Global Attractors of Non-autonomous Difference Equations

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Abstract. The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations. The results obtained are applied to the study of a triangular economic growth model $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ recently developed in S. Brianzoni, C. Mammana and E. Michetti [1].

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1 Introduction

Global attractors play a very important role in the qualitative study of difference equations (both autonomous and non-autonomous). The present work is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n \omega)u_k + F(u_k, \sigma^n \omega), \tag{1}$$

where Ω is a metric space (generally speaking non-compact), $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time $\mathbb{Z}_+$, $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness". An analogous problem has been studied in D. Cheban and C. Mammana [5] when the space Ω is compact.

The results obtained are applied to the study of a class of triangular maps $T = (T_1, T_2)$ describing an economic growth model in capital accumulation and population growth rate as recently proposed by S. Brianzoni, C. Mammana and E. Michetti [1] [1].

2 Global attractors of dynamical systems

2.1 Triangular maps and non-autonomous dynamical systems

Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ its Cartesian product. Recall that a continuous map $F : X \to X$ is called triangular if

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[1]The authors consider the neoclassical one–sector growth model with differential savings as in V. Bohm and L. Kaas [3], while assuming CES production function and the labour force dynamic described by the Beverton–Holt equation (see [2]), that has been largely studied in [7] and [8].

there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n), \\ \omega_{n+1} = g(\omega_n) \end{cases} \qquad (2)$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all non-negative integer numbers.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n \omega) \ (\omega \in \Omega), \qquad (3)$$

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

(i) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;

(ii) $\varphi(n+m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n \omega$;

(iii) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ is continuous.

Denote by $(\Omega, \mathbb{Z}_+, \sigma)$ the semi-group dynamical system generated by the positive powers of map $g : \Omega \to \Omega$, i.e. $\sigma(n, \omega) := g^n \omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Recall [4,9] that a triple $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ (or briefly φ) is called a cocycle over the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W if the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \to \Omega$ possesses the properties (i)-(iii).

Let $X := W$ and $(X, \mathbb{Z}_+, \pi)$ be a dynamical system on X, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then $(X, \mathbb{Z}_+, \pi)$ is called [9] a skew-product dynamical system, generated by the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$.

Taking into consideration this fact, we can study triangular maps in the framework of cocycles with discrete time.

2.2   Global attractors of autonomous dynamical systems

A dynamical system $(X, \mathbb{T}, \pi)$ is called compact dissipative if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t \to +\infty} \rho(xt, K) = 0; \qquad (4)$$

for all $x \in X$ and the equality (4) takes place uniformly w.r.t. x on the compact subsets from X.

For compact dissipative dynamical system $(X, \mathbb{T}, \pi)$ we denote by

$$J := \Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K},$$

then the set J does not depend on the choice of the attractor K and is characterized by the properties of the dynamical system $(X, \mathbb{T}, \pi)$. The set J is called a Levinson center of the dynamical system $(X, \mathbb{T}, \pi)$.

Let E be a finite-dimensional Banach space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber E (or shortly φ).

A cocycle φ is called:

– dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t \to +\infty} |\varphi(t, u, \omega)| \leq r \tag{5}$$

for all $\omega \in \Omega$ and $u \in E$;

– uniform dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t \to +\infty} \sup_{\omega \in \Omega', |u| \leq R} |\varphi(t, u, \omega)| \leq r$$

for all compact subsets $\Omega' \subseteq \Omega$ and $R > 0$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $x \in X$. Denote by $\omega_x := \cap_{t \geq 0} \overline{\cup_{\tau \geq t} \pi(\tau, x)}$ the ω-limit set of point x.

Theorem 1 ([6])**.** *If the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and the cocycle φ is uniform dissipative, then the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compact dissipative.*

2.3   Global attractors of quasi-linear triangular systems

Consider a difference equation

$$u_{n+1} = f(u_n, \sigma^n \omega) \ (\omega \in \Omega). \tag{6}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (6) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (6) is said to be dissipative (respectively, uniformly dissipative), if a cocycle φ, generated by equation (6), is dissipative (respectively, uniformly dissipative), i.e. there exists a positive number r such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \leq r \ \ (\text{respectively, } \limsup_{n \to +\infty} \sup_{\omega \in \Omega', |u| \leq R} |\varphi(n, u, \omega)| \leq r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$ and $\Omega' \in C(\Omega)$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k + F(u_k, \sigma^n \omega), \tag{7}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness".

