Almost Periodic Solutions of Linear Alomost Periodic Equations: **Favard's Theory** *

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*__Jean Favard__ (1902–1965) was a French mathematician. Professor (Sorbonna and Ecole Plytechnique, Paris, France). His works has a great influence on contemporary mathematics. He also was a President of the French Mathematician Sosiety (1946).

1.- INTRODUCTION

The well known Favard's Theorem states that the linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution. The main assumption in this theorem is the separation among bounded solutions of homogeneous equations

$$x' = B(t)x, \tag{2}$$

where $B \in H(A) := \{B \mid B(t) = \lim_{n \to +\infty} A(t+t_n)\}$. If there are bounded solutions which are non-separated, sometimes almost periodic solutions do not exist (R. Johnson, R. Ortega and M. Tarallo, V. Zhikov and B. Levitan).

In this talk we will give a short survey of works dedicated to the different generalizations of this theorem. Actually this field of differential equations is called the **Favard's theory**.

2.- Almost Periodic Functions

Let E be a Banach space with norm $|\cdot|$ (for example, E is a real n–dimensional euclidian space $\mathbb{R}^n$ with the norm $|x| := (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$, where $x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$). Denote by $C(\mathbb{R}, E)$ the space of all continuous functions $f : \mathbb{R} \mapsto E$ equipped with the compact-open topology. This topology may be defined, for example, by the following metric

$$d(f_1, f_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\max\limits_{|t| \leq n} |f_1(t) - f_2(t)|}{1 + \max\limits_{|t| \leq n} |f_1(t) - f_2(t)|}.$$

By $C_b(\mathbb{R}, E)$ we will denote the space of all continuous and bounded on $\mathbb{R}$ functions equipped with the sup-norm

$$||f|| := \sup_{t \in \mathbb{R}} |f(t)|.$$

The space $C_b(\mathbb{R}, E)$ with the norm $||\cdot||$ is a Banach space.

Definition 1 *(**Bohr H., 1923**) A function $f \in C(\mathbb{R}, E)$ is called almost periodic, if for any $\varepsilon > 0$ there exists a positive number $l = l(\varepsilon)$ such that on every segment $[a, a + l]$ of the length l contains at least one number $\tau \in [a, a + l]$ such that*

$$|f(t + \tau) - f(t)| < \varepsilon \qquad \textit{for all} \quad t \in \mathbb{R}.$$

Remark 2 *1. Every periodic function is almost periodic.*

2. The function $f(t) := \sin t + \sin \sqrt{2}t$ $(t \in \mathbb{R})$ is almost periodic, but it does not periodic.

3.- Some properties of Almost Periodic Functions

Denote by $AP(\mathbb{R}, E)$ the set of all almost periodic functions $f : \mathbb{R} \mapsto E$.

- Every almost periodic function f is uniformly continuous and its range $R_f := f(\mathbb{R})$ is relatively compact, in particularly, $f \in C_b(\mathbb{R}, E)$.

- The function $f := \alpha f_1 + \beta f_2$ is almost periodic for all $f_1, f_2 \in AP(\mathbb{R}, E)$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), i.e., $AP(\mathbb{R}, E)$ is a linear space.

- If the sequence $\{f_n\} \subset AP(\mathbb{R}, E)$ converges to f uniformly with respect to $t \in \mathbb{R}$, then f is also almost periodic. Thus the space $AP(\mathbb{R}, E)$ equipped with the norm sup is a Banach space (it is a subspace of the Banach space $C_b(\mathbb{R}, E)$).

- Let $g \in AP(\mathbb{R}, E)$ and $h \in AP(\mathbb{R}, \mathcal{P})$ ($\mathcal{P} : \mathbb{R}$ or $\mathbb{C}$), then the function $f(t) := h(t) \cdot g(t)$ (for all $t \in \mathbb{R}$) is also almost periodic.

- (H-**class of almost periodic function**) Denote by f_τ the τ–shift of f (i.e., $f_\tau(t) := f(t + \tau)$ for all $t \in \mathbb{R}$). The set $H(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}\}}$ is said to be H–class of the almost periodic function f, where by bar is denoted the closure in the space $C_b(\mathbb{R}, E)$. It is clear that $H(f) \subset AP(\mathbb{R}, E)$.

- (**Minimality.**) For all almost periodic function f the set $H(f)$ is minimal, i.e., $H(g) = H(f)$ for all $g \in H(f)$.

- (**Ergodicity.**) For every almost periodic function f there exists an uniform average, i.e., there exists a limit

$$M(f) := \lim_{T \to +\infty} \frac{1}{T} \int_s^{s+T} f(k)dk$$

uniformly with respect to $s \in \mathbb{R}$.

