COMPACT GLOBAL ATTRACTORS OF CONTROL SYSTEMS

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ABSTRACT. The paper is dedicated to the study of the problem of existence of compact global attractors of control systems and to description of its structure. In particular, sufficient conditions of the existence of chaotic attractor for some classes of switching systems are given. We study this problem in the framework of non-autonomous dynamical systems (cocyles).

Dedicated to my teacher
Professor B. A. Shcherbakov on his 85th birthday

1. Introduction

The aim of this paper is the study of the problem of existence of compact global attractors of control systems (see, for example, Bobylev, Emel'yanov and Korovin [1], Bobylev, Zalozhnev and Klykov [2], Cheban and Mammana [8, 9], Emel'yanov, Korovin and Bobylev [12], Kloeden [16] and the references therein).

Let E and F be two finite dimensional Hilbert spaces and $U \subset F$ be a compact subset. Denote by $\mathcal{U} := \{u : \mathbb{R} \mapsto U, \text{ measurable}\} \subset L_\infty(\mathbb{R}, F)$, equipped with the weak* topology of $L_\infty(\mathbb{R}, F) = (L_1(\mathbb{R}, F))^*$. This space is compact and metrizable [11, Ch.4]. On the space $\mathcal{U}$ is defined shift dynamical system $(\mathcal{U}, \mathbb{R}, \sigma)$ [11, Ch.4], where $\sigma(t, \varphi)(s) := \varphi(s + t)$ for all $t, s \in \mathbb{R}$ and $\varphi \in \mathcal{U}$.

Let $\mathcal{S}$ be a closed and invariant (with respect to translations) subset of $\mathcal{U}$. Consider a control dynamical system governed by the differential equation

$$x' = f(x; u(t)) \quad (x \in E, u \in \mathcal{S}). \tag{1}$$

The control system (1) is called regular on $\mathcal{S}$, if for all $u \in \mathcal{S}$ and $x \in E$ the equation (1) has a unique solution $\varphi(t, x, u)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x, u) = x$ and the mapping $\varphi : \mathbb{R}_+ \times E \times \mathcal{U} \mapsto E$ is continuous.

In the book [11] there is the condition of regularity control affine systems of the form

$$x'(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), \; u \in \mathcal{U}.$$

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It easy to see that the regular control system (1) generates a cocycle $\langle E, \varphi, (\mathcal{S}, \mathbb{R}, \sigma)\rangle$. Thus the system (1) can be studied in the framework of non-autonomous (cocycle) dynamical system. In particular, we can apply some of our general results to the study of global attractors of control system (1). Below we give some results of this type.

The appearance of this paper was stimulated by the works of Cheban and Mammana [8, 9] and Kloeden [16].

The work of Cheban and Mammana [8, 9] is dedicated to the study of compact global attractors of difference inclusions [17] and description of its structure. Let W be a metric space, $\mathcal{M} := \{f_i \ : \ i \in I\}$ be a family of continuous mappings of W into itself and $(W, f_i)_{i\in I}$ be the family of discrete dynamical systems, where (W, f) is a discrete dynamical system generated by positive powers of a continuous map $f : W \to W$. On the space W we consider a discrete inclusion

$$u_{t+1} \in F(u_t)$$

associated to $\mathcal{M} := \{f_i \ : \ i \in I\}$ $(DI(\mathcal{M}))$, where $F(u) = \{f(u) \ : f \in \mathcal{M}\}$ for all $u \in W$.

A solution of the difference inclusion $DI(\mathcal{M})$ is (see, for example, [13]) a sequence $\{\{x_j\} \mid j \geq 0\} \subset W$ such that

$$x_j = f_{i_j} x_{j-1}$$

for some $f_{i_j} \in \mathcal{M}$ (trajectory of $DI(\mathcal{M})$), i.e.

$$x_j = f_{i_j} f_{i_{j-1}} ... f_{i_1} x_0 \text{ all } f_{i_k} \in \mathcal{M}.$$

We can consider that it is a discrete control problem, where at each moment of time j we can apply a control from the set $\mathcal{M}$, and $DI(\mathcal{M})$ is the set of possible trajectories of the system.

Let $m \in \mathbb{N}$ $(m \geq 2)$, $\mathfrak{C} := \{c_1, c_2, \ldots, c_m\} \subseteq U$ and $S(\mathbb{R}, \mathfrak{C})$ denote the set of piecewise constant functions $u(t)$ defined on $\mathbb{R}$ that assume values of set $\mathfrak{C}$ (i.e. $u \in S(\mathbb{R}, \mathfrak{C})$ if and only if there is a increasing sequence $\{t_k(u)\}_{k\in\mathbb{Z}}$ such that $u(t) = c_{i_k}$ for all $t \in (t_k(u), t_{k+1}(u))$), continuous from the right and with the limit from the left on $\mathbb{R}$. Consider the set of control dynamical systems (1) with control of class $S(\mathbb{R}, \mathfrak{C})$. These systems constitute a continual set. Particularly important among all systems of this set are m system

$$x' = f(x; c_i) \quad (x \in E, i = 1, 2, \ldots, m). \tag{2}$$

Let $\tau > 0$, $\mathcal{R} = \mathbb{R}$ or $\mathbb{R}_+$ and $\mathcal{Z} = \mathcal{R} \bigcap \mathbb{Z}$. By $S_\tau(\mathcal{R}, \mathfrak{C})$ we denote the subset of $S(\mathcal{R}, \mathfrak{C})$ consisting from the functions $u \in S(\mathcal{R}, \mathfrak{C})$ with the set $\{t_k(u)\}_{k\in\mathcal{Z}}$ of points of discontinuity satisfying the condition $t_{k+1}(u) - t_k(u) \geq \tau$ for all $k \in \mathcal{Z}$.

Remark 1.1. *Below, without loss of generality, we can suppose that* $t_0(u) = \inf\{t_k(u) \mid$ *such that* $t_k(u) \geq 0\}$.

Denote by

$$d(u_1, u_2) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{d_n(u_1, u_2)}{1 + d_n(u_1, u_2)} \tag{3}$$

for all $u_1, u_2 \in S(\mathcal{R}, \mathfrak{C})$, where $d_n(u_1, u_2) := \int_{a_n}^{b_n} |u_1(t) - u_2(t)|dt$ for all $n \in \mathbb{N}$, where $[a_n, b_n] = [-n, n] \bigcap \mathcal{R}$. By (3) is defined a distance on the space $S(\mathbb{R}, \mathfrak{C})$. In the work of Kloeden [16] is studied the problem of existence of compact pullback attractors of switching systems (1) with $\mathcal{S} = S_\tau(\mathbb{R}, \mathfrak{C})$.

Theorem 1.2. [16] $(S_\tau(\mathbb{T}, \mathfrak{C}), d)$ *is a compact metric space.*

This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of set-valued dynamical systems which we use in our paper.

Section 3 is dedicated to the study of upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems. They play a very important role in the study of non-autonomous dynamical systems. We give the sufficient conditions which guarantee the existence of a unique globally asymptotically (but, generally speaking, not exponentially) stable invariant section (Theorem 3.3 - main result of paper). This theorem generalizes the main results from the work [8, 9]. Analogous statements for non-autonomous dynamical systems, when the base dynamical system $(Y, \mathbb{T}_2, \sigma)$ is invertible, are known (see, for example, [6, Ch.2] and [21]).

In Section 4 we apply our main result to the study the compact global attractors of switching systems. We give also the description the structure of global attractors for this type of control systems. In particular, we indicate the conditions of existence of chaotic attractor of switching systems.