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k.$$

Theorem 2. *Suppose that the following conditions hold:*

1. *there are positive numbers N and $q < 1$ such that*

$$\|U(n,\omega)\| \le Nq^n \qquad (n \in \mathbb{Z}_+); \tag{8}$$

2. *$|F(u,\omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1-q)N^{-1})$ for all $u \in E$ and $\omega \in \Omega$.*

Then equation (7) is uniform dissipative and

$$|\varphi(n,u,\omega)| \le (q + DN)^{n-1}qN|u| + \frac{CN}{q-1}(q^{n-1} - 1). \tag{9}$$

Proof. This statement can be proved using the same type of arguments as in the proof of Theorem 5.2 from [5] (see also [6]) and we omit the details. $\square$

Theorem 3. *Let $(\Omega, \mathbb{Z}_+, \sigma)$ be a compact dissipative dynamical system and φ be a cocycle generated by equation (7). Under the conditions of Theorem 2 the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, generates by cocycle φ admits a compact global attractor.*

Proof. This statement follows directly from Theorems 1 and 2. $\square$

Theorem 4 ([6])**.** *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

1. *the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and J_Ω its Levinson center;*

2. *positive numbers N and $q < 1$ exist such that inequality (8) holds;*

3. *$C > 0$ exists such that $|F(0,\omega)| \le C$ for all $\omega \in \Omega$;*

4. *$|F(u_1,\omega) - F(u_2,\omega)| \le L|u_1 - u_2|$ $(0 \le L < N^{-1}(1-q))$ for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

   Then

1. *the equation (7) (the cocycle φ generated by this equation) admits a compact global attractor;*

2. *there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that*

$$|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \le \mathcal{N}\nu^n|u_1 - u_2| \tag{10}$$

for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$.

3   Non-Autonomous Dynamical Systems with Convergence

$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h \rangle$ is said to be convergent if the following conditions are valid:

1. the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;

2. the set $J_X \bigcap X_y$ contains at most one point for all $y \in J_Y$, where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$).

Theorem 5 ([4, Ch.II]). *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

1. *the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is compact dissipative and J_Y its Levinson center;*

2. *there exists a homomorphism γ from $(Y, \mathbb{T}_2, \sigma)$ to $(X, \mathbb{T}_1, \pi)$ such that $h \circ \gamma = Id_Y$;*

3. *$\lim\limits_{t \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ for all $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$).*

Then

1. *the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and $\gamma(J_Y) = J_X$;*

2. *J_y consists a single point $\gamma(y)$ for all $y \in J_Y$.*

Theorem 6. *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

1. *the dynamical system $(\Omega, \mathbb{Z}, \sigma)$ is compact dissipative and J_Ω its Levinson center;*

2. *there exist positive numbers N and $q < 1$ such that inequality (8) holds;*

3. *there exists $C > 0$ such that $|F(0, \omega)| \le C$ for all $\omega \in \Omega$;*

4. *$|F(u_1, \omega) - F(u_2, \omega)| \le L|u_1 - u_2|$ ($0 \le L < N^{-1}(1 - q)$) for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

Then

1. *the equation (7) (the cocycle φ generated by this equation) admits a compact global attractor $\{I_\omega \mid \omega \in J_\Omega\}$ and I_ω consists of a single point u_ω (i.e. $I_\omega = \{u_\omega\}$) for all $\omega \in J_\Omega$;*

2. *the mapping $\omega \mapsto u_\omega$ is continuous and $\varphi(t, u_\omega, \omega) = u_{\sigma(t, \omega)}$ for all $\omega \in J_\Omega$ and $t \in \mathbb{Z}$;*

3. there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that

$$|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \leq \mathcal{N}\nu^n|u_1 - u_2| \qquad (11)$$

for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$;

4.

$$|\varphi(n, u, \omega) - u_{\sigma^n\omega}| \leq \mathcal{N}\nu^n|u - u_\omega| \qquad (12)$$

for all $u \in E$, $\omega \in J_\Omega$ and $n \in \mathbb{Z}_+$.

