}$ where $\Sigma_M := \{\pi(t, x) : x \in M, t \in \mathbb{T}\}$.*

Lemma 5.18. [6, 9] *If* $\mathfrak{L}^{+\infty}_{y,q} \subseteq \mathfrak{L}^{+\infty}_{x}$*, then there exists a single point* $p \in \omega_x := \cap_{t\ge 0}\overline{\cup_{\tau \ge t} x\tau}$ *such that* $\mathfrak{L}^{+\infty}_{y,q} \subseteq \mathfrak{L}^{+\infty}_{x,p}$.

Theorem 5.19. [6, 9] *If a point* x *is comparable with* y *w.r.t. the set* M*, then there exists a continuous mapping* $h : \sigma(\mathbb{T}, \Sigma_M) \to \omega_x$ *satisfying the condition*

$$h(\sigma(t, q)) = \pi(t, h(q)) \tag{18}$$

for all $q \in \sigma(\mathbb{T}, \Sigma_M)$ *and* $t \in \mathbb{T}$.

Let a point x be comparable with y w.r.t. M. Note that at the point of view of applications (see, for example, [4, 29, 30]) the following cases are the most important.

1. $\mathfrak{L}^{+\infty}_{y,y} \subseteq \mathfrak{L}^{+\infty}_{x,x}$.

As it is shown in [29, 30], the inclusion $\mathfrak{L}^{+\infty}_{y,y} \subseteq \mathfrak{L}^{+\infty}_{x,x}$ takes place if and only if $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. As it was mentioned in section 1.2 of the chapter I[9], the inclusion $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ takes place if and only if x is comparable by recurrence with y.

2. $\mathfrak{L}^{+\infty}_{y} \subseteq \mathfrak{L}^{+\infty}_{x}$ and $\mathfrak{L}^{+\infty}_{y,y} \subseteq \mathfrak{L}^{+\infty}_{x,x}$.

Assume $\mathfrak{M}^{+}_{y} = \{\{t_n\} : \{t_n\} \in \mathfrak{M}_y,\ t_n \in \mathbb{T}_+\}$.

Definition 5.20. *We will call the point* x *strongly comparable (in positive direction) with* y *if* $\mathfrak{L}^{+\infty}_{y,y} \subseteq \mathfrak{L}^{+\infty}_{x,x}$ *and* $\mathfrak{L}^{+\infty}_{y} \subseteq \mathfrak{L}^{+\infty}_{x}$.

The next theorem takes place.

Theorem 5.21. [6, 9] *The following statements are equivalent:*

1) *The point* x *is strongly comparable with* y.
2) *There exists a continuous mapping* $h : H^+(y) \to H^+(x)$ $(H^+(x) := \overline{\{xt \mid t \ge 0\}})$ *satisfying the condition* (18) *for all* $q \in H^+(y)$ *and* $t \in \mathbb{T}_+$, *and besides* $h(y) = x$.
3) $\mathfrak{M}^{+}_{y} \subseteq \mathfrak{M}^{+}_{x}$.

Remark 5.22. *From Theorem* 5.21 *and from the results of the works* [29, 30] *follows that the strong comparability of the point* x *with* y *is equivalent to their uniform comparability if the point* y *is st.* L^+. *In general case these notions are apparently different* (*though we do not know the according example*).

3. $\mathfrak{L}^{+\infty}_{y} \subseteq \mathfrak{L}^{+\infty}_{x}$.

Definition 5.23. *We will say that the point* x *is comparable in limit (in positive direction) with the point* y *if* $\mathfrak{L}^{+\infty}_{y} \subseteq \mathfrak{L}^{+\infty}_{x}$.

5.4. Comparability of Motions by the Character of Recurrence of V-Monotone NDS.

Definition 5.24. (X, ρ) *is called* [24] *a metric space with segments if for any* $x_1, x_2 \in X$ *and* $\alpha \in [0, 1]$, *the intersection of* $B[x_1, \alpha r]$ *(the closed ball centered at* x *with radius* αr*, where* $r = \rho(x_1, x_2)$*) and* $B[x_2, (1-\alpha)r]$ *has a unique element* $S(\alpha, x_1, x_2)$.

Definition 5.25. *The metric space (X,ρ) is called* [24] *strict-convex if (X,ρ) is a metric space with segments, and for any $x_1,x_2,x_3\in X,\ x_2\neq x_3,$ and $\alpha\in(0,1)$, the inequality $\rho(x_1,S(a,x_2,x_3))<\max\{\rho(x_1,x_2),\rho(x_1,x_3)\}$ holds.*

Definition 5.26. *A subset C of X is said to be metric-convex (see* [24]*) if $S(\alpha,x_1,x_2)\in C$ for any $\alpha\in(0,1)$ and $x_1,x_2\in C$.*

Definition 5.27. *A Banach space X is said to be:*

(i) *uniformly convex, if for each $0<\varepsilon\leq 2$ there exists $\delta(\varepsilon)>0$ such that $|x|,|y|\leq 1$ and $|x-y|\geq\varepsilon$ implies $|x+y|\leq 2(1-\delta(\varepsilon))$;*
(ii) *strictly convex, if for any $x,y\in X$ with $|x|=|y|=1$ and $x\neq y$, $|\lambda x+(1-\lambda)y|<1$ for $\lambda\in(0,1)$.*

Remark 5.28. *1. Uniformly convex Banach spaces are strictly convex, but the converse is not true.*

2. If $(X,|\cdot|)$ is a strictly convex Banach space, then the metric space (X,ρ) ($\rho(x_1,x_2):=|x_1-x_2|$) is metric-convex.

