# THE RELATIONSHIP BETWEEN PULLBACK, FORWARD AND GLOBAL ATTRACTORS OF NONAUTONOMOUS DYNAMICAL SYSTEMS

### DAVID N. CHEBAN, PETER E. KLOEDEN, AND BJÖRN SCHMALFUSS

ABSTRACT. Various types of attractors are considered and compared for nonautonomous dynamical systems involving a cocycle state space mapping that is driven by an autonomous dynamical system on a compact metric space. In particular, conditions are given for a uniform pullback attractor of the cocycle mapping to form a global attractor of the associated autonomous skew-product semi-dynamical system. The results are illustrated with several examples that are generated by differential equations on a Banach space with a uniformly dissiptative structure induced by a montone operator.

### 1. INTRODUCTION

Nonautonomous dynamical systems can often be formulated in terms of a cocycle mapping for the dynamics in the state space that is driven by an autonomous dynamical system in what is called a parameter space. Traditionally the driving system is topological and the resulting cartesian product system forms an autonomous semi-dynamical system that is known as a skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such nonautonomous dynamical systems via the associated skew-product flow [4, 6, 7, 10, 12, 20, 30, 34].

Recent developments in random dynamical systems give rise to non-autonomous dynamical systems for which the driving autonomous system is measure theoretic rather than topological. A new type of attractor, called a pullback attractor, was proposed and investigated for **nonautonomous or these** random dynamical systems [11, 13, 17, 26, 27]. Essentially, it consists of a parametrized family of nonempty compact subsets of the state space that are mapped onto each other by the cocycle mapping as the parameter is changed by the underlying driving system. Pull back attraction **describing this attractors** to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. **The deeper** 

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reason for this procedure is that a cocycle can be interpreted as a mapping between the fibers of a fiber bundle where the image fiber is fixed. (The kernels of a global attractor of the skew-product flows considered in [12] are very similar). This differs from the more conventional forward convergence where the parameter value of the limiting object also evolves with time, in which case the parametrized family could be called a forward attractor.

Pullback attractors and forward attractors can, of course, be defined for nonautonomous dynamical systems with a topological driving system [22, 23, 24]. In fact, when the driving system is the shift operator on the real line, forward attraction to a time varying solution, say, is the same as the attraction in Lyapunov asymptotic stability. The situation of a compact parameter space is dynamically more interesting as the associated skew-product flow may then have a global attractor. The relationship between the global attractor of the skew-product system and the pullback and forward attractors of the cocycle system is investigated in this paper. We also note that forward attractors are stronger than global attractors if we suppose a *compact* set of nonautonomous perturbations. An example is presented in which the cartesian product of the component subsets of a pullback attractor is not a global attractor of the skew-product flow. This set is, however, is a maximal compact invariant subset of the skew-product flow. By a generalization of some stability results of Zubov [35] it is asymptotically stable. Thus a pullback attractor always generates a local attractor of the skew-product system, but this need not be a global attractor. If, however, the pullback attractor generates a global attractor in the skew-product flow and if, in addition, its component subsets depend lower continuously on the parameter, then the pullback attractor is also a forward attractor.

Several examples illustrating these results are presented in the final section.

Peter, please emphasize in the introduction the meaning of compact sets P

### 2. Nonautonomous Dynamical Systems and their Attractors

A general nonautonomous dynamical system is defined here in terms of a cocycle mapping  $\phi$  on a state space U that is driven by an autonomous dynamical system  $\sigma$  acting on a base space P, which will be called the parameter space. In particular, let  $(U, d_U)$  be a complete metric space, let  $(P, d_P)$  be a compact metric space and let  $\mathbb{T}$ , the time set, be either  $\mathbb{R}$  or  $\mathbb{Z}$ . Peter, say same word here about our Towards... article. I have not understood exatly what the referee I want.

An autonomous dynamical system  $(P, \mathbb{T}, \sigma)$  on P consists of a continuous mapping  $\sigma : \mathbb{T} \times P \to P$  for which the  $\sigma_t = \sigma(t, \cdot) : P \to P, t \in \mathbb{T}$ , form a group of homeomorphisms on P under composition over  $\mathbb{T}$ , that is, satisfy

$$\sigma_0 = \mathrm{id}_P, \qquad \sigma_{t+\tau} = \sigma_t \circ \sigma_\tau$$

for all  $t, \tau \in \mathbb{T}$ . In addition, a continuous mapping  $\phi : \mathbb{T}^+ \times U \times P \to U$  is called a cocycle with respect to an autonomous dynamical system  $(P, \mathbb{T}, \sigma)$  if it satisfies

$$\phi(0, u, p) = u, \qquad \phi(t + \tau, u, p) = \phi(t, \phi(\tau, u, p), \sigma_{\tau} p)$$

 $\mathbf{2}$ 

for all  $t, \tau \in \mathbb{T}^+$  and  $(u, p) \in U \times P$ . **Definition 2.1.** The triple  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  is called a nonautonomous or cocycle dynamical system on the state space U.

Let  $(X, d_X)$  be the cartesian product of  $(U, d_U)$  and  $(P, d_P)$ . Then the mapping  $\pi$ :  $\mathbb{T}^+ \times X \to X$  defined by

$$\pi(t, (u, p)) := (\phi(t, u, p), \sigma_t p)$$

forms a semi–group on X over  $\mathbb{T}^+$  [29].

**Definition 2.2.** The autonomous semi-dynamical system  $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$  is called the skew-product dynamical system associated with the cocycle dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ .

For example, let U be a Banach space and let the space  $C = C(\mathbb{R} \times U, U)$  of continuous functions  $f : \mathbb{R} \times U \to U$  be equipped with the compact open topology. Consider the autonomous dynamical system  $(C, \mathbb{R}, \sigma)$ , where  $\sigma$  is the shift operator on C defined by  $\sigma_t f(\cdot, \cdot) := f(\cdot + t, \cdot)$  for all  $t \in \mathbb{T}$ . Let P be the hull H(f) of a given functions  $f \in C$ , that is,

$$P = H(f) := \overline{\bigcup_{t \in \mathbb{R}} \{f(\cdot + t, \cdot)\}},$$

and denote the restriction of  $(C, \mathbb{R}, \sigma)$  to P by  $(P, \mathbb{R}, \sigma)$ . Let  $F : P \times U \to U$  be the continuous mapping defined by F(p, u) := p(0, u) for  $p \in P$  and  $u \in U$ . Then, under appropriate restrictions on the given function  $f \in C$  (see Sell [29]) defining P, the differential equation

(1) 
$$u' = p(t, u) = F(\sigma_t p, u)$$

generates a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{R}, \sigma) \rangle$ , where  $\phi(t, p, u)$  is the solution of (1) with the initial value u at time t = 0.

Let dist<sub>Y</sub> denote the Hausdorff distance (semi-metric) between two nonempty sets of a metric space  $(Y, d_Y)$ , that is,

$$\operatorname{dist}_Y(A, B) = \sup_{a \in A} \inf_{b \in B} d_Y(A, B),$$

and let  $\mathcal{D}(U)$  be either  $\mathcal{D}_c(U)$  or  $\mathcal{D}_b(U)$ , classes of sets containing either the compact subsets or the bounded subsets of the metric space  $(U, d_U)$ .

The definition of a global attractor for an autonomous semi-dynamical system  $(X, \mathbb{T}^+, \pi)$  is well known. Specifically, a nonempty compact subset  $\mathcal{A}$  of X which is  $\pi$ -invariant, that is, satisfies

(2) 
$$\pi(t, \mathcal{A}) = \mathcal{A} \text{ for all } t \in \mathbb{T}^+,$$

is called a global attractor for  $(X, \mathbb{T}^+, \pi)$  with respect to  $\mathcal{D}(X)$  if

(3) 
$$\lim_{t \to \infty} \operatorname{dist}_X(\pi(t, D), \mathcal{A}) = 0$$

for every  $D \in \mathcal{D}(X)$ . Conditions for the existence of such global attractors and examples can be found in [2, 7, 19, 33, 34]. Of course, semi-dynamical systems need not be a skew-product systems, but when they are, the following definition

will be used.