Proof. Let $\langle E, \varphi, (\Omega, \mathbb{Z}, \sigma) \rangle$ be the cocycle generated by equation (7) and $C_b(\Omega, E)$ be the space of all continuous and bounded functions $\mu : \Omega \mapsto E$ equipped with the sup-norm. For every $n \in \mathbb{Z}_+$ we define the mapping $S^n : C_b(\Omega, E) \mapsto C_b(\Omega, E)$ by equality $(S^n\mu)(\omega) := \varphi(n, \mu(\sigma(-n, \omega)), \sigma(-n, \omega))$ for all $\omega \in \Omega$. It easy to verify that the family of mappings $\{S^n \mid n \in \mathbb{Z}_+\}$ forms a commutative semigroup. From the inequality (9) it follows that $S^n\mu \in C_b(\Omega, E)$ for every $\mu \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$. On the other hand from the inequality (10) we have

$$\|S^n\mu_1 - S^n\mu_2\| \leq \mathcal{N}\nu^n\|\mu_1 - \mu_2\|$$

for all $\mu_1, \mu_2 \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$, where $\mathcal{N} := \frac{qN}{q+LN}$ and $\nu := q + LN$. Under the conditions of Theorem $\nu = q + LN < q + 1 - q = 1$ and, consequently, the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ is contracting. Thus there exists a unique fixed point $\mu \in C_b(\Omega, E)$ of the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ and hence

$$\mu(\sigma(n, \omega)) = \varphi(n, \mu(\omega), \omega)$$

for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Let $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h \rangle$ be the non-autonomous dynamical system associated by cocycle φ (i.e. $X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \mapsto \Omega$). Under the conditions of Theorem by Theorem 4 we have $\rho(x_1t, x_2t) \leq \mathcal{N}e^{-\nu t}\rho(x_1, x_2)$ for all $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$). Since $\gamma := (\mu, Id_\Omega)$ is an invariant section of the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h \rangle$, then according to Theorem 5 the dynamical system $(X, \mathbb{Z}_+, \pi)$ is compactly dissipative, its Levinson center $J_X = \gamma(J_\Omega)$ and $J_\omega := J \cap X_\omega$ ($X_\omega := h^{-1}(\omega)$) consists a single point $\gamma(\omega)$, i.e. $J_\omega = \{\gamma(\omega)\}$ for all $\omega \in \Omega$. Taking into consideration that the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compact dissipative, $J_\omega = I_\omega \times \omega$ and $\gamma = (\mu, Id_\Omega)$ we obtain $I_\omega = \mu(\omega)$ for all $\omega \in J_\Omega$. $\qquad \square$

4   Economic Application

4.1   The model

Dynamic economic growth models have often considered the standard, one-sector neoclassical Solow model (see S. R. Solow [10]). V. Bohm and L. Kaas [3] considered the role of differential savings behavior between workers and shareholders and its

effects with regard to the stability of stationary steady states within the framework of the discrete-time Solow growth model. More recently, S. Brianzoni, C. Mammana and E. Michetti [1] have proposed a discrete-time version of the Solow growth model with differential savings as formalized by V. Bohm and L. Kaas [3] while considering two different assumptions. Firstly they assume the CES production function. Secondly they assume the labor force growth rate not being constant, in particular they consider a model for density dependent population growth described by the Beverton-Holt equation (see [2]).

The resulting system $(T, \mathbb{R}_+^2)$ describing capital accumulation k and population n dynamics of the model studied in S. Brianzoni, C. Mammana and E. Michetti [1], where $T = (T_1, T_2)$, is given by

$$T_1(k, n) = \frac{(1 - \delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)}{1 + n}$$

and

$$T_2(n) = \frac{rhn}{h + (r - 1)n}$$

for all $(k, n) \in \mathbb{R}_+^2$. In the model, $\delta \in (0, 1)$ is the depreciation rate of capital, $s_w \in (0, 1)$ and $s_r \in (0, 1)$ are the constant saving rates for workers and shareholders respectively[2], $\rho \in (-\infty, 1), \rho \neq 0$ is a parameter related to the elasticity of substitution between the production factors given by $1/(1 - \rho)$, $h > 0$ is the carrying capacity (for example resource availability) and $r > 1$ is the inherent growth rate (such a rate is determined by life cycle and demographic properties such as birth rates etc.). The Beverton-Holt T_2 have been studied extensively in J. V. Cushing and S. V. Henson [7,8].