- (**Denumerable modulus.**) There exists a denumerable set $\Lambda(f) \subset \mathbb{R}$ such that

$$a(\lambda) :== \lim_{T \to +\infty} \frac{1}{T} \int_s^{s+T} f(t)e^{i\lambda t}dt \neq 0$$

if and only if $\lambda \in \Lambda(f)$. The set $\Lambda(f)$ is called a modulus of frequencies for the function f.

- For every almost periodic function f corresponds a unique trigonometric (Fourier) series

$$f(t) \sim \sum_{k=1}^{+\infty} x_k e^{i\lambda_k t},$$

where $\lambda_k \in \Lambda(f)$ and $x_k := a(-\lambda_k)$.

- (**Approximation theorem.**) If f is almost periodic, then for all $\varepsilon > 0$ there exists a trigonometric polynomial

$$T_\varepsilon(t) = \sum_{k=1}^{N(\varepsilon)} x_k e^{i\lambda_k t}$$

($N(\varepsilon) \in \mathbb{N}$, $\lambda_k \in \mathbb{R}$ and $x_k \in E$) such that $|f(t) - T_\varepsilon(t)| < \varepsilon$ for all $t \in \mathbb{R}$ and vice versa.

- (**Bochner's theorem**) The function $f \in C(\mathbb{R}, E)$ is almost periodic if and only if the set $\{f_\tau : \tau \in \mathbb{R}\}$ is a relatively compact subset in the space $C_b(\mathbb{R}, E)$.

Remark 3 *1. A particular case of almost period functions before it was studied by Lituanian mathematician P. Bolh (1893).*

Definition 4 *An almost periodic function is called quasi-periodic if its modulus of the frequencies $\Lambda(f)$ admits a finite base (or finitely generated),*

i.e., the re exists a finite set $\{\nu_1, \nu_2, \ldots, \nu_m\}$ *such that:*

a. $\{\nu_1, \nu_2, \ldots, \nu_m\}$ *is rationally independent, i.e., the linear combination* $r_1\nu_1 + r_2\nu_2 + \ldots + r_m\nu_m$ *with rational numbers* $r_1, r_2, \ldots, r_m$ *is equal to zero if and only if* $r_1 = r_2 = \ldots = r_m = 0$*;*

b. *for all* $\lambda \in \Lambda(f)$ *there exists rational numbers* $r_1, r_2, \ldots, r_m \in \mathbb{Q}$ *such that* $\lambda = k_1\nu_1 + k_2\nu_2 + \ldots + k_m\nu_m$.

The numbers $\nu_1, \nu_2, \ldots, \nu_m$ *is called the spectrum of frequencies (or a basis of the frequencies) of the quasi-periodic function* f.

2. A continuous function $f \in C(\mathbb{R}, E)$ *is quasi-periodic with the base of frequencies* $\nu := (\nu_1, \nu_2, \ldots, \nu_m)$ *if and only if there exists a continuous function* $\Phi : \mathcal{T}^m \mapsto E$ *such that* $f(t) = \Phi(\nu t)$ *for all* $t \in \mathbb{R}$*, where* $\mathcal{T}^m$ *is a standard* m*-torus* ($\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$) *and* $\nu t := (e^{i\nu_1 t}, e^{i\nu_2 t}, \ldots, e^{i\nu_m t})$ *for all* $t \in \mathbb{R}$.

3. A finite some $f(t) := p_1(t) + p_2(t) + \ldots + p_m(t)$ of periodic functions p_i $(i = 1, 2, \ldots, m)$ is a quasi-periodic function.

4.- Favard's theorem

Along with equation (1) we consider also the corresponding homogeneous equation

$$x' = A(t)x. \tag{3}$$

A nontrivial bounded on $\mathbb{R}$ solution $x(t)$ is called separated from zero, if $\inf\limits_{t\in\mathbb{R}} |x(t)| > 0$.

Let $H(A)$ be the $H-$class for the matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$, where $[\mathbb{R}^n]$ is the space of all $n \times n-$matrices with the norm $||A|| := (\sum\limits_{i,j=1}^{n} |a_{ij}|^2)^{1/2}$.

Theorem 5 *(**J. Favard, 1928**) Let $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be an almost periodic matrix-function (i.e., its elements $a_{ij}(t)$ are almost periodic functions for a every $i, j = 1, 2, \ldots, n$) and $f \in C(\mathbb{R}, \mathbb{R}^n)$ be an almost periodic function. Suppose that equation (1) admits a bounded on $\mathbb{R}$ solution φ.*

If the bounded on $\mathbb{R}$ solutions of equation (3) and all limiting equation

$$y' = B(t)y \quad (B \in H(A)) \tag{4}$$

are separated from zero, then equation (1) admits at least one almost periodic solution.