3. If M is a convex subset of strictly convex Banach space $(X,|\cdot|)$, then the metric space (M,ρ) ($\rho(x_1,x_2):=|x_1-x_2|$) is metric-convex.

4. Every convex closed subset X of the Hilbert space H equipped with metric $\rho(x_1,x_2)=|x_1-x_2|$ is strictly metric-convex.

For any subset C of X we denote by $\overline{conv}C$ the closed convex envelope of C, i.e. $\overline{conv}C$ is the intersection of all closed, metric-convex sets containing C.

Lemma 5.29. [10] *Let (M,ρ) be a compact strictly metric-convex space and $\mathcal{E}$ be a compact sub-semigroup of isometries of the semi-group M^M (i.e. $E\subseteq M^M$ and $\rho(\xi x_1,\xi x_2)=\rho(x_1,x_2)$ for all $\xi\in E$ and $x_1,x_2\in M$). Then there exists a common fixed point $\bar{x}\in M$ of E, i.e. $\xi(\bar{x})=\bar{x}$ for all $\xi\in E$.*

Theorem 5.30. *Suppose that the following conditions hold:*

(i) *$\omega\in\Omega$ be a two-sided Poisson stable point;*
(ii) *$\langle(X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ be a V-monotone non-autonomous dynamical system and*
    (a) *there are two functions $a,b\in\mathcal{K}$ such that*
        (i) *$Im(a)=Im(b)$, where $Im(a)$ is the set of the values of $a\in\mathcal{K}$;*
        (ii) *$a(\rho(x_1,x_2))\leq V(x_1,x_2)\leq b(\rho(x_1,x_2))$ for all $x_1,x_2\in X$ ($h(x_1)=h(x_2)$);*
    (b) *$V(x_1,x_2)=V(x_2,x_1)$ for all $(x_1,x_2)\in X\dot{\times}X$;*
    (c) *$V(x_1,x_2)\leq V(x_1,x_3)+V(x_3,x_2)$ for all $x_1,x_2,x_3\in X$ with the condition $h(x_1)=h(x_2)=h(x_3)$;*
(iii) *the space (X_ω,V_ω) is strictly metric-convex for all $\omega\in\Omega$, where $X_\omega:=h^{-1}(\omega):=\{x\in X\mid h(x)=\omega\}$ $(\omega\in\Omega)$ and $V_\omega:=V|_{X_\omega\times X_\omega}$;*
(iv) *the NDS $\langle(X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ admits a conditionally precompact invariant set $J\subseteq X$ such that the set J_ω is metric-convex.*

Then there exists at least one point $x\in J_\omega$ such that $\mathfrak{N}_\omega\subseteq\mathfrak{N}_x$.

Proof. According to Lemma 3.5 the NDS $\langle(X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ is uniformly stable in the positive direction. By Theorem 4.10 under the conditions of Theorem 5.30 the semi-group dynamical system $(X,\mathbb{T},\pi)$ defines on J a group dynamical system $(J,\mathbb{S},\pi)$. Then under the conditions of Theorem 5.30, $E_\omega\neq\emptyset$ is a compact sub-semigroup of the Ellis semigroup $E(J,\mathbb{S},\pi)$. According to Theorem 5.13, we have $V(\xi(x_1),\xi(x_2)) = V(x_1,x_2)$ for all $(x_1,x_2)\in J_\omega\times J_\omega$ and $\xi\in E(J,\mathbb{S},\pi)$ and consequently, under the conditions of Theorem 5.30 we have a strictly metric-convex (with respect to the complete metric $V_\omega := V|J_\omega\times J_\omega$) compact set J_ω and a compact semigroup of isometries E_ω acting on J_ω. Applying Lemma 5.29, we obtain a common fixed point $x\in J_\omega$. Now we will prove that $\mathfrak{N}_\omega\subseteq\mathfrak{N}_x$. In fact, let $\{t_n\}\in\mathfrak{N}_\omega$. Under the conditions of Theorem the set $Q:=\overline{\cup\{\pi(t_n,J_\omega)\mid n\in\mathbb{N}\}}$ is compact. If x_1 and x_2 are two accumulation points of the sequence $\{xt_n\}$, then there are $\{t_n^i\}\subseteq\{t_n\}$ such that $\{xt_n^i\}\to x_i$ $(i=1,2)$. On the other hand since the set $Q:=\overline{\cup\{\pi(t_n,J_\omega)\mid n\in\mathbb{N}\}}$ is compact we may suppose that the sequences $\{\pi^{t_n^i}\}$ $(\pi^{t_n}: J_\omega\mapsto Q)$ is pointwise convergent and ξ_i $(i=1,2)$ is its limit. Then $\xi_i(x)=x_i$ and $x_1=\xi_1(x)=x=\xi_2(x)=x_2$. Thus the sequence $\{xt_n\}$ is relatively compact and admits a unique accumilation point and, consequently, it is convergent. The theorem is proved. □