4.2   Existence of an attractor for $\rho \in (-\infty, 0)$

Theorem 7. *If $\rho < 0$, then the dynamical system $(\mathbb{R}_+^2, T)$ admits a compact global attractor.*

Proof. Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ

$$T_1(k, n) = \frac{1}{1 + n}\left[(1 - \delta)k + (k^{-\lambda} + 1)^{\frac{1+\lambda}{-\lambda}}(s_w + s_r k^{-\lambda})\right]$$

$$= \frac{1}{1 + n}\left[(1 - \delta)k + \frac{k}{(1 + k^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w k^\lambda}{1 + k^\lambda}\right]. \tag{13}$$

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[2]The authors also assume $s_w \neq s_r$ since the standard growth model of R. V. Solow [10] is obtained if the two savings propensities are equal.

Note that $\dfrac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \longrightarrow 1$ as $k \longrightarrow +\infty$, $\dfrac{s_r + s_w k^\lambda}{1 + k^\lambda} \longrightarrow s_w$ as $k \longrightarrow +\infty$ and, consequently, there exists $M > 0$ such that

$$\left| \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} - \frac{s_r + s_w k^\lambda}{1 + k^\lambda} \right| \leq M, \tag{14}$$

for all $k \in [0, +\infty)$.

Since $0 \leq \dfrac{1}{1+n} \leq 1$ for all $n \in \mathbb{R}_+$, then from (13) and (14) we obtain

$$0 \leq T_1(k, n) \leq \alpha k + M \tag{15}$$

for all $n, k \in \mathbb{R}_+$, where $\alpha := 1 - \delta > 0$.

Since the map T is triangular, to prove this theorem it is sufficient to apply Theorem 3. Theorem is proved. □

Remark 1. 1. It is easy to see that the previous theorem is true also for $\delta = 1$ because in this case $\alpha = 1 - \delta = 0$ and from (15) we have $T_1(k, n) \leq M, \forall k, n \in \mathbb{R}_+$. Now it is sufficient to refer to Theorem 1.

2. If $\delta = 0$ the problem is open.

According to Theorem 7, it is possible to conclude that if the elasticity of substitution between the two production factors (capital and labor) is positive and lesser than one (that is $\rho < 0$), capital and population dynamics cannot be explosive so economic patterns are bounded.

4.3   Existence of an attractor for $\rho \in (0, 1)$ and $s_r < \delta$

The dynamical system $(X, \mathbb{T}, \pi)$ we will call:

– locally completely continuous if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi^l B(p, \delta)$ is relatively compact;

– weakly dissipative if a nonempty compact $K \subseteq X$ exists such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $x\tau \in B(K, \varepsilon)$. In this case we will call K a weak attractor.

Note that every dynamical system $(X, \mathbb{T}, \pi)$ defined on the locally compact metric space X is locally completely continuous.

Theorem 8 ([4]). *For the locally completely continuous dynamical systems the weak and compact dissipativity are equivalent.*

Theorem 9. *If $\rho \in (0, 1)$ and $s_r < \delta$, then the mapping T admits a compact global attractor.*

Proof. If $\rho \in (0,1)$ and $k > 0$ we have

$$T_1(k,n) = \frac{1}{1+n}\left[(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)\right]$$

$$= \frac{1}{1+n}\left[(1-\delta)k + s_r k + \theta(k)k\right] \tag{16}$$

where $\theta(k) := \dfrac{(k^\rho + 1)^{\frac{1}{\rho}}}{k(1+k^\rho)}(s_w + s_r k^\rho) - s_r \to 0$ as $k \to +\infty$. In fact $\dfrac{(k^\rho + 1)^{\frac{1}{\rho}}}{k} \to 1$

as $k \to +\infty$ while $\dfrac{(s_w + s_r k^\rho)}{1 + k^\rho} \to s_r$ as $k \to +\infty$ and, consequently,

$$\frac{\dfrac{(k^\rho + 1)^{\frac{1}{\rho}}}{1 + k^\rho}(s_w + s_r k^\rho)}{s_r k} = \frac{(k^\rho + 1)^{\frac{1}{\rho}}}{k}\frac{(s_w + s_r k^\rho)}{s_r(k^\rho + 1)} \to 1$$

as $k \to +\infty$, i.e. $\dfrac{(k^\rho + 1)^{\frac{1}{\rho}}}{1 + k^\rho}(s_w + s_r k^\rho) = s_r k + \theta(k)k$. From (16) we have

$$T_1(k,n) = \frac{1}{1+n}\left[(1 - \delta + s_r)k + \theta(k)k\right]$$

for all $(k,n) \in \mathbb{R}_+^2$ with $k > 0$.