2. Set-Valued Dynamical Systems and Their Compact Global Attractors

Let (X, ρ) be a complete metric space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a semi-group of additive group $\mathbb{S}$. If $A \subseteq X$ and $x \in X$, then we denote by $\rho(x, A)$ the distance from the point x to the set A, i.e. $\rho(x, A) := \inf\{\rho(x, a) : a \in A\}$. We note note by $B(A, \varepsilon)$ an ε-neighborhood of the set A, i.e. $B(A, \varepsilon) := \{x \in X : \rho(x, A) < \varepsilon\}$, by $K(X)$ we denote the family of all non-empty compact subsets of X. To every point $x \in X$ and number $t \in \mathbb{T}$ we associate a closed compact subset $\pi(t, x) \in K(X)$. So, if $\pi(P, A) = \bigcup\{\pi(t, x) : t \in P, x \in A\} (P \subseteq \mathbb{T})$, then

(i) $\pi(0, x) = x$ for all $x \in X$;
(ii) $\pi(t_2, \pi(t_1, x)) = \pi(t_1 + t_2, x)$ for all $x \in X$;
(iii) $\lim\limits_{x \to x_0, t \to t_0} \beta(\pi(t, x), \pi(t_0, x_0)) = 0$ for all $x_0 \in X$ and $t_0 \in \mathbb{T}$, where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-deviation of the set $A \subseteq X$ from the set $B \subseteq X$.

In this case it is said [20] that there is defined a set-valued semi-group dynamical system.

Let $\mathbb{T} = \mathbb{S}$ and be fulfilled the next condition:

(i) if $p \in \pi(t, x)$, then $x \in \pi(-t, p)$ for all $x, p \in X$ and $t \in \mathbb{T}$.

Then it is said that there is defined a set-valued group dynamical system $(X, \mathbb{T}, \pi)$ or a bilateral (two-sided) dynamical system.

Let $\mathbb{T}' \subset \mathbb{S}$ ($\mathbb{T} \subset \mathbb{T}'$). A continuous mapping $\gamma_x : \mathbb{T} \to X$ is called a motion of the set-valued dynamical system $(X, \mathbb{T}, \pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}'$, if

a. $\gamma_x(0) = x$;
b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ for all $t_1, t_2 \in \mathbb{T}'$ $(t_2 > t_1)$.

The set of all motions of $(X, \mathbb{T}, \pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\mathcal{F}_x(\pi)$ and $\mathcal{F}(\pi) := \bigcup\{\mathcal{F}_x(\pi) \mid x \in X\}$ (or simply $\mathcal{F}$).

The trajectory $\gamma \in \mathcal{F}(\pi)$ defined on $\mathbb{S}$ is called a full (entire) trajectory of the dynamical system $(X, \mathbb{T}, \pi)$.

Denote by $\Phi(\pi)$ the set of all full trajectories of the dynamical system $(X, \mathbb{T}, \pi)$ and $\Phi_x(\pi) := \mathcal{F}_x(\pi) \bigcap \Phi(\pi)$.

Theorem 2.1. [20] *Let $(X, \mathbb{T}, \pi)$ be a semi-group dynamical system and X be a compact and invariant set (i.e. $\pi^t X = X$ for all $t \in \mathbb{T}$, where $\pi^t := \pi(t, \cdot)$). Then*

(i) *$\mathcal{F}(\pi) = \Phi(\pi)$, i.e. every motion $\gamma \in \mathcal{F}_x(\pi)$ can be extended on $\mathbb{S}$ (this means that there exists $\tilde{\gamma} \in \Phi_x(\pi)$ such that $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in \mathbb{T}$);*
(ii) *there exists a group (generally speaking set-valued) dynamical system $(X, \mathbb{S}, \tilde{\pi})$ such that $\tilde{\pi}|_{\mathbb{T}\times X} = \pi$.*

A system $(X, \mathbb{T}, \pi)$ is called [5, 6] compactly dissipative, if there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t M, K) = 0;$$

for all $M \in K(X)$.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a compact set attracting every compact subset of X. Let us set

$$J := \omega(K) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi^\tau K}. \quad (4)$$

The set J is called a center of Levinson of the compact dissipative system $(X, \mathbb{T}, \pi)$.

3. Upper Semi-Continuous Invariant Sections of Non-Autonomous Dynamical Systems

This section is dedicated to the study of upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems. We give the sufficient conditions which guaranty the existence a unique globally asymptotically stable invariant section.

Let X be a metric space and Y be a topological space. The set-valued mapping $\gamma : Y \to K(X)$ is said to be upper semi-continuous (or β-continuous), if $\lim_{y\to y_0} \beta(\gamma(y), \gamma(y_0)) = 0$ for all $y_0 \in Y$.

Let (X, h, Y) be a fibre bundle [3, 14]. The mapping $\gamma : Y \to K(X)$ is called a section (selector) of the fibre bundle (X, h, Y), if $h(\gamma(y)) = y$ for all $y \in Y$.

Remark 3.1. *Let $X := W \times Y$. Then $\gamma : Y \to X$ is a section of the fibre bundle (X, h, Y) ($h := pr_2 : X \to Y$), if and only if $\gamma = (\psi, Id_Y)$ where $\psi : W \to K(W)$.*

Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two dynamical systems. The mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (respectively homeomorphic) and $h(\pi(x, t)) = \sigma(h(x), t)$ ($t \in \mathbb{T}_1,\ x \in X$).

A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where h is a homomorphism of $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ and (X, h, Y) is a fibre bundle [3, 14], is called a non-autonomous dynamical system.

A mapping $\gamma : Y \to X$ is called an invariant section of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, if it is a section of the fibre bundle (X, h, Y) and $\gamma(Y)$ is an invariant subset of the dynamical system $(X, \mathbb{T}_1, \pi)$ (or, equivalently,

$$\bigcup \{\pi^t \gamma(q) \ : \ q \in (\sigma^t)^{-1}(\sigma^t y)\} = \gamma(\sigma^t y)$$

for all $t \in \mathbb{T}_1$ nd $y \in Y$).

Lemma 3.2. [10] *Let $\omega : \mathbb{T}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and there exists a positive number $t_0 \in \mathbb{S}_+$ such that*

a. *$\omega(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotone increasing;*
b. *$\omega(t_0, r) < r$ for all $r > 0$;*
c. *the mapping $\omega(t_0, \cdot)$ is continuous on $\mathbb{R}_+$;*
d. *$\omega(t + \tau, r) \le \omega(t, \omega(\tau, r))$ for all $t, \tau \in \mathbb{T}_+$ and $r \in \mathbb{R}_+$;*
e. *for every $r \in \mathbb{R}_+$ the mapping $\omega(\cdot, r) : \mathbb{T}_+ \to \mathbb{R}_+$ is continuous.*

Then the following statements hold:

(i) *the equality*

$$\lim_{t\to+\infty} \omega(t, r) = 0 \tag{5}$$

takes place for all $r \in \mathbb{R}_+$;

(ii) *if the mapping $\omega : \mathbb{T}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, then the equality (5) holds uniformly with respect to r on every compact subset from $\mathbb{R}_+$.*

Theorem 3.3. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

(i) *the space Y is compact;*
(ii) *Y is invariant, i.e. $\sigma^t Y = Y$ for all $t \in \mathbb{T}_2$;*
(iii) *the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is contracting in the extended sense, i.e. there exists a continuous function $\omega : \mathbb{T}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\rho(\pi(t, x_1), \pi(t, x_2)) \le \omega(t, \rho(x_1, x_2)) \tag{6}$$

for all $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$) and $t \in \mathbb{T}_1$ and the following conditions hold:

a. *$\omega(t,\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotone increasing for all $t > 0$;*
b. *there exists some positive $t_0 \in \mathbb{S}_+$ such that $\omega(t_0, r) < r$ for all $r > 0$;*
c. *$\omega(t + \tau, r) \le \omega(t, \omega(\tau, r))$ for all $t, \tau \in \mathbb{T}_+$ and $r \in \mathbb{R}_+$.*