Suppose that the skew-product dynamical system  $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$  has a global attractor  $\mathcal{A}$ . Then we will call the set  $\mathcal{A}$  the *global attractor* with respect to  $\mathcal{D}(U)$  of the associated nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ .

Note since P is compact it is sufficient and necessary to study the convergence 3 for sets  $D \times P \subset X$ ,  $D \in \mathcal{D}(U)$ .

Definition 2.3. The global attractor  $\mathcal{A}$  with respect to  $\mathcal{D}(U)$  of the skew--product dynamical system  $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$  will be called the *global* attractor with respect to  $\mathcal{D}(U)$  of the associated nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ .

Other types of attractors, in particular pullback attractors, that consist of a family of nonempty compact subsets of the state space of the cocycle mapping have been proposed for nonautonomous or random dynamical systems [13, 14, 24, 27, 31]. The main task of this article is to investigate connections of different types of attractors.

**Definition 2.4.** Let  $\widehat{A} = \{A(p)\}_{p \in P}$  be a family of nonempty compact sets of U for which  $\bigcup_{p \in P} A(p)$  is pre-compact and let  $\widehat{A}$  be  $\phi$ -invariant with respect to a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , that is, satisfies

(4) 
$$\phi(t, A(p), p) = A(\sigma_t p)$$
 for all  $t \in \mathbb{T}^+$ ,  $p \in P$ .

The family  $\widehat{A}$  is called a pullback attractor of  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  with respect to  $\mathcal{D}(U)$  if

(5) 
$$\lim_{t \to \infty} \operatorname{dist}_U(\phi(t, D, \sigma_{-t}p), A(p)) = 0$$

for any  $D \in \mathcal{D}(U)$  and  $p \in P$ , or a uniform pullback attractor if the convergence (5) is uniform in  $p \in P$ , that is, if

$$\lim_{t\to\infty}\sup_{p\in P}{\rm dist}_U(\phi(t,D,\sigma_{-t}p),A(p))=0.$$

The family  $\widehat{A}$  is called a forward attractor if the forward convergence

$$\lim_{t \to T} \operatorname{dist}_U(\phi(t, D, p), A(\sigma_t p)) = 0$$

holds instead of the pullback convergence (5), or a uniform forward attractor if this forward convergence is uniform in  $p \in P$ , that is, if

$$\lim_{t\to\infty}\sup_{p\in P}{\rm dist}_U(\phi(t,D,p),A(\sigma_tp))=0.$$

It follows directly from the definition that a pullback attractor is unique. Obviously, any uniform pullback attractor is also a uniform forward attractor, and vice versa.

If  $\widehat{A}$  is a forward attractor for the nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , then ([7] Lemma 4.2) the subset

(6) 
$$\mathcal{A} = \bigcup_{p \in P} \left( A(p) \times \{p\} \right)$$

of X is the global attractor for the skew-product dynamical system  $(X, \mathbb{T}^+, \pi)$ . A weaker result holds when  $\widehat{A}$  is a pullback attractor. The inverse property is not

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true in general. Since the global attractor is unique so is the forward attractor.

Although we could formulate for weaker assumptions we will restrict ourselves to the case that  $\bigcup_{p\in P}A(p)$  is compact and  $\mathcal{D}(U)$  consists of compact sets.

Our considerations can be embedded into the more general theory of pullback attractors with a domain of attraction  $\mathcal{D}$  consisting of family of sets  $D = \{D(p)\}_{p \in P}$  such that  $\bigcup_{p \in P} D(p)$  is pre-compact in U, see [31].

The following existence result for pull back and attractors is adapted from [14, 21].

**Theorem 2.5.** Let  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , with P compact be a nonautonomous dynamical system and suppose that there exists a family of nonempty sets  $C = \{C(p)\}_{p \in P}$ ,  $\bigcup_{n \in P} C(p)$  pre-compact such that

$$\lim_{t \to 0} \operatorname{dist}_U(\phi(t, D, \sigma_{-t}p), C(p)) = 0$$

for any bounded subset D of U and any  $p \in P$ . Then there exists a pullback attractor.

I took here only compact sets because David proposed to restrict ourselves to compact sets.

A related result is given by Theorem 4.3.4 in [7]: if the skew-product system  $(X, \mathbb{T}^+, \pi)$  has a global attractor  $\mathcal{A}$ , then the nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  has a pullback attractor. The proof is based on the fact that the identical sets  $C(p) \equiv \operatorname{pr}_U \mathcal{A}$  satisfy the assumptions of the previous theorem.

Alternatively, conditions can be given on the nonautonomous dynamical system to ensure the existence of a global attractor of the associated skew-product system. The following theorem is from [7].

**Theorem 2.6.** Let  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , P compact, be a nonautonomous dynamical system for which

(i)  $\phi$  is asymptotically compact, that is, for every bounded positive invariant set D and  $p \in P$ , there exists a compact set C such that

$$\lim_{t \to \infty} \operatorname{dist}_U(\phi(t, D, p), C) = 0,$$

David do we need this if we restrict ourselves to  $\mathcal{D}(U) = \text{copact sets.}$  (ii) there exists a bounded set  $B_0$  that absorbs bounded subsets, that is, for every  $p \in P$  and  $D \in \mathcal{D}_b(U)$  there exists a  $T_{p,D} \geq 0$  such that

$$\phi(t, D, p) \subset B_0$$
 for all  $t \ge T_{p, D}$ 

Then the skew-product system  $(X, \mathbb{T}^+, \pi)$  has a unique global attractor that attractssets from  $\mathcal{D}_b(U)$ .

## We now continue to derive properties of pullback attractors an skew– product dynamical systems.

**Lemma 2.7.** If  $\widehat{A}$  is a pullback attractor of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , where P is compact, then the subset  $\mathcal{A}$  of X defined by (6) is the maximal  $\pi$ -invariant compact set of the associated skew-product dynamical system  $(X, \mathbb{T}^+, \pi)$ .

*Proof.* The  $\pi$ -invariance follows from the  $\phi$ -invariance of  $\widehat{A}$  via

$$\pi(t,\mathcal{A}) = \bigcup_{p \in P} (\phi(t,A(p),p),\sigma_t p) = \bigcup_{p \in P} (A(\sigma_t p),\sigma_t p) = \mathcal{A}.$$

Now  $\mathcal{A} \subset \bigcup_{p \in P} A(p) \times P$ , where P is compact and  $\bigcup_{p \in P} A(p)$  is pre-compact, so  $\mathcal{A}$  is pre-compact. Hence  $\mathcal{B} := \overline{\mathcal{A}}$  is compact, from which it follows that

$$B(p) := \{u : (u, p) \in \mathcal{B}\}$$

is a compact set in U for each  $p \in P$  and that the set

$$\bigcup_{p\in P} B(p)\subset \mathrm{pr}_1\mathcal{B}$$

is pre–compact. On the other hand,  $\mathcal{B}$  is  $\pi$ -invariant since

$$\pi(t,\mathcal{B}) = \pi(t,\bar{\mathcal{A}}) = \pi(t,\mathcal{A}) = \bar{\mathcal{A}} = \mathcal{B}$$

for the continuous mapping  $\pi(t, \cdot)$ . In addition,  $\phi(t, B(p), p) = B(\sigma_t p)$  holds, that is, the B(p) are  $\phi$ -invariant, since

$$\pi(t,\mathcal{B}) = \bigcup_{p \in P} (\phi(t,B(p),p),\sigma_t p) = \mathcal{B} = \bigcup_{p \in P} (B(\sigma_t p),\sigma_t p)$$

and  $\sigma_t p = \sigma_t \hat{p}$  implies that  $p = \hat{p}$  for the homeomorphism  $\sigma_t$ . The construction shows  $B(p) \supset A(p)$ . By the  $\phi$ -invariance of the B(p) and the pullback attraction property it follows then that B(p) = A(p) such that  $\mathcal{A} = \mathcal{B}$ . Hence  $\mathcal{A}$  is compact.