Since $s_r < \delta$ then $\alpha := 1 - \delta + s_r < 1$. Let $R_0 > 0$ be a positive number such that

$$|\theta(k)| < \frac{1-\alpha}{2}, \tag{17}$$

for all $k > R_0$. Note that for every $(k_0, n_0) \in \mathbb{R}_+^2$, with $k_0 > R_0$, the trajectory $\{T^t(k,n) \mid t \in \mathbb{Z}_+\}$ starting from point (k_0, n_0) at the initial moment $t = 0$, at least one time intersects the compact $K_0 := [0, h_0] \times [0, R_0]$, $(h_0 > h)$. In fact, if we suppose that this statement is false, then exists a point $(k_0, n_0) \in \mathbb{R}_+^2 \setminus K_0$ exists such that

$$(k_t, n_t) := T^t(k_0, n_0) \in \mathbb{R}_+^2 \setminus K_0 \tag{18}$$

for all $t \in \mathbb{Z}_+$. Taking into consideration that $n_t \to h$ (or 0) as $t \to +\infty$, we obtain from (18) that $k_t > R_0$ for all $t \geq t_0$, where t_0 is a sufficiently large number from $\mathbb{Z}_+$. Without loss of generality, we may suppose that $t_0 = 0$ (if $t_0 > 0$ then we start from the initial point $(n_{t_0}, k_{t_0}) := T^{t_0}(n_0, k_0)$, where $T^{t_0} := T \circ T^{t_0 - 1}$ for all $t_0 \geq 2$). Thus we have

$$k_t > R_0 \tag{19}$$

for all $t \geq 0$ and

$$k_{t+1} = \frac{1}{1+n}[\alpha k_t + \theta(k_t)k_t] \tag{20}$$

From (17) and (20) we obtain

$$k_{t+1} \leq \alpha k_t + \frac{1-\alpha}{2}k_t = \frac{1+\alpha}{2}k_t \tag{21}$$

since $\dfrac{1}{1+n} \leq 1$ for all $t \geq 0$. From (21) we have

$$k_t \leq \left(\frac{1+\alpha}{2}\right)^t k_0 \to 0 \text{ as } t \to +\infty, \qquad (22)$$

but (19) and (22) are contradictory. The obtained contradiction proves the statement. Let $(k_0, n_0) \in \mathbb{R}_+^2$ now be an arbitrary point.

(a) If $k_0 < R_0$ and $k_t \leq R_0$ for all $t \in \mathbb{N}$, then $\limsup\limits_{t \to +\infty} k_t \leq R_0$;

(b) If there exists $t_0 \in \mathbb{N}$ such that $k_{t_0} > R_0$, then there exists $\tau_0 \in \mathbb{N}$ $(\tau_0 > t_0)$ such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$ (see the proof above).

Thus we have proved that for all $(k_0, n_0) \in \mathbb{R}_+^2$ there exists $\tau_0 \in \mathbb{N}$ such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$. According to Theorem 8 the dynamic system $(\mathbb{R}_+^2, T)$ admits a compact global attractor. The theorem is proved. $\qquad\square$

4.4   Structure of the attractor

A fixed point $p \in X$ of dynamical system $(X, \mathbb{T}, \pi)$ is called

– Lyapunov stable if for arbitrary positive number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, p) < \delta$ implies $\rho(\pi(t, x), p) < \varepsilon$ for all $t \geq 0$;

– attracting if there exists $\delta_0 > 0$ such that $\lim\limits_{t \to +\infty} \rho(\pi(t, x), p) = 0$ for all $x \in B(p, \delta_0) := \{x \in X \mid \rho(x, p) < \delta_0\}$;

– asymptotically stable if it is Lyapunov stable and attracting.