Then

(i) *there exists a unique invariant section $\gamma \in \Gamma(Y,X)$ of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$;*
(ii) *the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is compactly dissipative and its Levinson center $J = \gamma(Y)$;*
(iii) *the set J is invariant with respect to dynamical system $(X, \mathbb{T}_1, \pi)$, i.e. $\bigcup\{\pi^t J_q \ : \ q \in (\sigma^t)^{-1}(\sigma^t y)\} = J_{\sigma(t,y)}$ for all $t \in \mathbb{T}_1$ and $y \in Y$;*
(iv) *if $(Y, \mathbb{T}_2, \sigma)$ is a group-dynamical system (i.e. $\mathbb{T}_2 = \mathbb{S}$), then the unique invariant section γ of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is one-valued (i.e. $\gamma(y)$ consists a single point for any $y \in Y$) and*

$$\rho(\pi(t,x), \pi(t, \gamma(h(x)))) \le \omega(t, \rho(x, \gamma(h(x)))) \qquad (7)$$

for all $x \in X$ and $t \in \mathbb{T}$.

Proof. Since the space Y is compact and invariant, then according to Theorem 2.1 the semi-group dynamical system $(Y, \mathbb{T}, \sigma)$ can be prolonged to a group set-valued dynamical system $(Y, \mathbb{S}, \tilde{\sigma})$ (this means that $\tilde{\sigma}(s,y) = \sigma(s,y)$ for all $(s,y) \in \mathbb{T} \times Y$).

Let us denote by $\alpha : K(X) \times K(X) \to \mathbb{R}_+$ the Hausdorff distance on $K(X)$ and $d : \Gamma(Y,X) \times \Gamma(Y,X) \to \mathbb{R}_+$ is the function defined by the equality

$$d(\gamma_1, \gamma_2) := \sup_{y \in Y} \alpha(\gamma_(y), \gamma_2(y)). \qquad (8)$$

It is easy to verify that by equality (8) there is defined a distance on $\Gamma(Y,X)$. Then (see [8]) the metric space $(\Gamma(Y,X), d)$ is complete.

Let $t \in \mathbb{T}_1$, by S^t we denote the mapping of $\Gamma(Y,X)$ in itself defined by the equality $(S^t\gamma)(y) = \pi(t, \gamma((\sigma^t)^{-1}y))$ for all $t \in \mathbb{T}_1$, $y \in Y$ and $\gamma \in \Gamma(Y,X)$. It is easy to see that $S^t\gamma \in \Gamma(Y,X)$, $S^t \circ S^\tau = S^{t+\tau}$ for all $t, \tau \in \mathbb{T}_1$ and $\gamma \in \Gamma(Y,X)$ and, hence, $\{S^t\}_{t \in \mathbb{T}_1}$ forms a commutative semi-group. Besides, from inequality (6) and the definition of the metric d, under the conditions of Theorem, the next inequality follows:

$$d(S^t\gamma_1, S^t\gamma_2) \le \omega(t, d(\gamma_1, \gamma_2)) \qquad (9)$$

for all $t \in \mathbb{T}_1$ and $\gamma_i \in \Gamma(Y,X)$ $(i = 1, 2)$. To prove the inequality (9) it is sufficient to show that

$$\alpha(\pi^t\gamma_1(\sigma^{-t}y), \pi^t\gamma_2(\sigma^{-t}y) \le \omega(t, d(\gamma_1, \gamma_2))$$

for all $y \in Y$, where $\sigma^{-t}y := \{q \in Y \mid \sigma(t,q) = y\}$.

Let $v \in \pi^t\gamma_2(\sigma^{-t}y)$ be an arbitrary element, then there is $q \in \sigma^{-t}y$ and $x_2(y) \in \gamma_2(q)$ so that $v = \pi^t x_2(y)$. We choose $x_1(y) \in \gamma_1(q)$ such that

$$\rho(x_1(y), x_2(y)) \le \alpha(\gamma_1(q), \gamma_2(q)) \le d(\gamma_1, \gamma_2)$$

(by compactness of $\gamma_i(q)$ $(i = 1, 2)$ obviously an such $x_1(y)$ there exists and additionally $h(x_1(y)) = h(x_2(y)) = q$). Then we have

$$\rho(\pi^t x_1(y), \pi^t x_2(y)) \leq \omega(t, \rho(x_1(y), x_2(y))) \leq \omega(t, d(\gamma_1, \gamma_2)),$$

i.e. for all $v \in \pi^t\gamma_2(\sigma^{-t}y)$ there exists $u := \pi^t x_1(y) \in \pi^t\gamma_1(\sigma^{-t}y)$ so that $\rho(u, v) \leq \omega(t, d(\gamma_1, \gamma_2))$. This means that $\beta(\pi^t\gamma_1(\sigma^{-t}y), \pi^t\gamma_2(\sigma^{-t}y)) \leq \omega(t, d(\gamma_1, \gamma_2))$. Analogously, can be established the inequality $\beta(\pi^t\gamma_2(\sigma^{-t}y), \pi^t\gamma_1(\sigma^{-t}y)) \leq \omega(t, d(\gamma_1, \gamma_2))$ and, consequently, $\alpha(\pi^t\gamma_1(\sigma^{-t}y), \pi^t\gamma_2(\sigma^{-t}y)) \leq \omega(t, d(\gamma_1, \gamma_2))$ for all $y \in Y$ and $t \in \mathbb{T}_1$. Thus, we have

$$d(S^t\gamma_1, S^t\gamma_2) \leq \omega(t, d(\gamma_1, \gamma_2)) \tag{10}$$

for all $t \in \mathbb{T}_1$ and $\gamma_1, \gamma_2 \in \Gamma(Y, X)$. From the inequality (10) it follows that the mapping S^{t_0} of the space $\Gamma(Y, X)$ to itself satisfy all the conditions of Theorem 3.7 from [10] and, hence, it has a unique fixed point. Since $\{S^t\}_{t\in\mathbb{T}_1}$ is commutative, then there exists a unique common stationary point γ which is an invariant section of non-autonomous dynamical system $\langle(X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, i.e. $\gamma(Y) \subset X$ is an invariant set of the dynamical system $(X, \mathbb{T}_1, \pi)$.

Let us denote by $K := \gamma(Y)$, then K is a nonempty compact and invariant set of the dynamical system $(X, \mathbb{T}_1, \pi)$. From the inequality (6) and Lemma 3.2 it follows that

$$\lim_{t\to+\infty} \rho(\pi^t M, K) = 0 \tag{11}$$

for all $M \in K(X)$ and, consequently, the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center $J \subseteq K$. On the other hand, $K \subseteq J$, because the set $K = \gamma(Y)$ is compact and invariant, but J is a maximal compact invariant set of $(X, \mathbb{T}_1, \pi)$. Thus we have $J = \gamma(Y)$.

Let now $\mathbb{T}_2 = \mathbb{S}$. Then we will show that the set $\gamma(y)$ contains a single point for any $y \in Y$. If we suppose that it is not true, then there are $y_0 \in Y$ and $x_1, x_2 \in \gamma(y_0)$ $(x_1 \neq x_2)$. Let $\phi_i \in \Phi_{x_i}$ $(i = 1, 2)$ be such that $\phi_i(\mathbb{S}) \subseteq J$. Then we have

$$\pi^t(\phi_i(-t)) = x_i \ (i = 1, 2) \tag{12}$$

for all $t \in \mathbb{T}_1$. Note that from the inequality (6) and the equality (12) it follows that

$$\begin{aligned} \rho(x_1, x_2) = \rho(\pi^t(\phi_1(-t)), \pi^t(\phi_2(-t))) \leq \\ \omega(t, \rho(\phi_1(-t), \phi_2(-t))) \leq \omega(t, C) \end{aligned} \tag{13}$$

for all $t \in \mathbb{T}$, where $C := \sup\{\rho(\phi_1(s), \phi_2(s)) \ : \ s \in \mathbb{S}\}$. Passing to the limit in (13) as $t \to +\infty$ we obtain $x_1 = x_2$. The obtained contradiction proves our statement.