Remark 6 *In general case (without Favard's separation condition) Favard's theorem is not true. There exists a scalar linear almost periodic equation*

$$x' = a(t)x + f(t) \tag{5}$$

*all solution of which are bounded on $\mathbb{R}$, but does not exist any almost periodic solution of equation (5) (***V.V. Zhikov,1978***).*

5.- Generalizations of Favard's theorem

Definition 7 *(***B. Levitan, 1939***) A continuous function $f \in C(\mathbb{R}, E)$ is called Levitan almost periodic, if for every $\varepsilon > 0$ there exists a positive number $l = l(\varepsilon)$ such that on every segment $[a, a + l]$ $(a \in \mathbb{R})$ of length l there exists at least one number τ such that*

$$|f(t \pm \tau) - f(t)| < \varepsilon \quad \text{for all} \quad t \in [-1/\varepsilon, 1/\varepsilon].$$

Some properties of Levitan almost periodic functions

Denote by $LAP(\mathbb{R}, E)$ the family of all Levitan almost periodic functions $f \in C(\mathbb{R}, E)$.

- Every Bohr almost periodic function is a Levitan almost periodic function.

- The function $f(t) := \dfrac{1}{2 + \sin t + \sin \sqrt{2}t}$ is almost periodic in the sense of Levitan, but not almost periodic in the sense of Bohr, because it is not bounded on $\mathbb{R}$. Thus the space of Levitan almost periodic functions $LAP(\mathbb{R}, E)$ is more large than the space of Bohr almost periodic functions $AP(\mathbb{R}, E)$, i.e., $AP(\mathbb{R}, E) \subset LAP(\mathbb{R}, E)$.

- The function $f := \alpha f_1 + \beta f_2$ is Levitan almost periodic for all $f_1, f_2 \in LAP(\mathbb{R}, E)$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), i.e., $LAP(\mathbb{R}, E)$ is a linear space.

- If the sequence $\{f_n\} \subset LAP(\mathbb{R}, E)$ converges to f uniformly with respect to $t \in \mathbb{R}$, then f is also Levitan almost periodic.

- Let $g \in LAP(\mathbb{R}, E)$ and $h \in LAP(\mathbb{R}, \mathcal{P})$ ($\mathcal{P} = \mathbb{R}$ or $\mathbb{C}$), then the function $f(t) := h(t) \cdot g(t)$ (for all $t \in \mathbb{R}$) is also Levitan almost periodic.

- (H-**class of Levitan almost periodic function**) Denote by f_τ the τ–shift of (i.e., $f_\tau(t) := f(t + \tau)$ for all $t \in \mathbb{R}$). The set $H(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}\}}$ is said to be H–class of the Levitan almost periodic function f, where by bar is denoted the closure in the space $C(\mathbb{R}, E)$. Note that in the general case $H(f)$ not consists only from the Levitan almost periodic functions.

- (**Minimality.**) For all Levitan almost periodic function f the set $H(f)$ is minimal, i.e., $H(g) = H(f)$ for all $g \in H(f)$.

- For every Levitan almost periodic function f corresponds a trigonometric (Fourier) series (**not unique**)

$$f(t) \sim \sum_{k=1}^{+\infty} x_k e^{i\lambda_k t}.$$

Theorem 8 *(***B. Levitan, 1939***) Let $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be an almost periodic matrix-function and $f \in C(\mathbb{R}, \mathbb{R}^n)$ be an almost periodic function. Suppose that equation (1) admits a bounded on $\mathbb{R}$ solution φ. If the bounded on $\mathbb{R}$ solutions of equation (3) are separated from zero, then the equation (1) admits at least one Levitan almost periodic solution.*

Remark 9 *Is it true the Levitan's theorem without separation condition? (Open question.)*

The theorems of Favard and Levitan were generalized for infinite-dimensional systems in the works of **L. Amerio** (1960, for uniform convex Banach spaces) and **V. Zhikov** (1978, for arbitrary separable Banach spaces).

Zhikov and Levitan (1978) (and also Johnson (1978), Ortega and Tarallo (2006), Sell (1981))

constructed examples of scalar linear differential equations for which all the solutions are bounded, but none of them is almost periodic. In particular, the following result was established.

Theorem 10 *(***Ortega and Tarallo, 2006***) Let*

$$x' = A(t)x \tag{6}$$

be a linear differential equation with Bohr almost periodic coefficients, and for some $B \in H(A)$ each nontrivial bounded on $\mathbb{R}$ solution φ of the equation

$$y' = B(t)y \tag{7}$$

is homoclinic to zero, i.e.,

$$\lim_{|t| \to +\infty} |\varphi(t)| = 0. \tag{8}$$

Then, there exists an almost periodic (in the sense of Bohr) function $g : \mathbb{R} \mapsto \mathbb{R}^n$ such that equation

$$y' = B(t)y + g(t) \tag{9}$$

has bounded solutions, but none of them is Bohr almost periodic.

Theorem 11 *(***T. Caraballo and D. Cheban, 2009***) A linear differential equation (1) with Levitan almost periodic coefficients has a unique Levitan almost periodic solution, if it has at least one bounded solution, and the bounded solutions of the homogeneous equation (6) are homoclinic to zero.*

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