Theorem 10. *Suppose that $\rho < 0$ and one of the following conditions hold:*

1. *$s_w < \min\{\delta, s_r\}$ and $0 < \lambda < \lambda_0$, where λ_0 is a positive root of the quadratic equation $(s_r - s_w)\lambda^2 + (s_r - 2\delta)\lambda - \delta = 0$;*

2. *$s_r < s_w < \delta$.*

Then

1. *the dynamic system $(\mathbb{R}_+^2, T)$ admits a compact global attractor $J = \{(0, n) \mid 0 \leq n \leq h\}$;*

2. *for all point $x := (k, n) \in \mathbb{R}_+^2$ with $n > 0$ the ω-limit set ω_x of x consists a single fixed point $(0, h)$ of dynamical system $(\mathbb{R}_+^2, T)$;*

3. *the fixed point $(0, h)$ is asymptotically stable.*

Proof. Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ (see the proof of Theorem 9)

$$T_1(k, n) = \frac{1}{1+n} \left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_w + s_r k^\lambda}{1+k^\lambda} \right].$$

Denote by

$$f(k) := \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_w + s_r k^\lambda}{1+k^\lambda},$$

then

$$f'(k) = \frac{s_w + (-s_w \lambda + (\lambda+1)s_r)k^\lambda}{(1+k^\lambda)^{2+1/\lambda}}.$$

It easy to verify that under the conditions of Theorem $f'(k) < s_w$ for all $k \geq 0$. Consider the non-autonomous difference equation

$$k_{t+1} = A(\sigma(t,n))k_t + F(k_t, \sigma(t,n)) \tag{23}$$

corresponding to triangular map $T = (T_1, T_2)$, where $A(n) := \dfrac{1}{n+1}$, $F(k, n) := \dfrac{1}{n+1} f(k)$ and $\sigma(t,n) := T_2^t(n)$ for all $t \in \mathbb{Z}_+$ and $n \in \mathbb{R}_+$. Under the conditions of the Theorem we can apply Theorem 6. By this Theorem the dynamical system $(\mathbb{R}_+^2, T)$ is compact dissipative with Levinson center J and there exists a unique continuous bounded function $\mu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $J = \{(\mu(n), n) \mid n \in [0, h]\}$. Since $F(n, 0) = 0$ for all $n \in \mathbb{R}_+$, then it easy to see that $\mu(n) = 0$ for all $n \in \mathbb{R}_+$.

Let $x = (k, n) \in \mathbb{R}_+^2$ and $n > 0$. Since the dynamical system $(\mathbb{R}_+^2, T)$ is compactly dissipative and its Levinson center $J = \cup\{J_n \mid 0 \leq n \leq h\}$, then $\omega_x \subseteq J$. Let $\tilde{x} = (\tilde{k}, \tilde{n}) \in \omega_x$, then there exists $t_m \to +\infty$ ($t_m \in \mathbb{Z}_+$) such that $T^{t_m}(k, n) \to (\tilde{k}, \tilde{n})$. It is evident that $\tilde{k} = 0$. Since $\lim\limits_{t \to +\infty} T_2^t n = h$ for all $n > 0$ we obtain $\tilde{n} = h$, i.e. $\tilde{x} = (0, h)$.

Now we will prove that the fixed point $(0, h)$ is stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_l \to 0$, $x_l := (k_l, n_l) \to (0, h)$ and $t_l \to +\infty$ (as $l \to +\infty$) such that $\rho(x_l, (0, h)) < \delta_l$ and

$$\rho(T^{t_l} x_l, (0, h)) \geq \varepsilon_0, \tag{24}$$

where $\rho(\cdot, \cdot)$ is the distance in $\mathbb{R}_+^2$. Since $T^{t_l} x_l = (\varphi(t_l, k_l, n_l), T_2^{t_l} n_l)$, where $\varphi(t, k, n)$ is the solution of equation (23) with initial condition $\varphi(0, k, n) = k$, and $n_l \to h$ by asymptotic stability of fixed point $h \in \mathbb{R}_+$ of dynamical system $(\mathbb{R}_+, T_2)$ we have $T_2^{t_l} n_l \to h$ as $l \to +\infty$. On the other hand by Theorem 6 we obtain

$$|\varphi(t_l, k_l, n_l) - \mu(T_2^{t_l})| \leq \mathcal{N} \nu^{t_l} |k_l - \mu(n_l)| = \mathcal{N} \nu^{t_l} |k_l| \to 0 \tag{25}$$

because $0 < \nu < 1$, $|k_l| \to 0$ and $t_l \to +\infty$. Taking into account that $\mu(n) = 0$ for all $n \geq 0$ we obtain $\mu(T_2^{t_l}) = 0$ for all $l \in \mathbb{N}$ and, consequently, $|\varphi(t_l, k_l, n_l)| \to 0$ as $l \to +\infty$, i.e.

$$\rho(T^{t_l} x_l, (0, h)) \to 0 \tag{26}$$

as $l \to +\infty$. The relations (24) and (26) are contradictory. The contradiction obtained proves our statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

When considering Theorem 10 it is possible to conclude that if shareholders save less than workers and the depreciation rate of capital is big enough or, if workers save less than shareholders and the elasticity of substitution between the two factors is close to zero, then the economic system will converge to the steady state $(0, h)$ which is characterized by no capital accumulation.