Corollary 5.31. *Under the conditions of Theorem 5.30 the point x is stationary (τ ($\tau>0$) - periodic, almost automorphic, almost periodic in the sense of Levitan, almost recurrent, Poisson stable), if the point $\omega\in\Omega$ is stationary (τ ($\tau>0$) - periodic, almost automorphic, almost periodic in the sense of Levitan, almost recurrent, Poisson stable).*

Definition 5.32. *A set $M\subset X$ is called minimal with respect to a dynamical system $(X,\mathbb{T},\pi)$ if it is nonempty, closed and invariant and if no proper subset of M has these properties.*

Theorem 5.33. *Suppose that the following conditions hold:*

- (i) *Ω is a compact minimal set;*
- (ii) *$\langle(X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ be a V-monotone non-autonomous dynamical system and*
  - (a) *$V(x_1,x_2)=V(x_2,x_1)$ for all $(x_1,x_2)\in X\dot{\times}X$;*
  - (b) *$V(x_1,x_2)\le V(x_1,x_3)+V(x_3,x_2)$ for all $x_1,x_2,x_3\in X$ with the condition $h(x_1)=h(x_2)=h(x_3)$;*
- (iii) *the space (X_ω,V_ω) is strictly metric-convex for all $\omega\in\Omega$;*
- (iv) *the NDS $\langle(X,\ \mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ admits a compact invariant set $J\subseteq X$ such that the set J_ω ($\omega\in\Omega$) is metric-convex.*

Then for all $\omega_0\in\Omega$ there exists at least one point $x_{\omega_0}\in J_{\omega_0}$ such that $\mathfrak{M}_{\omega_0}\subseteq\mathfrak{M}_{x_{\omega_0}}$.

Proof. According to Theorems 3.4 and 4.10, under the conditions of Theorem 5.13 the semi-group dynamical system $(X,\mathbb{T},\pi)$ defines on J a group dynamical system $(J,\mathbb{S},\pi)$. Let $\omega_0\in\Omega$ be an arbitrary point of Ω and $E=E(J,\mathbb{S},\pi)$ be the Ellis semigroup of the dynamical system $(J,\mathbb{S},\pi)$, i.e. $E(J,\mathbb{S},\pi)=\overline{\{\pi^t\mid t\in\mathbb{S}\ \}}$, where by bar we denote the close in J^J (J^J is equipped with the topology of Tihonoff). We denote by $E_{\omega_0}:=\{\xi\in E\mid \xi(J_{\omega_0})\subseteq J_{\omega_0}\}$. Then under the conditions of Theorem 5.13, $E_{\omega_0}\neq\emptyset$ is a compact sub-semigroup of Ellis semi-group E. According to

Theorem 4.10, we have $V(\xi(x_1),\xi(x_2)) = V(x_1,x_2)$ for all $(x_1,x_2) \in J_{\omega_0} \times J_{\omega_0}$ and consequently, under the conditions of Theorem 5.13 we have a strictly metric-convex (with respect to the complete metric $V_{\omega_0} := V|J_{\omega_0} \times J_{\omega_0})$ compact set J_{ω_0} and a compact semi-group of isometries E_{ω_0} acting on J_{ω_0}. Applying Lemma 5.29, we obtain a common fixed point $\bar{x}_{\omega_0} \in J_{\omega_0}$. We denote by $\Sigma := H(\bar{x}_{\omega_0}) = \overline{\{\bar{x}_{\omega_0}t \mid t \in \mathbb{S}\}}$. It is clear that Σ is a compact invariant set of $(J,\mathbb{S},\pi)$. Obviously, $\Sigma_{\omega_0} := \Sigma \bigcap J_{\omega_0} = \{x_{\omega_0}\}$. We will prove that $\Sigma_\omega := \Sigma \bigcap J_\omega$ contains a single point $\bar{x}_\omega$. It is evident that $\Sigma_\omega \neq \emptyset$ for all $\omega \in \Omega$, because Ω and Σ are compact invariant sets and Ω is minimal. Now we will prove that Σ_ω contains exactly one point. If we suppose the contrary, then there exist $\omega \in \Omega$ and $x_1, x_2 \in \Sigma_\omega$ such that $x_1 \neq x_2$. Since the set Ω is minimal, then there exists a sequence $\{t_n\} \to -\infty$ such that $\omega t_n \to \omega_0$ as $n \to +\infty$. Taking into consideration the compactness of Σ, we may suppose that the sequences $\{x_i t_n\}$ $(i = 1,2)$ are convergent. We denote by $x_i' = \lim_{n\to+\infty} x_i t_n$. It is clear that $x_i' \in \Sigma_{\omega_0}$ and, consequently, $x_1' = x_2'$. On the other hand, according to Theorem 4.10 the dynamical system $(J,\mathbb{S},\pi)$ is distal in negative direction and, consequently, $x_1' \neq x_2'$. The obtained contradiction proves our statement. Thus, we have a compact invariant set $\Sigma \subseteq J$ with the following property $\Sigma_\omega = \Sigma \bigcap J_\omega = \{x_\omega\}$ for all $\omega \in \Omega$. It is easy to verify that $\mathfrak{M}_{\omega_0} \subseteq \mathfrak{M}_{x_{\omega_0}}$. □