Thus, if $\mathbb{T}_2 = \mathbb{S}$, the unique fix point $\gamma \in \Gamma(Y, X)$ of the semi-group of operators $\{S^t\}_{t\in\mathbb{T}_1}$ is a single-valued function and, consequently, it is continuous. Finally, inequality (7) follows from (6), because $h(\gamma(h(x))) = (h \circ \gamma)(h(x)) = h(x)$ for all $x \in X$. The theorem is completely proved. □

Remark 3.4. *1. In the particular case, when $\omega(t, r) = \mathcal{N}e^{-\nu t}r$, Theorem 3.3 was proved by Cheban D. and Mammana C.* [8].

2. If $(Y, \mathbb{T}_2, \sigma)$ is a semi-group dynamical system (i.e. $\mathbb{T}_2 = \mathbb{R}_+$ or $\mathbb{Z}_+$), then the unique invariant section γ of the non-autonomous dynamical system $\langle(X, \mathbb{T}_1, \pi),$

$(Y,\mathbb{T}_2,\sigma),h\rangle$ is multi-valued (i.e. $\gamma(y)$ contains, generally speaking, more than one point [8]*).*

4. Applications

Below we apply our main result to the study of the compact global attractors of different classes of switching systems. We give also the description of the structure of global attractors for this type of control systems. In particular, we indicate the conditions of existence of chaotic global attractor of switching systems.

4.1. **Switching Systems on $\mathbb{R}$.** Consider a control dynamical system governed by the differential equation

$$x' = f(x;u) \quad (x \in E, u \in U \subset F). \tag{14}$$

Let $m \in \mathbb{N}$ $(m \geq 2)$, $\mathfrak{C} := \{c_1, c_2, \dots, c_m\} \subseteq F$ and $S(\mathbb{R},\mathfrak{C})$ denote the set of piecewise constant functions $u(t)$ defined on $\mathbb{R}$ that assume values of set $\mathfrak{C}$. Consider the set of control dynamical systems (14) with control of class $S(\mathbb{R},\mathfrak{C})$.

Let $\tau > 0$. By $S_\tau(\mathbb{R},\mathfrak{C})$ we denote the subset of $S(\mathbb{R},\mathfrak{C})$ consisting from the functions $u \in S(\mathbb{R},\mathfrak{C})$ with the set $\{t_k(u)\}_{k\in\mathbb{Z}}$ of points of discontinuity satisfying the condition $t_{k+1}(u) - t_k(u) \geq \tau$ for all $k \in \mathbb{Z}$.

Let σ be a mapping from $\mathcal{R}\times S_\tau(\mathcal{R},\mathfrak{C})$ into $S_\tau(\mathcal{R},\mathfrak{C})$ defined by equality $\sigma(t,u) := u_t$ for all $t \in \mathcal{R}$ and $u \in S_\tau(\mathcal{R},\mathfrak{C})$, where u_t is the t-shift of the function u, i.e. $u_t(s) := u(t+s)$ for all $s \in \mathcal{R}$. It easy to verify that: $\sigma(0,u) = u$, $\sigma(t_1+t_2,u) = \sigma(t_2,\sigma(t_1,u))$ and $\{t_k(u_t)\} = \{t_k(u)\} - t := \{t_k(u) - t \mid k \in \mathcal{Z}\}$ for all $t_1,t_2 \in \mathcal{R}$ and $u \in S_\tau(\mathcal{R},\mathfrak{C})$.

Theorem 4.1. [4, 16, 19] *The mapping $\sigma : \mathcal{R}\times S_\tau(\mathcal{R},\mathfrak{C}) \mapsto S_\tau(\mathcal{R},\mathfrak{C})$ is continuous and, consequently, $(S_\tau(\mathcal{R},\mathfrak{C}),\mathcal{R},\sigma)$ is a dynamical system on $S_\tau(\mathcal{R},\mathfrak{C})$.*

Denote by $\varphi(t,x,u)$ the solution of the equation (14) with initial condition $\varphi(0,x,u) = x$, assuming that a unique solution exists for all $t \geq 0$. Then the mapping φ : $\mathcal{R}\times E\times S_\tau(\mathcal{R},\mathfrak{C})$ possesses the semi-group (cocycle) property: $\varphi(t+s,x,u) = \varphi(s,\varphi(t,x,u),\sigma(t,u))$ for all $t,s \in \mathcal{R}$, $x \in E$ and $u \in S_\tau(\mathcal{R},\mathfrak{C})$.

Theorem 4.2. [16] *The mapping $\varphi : \mathcal{R}\times E\times S_\tau(\mathcal{R},\mathfrak{C}) \mapsto E$ is continuous.*

Thus the triple $\langle E,\varphi,(S_\tau(\mathcal{R},\mathfrak{C}),\mathcal{R},\sigma)\rangle$ is a cocycle under dynamical system $(S_\tau(\mathcal{R},$ $\mathfrak{C}),\ \sigma)$ with fiber E and, consequently, we can apply the results established in Section 3 to the study of switching system (14).

Lemma 4.3. *Let $\theta \in C(\mathbb{R},\mathbb{R})$ and it possesses the following properties:*

a. *θ is regular, i.e. the equation*

$$x' = \theta(x) \tag{15}$$

defines a semi-group dynamical system $(\mathbb{R},\mathbb{R}_+,\omega)$ on $\mathbb{R}$;

b. *there exists $r_0 > 0$ such that $x\theta(x) < 0$ for all $|x| > r_0$.*

Then

(i) *the dynamical system* $(\mathbb{R},\mathbb{R}_+,\omega)$ *is dissipative, i. e. there exists a positive number* r *such that*

$$\limsup_{t\to+\infty}|\omega(t,x)|<r \tag{16}$$

for all $x\in\mathbb{R}$ *and (16) takes place uniformly with respect to* x *on every bounded subset from* $\mathbb{R}$*;*

(ii) *the mapping* $\omega(t,\cdot):\mathbb{R}\mapsto\mathbb{R}$ *is monotone decreasing for every* $t\ge 0$.

Proof. To prove the first statement we note that the derivative of the function $V(x)=x^2$ along of the trajectories of $(\mathbb{R},\mathbb{R}_+,\omega)$ is negative for all $|x|>r_0$. Now it is sufficient to apply Theorem 5.5 and Remark 5.4 (item d.) from [6, Ch.5].

The second statement is a general property of scalar differential equations. □

Theorem 4.4. *Suppose that there exists a function* $\theta\in C(\mathbb{R},\mathbb{R})$ *with the properties a. and b. from Lemma 4.3 such that*

$$\langle x,f_i(x)\rangle\le\theta(|x|^2) \tag{17}$$

for all $i=1,2,\dots,m$ *and* $x\in E$*, where* $f_i(\cdot):=f(\cdot,c_i)$ $(i=1,2,\dots,m)$ *and* $\langle\cdot,\cdot\rangle$ *is the scalar product on* E.

Then the switching system (14) is dissipative, namely for all $R>o$ *there exists a constant* $L(R)>0$ *such that* $|\varphi(t,x,u)|<r$ *for all* $|x|\le R$, $t\ge L(R)$ *and* $u\in S_\tau(\mathbb{R},\mathfrak{C})$.

