LEVITAN ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF SECOND-ORDER MONOTONE DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is the study of problem of existence of Levitan almost periodic, almost automorphic, recurrent and Poisson stable solutions of seconde order differential equation

$$x'' = f(\sigma(t,y),x,x'), \quad (y \in Y) \tag{1}$$

where Y is a complete metric space and $(Y,\mathbb{R},\sigma)$ is a dynamical system (driving system). For equation (1) with increasing (with respect to second variable) function f the existence at least one quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent, Levitan almost periodic, almost recurrent, Poisson stable) solution of (1) is proved under the condition that (23) admits at least one bounded on the real axis solution together with its first derivative.

1. Introduction

The aim of this paper is the study of problem of existence of Levitan almost periodic, almost automorphic, recurrent and Poisson stable solutions of seconde order differential equation

$$x'' = f(\sigma(t,y),x,x'), \quad (y \in Y) \tag{2}$$

where Y is a complete metric space and $(Y,\mathbb{R},\sigma)$ is a dynamical system (driving system).

The problem of Bohr almost periodic solutions of equation

$$x'' = f(t,x,x') \tag{3}$$

with Bohr almost periodic right hand-side f with respect to time, uniformly with respect variables x,x' on every compact from $\mathbb{R}^2$ was studied by C. Corduneanu [9] (see also [1]) and he established that if $f'_x(t,x,u) \ge k > 0$ for all $(t,x,u) \in \mathbb{R}^3$, then equation (3) admits a unique almost periodic solution.

In the case when the function $f(t,x,u)$ is only increasing (in the large sense) Z. Opial [15] studied this problem and he established the following result.

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Theorem 1.1. *(Z. Opial [15]) Suppose that the following conditions are fulfilled:*

- (i) *$f \in C(\mathbb{R}^3, \mathbb{R})$ and it is increasing in the large sense with respect to variable x, i.e. the inequality $x_1 \le x_2$ implies $f(t, x_2, u) \le f(t, x_1, u)$ for all $u, t \in \mathbb{R}$;*
- (ii) *for all $r > 0$ there exists a number $L(r) > 0$ such that $|f(t, x_1, u_1) - f(t, x_2, u_2)| \le L(r)(|x_1 - x_2| + |u_1 - u_2|)$ for all $|x_i|, |u_i| \le r$ $(i = 1, 2)$ and $t \in \mathbb{R}$.*

The the following statements hold:

- (i) *If equation (3) admits a bounded on $\mathbb{R}$ solution together with its derivative, then this equations admits at least one almost periodic solution.*
- (ii) *If $u(t)$ and $v(t)$ are two almost periodic solutions of equation (3) then there exists a constant $c \in \mathbb{R}$ such that $u(t) - v(t) = c$ for all $t \in \mathbb{R}$.*
- (iii) *If the the function f is strict increasing with respect to $x \in \mathbb{R}$, then equation (3) admits at most one almost periodic solution.*

By P. Cieutat [8] was studied the bounded and almost periodic solutions of Liénard equation

$$x'' + f(x)x' + g(x) = p(t), \tag{4}$$

where $p : \mathbb{R} \mapsto \mathbb{R}$ is a almost periodic function, $f(x) \ge 0$ and g is a strictly decreasing function. He proved that every bounded on $\mathbb{R}_+$ solution is asymptotically almost periodic and there exists a unique almost periodic solution of equation (4). The model of equation (4) is

$$x'' + cx' + 1/x^\alpha = p(t), \quad (x \in (0, +\infty)) \tag{5}$$

where $c \ge 0$, $\alpha > 0$ and p is almost periodic.

In the work of P. Cieutat, S. Fatajou and G. M. N'Guerekata [10] was studied the problem of existence of almost automorphic solutions of equation (4) with almost automorphic forcing term p. It was proved the asymptotically almost automorphy of every bounded on $\mathbb{R}_+$ solution and the existence a unique almost automorphic solution of equation (4).

In the periodical case (p is periodic) the dynamics of equation (5) intensively was studied by P. Martinez-Amores and P. J. Torres [14] and J. Campos and P. J. Torres [3].

Desheng Li and Jinqiao Duan [13] study the structure of the set of bounded solution for equation (2). In particularly, it was proved the existence a unique periodic (respectively, quasi-periodic, almost periodic) solution of equation (2) if the point $y \in Y$ is so and the function f is strictly increasing with respect to second variable.

We note that in the all of the cited above works (with the exeption of [15]) figures an assumption of strict monotony. We consider equation (2) with the function f monotone increasing with respect to seconde variable in the large sense. All our results will be formulated and proved for this case which includes, of course, also the case of strict increasing too.

This paper is organized as follows.

In Section 2 we collect some notions, facts and constructions from theory of dynamical systems which we use in this paper.

Section 3 is dedicated to the study of a special class of non-autonomous dynamical systems (NDS): so called NDS with convergence. The main result in this section is Theorem 3.10 which give sufficient conditions of convergence of NDS.

In Section 4 we apply Theorem 3.10 to the study the dynamics of scalar one-dimensional equation $x' = f(\sigma(t, y), x)$ $(y \in Y)$ with pseudo recurrent base $(Y, \mathbb{R}, \sigma)$ (driving system). The main result of this section is Theorem 4.2.

Levitan almost periodic and almost automorphic solutions of second order equations $x'' = f(\sigma(t, y), x, x'')$ with monotone increasing f (in the large sense) are studied in the Section 5. The main results of this section are Theorem 5.4 and Corollary 5.5.

Section 6 is dedicated to the study the problem of quasi-periodic, Bohr almost periodic and recurrent (in the sense of Birhoff) for the equation $x'' = f(\sigma(t, y), x, x'')$ with monotone increasing f (in the large sense). The main results are Theorem 6.1 and Corollary 6.2.

In Section 7 we discus some generalizations of our main results (Theorems 5.4 and 6.1). One of this type results is given in Theorem 7.1 (see also Corollaries 7.2 and 7.3).

2. Almost Periodic and Almost Automorphic Motions of Dynamical Systems

2.1. **Recurrent, Almost Periodic and Almost Automorphic Motions.** Let X be a complete metric space, $\mathbb{R}$ $(\mathbb{Z})$ be a group of real (integer) numbers, $\mathbb{R}_+$ $(\mathbb{Z}_+)$ be a semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ $(\mathbb{S}_+ \subseteq \mathbb{T})$ be a sub-semigroup of the additive group $\mathbb{S}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system.

A number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ shift (respectively, almost period) of x, if $\rho(x\tau, x) < \varepsilon$ (respectively, $\rho(x(\tau + t), xt) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon > 0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

Denote $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T}\ :\ \text{such that } \{\pi(t_n, x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ is called Poisson stable in the positive direction if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to +\infty$ as $n \to \infty$.