To prove that the compact invariant set  $\mathcal{A}$  is maximal, let  $\mathcal{A}'$  be any other compact invariant set the of skew-product dynamical system  $(X, \mathbb{T}^+, \pi)$ . Then  $\widehat{\mathcal{A}}' = \{A'(p)\}_{p \in P}$  is a family of compact  $\phi$ -invariant subsets of U and by pullback attraction

as  $t \to +\infty$ , where  $K = \overline{\bigcup_{p \in P} A'(p)}$  is compact. Hence  $A'(p) \subseteq A(p)$  for every  $p \in P$ , i.e.  $\widehat{A}' \subseteq \widehat{A}$ , which means  $\mathcal{A}$  is maximal for  $(X, \mathbb{T}^+, \pi)$ .  $\Box$ 

A set valued mapping  $M = \{M(p)\}_{p \in P}$  for  $M(p) \in U$  is called is *upper semi-continuous* if

$$\lim_{n \to p_0} d_X(M(p), M(p_0)) = 0 \quad \text{for any } p_0 \in P.$$

We call such a set valued mapping lower sem-continuous if

$$\lim_{p \to p_0} d_X(M(p_0), M(p)) = 0 \quad \text{for any } p_0 \in P$$

*M* is called *continuous* if it is both upper and lower semi-continuous. *M* is upper semi-continuous if and only if the graph of is closed in  $U \times P$ . *add a reference* It follows straightforwardly from Lemma 2.7:

**Corollary 2.8.** The set valued mapping  $p \to A(p)$  formed with the components sets of a pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$  of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  where P is compact is upper semi-continuous.

The following example shows that, in general, a pullback attractor need not also be a forward attractor nor form a global attractor of the associated skew–product dynamical system.

An example. Let f be the function on  $\mathbb{R}$  defined by

$$f(t) = -\left(\frac{1+t}{1+t^2}\right)^2, \qquad t \in \mathbb{R},$$

and let  $(P, \mathbb{R}, \sigma)$  be the autonomous dynamical system P = H(f), the hull of f in  $C(\mathbb{R}, \mathbb{R})$ , with the shift operator  $\sigma$ . Note that

$$P = H(f) = \bigcup_{h \in \mathbb{R}} \{f(\cdot + h)\} \cup \{0\}.$$

Finally, let *E* be the evaluation functional on  $C(\mathbb{R}, \mathbb{R})$ , that is  $E(p) = p(0) \in \mathbb{R}$ . Lemma 2.9. The functional

$$\gamma(p) = -\int_0^\infty e^{-\tau} E(\sigma_\tau p) \, d\tau = -\int_0^\infty e^{-\tau} p(\tau) \, d\tau$$

is well defined and continuous on P, and the function of  $t \in \mathbb{R}$  given by

$$\gamma(\sigma_t p) = -e^t \int_t^\infty e^{-\tau} p(\tau) \, d\tau = \begin{cases} \frac{1}{1 + (t+h)^2} & : \ p = \sigma_h f \\ 0 & : \ p = 0 \end{cases}$$

is the unique solution of the differential equation

$$x' = x + E(\sigma_t p) = x + p(t)$$

that exists and is bounded for all  $t \in \mathbb{R}$ .

The proof is by straightforward calculation, so will be omitted.

Consider now the nonautonomous differential equation

(7) 
$$u' = g(\sigma_t(p), u),$$

where

$$g(p,u) := \begin{cases} -u - E(p)u^2 & : & 0 \le u\gamma(p) \le 1, \ p \ne 0 \\ -\frac{1}{\gamma(p)} \left(1 + \frac{E(p)}{\gamma(p)}\right) & : & 1 < u\gamma(p), \ p \ne 0 \\ -u & : & 0 \le u, \ p = 0 \end{cases}$$

It is easily seen that this equation has a unique solution passing through any point  $u \in U = \mathbb{R}^+$  at time t = 0 defined on  $\mathbb{R}$ . These solutions define a cocycle mapping

(8) 
$$\phi(t, u_0, p) = \begin{cases} \frac{u_0}{e^t(1 - u_0\gamma(p)) + u_0\gamma(\sigma_t p)} & : & 0 \le u_0\gamma(p) \le 1, \ p \ne 0 \\ u_0 + \frac{1}{\gamma(\sigma_t p)} - \frac{1}{\gamma(p)} & : & 1 < u_0\gamma(p), \ p \ne 0 \\ e^{-t}u_0 & : & 0 \le u_0, \ p = 0 \end{cases}$$

According to the construction, the cocycle mapping  $\phi$  admits as its only invariant sets  $A(p) = \{0\}$  for  $p \in P$ . To see that the  $A(p) = \{0\}$  form a pullback attractor, observe that check whether the following formula is o.k.

$$\phi(t, u_0, \sigma_{-t}p) = \begin{cases} \frac{u_0}{e^t(1 - u_0\gamma(\sigma_{-t}p)) + u_0\gamma(p)} & : & 0 \le u_0\gamma(p) \le 1, \ p \ne 0 \\ u_0 + \frac{1}{\gamma(p)} - \frac{1}{\gamma(\sigma_{-t}p)} & : & 1 < u_0\gamma(p), \ p \ne 0 \\ e^{-t}u_0 & : & 0 \le u_0, \ p = 0 \end{cases}$$

In particular, note that  $t \to \gamma(\sigma_t p)^{-1}$  is a solution of the differential equation (7). Since  $\gamma(\sigma_{-t}p)^{-1}$  tends to  $+\infty$  subexponentially fast for  $t \to \infty$ , it follows that

$$\phi(t, u, \sigma_{-t}p) \le \frac{1}{2}L e^{-\frac{1}{2}t}$$

for any  $u \in [0, L]$  for any  $L \ge 0$  and  $p \in P$  provided t is sufficiently large. Consequently  $\widehat{A} = \{A(p)\}_{p \in P}$  with  $A(p) = \{0\}$  for all  $p \in P$  is a pullback attractor for  $\phi$ . In view of (8), the stable set  $W^s(\mathcal{A}) := \{x \in X \mid \lim_{t \to +\infty} dist_X(\pi^t x, A) = 0\}$  of  $\mathcal{A}$ ,

that is, the set of all points in X that are attracted to  $\mathcal{A}$  by  $\pi$ , is given by

$$W^{s}(\mathcal{A}) = \{(u, p) : p \in P, u \ge 0, u \gamma(p) < 1\} \neq X.$$

Hence the cocycle mapping  $\phi$  in this example admits a pullback attractor that is neither a forward attractor for  $\phi$  nor a global attractor of the associated skew– product flow.

Other examples for different kinds of attractors are given by Scheutzow [28] for random dynamical systems generated by one dimensional stochastic differential equations. However, these considerations are based on the theory of Markov processes.

2.1. Existence of pullback attractors. Pullback attractors for a different class of dynamical systems  $(\mathbb{T}, P, \sigma)$  have been considered in [1, 14, 13, 17, 21, 27], which are all concerned with random dynamical systems that involve a measurable rather than topological driving system  $(\mathbb{T}, P, \sigma)$ . To handle the nonuniformities that are ubiquitous in such systems, the pullback attraction (5) was generalised in [17] to be respect to a prescribed family of p--dependent of sets  $D = \{D(p)\}_{p \in P}$ , namely

$$\lim_{t \to \infty} \operatorname{dist}_U(\phi(t, D(\sigma_{-t}p), \sigma_{-t}p), A(p)) = 0.$$

The family of sets D does not have to be *uniformly* bounded here. To apply general results on existence and uniqueness to the topological, in particular *compact*, dynamical systems under consideration in the present paper, convergence with respect to families  $D = \{D(p)\}_{p \in P}$  for which  $\bigcup_{p \in P} D(p)$  is bounded or pre--compact in U could be investigated. That would, however, be equivalent to convergence for p--independent sets D which are bounded or pre--compact, so nothing new would be gained. Sufficient conditions ensuring the existence of pullback attractors can be found in the references above.

# 3. Asymptotic stability in $\alpha$ -condensing semi-dynamical systems

To continue to investigate general relations between pull back attractors and skew-product flows we have to derive some results from the general stability theory. We start with some definitions.