Corollary 5.34. *Under the conditions of Theorem 5.13 the point x_{ω_0} is stationary (τ ($\tau > 0$) - periodic, quasi-periodic, almost periodic, almost automorphic, recurrent), if the point $\omega \in \Omega$ is stationary (τ ($\tau > 0$) - periodic, quasi-periodic, almost periodic, almost automorphic, recurrent).*

6. On the Structure of Bounded Motions V-Monotone NDS

Theorem 6.1. *Let $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{S},\sigma),h\rangle$ be a V - monotone non-autonomous dynamical system and the following conditions hold:*

(i) *$V(x_1,x_2) = V(x_2,x_1)$ for all $(x_1,x_2) \in X\dot{\times}X$;*
(ii) *$V(x_1,x_2) \leq V(x_1,x_3) + V(x_3,x_2)$ for all $x_1,x_2,x_3 \in X$ with condition $h(x_1) = h(x_2) = h(x_3)$;*
(iii) *the space (X_ω, V_ω) is strict metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X \mid h(x) = \omega\}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.*

If $\gamma_{x_i} \in \Phi_{x_i}$ ($i = 1,2$), $x_1,x_2 \in X_\omega$ $(\omega \in \Omega)$ and $V(\gamma_{x_1}(t),\gamma_{x_2}(t)) = V(x_1,x_2)$ for all $t \in \mathbb{S}$, then the function $\gamma : \mathbb{S} \to X$ $(\gamma(t) := S(\alpha,\gamma_{x_1}(t),\gamma_{x_2}(t))$ for all $t \in \mathbb{S})$ defines an entire trajectory of dynamical system $(X,\mathbb{T},\pi)$.

Proof. Let $\omega \in \Omega$, $\alpha \in [0,1]$ and $x_1,x_2 \in X_\omega$. We denote by $x := S(\alpha,x_1,x_2)$. Let $\gamma_{x_i} \in \Phi_{x_i}$ $(i = 1,2)$. Consider the function $\gamma : \mathbb{S} \to J$ defined by equality

$$\gamma(t) := S(\alpha,\gamma_{x_1}(t),\gamma_{x_2}(t)) \tag{19}$$

for all $t \in \mathbb{S}$. We will show that γ is an entire trajectory of $(X,\mathbb{T},\pi)$ with condition $\gamma(0) = x$. In fact, under the conditions of Theorem

$$V(\gamma_1(t),\gamma_2(t)) = V(x_1,x_2) = d$$

for all $t \in \mathbb{S}$. Since $V(\gamma_1(t), \gamma(t)) = \alpha d$ for all $t \in \mathbb{S}$ and $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h\rangle$ is V - monotone, we have

$$V(\gamma_1(t+\tau), \pi^t\gamma(\tau)) = V(\pi^t\gamma_1(\tau), \pi^t\gamma(\tau)) \le V(\gamma_1(\tau), \gamma(\tau)) \le \alpha d$$

and

$$V(\gamma_2(t+\tau), \pi^t\gamma(\tau)) = V(\pi^t\gamma_2(\tau), \pi^t\gamma(\tau)) \le V(\gamma_2(\tau), \gamma(\tau)) \le (1-\alpha)d$$

and, consequently,

$$\pi^t\gamma(\tau) \in S(\alpha, \gamma_1(t+\tau), \gamma_2(t+\tau))$$

and so $\pi^t\gamma(\tau) = \gamma(t+\tau)$ for all $\tau \in \mathbb{S}$ and $t \in \mathbb{T}$. Now we will prove that the function γ is continuous. It is clear that γ is continuous on $\mathbb{T}$. Let $t_0 \in \mathbb{S}$, $t_0 \le 0$ and $t = t_0 + h$ $(|h| < \delta,\ \delta > 0)$, then we have

$$\rho(\gamma(t_0+h), \gamma(t_0)) = \rho(\pi^{t_0+|t_0|+\delta+h}\gamma(-|t_0|-\delta), \pi^{t_0+|t_0|+\delta}\gamma(-|t_0|-\delta)). \tag{20}$$

Passing to limit in (20) as $h \to 0$ we obtain $\lim_{h\to 0} \gamma(t_0+h) = \gamma(t_0)$. □

Theorem 6.2. *Under the conditions of Theorem 6.1 if in additionally*

(i) *there exists a function* $a \in \mathcal{K}$ *with property* $\lim_{t\to+\infty} a(t) = +\infty$ *such that* $a(\rho(x_1, x_2)) \le V(x_1, x_2)$ *for all* $(x_1, x_2) \in X\dot{\times}X$*;*
(ii) $\gamma_{x_1} \in \Phi_{x_1}$ *(*$x_1 \in X_\omega$*) is bounded (i.e. the set* $\gamma_{x_1}(\mathbb{S})$ *is bounded in* X*);*
(iii) $\gamma_{x_2} \in \Phi_{x_2}$ *(* $x_2 \in X_\omega$*) and* $V(\gamma_{x_1}(t), \gamma_{x_2}(t)) = V(x_1, x_2)$ *for all* $t \in \mathbb{S}$*.*