Let $(X, \mathbb{T}, \pi)$ be a two-sided dynamical system (i.e., $\mathbb{T} = \mathbb{S}$). A point $x \in X$ is called Poisson stable in the negative direction if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$

such that $t_n \to -\infty$ as $n \to \infty$. The point $x \in X$ is called Poisson stable if it is Poisson stable in the both directions.

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be pseudo recurrent if X is compact and every point $x \in X$ is Poisson stable.

A point $x \in X$ is called [20, 22] pseudo recurrent if the dynamical system $(H(x), \mathbb{T}, \pi)$ is pseudo recurrent, where $H(x) := \overline{\{\pi(t, x) : \ t \in \mathbb{T}\}}$.

Remark 2.1. *Every recurrent point is pseudo recurrent, but there exists a pseudo recurrent points which are not recurrent* [20, 22].

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$, i.e., $\sigma(t, \nu) := (\nu_1 t, \nu_2 t, \dots, \nu_m t)$ for all $t \in S$ and $\nu \in \mathcal{T}^m$.

A point $x \in X$ is called quasi-periodic with the frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ such that $\pi(t, x) := \Phi(\omega t)$ for all $t \in \mathbb{T}$, where $\omega t := \sigma(t, \omega)$ and $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called Levitan almost periodic [12], if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.2. *1. Every almost automorphic point is Levitan almost periodic.*

2. A Levitan almost periodic point is almost automorphic if and only if it is stable in the sense of Lagrange.

Remark 2.3. *Let $x_i \in X_i$ $(i = 1, 2, \dots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then the point $x := (x_1, x_2, \dots, x_m)) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic in the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t, x) := (\pi_1(t, x_1), \pi_2(t, x_2), \ \dots, \ \pi_m(t, x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \dots, x_m) \in X$.*

A point $x \in X$ is called stable in the sense of Lagrange (st.L), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called almost automorphic [12, 18] in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y, \mathbb{T}, \sigma)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$ and an almost periodic in the sense of Bohr point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

2.2. **Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** Below we indicate one general method of construction of dynamical systems on the space of continuous functions. In this way we will get many well known dynamical systems on the functional spaces (see, for example, [2, 17, 20]).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X, Y be a complete pseudo metric space and $\mathcal{P}$ be a family of pseudo metrics on Y. We denote by $C(X, Y)$ the family of all continuous functions $f : X \to Y$ equipped with a compact-open topology. This topology is given by the following family of pseudo metrics $\{d_K^p\}$ $(p \in \mathcal{P},\ K \in C(X))$, where

$$d_K^p(f,g) := \sup_{x\in K} p(f(x), g(x))$$

and $C(X)$ a family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X,Y) \to C(X,Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau,x))$ $(x \in X)$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\forall \tau_1, \tau_2 \in \mathbb{T}$ $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1+\tau_2}$;
c. $\forall \tau \in \mathbb{T}$ σ_τ is continuous.

Lemma 2.4. [4] *The mapping $\sigma : \mathbb{T} \times C(X,Y) \to C(X,Y)$, defined by the equality $\sigma(\tau, f) := \sigma_\tau f$ $(f \in C(X,Y),\ \tau \in \mathbb{T})$, is continuous.*

Corollary 2.5. *The triple $(C(X,Y), \mathbb{T}, \sigma)$ is a dynamical system on $C(X,Y)$.*

Consider now some examples of dynamical systems of the form $(C(X,Y), \mathbb{T}, \sigma)$, useful in the applications.

Example 2.6. Let $X = \mathbb{T}$ and we denote by $(X, \mathbb{T}, \pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t,x) := x + t$. The dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ is called *Bebutov's dynamical system* [2, 17, 20] (a dynamical system of translations, or shifts dynamical system).

We will say that the function $\varphi \in C(\mathbb{T}, Y)$ possesses a property (A), if the motion $\sigma(\cdot, \varphi) : \mathbb{T} \to C(\mathbb{T}, Y)$, generated by this function, possesses this property in the dynamical system of Bebutov $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$, generated by the function φ. As property (A) we can take periodicity, quasi-periodicity, almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability etc.

Example 2.7. Let $X := \mathbb{T} \times W$, where W is some metric space and by $(X, \mathbb{T}, \pi)$ we denote a dynamical system on X defined in the following way: $\pi(t,(s,w)) := (s+t, w)$. Using the general method proposed above we can define on $C(\mathbb{T} \times W, Y)$ a dynamical system of translations $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

The function $f \in C(\mathbb{T} \times W, Y)$ is called almost periodic (quasi-periodic, recurrent, almost automorphic, etc) with respect to $t \in \mathbb{T}$ uniform on w on every compact from W, if the motion $\sigma(\cdot, f)$ is almost periodic (quasi-periodic, recurrent, almost automorphic, etc.) in the dynamical system $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

Remark 2.8. *Let W be a compact metric space, then the topology on $C(W,Y)$ is metrizable. For example by the equality*

$$d(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1 + d_k(f,g)}$$

there is defined a complete metric on the space $C(W,X)$ which is compatible with the compact-open topology on $C(W,X)$, where $d_k(f,g) := \max_{|t|\le k,\ x\in W} \rho(f(t,x), g(t,x))$. The space $C(\mathbb{T} \times W, Y)$ is topologically isomorphic to $C(\mathbb{T}, C(W,Y))$ [20], and also the shifts dynamical systems $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$ and $(C(\mathbb{T}, C(W,Y)), \mathbb{T}, \sigma)$ are dynamically isomorphic.

2.3. **Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}_+$).

A triplet $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system.

Let $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on Y, W be a complete metric space and φ be a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in W, possessing the following properties:

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then the triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ (or shortly φ) is called [17] a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with the fiber W.

Let $X := W \times Y$ and let us define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as follows: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [17] and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, hence, $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ $(X := W \times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of this type.

Example 2.9. Consider the system of differential equations

$$\left\{ \begin{array}{ll} u' = & F(y, u) \\ y' = & G(y), \end{array} \right. \tag{6}$$

where $Y \subseteq E^m$ (for example, $Y = \mathcal{T}^m$ is a m–torus), $G \in C(Y, E^n)$ and $F \in C(Y \times E^n, E^n)$. Suppose that for the system (6) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(Y, \mathbb{R}_+, \sigma)$ a dynamical system on Y generated by the second equation of the system (6) and by $\varphi(t, u, y)$ we denote the solution of the equation

$$u' = F(\sigma(t, y), u)$$

passing through the point $u \in E^n$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times Y \to E^n$ satisfies the conditions a. and b. from definition of cocycle and, consequently, system (6) generates a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ (where $X := E^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$).

Example 2.10. Let $(Y, \mathbb{R}, \sigma)$ be a dynamical system on the metric space Y. We consider the equation

$$u' = F(\sigma(y, t), u) \quad (y \in Y), \tag{7}$$

where $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$. Suppose that for equation (7) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ (respectively, the cocycle $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$), where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi, \sigma)$, $\varphi(\cdot, x, y)$ is the solution of (7) and $h := pr_2 : X \to Y$ is generated by equation (7).