Proof. Denote by $(E,\mathbb{R}_+,\pi_i)$ $(i=1,2,\dots,m)$ the dynamical system generated by equation (2). Let now $u\in S_\tau(\mathbb{R},\mathfrak{C})$, $\{t_k(u)\}_{k\in\mathbb{Z}}$ the set of points of discontinuity of u and $t\in\mathbb{R}_+$. Then there exists $k\in\mathbb{Z}_+$ such that $t_k(u)\le t<t_{k+1}(u)$ and $u(t)=c_{i_k}$ (for all $t\in[t_k,t_{k+1})$), hence, we have the equality

$$\varphi(t,x,u)=\pi_{i_k}(t-t_k,\varphi(t_k,x,u)). \tag{18}$$

According to the condition (17) we obtain

$$\frac{d|\pi_{i_k}(t,x)|^2}{dt}\le\theta(|\pi_{i_k}(t,x)|^2)$$

and, consequently,

$$|\pi_{i_k}(t,x)|^2\le\omega(t,|x|^2) \tag{19}$$

for all $t\ge0$ and $x\in E$, where $\omega(t,x)$ is the solution of equation (15) with initial condition $\omega(0,x)=x$.

From (18) and (19) we have

$$|\varphi(t,x,u)|^2=|\pi_{i_k}(t-t_k,\varphi(t_k,x,u))|^2\le\omega(t-t_k,|\varphi(t_k,x,u)|^2) \tag{20}$$

for all $t\in(t_k,t_{k+1})$, $x\in E$ and $u\in S_\tau(\mathbb{R},\mathfrak{C})$. Denote by $a_k(x,u):=|\varphi(t_k(u),x,u)|^2$ (for all $k\in\mathbb{Z}_+$), then by inequality (20) we obtain

$$a_{k+1}(x,u)\le\omega(t_{k+1}(u)-t_k(u),a_k(x,u))$$

for all $k \in \mathbb{N}$ and $(x,u) \in E \times S_\tau(\mathbb{R},\mathfrak{C})$. We note that

$$
\begin{aligned}
(21)\qquad & a_0(x,u) = |\varphi(t_0(u),x,u)|^2 \le \omega(t_0(u),|x|^2)\\
& a_1(x,u) \le \omega(t_1(u)-t_0(u),|\varphi(t_0,x,u)|^2)\\
& \le \omega(t_1(u)-t_0(u),\omega(t_0(u),|x|^2)) = \omega(t_1(u),|x|^2)\\
& \dots\\
& a_{k+1}(x,u) \le \omega(t_{k+1}(u)-t_k(u),a_k(x,u))\\
& \le \omega(t_{k+1}(u)-t_k(u),\omega(t_k(u),|x|^2)) = \omega(t_{k+1}(u),|x|^2).
\end{aligned}
$$

Now, using the (18), (20) and (21) we receive

$$
\begin{aligned}
(22)\qquad & |\varphi(t,x,u)|^2 \le \omega(t-t_k(u),|\varphi(t_k(u),x,u)|^2) \le \omega(t-t_k(u),a_k(x,u))\\
& \le \omega(t-t_k(u),\omega(t_k(u),|x|^2)) = \omega(t,|x|^2)
\end{aligned}
$$

for all $t \ge 0$ and $(x,u) \in E \times S_\tau(\mathbb{R},\mathfrak{C})$. From the inequality (22) and Lemma 4.3 it follows that for all $R > 0$there exists a constant $L(R) > 0$ such that $|\varphi(t,x,u)| < r$ for all $|x| \le R$, $t \ge L(R)$ and $u \in S_\tau(\mathbb{R},\mathfrak{C})$. □

As an example of Theorem 4.4 we consider the switching of nonlinear equations

$$
(23)\qquad x' = -|x|x + c_i \quad (i = 1,2,\dots,m)
$$

which satisfy of condition (17) with $\theta(x) := -x^{3/2} + Cx^{1/2}$ $(x \in \mathbb{R}_+)$, where $C := \max_{1\le i\le m} |c_i|$.

A cocycle φ over $(Y,\mathbb{S},\sigma)$ with the fiber W is said to be compactly dissipative, if there exits a nonempty compact $K \subseteq W$ such that

$$
(24)\qquad \lim_{t\to+\infty} \sup\{\beta(U(t,y)M,K) \mid y \in Y\} = 0
$$

for any $M \in C(W)$.

Theorem 4.5. [6] *Let Y be compact, $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (24), then:*

1. *$I_y = \omega_y(K) \ne \emptyset$, is compact, $I_y \subseteq K$ and*
$$\lim_{t\to+\infty} \beta(U(t,y^{-t})K,I_y) = 0$$
for every $y \in Y$;
2. *$U(t,y)I_y = I_{yt}$ for all $y \in Y$ and $t \in \mathbb{S}_+$;*
3. $$\lim_{t\to+\infty} \beta(U(t,y^{-t})M,I_y) = 0$$
for all $M \in C(W)$ and $y \in Y$;
4. $$\lim_{t\to+\infty} \sup\{\beta(U(t,y^{-t})M,I) \mid y \in Y\} = 0$$
for any $M \in C(W)$, where $I := \cup\{I_y \mid y \in Y\}$;
5. *$I_y = pr_1 J_y$ for all $y \in Y$, where J is the Levinson center of $(X,\mathbb{T}_+,\pi)$, and hence $I = pr_1 J$;*
6. *the set I is compact.*

Remark 4.6. *The theorem 4.5 is true also, when $(Y, \mathbb{S}_+, \sigma)$ is a semi-group dynamical system, but Y is invariant, i.e. $\sigma(t, Y) = Y$ for all $t \in \mathbb{S}_+$ (see* [7] *and also* [15]*).*

Theorem 4.7. *Under the conditions of Theorem 4.4 the cocycle φ, generates by control system (14), admits a compact global attractor.*

Proof. This statement follows from Theorems 4.4 and 4.5. □

4.2. **Singleton Global Attractors of Switching Systems.**

Lemma 4.8. [10] *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a function satisfying the following conditions:*

- (H1) $f(0) = 0$;
- (H2) $f(t) > 0$ *for all* $t > 0$;
- (H3) f *is locally Lipschitz;*
- (H4) f *satisfies the condition of Osgud, i.e.* $\int_0^{\varepsilon} \frac{ds}{f(s)} = +\infty$ *for all* $\varepsilon > 0$.

Then the equation

$$u' = -f(u)$$

admits a unique solution $\omega(t, r)$ with initial condition $\omega(0, r) = r$ and the mapping $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ possesses the following properties:

- (i) *the mapping* $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ *is continuous;*
- (ii) $\omega(t, r) < r$ *for all* $r > 0$ *and* $t > 0$;
- (iii) *for all* $t \in \mathbb{R}_+$ *the mapping* $\omega(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is increasing;*
- (iv) $\omega(0, t) = 0$ *for all* $t \in \mathbb{R}_+$;
- (v) $\lim\limits_{t \to +\infty} \sup\limits_{0 \le r \le r_0} \omega(t, r) = 0$ *for all* $r_0 > 0$.

Theorem 4.9. *Suppose that there exists a function $\theta \in C(\mathbb{R}, \mathbb{R})$ with the properties (H1)-(H4) from Lemma 4.8 such that*

$$\langle x_1 - x_2, f_i(x_1) - f_i(x_2) \rangle \le -\theta(|x_1 - x_2|^2) \tag{25}$$

for all $i = 1, 2, \dots, m$ and $x_1, x_2 \in E$, where $f_i(\cdot) := f(\cdot, c_i)$ $(i = 1, 2, \dots, m)$.