Then

(i) γ_{x_2} *is bounded too;*
(ii) *the function* $\gamma : \mathbb{S} \to X$ $(\gamma(t) := S(\alpha, \gamma_{x_1}(t), \gamma_{x_2}(t))$ for all $t \in \mathbb{S})$ *defines a bounded entire trajectory of dynamical system* $(X, \mathbb{T}, \pi)$*.*

Proof. Note that $a(\rho(\gamma_{x_1}(t), \gamma_{x_2}(t))) \le V(\gamma_{x_1}(t), \gamma_{x_2}(t) = V(x_1, x_2)$ for all $t \in \mathbb{S}$. From the last inequality we obtain

$$\rho(\gamma_{x_1}(t), \gamma_{x_2}(t))) \le a^{-1}(V(x_1, x_2)) \tag{21}$$

for all $t \in \mathbb{S}$. Taking into account the boudedness of γ_{x_1} and (21) we ontain the boundedness of γ_{x_2}.

Let $\omega \in \Omega,\ x_i \in X_\omega, \gamma_{x_i} \in \Phi_{x_i}$ $(i = 1, 2)$ $\alpha \in (0, 1)$ and $\gamma(t) := S(\alpha, \gamma_{x_1}(t), \gamma_{x_2}(t))$ for all $t \in \mathbb{S}$, then $\gamma(0) = x := S(\alpha, x_1, x_2)$ and by Theorem 6.1 γ is a bounded entire trajectory of dynamical system $(X, \mathbb{T}, \pi)$. □

7. Applications

Lemma 7.1. *Let* $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ *be a cocycle,* $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ *be the NDS generated by cocycle* φ *(*$X = W \times \Omega$*,* $\pi := (\varphi, \sigma)$*,* $h := pr_2 : X \mapsto \Omega$ *and* $\mathbb{T}_1 \subseteq \mathbb{T}_2$*) and* $\varphi(\mathbb{T}_1, u_0, \omega_0)$ *(*$(u_0, \omega_0) \in W \times \Omega$*) be relatively compact subset of* W*. Then the set* $J := H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{T}_1\}}$ *is conditionally relatively compact with respect to NDS* $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$*, where* $x_0 := (u_0, \omega_0) \in X$*.*

Proof. Let $K \subseteq \Omega$ be an arbitrary compact subset of Ω and $\{x_n\} \subseteq h^{-1}(K) \cap J$ ($x_n = (u_n, \omega_n)$). Since $h(x_n) = \omega_n \in K$, then the sequence $\{\omega_n\}$ is relatively compact. For $x_n \in J = H(x_0)$ there exists $t_n \in \mathbb{T}_1$ such that

$$\max\{\rho(\omega_0, \sigma(t_n, \omega_0), \omega_n), \rho(\varphi(t_n, u_0, \omega_0), u_n)\} \leq \frac{1}{n} \tag{22}$$

($n \in \mathbb{N}$). From the inequality (22) it follows that the sequence $\{u_n\}$ is relatively compact, because $\{\varphi(t_n, u_0, \omega_0)\}$ is so. Thus the sequence $\{x_n\} := \{(u_n, \omega_n)\}$ is relatively compact. □

7.1. **Almost Periodic and Almost Automorphic Solutions of Forced Vectorial Liénard Equations.** Let $(\Omega, \mathbb{R}, \sigma)$ be a two-sided dynamical system. Consider the following vectorial Liénard equation

$$u''(t) + \frac{d}{dt}[\nabla F(u(t))] + Cu(t) = f(\omega t), \tag{23}$$

where $\omega t := \sigma(t, \omega)$, $f \in C(\Omega, E^m)$, $C : E^m \mapsto E^m$ is a symmetric and nonsingular linear operator, and ∇F denotes the gradient of the convex function F on E^m. If the operator C is positive definite, then by introducing the product space $E^m \times E^m \simeq E^{2m}$ endowed with the inner product associated to the quadratic form Q given by

$$Q(u, v) := |u|^2 + \langle C^{-1}v, v\rangle,$$

the equation (23) can be put the form

$$U'(t) + G(\omega t, U(t)) = 0, \tag{24}$$

where $G \in C(\Omega \times E^{2m}, E^{2m})$ and the partial function $G(\omega, \cdot, \cdot)$ is monotone for each $\omega \in \Omega$ with respect to inner product associated to Q, i.e. for each $\omega \in \Omega$

$$\begin{aligned} &\langle G_1(\omega, u_1, v_1) - G_1(\omega, u_2, v_2), u_1 - u_2\rangle + \\ &\langle C^{-1}(G_2(\omega, u_1, v_1) - G_2(\omega, u_2, v_2)), v_1 - v_2\rangle \geq 0, \end{aligned} \tag{25}$$

for all u_1, u_2, v_1 and $v_2 \in E^m$. Indeed, by letting $v(t) := u'(t) + \nabla F(u(t))$; $U(t) := (u(t), v(t))$, equation (23) reduces to

$$U'(t) + \nabla\Phi(U(t)) + JU(t) = \mathcal{F}(\omega t)$$

where $\Phi(u, v) := F(u)$, $J := \begin{pmatrix} 0 & -I \\ C & 0 \end{pmatrix}$ and $\mathcal{F}(\omega) := (0, f(\omega))$.