Example 2.11. We consider the equation

$$u' = f(t, u), \tag{8}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with equation (8) consider the family of equations

$$u' = g(t, u), \tag{9}$$

where $g \in H(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}\}}$ and f_τ is the τ-shift of f with respect to time, i.e., $f_\tau(t, u) := f(t + \tau, u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$. Suppose that the function f is regular [17], i.e., for all $g \in H(f)$ and $u \in \mathbb{R}^n$ there exists a unique solution $\varphi(t, u, g)$ of equation (9). Denote by $Y = H(f)$ and $(Y, \mathbb{R}, \sigma)$ a shift dynamical system on Y induced by Bebutov's dynamical system $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$. Now the family of equations (9) may by written as (7) if we take as quality of the mapping $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ defined by equality $F(g, u) := g(0, u)$ for all $g \in H(f)$ and $u \in \mathbb{R}^n$.

A solution $\varphi(t, u, y)$ of equation (7) is called [20, 22] compatible (respectively, uniformly compatible) by character of recurrence if $\mathfrak{N}_y \subseteq \mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_\varphi$), where $\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) is a set of all sequences $\{t_n\} \subset \mathbb{R}$ such that $\{\varphi(t + t_n, u, y\}$ converges to $\varphi(t, u, y)$ (respectively, $\{\varphi(t + t_n, u, y\}$ converges) in the space $C(\mathbb{T}, \mathbb{R}^n)$.

Remark 2.12. *The sequence $\{\varphi(t + t_n, u, y)\}$ converges to function ψ in the space $C(\mathbb{T}, \mathbb{R}^n)$ if and only if $\{\varphi(t_n, u, y)\}$ converges to $\psi(0)$.*

Theorem 2.13. *The following statements hold:*

1. Let $y \in Y$ be a stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable) point. If $\varphi(t, u, y)$ is a compatible solution of equation (7), then $\varphi(t, u, y)$ is so.

2. Let $y \in Y$ be a stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) point. If $\varphi(t, u, y)$ is a uniformly compatible solution of equation (7), then $\varphi(t, u, y)$ is so.

Example 2.14. Let us consider a differential equation of the second order

$$x'' = f(\sigma(t, y), x, x'), \ (y \in Y) \tag{10}$$

where $f \in C(Y \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, and give a criterion of the existence of Levitan almost periodic and almost automorphic solutions for this equation. Below we will suppose that the function f is regular, i.e., for all $y \in Y$ and $x, y \in \mathbb{R}^n$ the equation (10) admits a unique solution $\varphi(t, x, x', y)$ defined on $\mathbb{R}_+$ with the initial conditions $\varphi(0, x, x', y) = x$ and $\varphi'(0, x, x', y) = x'$.

As we know, we can reduce the equation (10) to the equivalent system

$$\begin{cases} u' = v \\ v' = f(\sigma(t, y), u, v) \end{cases} \tag{11}$$

($y \in Y$) or to the equation

$$z' = F(\sigma(t,y),z) \tag{12}$$

on the product space $\mathbb{R}^n \times \mathbb{R}^n$, where $z := (u,v)$ and $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$ is the function defined by the equality $F(y,z) := (v, f(y,u,v))$ for all $\omega \in \Omega$ and $z := (u,v) \in \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 2.15. [20] *Let $\varphi \in C(\mathbb{R},\mathbb{R}^n)$be a continuously differentiable function. If its derivative $\varphi' \in C(\mathbb{R},\mathbb{R}^n)$ is uniformly continuous on $\mathbb{R}$, then φ' is uniformly comparable by character of recurrence with φ, i.e., $\mathfrak{M}_{\varphi} \subseteq \mathfrak{M}_{\varphi'}$.*

Lemma 2.16. *Suppose that the following conditions hold:*

(i) *Y is compact;*
(ii) *$f \in C(Y \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ is regular;*
(iii) *$\varphi(t,x_0,x_0',y)$ is a solution of equation (10) defined and bounded on $\mathbb{R}$ together with its derivative $\varphi'(t,x_0,x_0',y)$.*

Then the following two statements are equivalent:

a. *a solution $\varphi(t,x_0,x_0',y)$ of equation (10) is compatible (respectively, uniform compatible) by character of recurrence with the right-hand site;*
b. *a solution $(\varphi(t,x_0,x_0',y), \varphi'(t,x_0,x_0',y))$ of equation (11) is compatible (respectively, uniform compatible) by character of recurrence with the right-hand site.*

Proof. The implication b.$\longrightarrow$ a. is evident. Thus to prove the lemma it is sufficient to establish the inverse implication. Let $\varphi(t,x_0,x_0',y)$ be a solution of equation (10) defined and bounded on $\mathbb{R}$ together with its derivative $\varphi'(t,x_0,x_0',y)$, then $\mathfrak{N}_y \subseteq \mathfrak{N}_{\varphi}$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi}$). We need to show that then we will have also the inclusion $\mathfrak{N}_y \subseteq \mathfrak{N}_{\varphi'}$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi'}$). In fact. Let $\{t_n\} \in \mathfrak{N}_y$ (respectively, $\{t_n\} \in \mathfrak{M}_y$), then the sequence $\{\sigma(t_n,y)\}$ converges to y (respectively, the sequence $\{\sigma(t_n,y)\}$ converges to some point $\tilde{y} \in Y$), then the functional sequence $\{f(\sigma(t+t_n,y),u,v)\}$ converges to $f(\sigma(t,y),u,v)$ (respectively, to $f(\sigma(t,\tilde{y},u,v))$ uniformly with respect to t on every compact subset from $\mathbb{R}$ and $u,v \in Q := \overline{\varphi(\mathbb{R},x_0,x_0',y)} \times \overline{\varphi'(\mathbb{R},x_0,x_0',y)}$. Since $\mathfrak{N}_y \subseteq \mathfrak{N}_{\varphi}$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi}$), then the sequence $\{\varphi(t_n,x_0,x_0',y)\}$ converges to x_0 (respectively, to some point $\tilde{x_0} \in \mathbb{R}$). Since the function $f \in C(Y \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ is regular, then the functional sequence $\{\varphi(t+t_n,x_0,x_0',y)\}$ converges to function $\varphi(t,x_0,x_0',y)$ (respectively, to $\varphi(t,\tilde{x_0},\tilde{x_0}',\tilde{y})$) uniformly with respect to t on every compact subset from $\mathbb{R}$. Note that under the conditions of Lemma, the second derivative $\varphi''(t,x_0,x_0',y)$ of the function $\varphi(t,x_0,x_0',y)$ is bounded on $\mathbb{R}$ and, consequently, the first derivative $\varphi'(t,x_0,x_0',y)$ is uniformly continuous in $t \in \mathbb{R}$. Thus according to Theorem 2.15 the first derivative $\varphi'(t,x_0,x_0',y)$ is comparable (respectively, uniformly comparable) by character of recurrence and, consequently, the sequence $\varphi'(t+t_n,x_0,x_0',y)$ converges to $\varphi'(t,x_0,x_0',y)$ (respectively, $\varphi(t,\tilde{x_0},\tilde{x_0}',\tilde{y})$). The lemma is proved. $\square$