Then the following statements hold:

- (i)
$$|\varphi(t, x_1, u) - \varphi(t, x_1, u)|^2 \le \omega(t, |x_1 - x_2|^2)$$
for all $t \ge 0$ and $(x_i, u) \in E \times S_\tau(\mathbb{R}, \mathfrak{C})$ $(i = 1, 2)$;
- (ii) *there exists a continuous function $\nu \in C(S_\tau(\mathbb{R}, \mathfrak{C}), E)$ such that $\varphi(t, \nu(u), u) = \nu(\sigma(t, u))$ for all $t \in \mathbb{R}$ and $u \in S_\tau(\mathbb{R}, \mathfrak{C})$;*
- (iii) *$\{I_u \mid u \in S_\tau(\mathbb{R}, \mathfrak{C})\}$, where $I_u := \{\nu(u)\}$, is a compact forward attractor of switching system (14).*

Proof. Denote by $(E, \mathbb{R}_+, \pi_i)$ $(i = 1, 2, \dots, m)$ the dynamical system generated by equation (2). Let now $u \in S_\tau(\mathbb{R}, \mathfrak{C})$, $\{t_k(u)\}_{k \in \mathbb{Z}}$ the set of points of discontinuity of u and $t \in \mathbb{R}_+$. Then there exists $k \in \mathbb{Z}_+$ such that $t_k(u) \le t < t_{k+1}(u)$ and $u(t) = c_{i_k}$ (for all $t \in [t_k, t_{k+1})$), hence, we have the equality (18).

According to the condition (25) we obtain

$$\frac{d|\pi_{i_k}(t,x_1)-\pi_{i_k}(t,x_2)|^2}{dt} \le -\theta(|\pi_{i_k}(t,x)-\pi_{i_k}(t,x_2)|^2)$$

and, consequently,

$$|\pi_{i_k}(t,x_1)-\pi_{i_k}(t,x_2)|^2 \le \omega(t,|x_1-x_2|^2) \tag{26}$$

for all $t \ge 0$ and $x_1, x_2 \in E$, where $\omega(t,x)$ is the solution of equation $x' = -\theta(x)$ with initial condition $\omega(0,x) = x$.

From (18) and (26) we have

$$\begin{aligned} |\varphi(t,x_1,u)-\varphi(t,x_2,u)|^2 = |\pi_{i_k}(t-t_k,\varphi(t_k,x_1,u)) - \\ \pi_{i_k}(t-t_k,\varphi(t_k,x_1,u))|^2 \le \omega(t-t_k,|\varphi(t_k,x_1,u)-\varphi(t_k,x_2,u)|^2) \end{aligned} \tag{27}$$

for all $t \in (t_k, t_{k+1})$, $x_1, x_2 \in E$ and $u \in S_\tau(\mathbb{R},\mathfrak{C})$. Denote by $b_k(x_1,x_2,u) := |\varphi(t_k(u),x_1,u)-\varphi(t_k(u),x_1,u)|^2$ (for all $k \in \mathbb{Z}_+$), then by inequality (27) we obtain

$$b_{k+1}(x_1,x_2,u) \le \omega(t_{k+1}(u)-t_k(u), b_k(x_1,x_2,u))$$

for all $k \in \mathbb{Z}_+$ and $(x_i,u) \in E \times S_\tau(\mathbb{R},\mathfrak{C})$ $(i = 1,2)$. We note that

$$\begin{aligned} & b_0(x_1,x_2,u) = |\varphi(t_0(u),x_1,u)-\varphi(t_0(u),x_1,u)|^2 \le \omega(t_0(u),|x_1-x_2|^2) \\ & b_1(x_1,x_2,u) \le \omega(t_1(u)-t_0(u),|\varphi(t_0,x_1,u)-\varphi(t_0,x_1,u)|^2) \\ & \le \omega(t_1(u)-t_0(u),\omega(t_0(u),|x_1-x_2|^2)) = \omega(t_1(u),|x_1-x_2|^2) \\ & \dots \\ & b_{k+1}(x_1,x_2,u) \le \omega(t_{k+1}(u)-t_k(u),b_k(x_1,x_2,u)) \\ & \le \omega(t_{k+1}(u)-t_k(u),\omega(t_k(u),|x_1-x_2|^2)) = \omega(t_{k+1}(u),|x_1-x_2|^2). \end{aligned} \tag{28}$$

Now, using the (18), (27) and (28) we receive

$$\begin{aligned} |\varphi(t,x_1,u)-\varphi(t,x_1,u)|^2 \le \omega(t-t_k(u),|\varphi(t_k(u),x_1,u) \\ -\varphi(t_k(u),x_1,u)|^2) \le \omega(t-t_k(u),b_k(x_1,x_2,u)) \\ \le \omega(t-t_k(u),\omega(t_k(u),|x_1-x_2|^2)) = \omega(t,|x_1-x_2|^2) \end{aligned} \tag{29}$$

for all $t \ge 0$ and $(x_i,u) \in E \times S_\tau(\mathbb{R},\mathfrak{C})$ $(i = 1,2)$.

From the inequality (29) and Lemma 4.8 it follows that all the conditions of Theorem 3.3 are fulfilled. Now to finish the proof it is sufficient to apply Theorem 3.3. □

Remark 4.10. *In the case when $\theta(x) = -x$ Theorem 4.9 improve the result of P. Kloeden* [16]. *He proved the existence of singleton pullback attractor for the switching systems. Note that every forward attractor is a pullback attractor, the inverse is not true.*

As an example of Theorem 4.9 we consider the switching of nonlinear equations (23), which satisfy to the condition (25) with $\theta(x) := -x^{3/2}$ $(x \in \mathbb{R}_+)$.

4.3. **Switching Systems on $\mathbb{R}_+$.** Let $S(\mathbb{R}_+,\mathfrak{C})$ denote the set of piecewise constant functions $u(t)$ defined on $\mathbb{R}_+$ that assume values of set $\mathfrak{C} := \{c_1, c_2, \dots, c_m\} \subseteq F$. Consider the set of control dynamical systems (14) with control of class $S(\mathbb{R}_+,\mathfrak{C})$.

Theorem 4.11. *The following statements hold:*

(i) $S_\tau(\mathbb{R}_+,\mathfrak{C}) = \overline{Per(\sigma)}$, *where $Per(\sigma)$ is the set of all periodic points of $(S_\tau(\mathbb{R}_+,\mathfrak{C}),\mathbb{R}_+,\sigma)$ (i.e. $\varphi \in Per(\sigma)$, if there exists $h > 0$ such that $\sigma(h + t,\varphi) = \sigma(t,\varphi)$ for all $t \in \mathbb{R}_+$);*
(ii) $S_\tau(\mathbb{R}_+,\mathfrak{C})$ *is invariant, i.e.* $\sigma^t S_\tau(\mathbb{R}_+,\mathfrak{C}) = S_\tau(\mathbb{R}_+,\mathfrak{C})$ *for all* $t \in \mathbb{R}_+$.

Proof. We denote by $Per(\sigma)$ the set of all periodic points of the dynamical system $(S_\tau(\mathbb{R}_+,\mathfrak{C}),\mathbb{R}_+,\sigma)$. We will prove that $\overline{Per(\sigma)} = S_\tau(\mathbb{R}_+,\mathfrak{C})$, i.e. the set of all periodic points of $S_\tau(\mathbb{R}_+,\mathfrak{C})$ is dense in $S_\tau(\mathbb{R}_+,\mathfrak{C})$. In fact, if $\varphi \in S_\tau(\mathbb{R}_+,\mathfrak{C})$, then denote by φ_k the periodic point from $Per(\sigma)$ such that $\varphi_k(t) := \varphi(t)$ for all $t \in [0,k]$ (if $k \in \{t_k(\varphi)\}$, then we put $\varphi_k(k) := \varphi(k-0)$). It is easy to see that $\{\varphi_k\} \to \varphi$ in $S_\tau(\mathbb{R}_+,\mathfrak{C})$.