Lemma 7.2. [15] *Let I the interval $[0, +\infty)$ or the whole real line $\mathbb{R}$. Let $f(t) := f(\omega t)$ ($\forall t \in I$) bounded on I. If $u \in C^2(I, \mathbb{R}^n)$ is a bounded on I solution of equation (23), then u' and u'' are bounded on I too.*

Theorem 7.3. *The following statements hold:*

(i) *Let $\omega_0 \in \Omega$ be a τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) point. If the equation (23) admits a bounded on $\mathbb{R}$ solution u_0, then (23) has at least one τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) solution u such that $\mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_u$.*

(ii) *Let* $\omega_0 \in \Omega$ *be a* τ*-periodic (quasi-periodic, almost periodic in the sense of Bohr, recurrant in the sense of Birkhoff) point. If the equation (23) admits a bounded on* $\mathbb{R}$ *solution* u_0*, then (23) has at least one* τ*-periodic (quasi-periodic, almost periodic in the sense of Bohr, recurrant in the sense of Birkhoff) solution* u *such that* $\mathfrak{M}_{\omega_0} \subseteq \mathfrak{M}_u$*.*

Proof. Consider the equation (24). By Lemma 7.2 it admits a bounded on $\mathbb{R}$ solution $U_0(t) := (u_0(t), u_0'(t))$ $(t \in \mathbb{R})$.

Denote by φ the cocycle associated by equation (23), where $\varphi(t, x, \omega)$ is the solution of equation (23) with initial condition $\varphi(0, x, \omega) = x$ and $x = (u, v)$. Let $X = \Omega \times \mathbb{R}^n$, $(X, \mathbb{R}_+, \pi)$ be a skew-product dynamical system and $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be a nonautonomous dynamical system, generated by equation (24). Denote by $\mathcal{V} : X \to \mathbb{R}^+$ the function defined by equality $\mathcal{V}(\omega, (u_1, v_1), (u_2, v_2)) := \frac{1}{2}(\langle u_1 - u_2, u_1 - u_2\rangle + \langle C^{-1}(v_1 - v_2), v_1 - v_2\rangle)$ for all $(\omega, (u_i, v_i)) \in X := \Omega \times \mathbb{R}^n$ $(i = 1, 2)$, then

$$V(\omega t, \varphi(t, x_1, \omega), \varphi(t, x_2, \omega)) =$$
$$\frac{1}{2}(\langle \varphi_1(t, x_1, \omega) - \varphi_1(t, x_2, \omega)), \varphi_1(t, x_1, \omega) - \varphi_1(t, x_2, \omega)\rangle +$$
$$\langle C^{-1}(\varphi_2(t, x_1, \omega)) - \varphi_2(t, x_2, \omega), \varphi_2(t, x_2, \omega) - \varphi_2(t, x_2, \omega))\rangle$$

$(\varphi := (\varphi_1, \varphi_2))$. Since

$$\frac{dV(\omega t, \varphi(t, x_1, \omega), \varphi(t, x_2, \omega))}{dt} = \langle G_1(\omega t, \varphi_1(t, x_1, \omega)) - G_1(\omega t, \varphi_1(t, x_2, \omega)),$$
$$\varphi_1(t, x_1, \omega) - \varphi_1(t, x_2, \omega)\rangle + \langle C^{-1}(G_2(\omega t, \varphi_1(t, x_1, \omega)) - G_2(\omega t, \varphi_1(t, x_2, \omega))),$$
$$\varphi_2(t, x_1, \omega) - \varphi_2(t, x_2, \omega)\rangle,$$

then by (25) one has $V(\omega t, \varphi(t, x_1, \omega), \varphi(t, x_2, \omega)) \le V(\omega, x_1, x_2)$ for all $\omega \in \Omega$, x_1, $x_2 \in X$ and $t \in \mathbb{R}_+$.

Let u_0 be a bounded on $\mathbb{R}$ solution of equation (23), then by Lemmma 7.2 $U_0 := (u_0, u_0')$ is the bounded on $\mathbb{R}$ solution of equation (24). Let $x_0 := U_0(0) \in \mathbb{R}^n \times \mathbb{R}^n$, $V := \overline{co}(\varphi(\mathbb{R}, x_0, \omega_0))$ be the compact convex hull of the set $\varphi(\mathbb{R}, x_0, \omega_0)$ and $I_\omega := \{x \in V \mid$ such that $\varphi(t, x, \omega)$ is a solusion of (23) defined on $\mathbb{R}$ and $\varphi(\mathbb{R}, x, \omega) \subseteq V$ $\}$. It easy to veryfy that I_ω is a nonempty and compact subset of V. According to Theorem 6.2 the set I_{ω_0} is convex. Now to finish the proof of Theorem it is sufficient to refere Theorems 5.30 and 5.33 (see also Corollaries 5.31 and 5.34). □