Remark 2.17. *Not that if Y is not compact, then from the boundedness of $\varphi(t,x_0,x_0',y)$ on $\mathbb{R}$ together with its prime derivative $\varphi'(t,x_0,x_0',y)$ in general case does not imply the boundedness on $\mathbb{R}$ of the second derivative $\varphi''(t,x_0,x_0',y)$. In this case the equivalence of of the statements a. and b. of Lemmma 2.16 remains open.*

3. Non-Autonomous Dynamical Systems with Convergence

A dynamical system $(X, \mathbb{T}, \pi)$ is called point dissipative (respectively, compact dissipative), if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t \to +\infty} \rho(\pi(t, x), K) = 0 \tag{13}$$

for all $x \in X$ (respectively, the equality (13) holds uniformly with respect to x on every compact subset M from X).

$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be convergent if the following conditions are valid:

(i) the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;
(ii) the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$ where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and $J_X(J_Y)$ is the Levinson's center of the dynamical system $(X, \mathbb{T}_1, \pi)((Y, \mathbb{T}_2, \sigma))$.

Remark 3.1. *1. Note that convergent systems are in some sense the simplest dissipative dynamical systems. If $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a convergent non-autonomous dynamical system and J_X (respectively, J_Y) is a Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$), then J_X and J_Y are dynamically homeomorphic. Although the center of Levinson of a convergent system can by completely described, it may be sufficiently complicated. An example which illustrates the above comment is given in* [4, ChII].

2. If

(i) *Y is compact and invariant, then evidently $(Y, \mathbb{T}, \sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Y;*
(ii) *$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a convergent non-autonomous dynamical system, and J_X (respectively, J_Y) is a Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) and $J_Y = Y$, then J_X and Y are dynamically homeomorphic. In particularly, if the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, almost periodic, recurrent), then the point $x = h^{-1} \in J_X$ is so.*

A nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ is said to be uniformly stable in the positive direction on compacts of X if, for arbitrary $\varepsilon > 0$ and $K \subseteq X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that inequality $\rho(x_1, x_2) < \delta$ $(h(x_1) = h(x_2))$ implies that $\rho(\pi^t x_1, \pi^t x_2) < \varepsilon$ for $t \in \mathbb{T}_1$, where $\pi^t := \pi(t, \cdot)$.

Denote by $X \dot{\times} X = \{(x_1, x_2) \in X \times X \mid h(x_1) = h(x_2)\ \}$. If there exists the function $V : X \dot{\times} X \to \mathbb{R}_+$ with the following properties:

(i) V is continuous;
(ii) V is positive defined, i.e., $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$;
(iii) $V(x_1 t, x_2 t) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$ and $t \in \mathbb{T}_1^+ := \{t \in \mathbb{T}_1 \mid t \geq 0\}$,

then the nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \Theta), h\rangle$ is called (see [4] and [24], [12]) V - monotone.

Theorem 3.2. [5] *Every V - monotone st.L^+ nonautonomous dynamical system $\langle (X,\ \mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\Theta),h\rangle$ is uniformly stable in the positive direction on compacts from X.*

Let (X,h,Y) be a fiber space, i.e. X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X onto Y. The subset $M \subseteq X$ is said to be conditionally relatively compact, if the pre-image $h^{-1}(Y')\bigcap M$ of every relatively compact subset $Y' \subseteq Y$ is a relatively compact subset of X, in particularly $M_y := h^{-1}(y)\bigcap M$ is relatively compact for every y. The set M is called conditionally compact if it is closed and conditionally relatively compact.

Example 3.3. *Let K be a compact space, $X := K \times Y$, $h = pr_2 : X \to Y$, then the triplet (X,h,Y) be a fiber space, the space X is conditionally compact, but not compact.*

Theorem 3.4. [5] *Let $(X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma)\rangle$ be a NDS with the following properties:*

(i) *It admits a conditionally relatively compact invariant set J.*
(ii) *The NDS $\langle (X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ is positively uniformly stable on J;*
(iii) *every point $y \in Y$ is two-sided Poisson stable.*

Then

(i) *all motions on J may be continued uniquely to the left and define on J a two-sided dynamical system $(J,\mathbb{S},\pi)$, i.e., the semi-group dynamical system $(X,\mathbb{T},\pi)$ generates on J a two-sided dynamical system $(J,\mathbb{S},\pi)$;*
(ii) *for every $y \in Y$ there are two sequences $\{t_n^1\} \to +\infty$ and $\{t_n^2\} \to -\infty$ such that*
$$\pi(t_n^i,x) \to x\ (i = 1,2)$$
as $n \to \infty$ for all $x \in J_y$.

Corollary 3.5. *Let $(X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma)\rangle$ be a NDS with the following properties:*

(i) *It admits a conditionally relatively compact invariant set J.*
(ii) *The NDS $\langle (X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ is V–monotone;*
(iii) *every point $y \in Y$ is two-sided Poisson stable.*

Then

(i) *all motions on J may be continued uniquely to the left and define on J a two-sided dynamical system $(J,\mathbb{S},\pi)$, i.e., the semi-group dynamical system $(X,\mathbb{T},\pi)$ generates on J a two-sided dynamical system $(J,\mathbb{S},\pi)$;*
(ii) *for every $y \in Y$ there are two sequences $\{t_n^1\} \to +\infty$ and $\{t_n^2\} \to -\infty$ such that*
$$\pi(t_n^i,x) \to x\ (i = 1,2)$$
as $n \to \infty$ for all $x \in J_y$.

Proof. This statement directly it follows from Theorems 3.2 and 3.4. □

Denote by $\mathcal{K} := \{a \in C(\mathbb{R}_+,\mathbb{R}_+)|\ a(0) = 0,\ a \text{ is strict increasing}\}$.

Theorem 3.6. [5] *Suppose that the following conditions hold:*

(i) $y \in Y$ *is a two-sided Poisson stable point;*
(ii) *the NDS* $\langle (X,\ \mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ *admits a conditionally relatively compact invariant set* J*;*
(iii) $\langle (X,\ \mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ *be a* V*-monotone non-autonomous dynamical system and there are two functions* $a,b \in \mathcal{K}$ *such that*
  (a) $Im(a) = Im(b)$, *where* $Im(a) := a(\mathbb{R}_+)$ *is the domain of the values of* $a \in \mathcal{K}$;
  (b) $a(\rho(x_1,x_2)) \le V(x_1,x_2) \le b(\rho(x_1,x_2))$ *for all* $x_1,x_2 \in X$ $(h(x_1) = h(x_2))$.