From the fact established above it follows that $S_\tau(\mathbb{R}_+,\mathfrak{C})$ is invariant, i.e. $\sigma^t S_\tau(\mathbb{R}_+,\mathfrak{C}) = S_\tau(\mathbb{R}_+,\mathfrak{C})$. In fact, let $\varphi \in S_\tau(\mathbb{R}_+,\mathfrak{C})$, $t \in \mathbb{R}_+$ and $\{\varphi_k\} \subset Per(\sigma)$ be such that $\{\varphi_k\} \to \varphi$. Let $h_k > 0$ be such that $\sigma(h_k,\varphi_k) = \varphi_k$ and $h_k \to +\infty$. Then there exists $k_0 = k_0(t)$ such that $h_k \ge t$ for all $k \ge k_0$ and, consequently, we have

$$\varphi_k = \sigma(h_k,\varphi_k) = \sigma(t,\sigma(h_k - t,\varphi_k)) \tag{30}$$

for all $k \ge k_0$. Since the space $S_\tau(\mathbb{R}_+,\mathfrak{C})$ is compact we may suppose that the sequence $\{\sigma(\tau_k - t,\varphi_k)\}$ is convergent. Let $\overline{\varphi} := \lim\limits_{k\to+\infty} \sigma(h_k - t,\varphi_k)$, then from the equality (30) we obtain $\varphi = \sigma(t,\overline{\varphi})$, i.e. $\sigma^t S_\tau(\mathbb{R}_+,\mathfrak{C}) = S_\tau(\mathbb{R}_+,\mathfrak{C})$. □

Theorem 4.12. *Suppose that there exists a function $\theta \in C(\mathbb{R},\mathbb{R})$ with the properties a. and b. from Lemma 4.3 such that*

$$\langle x, f_i(x)\rangle \le \theta(|x|^2)$$

for all $i = 1, 2, \dots, m$ and $x \in E$, where $f_i(\cdot) := f(\cdot, c_i)$ $(i = 1, 2, \dots, m)$.

Then the switching system (14) is dissipative, namely for all $R > 0$ there exists a constant $L(R) > 0$ such that $|\varphi(t,x,u)| < r$ for all $|x| \le R$, $t \ge L(R)$ and $u \in S_\tau(\mathbb{R}_+,\mathfrak{C})$.

Proof. This statement can be proved with slight modification of the proof of Theorem 4.4. □

Theorem 4.13. *Under the conditions of Theorem 4.7 the cocycle φ, generated by control system (14), admits a compact global attractor.*

Proof. Consider the cocycle $\langle E,\varphi,(S_\tau(\mathbb{R}_+,\mathcal{C}),\mathbb{R}_+,\sigma)\rangle$ generated by control system (14). According to Theorems 1.2 and 4.11 $S_\tau(\mathbb{R}_+,\mathcal{C})$ is compact and invariant. By Theorem 4.7 the cocycle φ is dissipative and now to finish the proof it is sufficient to apply Theorem 4.5 and Remark 4.6. □

Theorem 4.14. *Suppose that there exists a function $\theta \in C(\mathbb{R},\mathbb{R})$ with the properties (H1)-(H4) from Lemma 4.8 such that*

$$\langle x_1 - x_2, f_i(x_1) - f_i(x_2)\rangle \le -\theta(|x_1 - x_2|^2)$$

for all $i = 1,2,\dots,m$ and $x_1, x_2 \in E$, where $f_i(\cdot) := f(\cdot, c_i)$ $(i = 1,2,\dots,m)$.

Then the following statements hold:

(i)
$$|\varphi(t,x_1,u) - \varphi(t,x_1,u)|^2 \le \omega(t, |x_1 - x_2|^2)$$
for all $t \ge 0$ and $(x_i,u) \in E \times S_\tau(\mathbb{R}_+,\mathfrak{C})$ $(i = 1,2)$;

(ii) *there exists a upper semi-continuous function $\nu : S_\tau(\mathbb{R}_+,\mathfrak{C}) \mapsto C(E)$ such that $\{I_u \mid u \in S_\tau(\mathbb{R}_+,\mathfrak{C})\}$, where $I_u := \{\nu(u)\}$, is a compact forward attractor of switching system (14).*

Proof. The first statement can be proved using the same reasoning as in the proof of Theorem 4.9. To prove the second statement it is sufficient to apply Theorem 3.3. □

Remark 4.15. *W'd like to stress that in contrast to Theorem 4.9 under the conditions of Theorem 4.14 the fibers I_u $(u \in S_\tau(\mathbb{R}_+,\mathcal{C}))$ contains more than one point.*

4.4. **Chaotic Attractors of Switching Systems.** Let $\Sigma_m := \{1,2,\dots,m\}^{\mathbb{Z}}$ and $\sigma : \Sigma_m \mapsto \Sigma_m$ be the mapping defined by equality $(\sigma\xi)(n) = \xi(n+1)$ for all $n \in \mathbb{Z}$ and $\xi \in \Sigma_m$. Denote by (Σ_m, σ) the discrete dynamical system generated by powers of the mapping σ.

We put $\mathcal{P} := \{\varphi \in S_\tau(\mathbb{R},\mathcal{C}) \mid \{t_k(\varphi)\} = \{k\tau\}_{k\in\mathbb{Z}}\}$.

Let (X_1, f_1) and (X_2, f_2) be two discrete dynamical systems. Recall that the dynamical systems (X_1, f_1) is called topological equivalent to (X_2, f_2) if there exists a homeomorphism $h : X_1 \mapsto X_2$ such that $h \circ f_1 = f_2 \circ h$.

Lemma 4.16. *The following statement hold:*

(i) *the mapping $h : \Sigma_m \mapsto S_\tau(\mathbb{R},\mathcal{C})$, defined by equality*

(31) $$h(\xi) = \varphi \ (\xi \in \Sigma_m, \text{ where } \varphi(t) := c_{\xi(k)} \text{ for all } t \in [k\tau,(k+1)\tau)),$$

is a bijective operator from Σ_m onto $\mathcal{P}$;

(ii) *the operator $h : \Sigma_m \mapsto \mathcal{P}$ defined by equality (31) is a homeomorphism;*

(iii) *the subset $\mathcal{P}$ is closed in $S_\tau(\mathbb{R},\mathcal{C})$;*

(iv) *the set $\mathcal{P}$ is invariant with respect to mapping T, where $(T\varphi)(t) := \varphi(t+\tau)$ for all $t \in \mathbb{R}$ and $\varphi \in \mathcal{P}$;*

(v) *the discrete dynamical systems $(\mathcal{P}, T)$ and (Σ_m, σ) are topological isomorphic.*

Proof. It is obviously that by equality (31) is defined correctly a mapping from Σ_m into $\mathcal{P} \subset S_\tau(\mathbb{R},\mathcal{C})$. Note that h is an injection. In fact, if $\xi_1, \xi_2 \in \Sigma_m$ and $\xi_1 \ne \xi_2$, then there exists $k_0 \in \mathbb{Z}$ such that $\xi_1(k_0) \ne \xi_2(k_0)$. Thus $\varphi_1 = h(\xi_1) \ne h(\xi_2) = \varphi_2$ because $\varphi_i(t) = c_{\xi_i(k_0)}$ for all $t \in [k\tau,(k+1)\tau)$ $(i = 1,2)$ and $c_{\xi_1(k_0)} \ne c_{\xi_2(k_0)}$. Let now $\varphi \in \mathcal{P}$ be an arbitrary element, then for all $k \in \mathbb{Z}$ there exists a unique

$i_k \in \{1, 2, \dots, m\}$ such that $\varphi(t) = c_{i_k}$ for all $t \in [k\tau, (k+1)\tau)$. It easy to see that $h(\xi) = \varphi$, where $\xi(k) := i_k$ for all $k \in \mathbb{Z}$, i.e. $h(\Sigma_m) = \mathcal{P}$.