Let  $(X, \mathbb{T}^+, \pi)$  be a semi–dynamical system. The  $\omega$ -limit set of a set M is defined to be

$$\boldsymbol{\omega}(M) = \bigcap_{\tau \ge 0} \overline{\bigcup_{t \ge \tau} \pi(t, M)}$$

A set M is called Lyapunov stable if for any  $\varepsilon > 0$  there exists an  $\delta = 0$ such that  $\pi(\mathcal{U}_{\delta}(M)) \subset \mathcal{U}_{\varepsilon}(U)$  for  $t \geq 0$ . M is called a *local attractor* if there exists a neighborhood  $\mathcal{U}(M)$  of M such that  $\mathcal{U}(M) \subseteq W^{s}(M)$ . A set Mwhich is Lyapunov stable and a local attractor is called *asymptotically* stable. Note that any asymptotically stable compact set M also attracts compact sets contained in  $\mathcal{U}(M)$ .

Recall that a  $\pi$ -invariant compact set M is said to be *locally maximal* if there exists a number  $\delta > 0$  such that any  $\pi$ -invariant compact set contained in the open  $\delta$ -neighborhood  $U_{\delta}(M)$  of M is in fact contained in M. In addition, a mapping  $\gamma^x : \mathbb{T} \to X$  is called an *entire trajectory through* x of the semi-dynamical system  $(X, \mathbb{T}^+, \pi)$  if

$$\pi(t,\gamma^x(\tau)) = \gamma^x(t+\tau)$$
 for all  $t \in \mathbb{T}^+, \tau \in \mathbb{T}, \gamma^x(0) = x$ 

Finally, the alpha limit set of an entire trajectory  $\gamma^x$  is defined by

$$\boldsymbol{\alpha}_{\gamma^x} = \{ y \in X : \exists \tau_n \to -\infty, \gamma^x(\tau_n) \to y \}.$$

Let  $\alpha$  be a measure of noncompactness on the bounded subsets of a complete metric space  $(Y, d_Y)$  ([19], Pages 13 ff.). Then  $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$  for all nonempty bounded subsets A and B with  $\alpha(A) = 0$  whenever A is pre-compact. An example is the Kuratowski measure of noncompactness defined by

 $\alpha(A) = \inf\{d : A \text{ has a finite cover of diameter } < d\}.$ 

An autonomous semi–dynamical system  $(X, \mathbb{T}^+, \pi)$  is called  $\alpha$ -condensing if  $\pi(t, B)$  is bounded and

$$\alpha(\pi(t,B)) < \alpha(B)$$

for all t > 0 for any bounded set B of X with  $\alpha(B) > 0$ . Remark. In the book [19] there are the a lot class of dynamical systems wich possesse with property. For example: every d.s. on the finite-dimensional space, every d.s. with compact  $\pi^t(t > 0)$  or  $\pi^t = m(t) + r(t)(m(t); X \to X$  is compact fo every t > 0 and  $r(t)x \to 0$  as  $t \to +\infty$  uniformly w.r.t. x on every bounded set X).

**Theorem 3.1.** Let M be a locally maximal compact set for an  $\alpha$ -condensing semidynamical system  $(X, \mathbb{T}^+, \pi)$ . Then M is Lyapunov stable if and only if there exists  $a \delta > 0$  such that

$$\boldsymbol{\alpha}_{\gamma^x} \cap M = \emptyset$$

for any entire trajectory  $\gamma^x$  through any  $x \in U_{\delta}(M) \setminus M$ .

Proof. A proof of the necessity direction was given by Zubov in [35] Theorem 7 for a locally compact space X. This proof remains also true for a non locally compact space under consideration here. Really, let M be a compact invariant set for  $(X, \mathbb{T}^+, \pi)$  stable in the positive direction. If we suppose that this assertion is not true, then there exist  $x \notin M, \gamma^x$  and  $\tau_n \to -\infty$  such that  $\rho(\gamma^x(\tau_n), M) \to 0$  as  $n \to \infty$ . Let  $0 < \varepsilon < \rho(x, M)$  and  $\delta(\varepsilon) > 0$  the corresponding positive number from stability of set M, then for sufficiently large n we have  $\rho(\gamma^x(\tau_n), M) < \delta(\varepsilon)$  and, consequently,  $\rho(\pi^t \gamma^x(\tau_n), M) < \varepsilon$  for all  $t \ge 0$ . In particulary for  $t = -\tau_n$  we have  $\rho(x, M) = \rho(\pi^{-\tau_n} \gamma^x(\tau_n), M) < \varepsilon$ . The obtained contradiction prove our assertion.

# David, add a short proof, or some arguments for the proof

For the sufficiency direction, consider first the case  $\mathbb{T}^+ = \mathbb{Z}^+$  and let  $U_{\delta_0}(M)$  be a neighborhood such that M is locally maximal in  $U_{\delta_0}(M)$ . Suppose that M is not Lyapunov stable, but that the other condition of the theorem holds. Then there exist an  $\varepsilon_0 > 0$  and sequences  $\delta_n \to 0$ ,  $x_n \in U_{\delta_n}(M)$ ,  $k_n \to \infty$  such that  $\pi(k, x_n)$  $\in U_{\varepsilon_0}(M)$  for  $0 \le k \le k_n - 1$  and  $\pi(k_n, x_n) \notin U_{\varepsilon_0}(M)$ . This  $\varepsilon_0$  has to be chosen sufficiently small such that

$$\operatorname{dist}_X(\pi(1, U_{\varepsilon_0}(M)), M) < \frac{\delta_0}{2}.$$

Define  $A = \{x_n\}$  and  $B = \bigcup_{n \in \mathbb{N}} \{\pi(k, x_n) | 0 \le k \le k_n - 1\}$ . Then  $\alpha(A) = 0$  since A is pre–compact. In addition,  $\pi(1, B) \subset U_{\delta_0}(M)$ , so  $\pi(1, B)$  is bounded. Suppose that B is not pre–compact, so  $\alpha(B) > 0$ . It follows by the properties of the measure of noncompactness for the non pre–compact set B that

$$\alpha(B) = \alpha(A \cup \pi(1, B) \cap B) \le \max(\alpha(A), \pi(1, B)) = \alpha(\pi(1, B)) < \alpha(B)$$

which is a contradiction. This shows that B is pre–compact. Now  $\tilde{\gamma}^{\tilde{x}}$  is an entire trajectory of the discrete–time semi–dynamical system above with  $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x}$  and  $\overline{\tilde{\gamma}^{\tilde{x}}(\mathbb{Z}^{-})} \subset \overline{B}$ . Thus the alpha limit set  $\boldsymbol{\alpha}_{\tilde{\gamma}^{\tilde{x}}}$  is nonempty, compact and invariant. In addition,  $\boldsymbol{\alpha}_{\tilde{\gamma}^{\tilde{x}}} \subset U_{\varepsilon_0}(M)$ , hence  $\boldsymbol{\alpha}_{\tilde{\gamma}^{\tilde{x}}} \subset M$  because M is a locally maximal invariant compact set. On the other hand,  $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x} \in U_{\varepsilon_0}(M) \setminus M$ , so  $\boldsymbol{\alpha}_{\tilde{\gamma}^{\tilde{x}}} \cap M = \emptyset$  holds by the assumptions. This contradiction proves the sufficiency of the condition in the discrete–time case.

Now let  $\mathbb{T}^+ = \mathbb{R}^+$  and suppose that  $\alpha_{\gamma^x} \cap M = \emptyset$  where  $x \notin M$  holds for the continuous–time system. Then it also holds for the restricted discrete–time system generated by  $\pi_1 := \pi(1, \cdot)$  because any entire trajectory  $\gamma^x$  of the restricted discrete–time system can be extended to an entire trajectory of the continuous–time system via

$$\gamma^x(t) = \pi(\tau, \gamma^x(n)), \quad n \in \mathbb{Z}, \quad t = n + \tau, \quad 0 < \tau < 1.$$

Consequently, the set M is Lyapunov stable with respect to the restricted discrete– time dynamical system generated by  $\pi_1$ . Since M is compact, for every  $\varepsilon > 0$  there exists a  $\mu > 0$  such that

$$d_X(\pi(t,x),M) < \varepsilon$$
 for all  $t \in [0,1], x \in U_\mu(M)$ .