Example 7.4. *We consider the equation*

$$x'' + p(x)x' + ax = f(\omega t), \tag{26}$$

where $p \in C(\mathbb{R}, \mathbb{R})$*,* $f \in C(\Omega, \mathbb{R})$ *and* a *is a positive number. Denote by* $y = x' + \mathcal{F}(x)$*, where* $\mathcal{F}(x) = \int_0^x p(s)ds$*, then we obtain the system*

$$\begin{cases} x' = y - \mathcal{F}(x) \\ y' = -ax + f(\omega t). \end{cases}$$

Theorem 7.5. *Suppose the following conditions hold:*

(i) $p(x) \ge 0$ *for all* $x \in \mathbb{R}$*;*
(ii) *the equation (26) admits a bounded on* $\mathbb{R}$ *solution.*

Then

(i) *if $\omega_0 \in \Omega$ is a τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) point, then the equation (26) has at least one τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) solution u such that $\mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_u$.*

(ii) *if $\omega_0 \in \Omega$ is a τ-periodic (quasi-periodic, almost periodic in the sense of Bohr, recurrant in the sense of Birkhoff) point, then (26) has at least one τ-periodic (quasi-periodic, almost periodic in the sense of Bohr, recurrant in the sense of Birkhoff) solution u such that $\mathfrak{M}_{\omega_0} \subseteq \mathfrak{M}_u$.*

Proof. Denote by $F(x) := \int_0^x \int_0^\eta p(s)dsd\eta$, then $p(x)x' = \frac{dF'(x)}{dt}$ and $F''(x) = p(x) \geq 0$. Now our statement it follows from Theorem 7.3. $\square$

7.2. **Dissipative diferential equation.** Let X be a real Banach space with the norm $|\cdot|$ and X^* its dual with the dual norm $|\cdot|$. The value of $f \in X^*$ will be denote by $\langle x, f\rangle$. Let $J : X \mapsto X^*$ be the duality mapping of X [1, 33], i.e., for $x \in X$, $J(x) := \{f \in X^* \mid \langle x, f\rangle = |x|^2 = |f|^2\}$.

Definition 7.6. *The mapping $F : X \mapsto X$ is called dissipative, if for any $x, y \in X$,*

$$\langle F(x) - F(y), f\rangle \leq 0 \tag{27}$$

for $f \in J(x - y)$.

If X is a Hilbert space, then for any $x \in X$, $J(x) = x$, hence (27) become

$$\langle F(x) - F(y), x - y\rangle \leq 0$$

for $x, y \in X$.

Lemma 7.7. [18] *Let $F : \Omega \times X \mapsto X$ be a continuous function and for each $\omega \in \Omega$ the partial mapping $F(\omega, \cdot) : X \mapsto X$ is dissipative in x. If $x(t)$ and $y(t)$ are two solutions on interval $(a, b) \subseteq \mathbb{R}$ of the equation*

$$x' = F(\omega t, x) \ (\omega \in \Omega), \tag{28}$$

then

$$|x(t) - y(t)| \leq |x(s) - y(s)|$$

for $a \leq s \leq t \leq b$.

Corollary 7.8. *Let $F : \Omega \times X \mapsto X$ be a continuous function and for each $\omega \in \Omega$ the partial mapping $F(\omega, \cdot) : X \mapsto X$ is dissipative in x. Then the problem Cauchy*

$$x' = F(\omega t, x), \ x(t_0) = x_0$$

admits at most one solution.

Theorem 7.9. *Suppose the following conditions hold:*

(i) *X is a real and strictly convex Banach space;*
(ii) *$F \in C(\Omega \times X, X)$;*
(iii) *for each $\omega \in \Omega$ the mapping $F(\omega, \cdot) : X \mapsto X$ is dissipative;*
(iv) *the equation (28) has a relatively compact on $\mathbb{R}$ solution $u(t)$ (i.e. $u(t)$ is a solution of equation (28) defined on $\mathbb{R}$ and $u(\mathbb{R})$ is relatively compact).*

Then

(i) *If $\omega_0 \in \Omega$ is a τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) point, then (28) has at least one τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) solutin $\bar{w}$ such that $\mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_{\bar{w}}$.*

(ii) *If $\omega_0 \in \Omega$ is a quasi-periodic (almost periodic in the sense of Bohr, recurrant in the sense of Birkhoff) point, then (28) has at least one quasi-periodic (almost periodic in the sense of Bohr, recurrent in the sense of Birkhoff) solution $\bar{w}$ such that $\mathfrak{M}_{\omega_0} \subseteq \mathfrak{M}_{\bar{w}}$.*