We will show that the mapping $h : \Sigma_m \mapsto \mathcal{P}$ defined by equality (31) is continuous. Let $\{\xi_n\} \to \xi$ as $n \to +\infty$, we will prove that $\varphi_n := h(\xi_n) \to h(\xi) := \varphi$. Fixe $l \in \mathbb{N}$ and we will show that $d_l(\varphi_n, \varphi) \to 0$ as $n \to +\infty$, where $d_l(\varphi_n, \varphi) := \int_{|t| \le l} |\varphi_n(t) - \varphi(t)|dt$. Denote by $k_1(l) := \min\{k \in \mathbb{Z} \ |k\tau \ge -l\}$, $k_1(l) := \max\{k \in \mathbb{Z} \ |k\tau \le l\}$ and $k(l) := \max\{1 + |k_1(l)|, 1 + |k_2(l)|\}$. Since $\{\xi_n\} \to \xi$ in Σ_m, then there exists $n(l) \in \mathbb{N}$ such that $\xi_n(k) = \xi(k)$ for all $n \ge n(l)$ and $|k| \le k(l)$. Hence $\varphi_n(t) = \varphi(t)$ for all $n \ge n(l)$ and $|t| \le l$, thus we have $d_l(\varphi_n, \varphi) = 0$ $(n \ge n(l))$, i.e. h is continuous.

Since Σ_m is compact and h is a continuous bijective operator, then it is a homeomorphism. Thus the set $\mathcal{P} = h(\Sigma_m)$ is a compact subset of $S_\tau(\mathbb{R}, \mathcal{C})$.

The set $\mathcal{P}$ is invariant this respect to T because $\{t_k(\varphi)\} = \{k\tau\}_{k \in \mathbb{Z}}$.

From the definition of the mappings h, σ and T it follows that $h(\sigma(\xi)) = T(h(\xi))$ for all $\xi \in \Sigma_m$. Lemma is proved. □

The set $S \subset W$ is

(i) nowhere dense, provided the interior of the closure of S is empty set, $int(cl(S)) = \emptyset$;
(ii) totally disconnected, provided the connected components are single points;
(iii) perfect, provided it is closed and every point $p \in S$ is the limit of points $q_n \in S$ with $q_n \neq p$.

The set $S \subset W$ is called a Cantor set, provided it is totally disconnected, perfect and compact.

Let (X, ρ) be a metric space and $(X, \mathbb{Z}_+, \pi)$ be a discrete dynamical system generated by positive powers of the map $f : X \to X$, i.e. $\pi(n, x) := f^n x$ for all $x \in X$ and $n \in \mathbb{Z}_+$, where $f^n := f^{n-1} \circ f$.

A subset $M \subseteq X$ is called transitive, if there exists a point $x_0 \in X$ such that $H(x_0) := \overline{\{\pi(n, x_0) \mid n \in \mathbb{Z}_+\}} = M$.

$\{p, q\} \subseteq X$ is called a Li-Yorke pair, if simultaneously

$$\liminf_{n \to +\infty} \rho(\pi(n, p), \pi(n, q)) = 0 \text{ and } \limsup_{n \to +\infty} \rho(\pi(n, p), \pi(n, q)) > 0.$$

A set $M \subseteq X$ is called scrambled, if any pair of distinct points $\{p, q\} \subseteq M$ is a Li-Yorke pair.

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be chaotic, if X contains an uncountable subset M satisfying the following conditions:

(i) the set M is transitive;
(ii) M is scrambled;
(iii) $\overline{P(M)} = M$, where $P(M) := \{x \in X \mid x \in \omega_x = \alpha_x\}$ and by bar we denote the closure in X.

Theorem 4.17. [8] *Let $(X,\mathbb{T},\pi)$ and $(\Omega,\mathbb{T},\sigma)$ be two dynamical systems and $\nu : X \to \Omega$ be a homeomorphism of $(\Omega,\mathbb{T},\sigma)$ onto $(X,\mathbb{T},\pi)$. Assume that $(\Omega,\mathbb{T},\sigma)$ is chaotic. Then the dynamical system $(X,\mathbb{T},\pi)$ is chaotic too.*

Corollary 4.18. *The discrete dynamical system $(\mathcal{P},T)$ is chaotic.*

Proof. This statement follows from Theorem 4.17, Lemma 4.16 and from the fact that (Σ_m,σ) is chaotic (see, for example, [18, 22]). □

Lemma 4.19. *Suppose the following conditions hold:*

(i) *$(X,\mathbb{R},\pi)$ is a dynamical system with continuous time;*
(ii) *M is a nonempty compact subset of X;*
(iii) *there exists $t_0 > 0$ ($t_0 \in \mathbb{R}$) such that M is invariant with respect to $f := \pi(t_0,\cdot)$, i.e. $f(M) = M$;*
(iv) *the discrete dynamical system (M,f) is chaotic.*

Then the dynamical system $(X,\mathbb{R},\pi)$ is chaotic too.

Proof. This statement follows directly from the corresponding definitions. □

Corollary 4.20. *The dynamical system $(S_\tau(\mathbb{R},\mathcal{C}),\mathbb{R},\sigma)$ with continuous time $\mathbb{R}$ is chaotic.*

Proof. This statement follows from Lemma 4.19 and Corollary 4.18. □

Theorem 4.21. *Under the conditions of Theorem 4.9 the following statements hold:*

(i) *the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ generated by switching system (14) is compactly dissipative;*
(ii) *Levinson center J of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ is chaotic;*
(iii) *the cocycle $\langle E,\varphi,(S_\tau(\mathbb{R},\mathcal{C}),\mathbb{R},\sigma)\rangle$ generated by switching system (14) is compactly dissipative;*
(iv) *the Levinson center $I := \cup\{I_u \mid u \in S_\tau(\mathbb{R},\mathcal{C})\}$, where I is the Levinson center of φ and $\{I_u \mid u \in S_\tau(\mathbb{R},\mathcal{C})\}$ is the global attractor of cocycle φ;*
(v) *$I = \overline{Per(\varphi)}$, where $Per(\varphi) := \{x \in E\ : \exists s > 0$ and $u \in S_\tau(\mathbb{R},\mathcal{C})$ such that $\sigma(s,u) = u$ and $\varphi(s,x,u) = x\}$.*

Proof. By Theorem 4.9, the cocycle φ generated by switching system (14) is compactly dissipative and, hence, the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ ($X := E \times S_\tau(\mathbb{R},\mathcal{C}), \pi := (\varphi,\sigma)$) is compactly dissipative too.

Now we will prove that the Levinson center J of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ possesses the properties listed in the theorem. For this aim, we note that by Lemma 4.16 the discrete dynamical systems $(\mathcal{P},T)$ and (Σ_m,σ) are topological isomorphic and, consequently (see, for example, [18, 22]), the dynamical system $(J,\mathbb{R},\pi)$ is chaotic too.

By Theorem 4.9 the Levinson center J of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ is dynamically isomorphic to $(S_\tau(\mathbb{R},\mathcal{C}),\mathbb{R},\sigma)$ and, consequently, $\overline{Per(\pi)} =$

J. Taking into consideration that $I = pr_1(J)$ we obtain the last statement of Theorem. □

Remark 4.22. *Note that the results presented above also are true in the case when the phase space E is an infinite dimensional Hilbert space.*

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