Let γ be a full trajectory of dynamical system $(X, \mathbb{T}, \pi)$. Denote by $\cap_{t \geq 0} \overline{\cup_{\tau \geq t} \gamma(\tau)} := \omega_\gamma$ (respectively, $\cap_{t \leq 0} \overline{\cup_{\tau \leq t} \gamma(\tau)} := \alpha_\gamma$).

Theorem 11. *Let $\rho \in (0, 1)$, $s_r < \delta$ and J be the Levinson center of dynamical system $(\mathbb{R}_+^2, T)$. Then the following statements hold:*

1. *J is connected;*

2. *$J = \cup \{J_n \mid 0 \leq n \leq h\}$, where $J_n := I_n \times \{n\}$ and $I_n := [a_n, b_n]$ ($a_n, b_n \in \mathbb{R}_+$);*

3. *dynamical systems $(\mathbb{R}_+, T_0)$ and $(\mathbb{R}_+, T_h)$ are compactly dissipative, where $T_0(k) := T(k, 0)$ and $T_h(k) := T(k, h)$ for all $k \in \mathbb{R}_+$;*

4. *$J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h := [a_h, b_h] \times \{h\}$) is the Levinson center of dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$);*

5. *there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of the dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$);*

6. *for all point $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$) and $\gamma \in \Phi_{x_0}$ we have $\omega_\gamma \subseteq J_h$ and $\alpha_\gamma \subseteq J_0$.*

Proof. Let $\rho \in (0, 1)$ and $s_r < \delta$, then by Theorem 9 the dynamical system $(\mathbb{R}_+^2, T)$ is compactly dissipative. Denote by J the Levinson center of $(\mathbb{R}_+^2, T)$, then by Theorem 1.33 [4] the set J is connected. Note that $J = \cup \{J_n \mid 0 \leq n \leq h\}$, where $J_n = I_n \times \{n\}$ and I_n is a compact subset of $\mathbb{R}_+$. According to Theorem 2.25 [4] the set I_n is connected and, consequently, there are $a_n, b_n \in \mathbb{R}_+$ such that $I_n = [a_n, b_n]$.

Since the set $\mathbb{R}_+ \times \{0\}$ (respectively, $\mathbb{R}_+ \times \{h\}$) is invariant with respect to dynamical system $(\mathbb{R}_+^2, T)$, then on the set $\mathbb{R}_+ \times \{0\}$ (respectively, on $\mathbb{R}_+ \times \{h\}$) is defined as a compactly dissipative dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$) and the set J_0 (respectively, J_h) is its Levinson center. Taking into account that T_0 (respectively, T_h) is a continuous mapping of $J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h = [a_h, b_h] \times \{h\}$) on itself, then there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of dynamical system $(\mathbb{R}_+, T_0)$ (respectively, $(\mathbb{R}_+, T_h)$).

Let $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$), $\gamma \in \Phi_{x_0}$ and $x = (k, n) \in \omega_\gamma$ (respectively, $x \in \alpha_\gamma$). Then there exists a sequence $\{t_m\} \subseteq \mathbb{Z}$ such that $t_m \to +\infty$ (respectively, $t_m \to -\infty$) such that $\gamma(t_m) \to x$ as $m \to +\infty$. Since $x_0 = (k_0, n_0)$,

$0 < n_0 < h$ and $pr_2(\gamma(t_m)) = T_2^{t_m}(n_0)$, then $\{T_1^{t_m}(n_0)\} \to h$ (respectively, $\{T_2^{t_m}(n_0)\} \to 0$) as $m \to +\infty$. On the other hand $x \in J$ and, consequently, $p_2(x) = h$ (respectively, $pr_2(x) = 0$). Analogously we can prove that $\omega_{x_0} \subseteq J_h$ for all $x_0 = (k_0, n_0) \in \mathbb{R}^2_+$ with $n_0 > 0$, where ω_{x_0} is the ω-limit set of point x_0. $\qquad\square$

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