Then $V(x_1t,x_2t) = V(x_1,x_2)$ *for all* $t \in \mathbb{S}$ *and* $x_1,x_2 \in J_y$.

Recall that the dynamical system $(X,\mathbb{T}_1,\pi)$ is called asymptotically compact if for every positively invariant bounded subset $M \subseteq X$ there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t\to+\infty} \beta(\pi(t,M),K) = 0.$$

Lemma 3.7. [6] *Let* $\langle (X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ *be a non-autonomous dynamical system and the following conditions be fulfilled:*

1) $(Y,\mathbb{S},\sigma)$ *is pseudo recurrent;*
2) $\gamma \in C(Y,X)$ *is an invariant section of the homomorphism* $h : X \mapsto Y$, *i.e.,* $h(\gamma(y)) = y$ *for all* $y \in Y$.

Then the autonomous dynamical system $(\gamma(Y),\mathbb{S},\pi)$ *is pseudo recurrent too.*

Denote by ω_x the ω–limit set of point x and by $\Omega_X := \overline{\{\omega_x|\ x \in X\}}$. Let $M \subseteq X$ we put

$$D^+(M) := \bigcap_{\varepsilon>0} \overline{\bigcup_{t\ge 0} \pi(t,B(M,\varepsilon))},$$

where $B(M,\varepsilon) := \{x \in X|\ \rho(x,M) < \varepsilon\}$.

A subset M
$subseteqX$ is called orbital stable if for arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\rho(x,M) < \delta$ implies $\rho(\pi(t,x),M) < \varepsilon$ for all $t \ge 0$.

Theorem 3.8. [4, ChI] *A point dissipative dynamical system* $(X,\mathbb{T},\pi)$ *on the complete metric space* X *is compact dissipative of and only if* $D^+(\Omega_X)$ *is compact and orbital stable.*

Corollary 3.9. *Let* $(X,\mathbb{T},\pi)$ *be point dissipative and* Ω_X *is orbital stable, then* $(X,\mathbb{T},\pi)$ *is compact dissipative and its Levinson center* J_X *coincides with* Ω_X*.*

Proof. If Ω_X is orbital stable, then $D^+(\Omega_X) = \Omega_X$and now to finish the proof it is sufficient to apply Theorem 3.8. □

Theorem 3.10. *Let* $\langle(X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ *be a non-autonomous dynamical systems and the following conditions be held:*

1. *the dynamical system* $(Y,\mathbb{S},\sigma)$ *is pseudo recurrent;*
2. *the dynamical system* $(X,\mathbb{T},\pi)$ *is asymptotically compact;*
3. *there exists a point* $x_0\in X_{y_0}$ *with relatively compact positive semi-trajectory* $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$*;*
4. *there exists a continuous function* $V: X\dot{\times}X\to\mathbb{R}_+$ *such that* $V(x_1t,x_2t)<V(x_1,x_2)$ *for all* $(x_1,x_2)\in X\dot{\times}X\setminus\Delta_X$ *and* $t>0$ $(t\in\mathbb{T}_2)$*, where* $\Delta_X:=\{(x,x):\ x\in X\}$*;*
5. *there are functions* $a,b\in\mathcal{K}$ *such that* $Im(a)=Im(b)$ *and* $a(\rho(x_1,x_2)\le V(x_1,x_2)\le b(\rho(x_1,x_2))$ *for all* $(x_1,x_2)\in X\dot{\times}X$*.*

Then the following statements take place:

(i) *the NDS* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ *is convergent;*
(ii) $J_X=\omega_{x_0}$*;*
(iii) $h(J_X)=Y$*.*

Proof. Since the point y_0 is Poisson stable and $\omega_{x_0}=H(y_0)=Y$, for all $y\in Y$ there exists a sequence $\{t_n\}\subseteq\mathbb{T}_2$ such that $t_n\to+\infty$ as $n\to\infty$ and $\{\sigma(t_n,y_0)\}\to y$. Consider the sequence $\{\pi(t_n,x_0)\}$. Under the conditions of Theorem this sequence may be considered convergent. Let p be its limit, then it is clear that $p\in\omega_{x_0}\bigcap X_y$. Thus we established that $h(\omega_0)=Y$.

At first we note that the set ω_{x_0} is compact and invariant and according to Corollary 3.5 on ω_{x_0} is defined a two-sided dynamical system $(\omega_{x_0},\mathbb{S},\pi)$ such that $\pi(t,x)=\gamma_x(t)$ for all $x\in\omega_{x_0}$ and $t\in\mathbb{R}_-$, where γ_x is a unique full trajectory of dynamical system $(X,\mathbb{T},\pi)$ passing through the point x at the initial moment $t=0$. We will show now that under the conditions of Theorem the set $\omega_{x_0}\bigcap X_y$ contains at most one point for all $y\in Y$. In fact, the set ω_{x_0} is compact, invariant and according to Theorem 3.6 (The principle of invariance for the NDS) we have $V(\pi(t,p_1),\pi(t,p_2))=V(p_1,p_2)$ for all $t\in\mathbb{S}$. But the last equality takes place only if $p_1=p_2$.

Let now x be an arbitrary point from X, $y:=h(x)$ and $p\in\in\omega_{x_0}\bigcap X_y$. According to the conditions 5. and 6. we have $a(\rho(xt,pt))\le V(xt,pt)\le V(x,p)\le b(\rho(x,p))$ for all $t\ge 0$ and, consequently, we obtain $\rho(xt,pt)\le a^{-1}(b(x_1,x_2))$ for all $t\ge 0$. Since $p\in L_X$ then from the last inequality we obtain that the set Σ^+_x is bounded. Taking into account that $(X,\mathbb{T},\pi)$ is asymptotically compact, then we may conclude that the point x is stable in the sense of Lagrange in the positive direction. It easy to show that $\omega_x\bigcap X_y$ contains a single point using the same arguments as above for the set ω_{x_0}. We will show that $\omega_x=\omega_{x_0}$. To this end denote by $M:=\omega_{x_0}\bigcup\omega_x$ and repeating the reasoning above for this set we we obtain that $M\bigcap X_y$ consists a singe point for all $y\in Y$. Thus we have $\omega_{x_0}\bigcup X_y=\omega_x\bigcup X_y=M\bigcup X_y$ for all $y\in Y$ and, consequently, $\omega_x=\omega_{x_0}$ foe all $x\in X$. This means that the dynamical system $(X,\mathbb{T},\pi)$ is point dissipative and $\Omega_X=M$, where $M:=\omega_{x_0}$. Now we will show that $(X,\mathbb{T},\pi)$ is compact dissipative. By Theorem 3.8 (see also Corollary 3.9) it is sufficient to establish that the set M is orbitally stable, i.e., for every $\varepsilon>0$ there exists a positive number $\delta(\varepsilon)$ such that $\rho(x,M)<\delta$ implies $\rho(\pi(t,x),M)<\varepsilon$ for all $t\ge 0$. If we suppose the contrary, then there are $\varepsilon_0>0$, $\delta_n\to 0$ $(\delta_n>0)$