In view of the first part of the proof above, there is a  $\delta > 0$  such that

$$d_X(\pi(n,x),M) < \min(\mu,\varepsilon)$$
 for  $x \in U_{\delta}(M)$  for  $n \in \mathbb{Z}^+$ .

The Lyapunov stability of M for the continuous dynamical system  $(X, \mathbb{R}^+, \pi)$  then follows from the semi–group property of  $\pi$ .

The next lemma will be needed to formulate the second main theorem of this section. Asymptotical stability here means (locally) Lyapunov stability and attracting.

**Lemma 3.2.** Let M be a compact subset of X that is positively invariant for a semi-dynamical system  $(X, \mathbb{T}^+, \pi)$ . Then M is asymptotically stable if and only if  $\omega(M)$  is locally maximal and asymptotically stable.

*Proof.* Suppose that M is asymptotically stable. Then there exists a closed positively invariant bounded neighborhood C of M contained in its stable set  $W^s(M)$ . The mapping  $\pi$  can be restricted to the complete metric space  $(C, d_X)$  to form a semi-dynamical system  $(C, \mathbb{T}^+, \pi)$ . Since M is a locally attracting set it attracts compact subsets of C. The assertion then follows by Theorems 2.4.2 and 3.4.2 in [19] because  $\omega(M) = \bigcap_{t \in \mathbb{T}^+} \pi(t, M)$ .

Suppose instead that  $\boldsymbol{\omega}(M)$  is asymptotically stable and locally maximal. Since M is compact,  $\boldsymbol{\omega}(M) = \bigcap_{t>0} \pi(t, M)$ . Hence there exist  $\eta > 0$  and  $\tau \in \mathbb{T}^+$  such that

$$\pi(\tau, M) \subset U_{\eta}(\boldsymbol{\omega}(M)) \subset W^{s}(\boldsymbol{\omega}(M)).$$

Now  $\pi^{-1}(\tau, U_{\eta}(\boldsymbol{\omega}(M)))$ , where  $\pi^{-1}$  denotes the pre-image of  $\pi(\tau, \cdot)$  for fixed  $\tau$ , is an open neighborhood of M and  $\pi(\tau, \pi^{-1}(\tau, U_{\eta})(\boldsymbol{\omega}(M))) \subset W^{s}(\boldsymbol{\omega}(M))$ . Hence for any  $x \in \pi^{-1}(\tau, U_{\eta}(\boldsymbol{\omega}(M))) \subset W^{s}(\boldsymbol{\omega}(M))$  we have that  $\pi(t, x)$  tends to  $\boldsymbol{\omega}(M)$  as  $t \to \infty$ , from which it follows that  $\pi(t, x)$  also tends to M because  $M \supset \boldsymbol{\omega}(M)$ .

Then, if M were not Lyapunov stable, there would exist  $\varepsilon_0 > 0$ ,  $\delta_n \to 0$ ,  $x_n \in U_{\delta_n}(M)$  and  $t_n \to \infty$  such that

(9) 
$$\operatorname{dist}_X(\pi(t_n, x_n), M) \ge \varepsilon_0.$$

For sufficiently large  $n_0$ , the set  $\overline{\{x_n\}}_{n \ge n_0}$  would then be contained in the pre-image  $\pi^{-1}(1, U_\eta(\boldsymbol{\omega}(M)))$ . Since M is compact, so is the set  $\overline{\{x_n\}}_{n \ge n_0}$ . This set would thus be attracted by  $\boldsymbol{\omega}(M) \subset M$ , which contradicts (9).

**Lemma 3.3.** Let M be a compact subset of X that is a positively invariant set for an asymptotically compact semi-dynamical system  $(X, \mathbb{T}^+, \pi)$ . Then the set M is asymptotically stable if and only if  $\omega(M)$  is locally maximal and Lyapunov stable. *Proof.* The necessity follows by Lemma 3.2. Suppose instead that  $\omega(M)$  is locally maximal and Lyapunov stable; it is automatically  $\pi$ -invariant since it is an omega limit set. Then for any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$\pi(t, U_{\delta}(\boldsymbol{\omega}(M))) \subset U_{\varepsilon}(\boldsymbol{\omega}(M)) \quad \text{for all } t \ge 0.$$

By the assumption of asymptotical compactness,  $\omega(U_{\delta}(\omega(M)))$  is nonempty and compact with

$$\lim_{t \to \infty} \operatorname{dist}_X(\pi(t, U_{\delta}(\boldsymbol{\omega}(M))), \boldsymbol{\omega}(U_{\delta}(\boldsymbol{\omega}(M)))) = 0$$

(see [19] Corollary 2.2.4.). Since  $\omega(M)$  is locally maximal,  $\omega(U_{\delta}(\omega(M))) \subset \omega(M)$  for sufficiently small  $\delta > 0$ , which means that  $\omega(M)$  is asymptotically stable. The conclusion then follows by Lemma 3.2.

**Corollary 3.4.** Let  $(X, \mathbb{T}^+, \pi)$  be asymptotically compact and let M be a compact  $\pi$ -invariant set. Then M is asymptotically stable if and only if M is locally maximal and Lyapunov stable.

Indeed,  $M = \omega(M)$  here, so just apply Lemma 3.3.

The next theorem is a generalization to infinite dimensional spaces and  $\alpha$ -condensing systems of Theorem 8 of Zubov [35] characterizing the asymptotic stability of a compact set.

**Theorem 3.5.** Let  $(X, \mathbb{T}^+, \pi)$  be an  $\alpha$ -condensing semi-dynamical system and let  $M \subset X$  be a compact invariant set. Then the set M is asymptotically stable if and only if

(i) M is locally maximal, and

(ii) there exists a  $\delta > 0$  such that  $\alpha_{\gamma^x} \cap M = \emptyset$  for any entire trajectory  $\gamma^x$  through any  $x \in U_{\delta}(M) \setminus M$ .

*Proof.* By Lemma 2.3.5 in [19] any  $\alpha$ -condensing semi-dynamical system is asymptotically compact, so the assertion follows easily from Theorem 3.1 and Corollary 3.4.

A cocycle mapping  $\phi$  of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  will be called  $\alpha$ -condensing if the set  $\phi(t, B, P)$  is bounded and

$$\alpha(\phi(t, B, P)) < \alpha(B)$$

for all t > 0 for any bounded subset B of U with  $\alpha(B) > 0$ .

**Lemma 3.6.** Suppose that the cocycle mapping  $\phi$  of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  is  $\alpha$ -condensing. Then the mapping  $\pi$  of the associated skew-product flow  $(P, \mathbb{T}, \sigma)$  is also  $\alpha$ -condensing.

*Proof.* Let  $M = \bigcup_{p \in P} (M(p) \times \{p\})$  be a bounded set in X. Then M can be covered by finitely many balls  $M_i \subset X$ ,  $i = 1, \dots, n$ , of largest radius  $\alpha(M) + \varepsilon$  for an arbitrary  $\varepsilon > 0$ . The sets  $\operatorname{pr}_1 M_i \subset U$ ,  $i = 1, \dots, n$ , cover  $\operatorname{pr}_1 M$ . The sets  $M_i$  are balls so  $\alpha(\operatorname{pr}_1 M_i) = \alpha(M_i) < \alpha(M) + \varepsilon$  for  $i = 1, \dots, n$ . It is easily seen that

$$\pi(t,M) = \bigcup_{p \in P} \{\pi(t,(M(p),p))\} = \bigcup_{p \in P} \{(\phi(t,M(p),p),\sigma_t p)\} \subset \phi(t,\operatorname{pr}_1 M, P) \times P.$$

Since  $\phi$  is  $\alpha$ -condensing, the set  $\phi(t, \mathrm{pr}_1 M, P)$  is bounded. Hence

$$\begin{array}{rcl} (10) & \alpha(\pi(t,M)) & \leq & \alpha(\phi(t,\mathrm{pr}_1M,P)\times P) \\ & \leq & \alpha(\phi(t,\mathrm{pr}_1M,P)) < \alpha(\mathrm{pr}_1M) \leq \alpha(M) & \text{for each } t > 0. \end{array}$$

The second inequality above is true by the compactness of P. Indeed, P can be covered by finitely many open balls  $P_i$  of arbitrarily small radius. Hence

$$\alpha(\phi(t, \mathrm{pr}_1 M, P) \times P) \le \max \alpha(\phi(t, \mathrm{pr}_1 M, P) \times P_i) \le \alpha(\phi(t, \mathrm{pr}_1 M, P)) + \varepsilon$$

for arbitrarily small  $\varepsilon > 0$ . The conclusion of the Lemma follows by (10).