Proof. Let $u_0 := u(0)$, $V := \overline{conv}\varphi(\mathbb{R}, u_0, \omega_0)$ $(\varphi(t, u_0, \omega_0) := u(t))$ $\tilde{X} := V \times \Omega$. Denote by $X := \{(u, \omega) \in \tilde{X} \mid$ such that the equation (28) admits a unique solution $\varphi(t, u, \omega) \in V$ $(\forall t \in \mathbb{R})$ $\}$. Note that X is a closed subset of $V \times \Omega$. In fact, let $(u, \omega) \in \bar{X}$ (by $\bar{X}$ is denoted the closure of X in $V \times \Omega$), then there exists a sequence $\{(u_n, \omega_n)\} \subseteq X$ such that $\{(u_n, \omega_n)\} \to (u, \omega)$ as $n \to \infty$. Thus the sequence $\{\omega_n\} \to \omega$ as $n \to \infty$ and $\varphi(\mathbb{R}, u_n, \omega_n) \subseteq V$. Now we will prove that the sequence $\{\varphi(t, u_n, \omega_n)\}$ converges to $\varphi(t, u, \omega)$ uniformly with respect to t on every interval $[-L, L] \subset \mathbb{R}$ $(L > 0)$. For this end we remark that $\varphi(t, u_n, \omega_n)' = F(\omega_n t, \varphi(t, u_n, \omega_n))$ $(\forall t \in \mathbb{R})$ and, consequently, $|\varphi(t, u_n, \omega_n)'| \le M(\omega, u, L)$ for all $t \in [-L, L]$, where $M(L) := \max\{|F(\omega, v)| \ : \omega \in \Omega, \ v \in K(L)\}$ and

$$K(L) := \overline{\cup\{\varphi(t, u_n, \omega_n) \mid t \in [-L, L], n \in \mathbb{N}\}}.$$

By Arzelà-Ascoli Theorem the sequence $\{\varphi(t, u_n, \omega_n)\}$ is relatively compact in compact-open topology. Note that every point of accumulation of this sequence is a solution of the equation (28) passing though the point $u \in V$ at the initial moment $t = 0$. According to Corollary 7.8 the equation (28) admits at most one solution with initial data $u(0) = u$. This means that the sequence $\{\varphi(t, u_n, \omega_n)\}$ is relatively compact and it has at most one point of accumulation and, consequently, it is convergent and $\lim\limits_{n \to \infty} \varphi(t, u_n, \omega_n) = \varphi(t, u, \omega)$ for all $t \in \mathbb{R}$, i.e., $(u, \omega) \in X$.

Now we can define on X a dynamical system as follow: $\pi(t, (u, \omega)) := (\varphi(t, u, \omega), \omega t)$ for all $(u, \omega) \in X$ and $t \in \mathbb{R}$. Let $h := pr_2 : X \mapsto \Omega$ be the mapping defined by equality $h(u, \omega) := \omega$. It easy to check that h is a homomorphism from $(X, \mathbb{R}, \pi)$ onto $(\Omega, \mathbb{R}, \sigma)$ and, consequently, the triplet $\langle (X, \mathbb{R}, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ is a non-autonomous dynamical system (generated by equation (28) and its solution $\varphi(t, u_0, \omega_0)$). Finally we rmark that the space X is conditionally relatively compact. To finish the proof of Theorem it is sufficient to apply Theorems 5.30 and 5.33 (see also Corollaries 5.31 and 5.34). ☐

Example 7.10. *In quality of example which illustrates this theorem we can consider the following equation*

$$u' = g(u) + f(\theta_t \omega),$$

where $f \in C(\Omega, \mathbb{R})$ *and*

$$g(u) = \begin{cases} (u+1)^2 & : \quad u < -1 \\ 0 & : \quad |u| \le 1 \\ -(u-1)^2) & : \quad u > 1. \end{cases}$$

7.3. **The second order equation** $w'' + B(\omega t, w') + Aw = f(\omega t)$**.** Let A be a nxn positive definite matrixe and $A^{1/2}$ its square root, $B \in C(\Omega \times R^n, R^n)$. We consider the second order equation

$$w'' + B(\omega t, w') + Aw = f(\omega t) \ (\omega \in \Omega). \tag{29}$$

Letting $u := A^{1/2}w, \ v := w'$, the equation (29) reduces to the first order equation

$$\begin{cases} u' = A^{1/2}v \\ v' = -A^{1/2}u - B(\omega t, v) + f(\omega t). \end{cases}$$

If for each $\omega \in \Omega$ the mapping $-B(\omega, \cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is dissipative, i.e.,

$$\langle B(\omega, u) - B(\omega, v), u - v \rangle \ge 0$$

for all $u, v \in \mathbb{R}^n$, then for each $\omega \in \Omega$, $F(\omega, u, v) := (A^{1/2}v, -A^{1/2}u - B(\omega, v) + f(\omega))$ is dissipative in $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Thus from Theorem 7.9 one has the following

Corollary 7.11. *Suppose the following conditions hold:*

(i) *A is a positive definite matrix;*
(ii) *$B : \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $f : \Omega \mapsto \mathbb{R}^n$ are continuous functions;*
(iii) *The equation (29) admits a bounded on $\mathbb{R}$ solution w such that w' is bounded too.*

Then the following statements hold:

(i) *If $\omega_0 \in \Omega$ is a τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) point, then (29) has at least one τ-periodic (almost automorphic, almost periodic in the sense of Levitan, almost recurrent, stable in the sense of Poisson) solution $\bar{w}$ such that $\mathfrak{N}_{\omega_0} \in \mathfrak{N}_{\bar{w}}$.*
(ii) *If $\omega_0 \in \Omega$ is a quasiperiodic (almost periodic in the sense of Bohr, recurant in the sense of Birkhoff) point, then (26) has at least one qusiperiodic (almost periodic, recurrent) solution $\bar{w}$ such that $\mathfrak{M}_{\omega_0} \in \mathfrak{M}_{\bar{w}}$.*

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