$x_n \in X$ and $t_n \to +\infty$ such that

$$\rho(x_n, M) < \delta_n \quad \text{and} \quad \rho(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{14}$$

Let $m_n \in M$ be a point such that $\rho(x_n, m_n) = \rho(x_n, M)$ and denote by $y_n := h(x_n)$. Since the set M is compact and taking into account (14) we may suppose that the sequences $\{x_n\}$, $\{y_n\}$ and $\{m_n\}$ are convergent. Let $\bar{x} := \lim\limits_{n\to\infty} x_n$ and $\bar{m} := \lim\limits_{n\to\infty} m_n$, then by (14) we have $\bar{x} = \bar{m}$. Denote by $m_{y_n} := M \bigcap X_{y_n}$ and taking into consideration the continuity of the mapping $y \mapsto m_y$ we obtain $\lim\limits_{n\to\infty} m_{y_n} = m_{\bar{y}}$, where $\bar{y} := h(\bar{m})$. Note that

$$\bar{m} = m_{\bar{y}} \tag{15}$$

and, consequently,

$$\rho(x_n, m_{y_n}) \leq \rho(x_n, m_n) + \rho(m_n, m_{y_n}). \tag{16}$$

From (15) and (16) it follows that

$$\rho(x_n, m_{y_n}) \to 0 \tag{17}$$

as $n \to \infty$. On the other hand we have

$$V(\pi(t_n, x_n), \pi(t_n, m_{y_n})) < V(x_n, m_{y_n}) \to 0 \tag{18}$$

as $n \to \infty$. It is clear that $\pi(t_n, m_{y_n}) = m_{\sigma(t_n, y_n)}$ and since the space Y is compact me may suppose that the sequence $\{\sigma(t_n, y_n)\}$ is convergent and denote its limit by $\tilde{y}$, then $\lim\limits_{n\to\infty} m_{\sigma(t_n,y_n)} = m_{\tilde{y}} \in M$. But the last equality contradict to inequality (14). The obtained contradiction prove our statement.

Since the set $\Omega_X = M$ is orbitally stable, then according to Theorem 3.8 and Corollary 3.9 the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and its Levinson center J_X coincides with $\Omega_X = M$. Since we established above that $J_X \bigcap X_y = M \bigcap X_y$ consists a single point for all $y \in Y$, then the NDS $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is a system with convergence. The theorem is completely proved. $\square$

4. First order differential equations

This section is dedicated to the study of scalar differential equation of the form

$$x' = f(\sigma(t, y), x) \quad (y \in Y), \tag{19}$$

where $f \in C(Y \times \mathbb{R}, \mathbb{R})$, Y is a complete metric space and $(Y, \mathbb{R}, \sigma)$ is a dynamical system.

A function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ is said to be decreasing in the large sense (respectively, strict increasing) with respect to variable $x \in \mathbb{R}$ if for all $x_1, x_2 \in \mathbb{R}$ and $y \in Y$ the inequality $x_2 > x_1$ implies $f(y, x_2) \leq f(y, x_1)$ (respectively, $f(y, x_2) < f(y, x_1)$).

Theorem 4.1. [5, 16, 21] *Suppose that the function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ is regular and monotone decreasing (in the large sense) with respect to variable $x \in \mathbb{R}$ and the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then the following statements hold:*

(i) *If (10) admits a bounded on $\mathbb{R}$ solution, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly decreasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Below we will give some results which refine and generalize a third statement of Theorem 4.1.

Theorem 4.2. *Suppose that the function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ is regular and strictly decreasing with respect to variable $x \in \mathbb{R}$ and the dynamical system $(Y, \mathbb{R}, \sigma)$ is pseudo recurrent. If (10) admits a bounded on $\mathbb{R}_+$, solution $\varphi(t, u_0, y)$, then it is convergent, i.e., the non-autonomous dynamical system generated by equation (10) is convergent.*

Proof. Let $\varphi(t, u, y)$ be a unique solution of equation (10) passing through point $u \in \mathbb{R}$ at the initial moment $t = 0$. Consider the non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ ($X := \mathbb{R} \times Y$, $\pi := (\varphi, \sigma)$ and $h = pr_2 : X \mapsto Y$) generated by (10). Consider the mapping $V : X \dot{\times} X \mapsto \mathbb{R}_+$ defined by equality

$$V((u_1, y), (u_2, y)) := \frac{|u_1 - u_2|^2}{2} \tag{20}$$

for all $u_1, u_2 \in \mathbb{R}$ and $y \in Y$. It easy to verify that under the conditions of Theorem we have $V(\pi(t, (u_1, y)), \pi(t, (u_2, y))) < V((u_1, y), (u_2, y))$ for all $t > 0$, $u_1, u_2 \in \mathbb{R}$ and $y \in Y$. Now to finish the proof of Theorem it is apply Theorem 3.10. □

Corollary 4.3. *Suppose that the function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ is regular and is strict decreasing with respect to variable $x \in \mathbb{R}$ and the point y is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent).*

Then if (10) admits a bounded on $\mathbb{R}_+$ solution, then it has at unique stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) solution which is globally uniformly asymptotically stable.

Proof. This statement it follows directly from Theorem 4.2 and Remark 3.1 (item 2 (ii)). □

Remark 4.4. *1. The analog of Theorem 4.2 (and also Corollary 4.3) take place if we replace the condition "f is strict decreasing" by "f is strict increasing". This case may be reduced to the considered case by time substitution $t \to -t$.*

2. Note that Theorem 4.2 and Corollary 4.3 remain true also for vectorial equation (system of equations). In fact, if $f \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and we replace the condition "f is strict decreasing" by the condition

$$\langle f(\omega, u_1) - f(\omega, u_2), u_1 - u_2 \rangle < 0$$

for all $\omega \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}^n$*, where* $\langle , \rangle$ *is the scalar product on the space* $\mathbb{R}^n$*.*

3. If the function $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *is continuously differentiable with respect to* $x \in \mathbb{R}$ *and*

$$\frac{\partial f}{\partial x}(\omega, x) \le -k < 0 \tag{21}$$

for all $\omega \in \Omega$ *and* $x \in' \mathbb{R}$*, then Theorem 4.2 and Corollary 4.3 take place without requirement that equation (10) admits at least one bounded on* $\mathbb{R}_+$ *solution. Since from condition (21) it follows*

$$\langle f(\omega, u_1) - f(\omega, u_2), u_1 - u_2 \rangle \le -k|u_1 - u_2|^2 \tag{22}$$

for all $\omega \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}$*. But condition (22) guaranties (see* [7]*) that equation (10) will be convergent.*

4. More in detail the multi-dimensional case we plane to study in one of our next publications.

5. Levitan almost periodic and almost automorphic solutions of second order differential equations

In this section we consider a scalar differential equation of the type (10), i.e., $n = 1$.