### 4. UNIFORM PULLBACK ATTRACTORS AND GLOBAL ATTRACTORS

It was seen earlier that the set  $\cup_{p \in P} (A(p) \times \{p\}) \subset X$  which was defined in terms of the pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$  of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  is the maximal  $\pi$ -invariant compact subset of the associated skew– product system  $(X, \mathbb{T}^+, \pi)$ , but need not be a global attractor. However, this set is always a *local* attractor under the additional assumption that the cocycle mapping  $\phi$  is  $\alpha$ -condensing.

**Theorem 4.1.** Let  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , P compact, be an  $\alpha$ -condensing dynamical system with a pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$  and define  $\mathcal{A} = \bigcup_{p \in P} (A(p) \times \{p\})$ . The n

(i) The  $\alpha$ -limit set  $\alpha_{\gamma^x}$  of any entire trajectory  $\gamma^x$  passing through  $x \in X \setminus A$  is empty.

(ii)  $\mathcal{A}$  is asymptotically stable with respect to  $\pi$ .

Proof. Suppose that there exists an entire trajectory  $\gamma^x$  through  $x = (u, p) \in X \setminus \mathcal{A}$  such that  $\alpha_{\gamma^x} \neq \emptyset$ . Then there exists a subsequence  $-\tau_n \to \infty$  such that  $\gamma^x(\tau_n)$  converges to a point in  $\alpha_{\gamma^x}$ . The set  $K = \operatorname{pr}_1 \overline{\bigcup_{n \in \mathbb{N}} \gamma^x(\tau_n)}$  is compact since  $\overline{\bigcup_{n \in \mathbb{N}} \gamma^x(\tau_n)}$  is compact. Also  $\widehat{A} = \{A(p)\}_{p \in P}$  is a pullback attractor, so

$$\lim_{n \to \infty} \operatorname{dist}_U(\phi(-\tau_n, K, \sigma_{\tau_n} p), A(p)) = 0$$

from which it follows that  $u \in A(p)$ . Hence  $(u, p) \in \mathcal{A}$ , which is a contradiction. This proves the first assertion.

By Lemma 3.6  $(X, \mathbb{T}^+, \pi)$  is  $\alpha$ -condensing. According to Lemma 2.7  $\mathcal{A}$  is a maximal compact invariant set of  $(X, \mathbb{T}^+, \pi)$  since  $\widehat{A}$  is a pullback attractor of the cocycle  $\phi$ . The second assertion then follows from Theorem 3.5 and from the first assertion of this theorem.  $\Box$ 

**Remark 4.2.** (i) The skew–product system in the example in Section 2 has only a *local* attractor associated with the pullback attractor.

(ii) If in addition to the assumptions of Theorem 4.1 the stable set  $W^s(\mathcal{A})$  of  $\mathcal{A}$  satisfies  $W^s(\mathcal{A}) = X$ , then  $\mathcal{A}$  is in fact a global attractor ([5], Lemma 7).

**Theorem 4.3.** Suppose that  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ , P compact, is a nonautonomous dynamical system with a pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$  and suppose that  $W^{s}(\mathcal{A}) = X$  where  $\mathcal{A} = \bigcup_{p \in P} (A(p) \times \{p\})$ .

If the mapping  $p \to A(p)$  is lower semi-continuous, then  $\widehat{A}$  is a uniform pullback attractor hence a uniform forward attractor.

*Proof.* Suppose that the uniform convergence

 $\lim_{t \to \infty} \sup_{p \in P} \operatorname{dist}_U(\phi(t, D, p), A(\sigma_t p)) = 0$ 

is not true for some  $D \in \mathcal{D}_c$ . Then there exist  $\varepsilon_0 > 0$ , a set  $D_0 \in \mathcal{D}_c$  and sequences  $t_n \to \infty$ ,  $p_n \in P$  and  $u_n \in D_0$  such that:

(11) 
$$\operatorname{dist}_{U}(\phi(t_{n}, u_{n}, p_{n}), A(\sigma_{t_{n}}p)) \geq \varepsilon_{0}.$$

Now P is compact and  $\mathcal{A}$  is a global attractor by Remark 4.2 (ii), so it can be supposed that the sequences  $\{\phi(t_n, u_n, p_n)\}$  and  $\{\sigma_{t_n}p\}$  are convergent. Let  $\bar{u} = \lim_{n\to\infty} \phi(t_n, u_n, p_n)$  and  $\bar{p} = \lim_{n\to\infty} \sigma_{t_n}p_n$ . Then  $\bar{u} \in A(\bar{p})$  because  $\bar{x} = (\bar{u}, \bar{p}) \in \mathcal{A}$ . On the other hand, by (11),

$$\varepsilon_0 \leq \operatorname{dist}_U(\phi(t_n, p_n, x_n), A(\sigma_{t_n} p_n)) \\ \leq \operatorname{dist}_U(\phi(t_n, p_n, x_n), A(\bar{p})) + \operatorname{dist}_U(A(\bar{p}), A(\sigma_{t_n} p_n)).$$

By the lower semi–continuity of  $p \to A(p)$  it follows then that  $\bar{u} \notin A(\bar{p})$ , which is a contradiction.

**Remark 4.4.** The example in Section 2 shows that Theorem 4.3 is in general not true without the assumption that  $W^s(\mathcal{A}) = X$ . In view of Corollary 2.8, the set valued mapping  $p \to A(p)$  will, in fact, then be continuous here.

### 5. Examples of uniform pullback attractors

Several examples illustrating the application of the above results, in particular of Theorem 4.3, are now presented. More complicated examples will be discussed in another paper.

5.1. **Periodic driving systems.** Consider a periodical dynamical system  $(P, \mathbb{T}, \sigma)$ , that is, for which there exists a minimal positive number T such that  $\sigma_T p = p$  for any  $p \in P$ .

**Theorem 5.1.** Suppose that a nonautonomous  $\alpha$ -condensing dynamical system  $\langle U, \phi, (P, \mathbb{R}, \sigma) \rangle$  with a periodical dynamical system  $(P, \mathbb{R}, \sigma)$  has a pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$ . Then  $\widehat{A}$  is a uniform forward attractor for  $\langle U, \phi, (P, \mathbb{R}, \sigma) \rangle$ .

*Proof.* Consider a sequence  $p_n \to p$ . By the periodicity of the driving system there exists a sequence  $\tau_n \in [0,T]$  such that  $p_n = \sigma_{\tau_n} p$ . By compactness, there is a convergent subsequence (indexed here for convenience like the full one)  $\tau_n \to \tau \in [0,T]$ . Hence

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} \sigma_{\tau_n} p = \sigma_{\tau} p$$

which means  $\tau = 0$  or T. Suppose that  $\tau = T$ . Then

$$\lim_{n \to \infty} \operatorname{dist}_U(A(p), A(p_n)) = \lim_{n \to \infty} \operatorname{dist}_U(A(p), \phi(\tau_n, A(p), p))$$
$$= \operatorname{dist}_U(A(p), \phi(T, A(p), p))$$
$$= \operatorname{dist}_U(A(p), A(\sigma_T p)) = 0$$

since  $\phi$  is continuous and  $A(p_n) = A(\sigma_{\tau_n} p) = \phi(\tau_n, A(p), p)$  by the  $\phi$ -invariance of  $\widehat{A}$ . Hence the set valued  $p \to A(p)$  is lower semi-continuous.

Now  $\phi(nT, u_0, p) = \phi(nT, u_0, \sigma_{-nT}p)$  since  $p = \sigma_{-nT}p$  by the periodicity of the driving system  $(P, \mathbb{R}, \sigma)$ . Hence from pullback convergence

$$\lim_{n \to \infty} \operatorname{dist}_U(\phi(nT, u_0, p), A(p)) = \lim_{n \to \infty} \operatorname{dist}_U(\phi(nT, u_0, \sigma_{-nT}p), A(p)) = 0$$

for any  $(u_0, p) \in U \times P$ . On the other hand

$$\sup_{s \in [0,T]} \operatorname{dist}_U(\phi(s+nT, u_0, p), A(\sigma_{s+nT}p))$$
  
= 
$$\sup_{s \in [0,T]} \operatorname{dist}_U(\phi(s, \phi(nT, u_0, p), \sigma_{nT}p), \phi(s, A(\sigma_{nT}p), \sigma_{nT}p)) = 0$$

by the cocycle property of  $\phi$  and the  $\phi$ -invariance of  $\widehat{A}$ . Hence

 $\lim_{t \to \infty} \operatorname{dist}_X((\phi(t, u_0, p), \sigma_t p), \mathcal{A}) = 0,$ 

where  $\mathcal{A} = \bigcup_{p \in P} (A(p) \times \{p\})$ . This shows that  $W^s(\mathcal{A}) = X$ . The result then follows by Theorem 4.3.

Consider the 2-dimensional Navier Stokes equation in the operator form

(12) 
$$\frac{du}{dt} + \nu Au + B(u) = f(t), \quad u(0) = u_0 \in H,$$

which can be interpreted as an evolution equation on the rigged space  $V \subset H \subset V'$ , where V and H are certain Banach spaces. In particular, here U = H, which is in fact a Hilbert space, for the phase space. Then, from [32] Chapter 3,

**Lemma 5.2.** The 2-dimensional Navier Stokes equation (12) has a unique solution  $u(\cdot, u_0, f)$  in C(0, T; H) for each initial condition  $u_0 \in H$  and forcing term  $f \in C(0, T; H)$  for every T > 0. Moreover,  $u(t, u_0, f)$  depends continuously on  $(t, u_0, f)$  as a mapping from  $\mathbb{R}^+ \times H \times C(\mathbb{R}, H)$  to H.

Now suppose that f is a periodic function in  $C(\mathbb{R}, H)$  and define  $\sigma_t f(\cdot) := f(\cdot + t)$ . Then  $P = \bigcup_{t \in \mathbb{R}} \sigma_t f$  is a compact subset of  $C(\mathbb{R}, H)$ . By Lemma 5.2 the mapping  $(t, u_0, p) \to \phi(t, u_0, p)$  from  $\mathbb{R}^+ \times H \times C(\mathbb{R}, H) \to H$  defined by  $\phi(t, u_0, p) = u(t, u_0, p)$  is continuous and forms a cocycle mapping with respect to  $\sigma$  on P. By [32] Theorem III.3.10 the mapping  $\phi$  is completely continuous and hence  $\alpha$ -condensing.

**Lemma 5.3.** The nonautonomous dynamical system  $\langle H, \phi, (P, \mathbb{R}, \sigma) \rangle$  generated by the Navier Stokes equation (12) with periodic forcing term in  $C(\mathbb{R}, H)$  has a pullback attractor.

Proof. The solution of the Navier Stokes equation satisfies an energy inequality

$$\|u(t)\|_{H}^{2} + \lambda_{1}\nu \int_{0}^{t} \|u(\tau)\|_{H}^{2}d\tau \leq \|u_{0}\|_{H}^{2} + \frac{1}{\nu}\int_{0}^{t} \|p(\tau)\|_{V'}^{2}d\tau$$

where  $\lambda_1$  is the smallest eigenvalue of A. It follows that the balls B(p) in H with center zero and square radii

$$R^{2}(p) = \frac{1}{\nu} \int_{-\infty}^{0} e^{\nu \lambda_{1} \tau} \|p(\tau)\|_{V'}^{2} d\tau$$

is a pullback attracting family of sets in the sense of Theorem 2.5. In particular,  $C(p) := \phi(1, B(\sigma_{-1}p), \sigma_{-1}p)$  satisfies all of the required properties of Theorem 2.5 because  $\phi(1, \cdot, p)$  is completely continuous.

This theorem and Theorem 5.1 give

**Theorem 5.4.** The nonautonomous dynamical system  $\langle H, \phi, (P, \mathbb{R}, \sigma) \rangle$  generated by the Navier Stokes equation (12) with periodic forcing term in  $C(\mathbb{R}, H)$  has a uniform pullback attractor which is also a uniform forward attractor.

**Remark 5.5.** See [16] for a related result involving a different type of nonautonomous attractor.

5.2. Pullback attractors with singleton component sets. Now pullback attractors with singleton component sets, that is with

$$A(p) = \{a(p)\}, \qquad a(p) \in U,$$

will be considered.

**Lemma 5.6.** Let  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  a nonautonomous dynamical system and let  $\widehat{A} = \{A(p)\}_{p \in P}$  be a pullback attractor with singleton component sets. Then the mapping  $p \to A(p)$  is continuous, hence lower semi-continuous.

*Proof.* This follows from Corollary 2.8 since the upper semi-continuity of a set valued mapping  $p \to A(p)$  reduces to continuity when the A(p) are single point sets.

It follows straightforwardly from this lemma and Theorem 4.3 that

**Theorem 5.7.** Suppose that  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  with compact P has at pullback attractor  $\widehat{A} = \{A(p)\}_{p \in P}$  with singleton component sets which generates a global attractor  $\mathcal{A} = \bigcup_{p \in P} A(p) \times \{p\}$ . Then  $\widehat{A}$  is a uniform pullback attractor and, hence, also a uniform forward attractor.

The previous theorem can be applied to differential equations on a Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  of the form

(13) 
$$u' = F(\sigma_t p, u)$$

where  $F \in C(P \times H, H)$  is uniformly dissipative, that is, there exist  $\nu \ge 2, \alpha > 0$ 

(14) 
$$\langle F(p, u_1) - F(p, u_2), u_1 - u_2 \rangle \le -\alpha ||u_1 - u_2||^{\nu}$$

for any  $u_1, u_2 \in H$  and  $p \in P$ .

**Theorem 5.8.** [9] The nonautonomous dynamical system  $\langle H, \varphi, (P, \mathbb{T}, \sigma) \rangle$  generated by (13) has a uniform pullback attractor that consists of singleton component subsets.

For example, the equation

$$u' = F(\sigma_t p, x) = -u|u| + g(\sigma_t p)$$

with  $g \in C(P, \mathbb{R})$  satisfies

$$\langle u_1 - u_2, F(\sigma_t p, u_1) - F(\sigma_t p, u_2) \rangle \le -\frac{1}{2} |u_1 - u_2|^2 (|u_1| + |u_2|) \le -\frac{1}{2} |u_1 - u_2|^3,$$

which is condition (14) with  $\alpha = \frac{1}{2}$  and  $\nu = 3$ .

The above considerations apply also to nonlinear nonautonomous partial differential equations with a uniform dissipative structure, such as the dissipative quasigeostrophic equations

(15) 
$$\omega_t + J(\psi, \omega) + \beta \psi_x = \nu \Delta \omega - r\omega + f(x, y, t) ,$$

with relative vorticity  $\omega(x, y, t) = \Delta \psi(x, y, t)$ , where  $J(f, g) = f_x g_y - f_y g_x$  is the Jacobian operator. This equation can be supplemented by homogeneous Dirichlet boundary conditions for both  $\psi$  and  $\omega$ 

(16) 
$$\psi(x, y, t) = 0, \qquad \omega(x, y, t) = 0 \quad \text{on } \partial D$$

and an initial condition,

$$\omega(x, y, 0) = \omega_0(x, y) \quad \text{on } D,$$

where D is an arbitrary bounded planar domain with area |D| and piecewise smooth boundary. Let U be the Hilbert space  $L_2(D)$  with norm  $\|\cdot\|$ .