Everywhere below in this paper we suppose that the function $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable x, i.e., if $u_1 \le u_2$ then $f(y, u_1, v) \le f(y, u_2, v)$ for all $y \in Y$ and $v \in \mathbb{R}$.

Lemma 5.1. [16] *Let* $u(t)$*,* $v(t)$ *be two solutions of equation (10) defined on* $\mathbb{R}$*. Then one of the following three cases is possible:*

- (i) *the function* $u(t) - v(t)$ *is monotone on the real axis* $\mathbb{R}$*;*
- (ii) $u(t) - v(t)$ *is positive on* $\mathbb{R}$ *and there exists a number* $t_0 \in \mathbb{R}$ *such that this function is non-decreasing on interval* $(t_0, +\infty)$ *and non-increasing on* $(-\infty, t_0)$*;*
- (iii) *the function* $u(t) - v(t)$ *is negative on* $\mathbb{R}$ *and there exists a number* $t_0 \in \mathbb{R}$ *such that it is non-increasing on interval* $(t_0, +\infty)$ *and non-decreasing on* $(-\infty, t_0)$*.*

Let $\varphi \in C(\mathbb{R}, \mathbb{R})$. Denote by $a_\varphi := \inf\{\varphi(t) |\ t \in \mathbb{R}\}$ and $b_\varphi := \sup\{\varphi(t) |\ t \in \mathbb{R}\}$.

Remark 5.2. *1.* $a_\varphi \le b_\varphi$ *for all* $\varphi \in C(\mathbb{R}, \mathbb{R})$*.*

2. The following inequality inequalities

$$a_\varphi \le a_\psi \le b_\psi \le b_\varphi \tag{23}$$

hold for all $\psi \in H(\varphi)$*.*

3. If the function φ *is recurrent, then the following equalities*

$$a_\varphi = a_\psi \quad and \quad b_\psi = b_\varphi \tag{24}$$

hold for all $\psi \in H(\varphi)$*.*

Theorem 5.3. [19, 23] *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a Poisson stable with respect to time $t \in \mathbb{R}$. If the equation*

$$x' = f(t, x) \tag{25}$$

admits a bounded on $\mathbb{R}$ solution φ, then it admits at least one Poisson stable (jointly with f) solution $\psi \in H(\varphi)$.

Theorem 5.4. *Suppose that the function $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$ and the point $y \in Y$ is Poisson stable. Then the following statements hold:*

(i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one compatible (by character of recurrence with the right-hand side) solution;*
(ii) *If $u(t)$ and $v(t)$ are two compatible solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ compatible solution.*

Proof. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be a bounded on $\mathbb{R}$, together with its derivative φ', solution of equation (10). To prove the first statement of Theorem by Lemma 2.16 it is sufficient to show that the function φ is comparable with y by character of recurrence, i.e., that the functional sequence $\{\varphi(t + t_n)\}$ converges to $\varphi(t)$ uniformly on every compact subset from $\mathbb{R}$ for every sequence $\{t_n\} \in \mathfrak{N}_y$. Consider the motion $\sigma(t, \phi)$ in the shift dynamical system (Bebutov's system) $(C(\mathbb{R}, \mathbb{R}), \mathbb{R}, \sigma)$. According to Theorem 5.3 the set $H(\phi) := \overline{\{\sigma(\tau, \phi) | \ \tau \in \mathbb{R}\}}$ contains at least one Poisson stable solution $\varphi \in H(\phi)$ (in fact the function φ and the point y are jointly Poisson stable) of equation (10). We will prove that the solution φ will be compatible. To this end we will show that equation (10) has at most one solution from $H(\varphi) \subseteq H(\phi)$. In fact, if $\psi \in H(\varphi)$ is a solution of equation (10) and $r(t) := \psi(t) - \varphi(t)$ for all $t \in \mathbb{R}$, then by Lemma 5.1 there exist limits

$$\lim_{t \to +\infty} r(t) = c_+, \quad \lim_{t \to -\infty} r(t) = c_- \tag{26}$$

and

$$|c_+| + |c_-| > 0. \tag{27}$$

Suppose, for example, that $c_+ > 0$. Then by jointly Poisson stability of point ω and the solution φ there exists a sequence $\{t_n\} \in \mathfrak{N}_y \cap \mathfrak{N}\varphi$ such that $t_n \to +\infty$ as $n \to \infty$. Without lost of generality we may suppose that the sequence $\{\psi(t + t_n)\}$ is convergent in the space $C(\mathbb{R}, \mathbb{R})$. Let $\bar{\psi}$ be its limit, i.e., $\bar{\psi}(t) = \lim_{t \to +\infty} \psi(t + t_n)$, then we have

$$\bar{\psi}(t) = \varphi(t) + c_+ \ \text{ for all } \ t \in \mathbb{R}. \tag{28}$$

From (23) and $\bar{\psi} \in H(\psi) \subseteq H(\varphi)$ we have

$$a_\varphi \le a_\psi \le a_{\bar{\psi}} \le b_{\bar{\psi}} \le b_\psi \le a_\varphi. \tag{29}$$

On the other hand from (28) we have $b_{\bar{\psi}} = b_\varphi + c_+$. From the last equality and (29) we obtain $C_+ \le 0$. The obtained contradiction proves our statement. Similarly can be considered the others cases.

Let now $u(t)$ and $v(t)$ be two compatible solutions of equation (10), then by Lemma 5.1 there exists a number $t_0 \in \mathbb{R}$ such that the function $r(t) := u(t) - v(t)$ is monotone on the one of the two intervals: $(-\infty, t_0)$ or $(t_0, +\infty)$. Consider, for example, the case when $r(t)$ is monotone on the interval $(-\infty, t_0)$. Since the solutions u and v are compatible and the point y is Poisson stable, then the function $r(t)$ is Poisson stable too. In particularly, it is Poisson stable in the negative direction. On the other hand this function is monotone on the interval $(-\infty, t_0)$ and, consequently, it is a constant. Thus $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant.