Theorem 5.9. Assume that

$$\frac{r}{2} + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)$$

and that the wind forcing f(x, y, t) is temporally almost periodic with its  $L^2(D)$ norm bounded uniformly in time  $t \in \mathbb{R}$  by

$$||f(\cdot, \cdot, t)|| \le \sqrt{\frac{\pi r}{3|D|}} \left[ \frac{r}{2} + \frac{\pi \nu}{|D|} - \frac{1}{2}\beta \left( \frac{|D|}{\pi} + 1 \right) \right]^{\frac{3}{2}}$$

Then the dissipative quasigeostrophic model (15)–(16) has a unique temporally almost periodic solution that exists for all time  $t \in \mathbb{R}$ .

The proof in [15] involves explicitly constructing a uniform pullback and forward absorbing ball in  $L^2(D)$  for the vorticity  $\omega$ , hence implying the existence of a uniform pullback attractor as well as a global attractor for the associate skew– product flow system for which the component sets are singleton sets. The parameter set P here is the hull of the forcing term f in  $L^2(D)$  and a completely continuous cocycle mapping  $\phi(t, u_0, p) = \omega(t, u_0, p)$  with respect to the shift operator  $\sigma$  on Pthat is continuous in all variables.

5.3. Distal dynamical systems. A function  $\gamma^{(u,p)} : \mathbb{R} \to U$  represents an entire trajectory  $\gamma^{(u,p)}$  of a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  if  $\gamma^{(u,p)}(0) = u \in U$  and  $\phi(t, \gamma^{(u,p)}(\tau), \sigma_{\tau}p) = \gamma^{(u,p)}(t+\tau)$  for  $t \geq 0$  and  $\tau \in \mathbb{R}$ . A nonautonomous dynamical system is called *distal on*  $\mathbb{T}^-$  if

$$\inf_{t\in\mathbb{T}^-} d_U\left(\gamma^{(u_1,p)}(t),\gamma^{(u_2,p)}(t)\right) > 0$$

for any entire trajectories  $\gamma^{(u_1,p)}$  and  $\gamma^{(u_2,p)}$  with  $u_1 \neq u_2 \in U$  and any  $p \in P$ . A nonautonomous dynamical system is said to be uniformly Lyapunov stable if for any  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon) > 0$  such that

$$d_U\left(\phi(t, u_1, p), \phi(t, u_2, p)\right) < \delta$$

for all  $u_1, u_2 \in U$  with  $d_U(u_1, u_2) < \varepsilon, p \in P$  and  $t \ge 0$ . Finally, an autonomous dynamical system  $(P, \mathbb{T}, \sigma)$  is called *minimal* if P does not contain proper compact subsets which are  $\sigma$ -invariant.

The following lemma is due to Furstenberg [18] (see also [3] Chapter 3 or [25] Chapter 7 Proposition 4).

**Lemma 5.10.** Suppose that a nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ is distal on  $\mathbb{T}^-$  and that  $(P, \mathbb{T}, \sigma)$  is minimal. In addition suppose that a compact subset  $\mathcal{A}$  of X is  $\pi$ -invariant with respect to the skew-product system  $(X, \mathbb{T}^+, \pi)$ . Then the mapping  $p \to A(p) := \{u \in U : (u, p) \in \mathcal{A}\}$  is continuous.

The following theorem gives the existence of uniform forward attractors.

**Theorem 5.11.** Suppose that the nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ is uniformly Lyapunov stable and that the skew-product system  $(X, \mathbb{T}^+, \pi)$  has a global attractor  $\mathcal{A} = \bigcup_{p \in P} A(p) \times \{p\}$ . Then  $\widehat{A} = \{A(p)\}_{p \in P}$  is a uniform forward attractor for  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$ .

*Proof.* Suppose that the nonautonomous dynamical system  $\langle U, \phi, (P, \mathbb{T}, \sigma) \rangle$  is not distal. Then there is a  $p_0 \in P$ , a sequence  $t_n \to \infty$  and entire trajectories  $\gamma^{(u_1, p_0)}$ ,  $\gamma^{(u_2, p_0)}$  with  $u_1 \neq u_2$  such that

$$\lim_{n \to \infty} d_U \left( \gamma^{(u_1, p_0)}(-t_n), \gamma^{(u_2, p_0)}(-t_n) \right) = 0.$$

Let  $\varepsilon = d_U(u_1, u_2) > 0$  and choose  $\delta = \delta(\varepsilon) > 0$  from the uniformly Lyapunov stability property. Then

$$d_U\left(\gamma^{(u_1,p_0)}(-t_n),\gamma^{(u_2,p_0)}(-t_n)\right) < \delta$$

for sufficiently large n. Hence

$$d_U\left(\phi(t,\gamma^{(u_1,p_0)}(-t_n),\sigma_{-t_n}p_0),\phi(t,\gamma^{(u_2,p_0)}(-t_n),\sigma_{-t_n}p_0)\right) < \varepsilon$$

for  $t \ge 0$  and, in particular,

$$r = d_U(u_1, u_2) = d_U\left(\phi(t_n, \gamma^{(u_1, p_0)}(-t_n), \sigma_{-t_n} p_0), \phi(t_n, \gamma^{(u_2, p_0)}(-t_n), \sigma_{-t_n} p_0)\right) < \varepsilon$$

for  $t = t_n$ , which is a contradiction. The nonautonomous dynamical system is thus distal, so  $p \to A(p)$  is continuous by Lemma 5.10. The result then follows from Theorem 4.3 since  $\{A(p)\}_{p \in P}$  generates a pullback attractor.

This theorem will now be applied to the nonautonomous differential equation (13) on a Hilbert space H, which is assumed to generate a cocycle  $\phi$  that is continuous on  $\mathbb{T}^+ \times P \times H$  and asymptotically compact.

**Theorem 5.12.** Suppose that  $F \in C(H \times P, H)$  satisfies the dissipativity conditions

(17) 
$$\langle F(u_1, p) - F(u_2, p), u_1 - u_2 \rangle \leq 0$$

(18) 
$$\langle F(u,p), u \rangle \leq -\mu(|u|)$$

for  $u_1, u_2, u \in H$  and  $p \in P$ , where  $\mu : [R, \infty) \to \mathbb{R}^+ \setminus \{0\}$ . Suppose also that (13) generates a cocycle  $\phi$  that is continuous and asymptotically compact. Finally, suppose that  $(P, \mathbb{T}, \sigma)$  is a minimal dynamical system.

Then the nonautonomous dynamical system  $\langle H, \phi, (P, \mathbb{T}^+, \sigma) \rangle$  has a uniform pull-back attractor.

*Proof.* It follows by the chain rule applied to  $||u||^2$  for a solution of (13) that  $||\phi(t, u, p)|| < ||u||$  for |u| > R, t > 0 and  $p \in P$ . Hence the nonautonomous dynamical system  $(X, \mathbb{T}^+, \pi)$  has a global attractor [8]. On the other hand, by (17),

$$\|\phi(t, u_1, p) - \phi(t, u_2, p)\| \le \|u_1 - u_2\|$$

for  $t \ge 0$ ,  $p \in P$  and  $u_1, u_2 \in H$ . Theorem 5.11 then gives the result.

The above theorem holds for a differential equation (13) on  $H = \mathbb{R}$  with

$$F(p,u) = \begin{cases} -(u+1) + g(p) & : \quad u < -1 \\ g(p) & : \quad |u| \le 1 \\ -(u-1) + g(p) & : \quad u > 1. \end{cases}$$

where  $g \in C(P, \mathbb{R})$ .

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(D. Cheban) State University of Moldova, Departement of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova

E-mail address, D. Cheban: cheban@usm.md

(P. Kloeden) Fachbereich Mathematik, Johann Wolfgang Goethe Universität, D–60054 Frankfurt am Main, Germany

*E-mail address*, P. Kloeden: kloeden@math.uni-frankfurt.de

(B. Schmalfuß) Department of Sciences, University of Applied Sciences, D-06217 Merseburg, Germany

*E-mail address*, B. Schmalfuß: schmalfuss@in.fh-merseburg.de