Finally, we will prove the third statement of Theorem. Suppose that the function f is strictly increasing with respect to variable $x \in \mathbb{R}$. If we suppose that equation (10) admits two different bounded on $\mathbb{R}$ solutions u and v, then the function

$$r(t) := u(t) - v(t) \quad (t \in \mathbb{R}) \tag{30}$$

admits the limits $c_{\pm} := \lim\limits_{t \to \pm\infty} r(t)$ and $|c_-| + |c_+| > 0$. Suppose, for example, that $c_+ > 0$ then we take a sequence $\{t_n\} \in \mathfrak{N}_y$ such that $t_n \to +\infty$ and the functional sequences $\{u(t + t_n)\}$ and $\{v(t + t_n)\}$ are convergent (since the functions u and v are bounded on $\mathbb{R}$ solutions of (10), then evidently, it is possible). Denote by $\bar{u}$ (respectively, $\bar{v}$) the limit of the sequence $\{u(t + t_n)\}$ (respectively, $\{v(t + t_n)\}$)). from equality (30) we have

$$\bar{u}(t) := \bar{v}(t) + c_+ \quad \text{for all} t \in \mathbb{R} \tag{31}$$

and, consequently, we obtain $f(\sigma(t, y), \bar{v}(t), \bar{v}'(t)) = f(\sigma(t, y), \bar{v}(t) + c_+, \bar{v}'(t))$ for all $t \in \mathbb{R}$. The last identity contradicts to the strict monotony of the function f with respect to variable x. The obtained contradiction completes the proof of Theorem. $\square$

Corollary 5.5. *Suppose that the function $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$ and the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable). Then the following statements hold:*

- (i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) solution;*
- (ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
- (iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) solution.*

Proof. This statement it follows from Theorem 5.4 and Theorem 2.13. $\square$

6. Quasi-periodic, almost periodic and recurrent solutions

In this section we suppose that Y is compact and $(Y,\mathbb{R},\sigma)$ is a minimal dynamical system.

Theorem 6.1. *Suppose that the function $f\in C(Y\times\mathbb{R}^2,\mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x\in\mathbb{R}$. Then the following statements hold:*

- (i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one uniformly compatible (by character of recurrence with the right-hand side) solution;*
- (ii) *If $u(t)$ and $v(t)$ are two uniformly compatible solutions of equation (10), then $u(t)-v(t)=c$ for all $t\in\mathbb{R}$, where $c\in\mathbb{R}$ is some constant;*
- (iii) *If the function f is strictly increasing with respect to variable $x\in\mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ uniformly compatible solution.*

Proof. Let $\varphi\in C(\mathbb{R},\mathbb{R})$ be a bounded on $\mathbb{R}$, together with its derivative φ', solution. To prove the first statement of Theorem by Lemma 2.16 it is sufficient to show that the function φ is uniformly comparable with ω by character of recurrence, i.e., that the functional sequence $\{\varphi(t+t_n)\}$ is convergent uniformly on every compact subset from $\mathbb{R}$ for every sequence $\{t_n\}\in\mathfrak{M}_y$. Denote by $X:=C(\mathbb{R},\mathbb{R})\times Y$ and $(X,\mathbb{R},\pi)$ the product dynamical system, i.e., $\pi(\tau,(\varphi,y)):=(\varphi_\tau,\sigma(\tau,y))$ for all $(\varphi,y)\in C(\mathbb{R},\mathbb{R})\times Y$ and $\tau\in\mathbb{R}$, where φ_τ is a τ–shift of the function φ ($\varphi_\tau(t):=\varphi(t+\tau)$ for all $t\in\mathbb{R}$). Consider the motion $\pi(t,(\varphi,y))$ in the product dynamical system $(X,\mathbb{R},\pi)$. Under the condition of Theorem this motion is stable in the sense of Lagrange, i.e., the set $H(\varphi,y):=\overline{\{\pi(\tau,(u\varphi,y))|\ \tau\in\mathbb{R}\}}$ is compact. According to Birkhoff's theorem the set $H(\varphi,y)$ contains at least one minimal set $\mathcal{M}\subseteq H(\varphi,y)$. Note that the mapping $h:=pr_2\ :\mathcal{M}\mapsto Y$ is an homomorphism of dynamical system $(H(\varphi,y),\mathbb{R},\pi)$ onto $(Y,\mathbb{R},\sigma)$ and, consequently, $\mathcal{M}_y:=\{(\psi,y)\ :\ (\psi,y)\in H(\varphi,y)\}$ is a nonempty compact subset of $H(\varphi,y)$. Now we will show that the set $\mathcal{M}_y$ consists a single point for every $y\in Y$. In fact, if we suppose the contrary then there exists a point $y_0\in Y$ such that $\mathcal{M}_{y_0}$ contains at least two different points (v_i,y_0) ($i=1,2$ and $v_1\neq v_2$). According to Theorem 5.4 without loss of generality we may suppose, for example, that v_1 is comparable by character of recurrence with the point y_0, i.e., $\mathfrak{N}_{y_0}\subseteq\mathfrak{N}_{v_1}$. On the other hand by Lemma 5.1 there exist limits $\lim\limits_{t\to\pm\infty}r(t)=c_\pm$ and $|c_-|+|c_+|>0$, where $r(t):=v_2(t)-v_1(t)$ for all $t\in\mathbb{R}$. Suppose for example that $c_->0$, then taking into account the fact that the point (v_1,y_0) is negatively Poisson's stable we have a sequence $\{t_n\}\in\mathfrak{N}_{v_1}\cap\mathfrak{N}_{\omega_0}$ such that $t_n\to-\infty$ as $n\to\infty$. We may suppose that the sequence $\{v_2(t+t_n)\}$ is convergent. Denote by $\bar{v}_2$ its limit, then we have $\bar{v}_2(t)=v_1(t)+c_-$ for all $t\in\mathbb{R}$ and, consequently, we have

$$a_{\bar{v}_2}=a_{v_1}+c_-. \tag{32}$$

But the functions $v_1,\bar{v}_2\in H(v_1)$ and the function v_1 is recurrent and, consequently, we have

$$a_{\bar{v}_2}=a_{v_1}=a_{v_2}. \tag{33}$$

From (32) and (33) it follows that $c_- = 0$. The obtained contradiction proves our statement. The other cases may be considered similarly. Thus we established that the set $\mathcal{M}_y$ consists a single point for all $\omega \in \Omega$. Let ϕ be a solution of equation (10) such that $\{(\phi, y)\} = \mathcal{M}_y$. Now it is easy to show that the solution ϕ is uniformly compatible. In fact, let $\{t_n\} \in \mathfrak{M}_y$, then the sequence $\{\sigma(t_n, y)\}$ converges. Denote by $\tilde{y}$ its limit. We will show that the sequence functional $\{\phi(t + t_n)\}$ is convergent too in the space $C(\mathbb{R}, \mathbb{R})$. If it is not true, then there exists at least two points ψ_i ($i = 1, 2$ and $\psi_1 \neq \psi_2$) of accumulation for this sequence. On the other hand it easy to see that $(\psi_i, \tilde{y}) \in \mathcal{M}_{\tilde{y}}$ $(i = 1, 2)$. The last inclusion contradicts to the fact that every subsets $\mathcal{M}_y \subseteq \mathcal{M}$ consists a single point for all $y \in Y$. The obtained contradiction proves the first statement of Theorem.

The second and third statements of Theorem follow from Theorem 5.4. $\square$

Corollary 6.2. *Suppose that the function $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$ and the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then the following statements hold:*

(i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. This statement it follows from Theorem 6.1 and Theorem 2.13. $\square$

Remark 6.3. *In the case when Y is a Bohr almost periodic minimal set, then Corollary 6.2 coincides with Opial's result* [16].

7. Some generalizations

Let now $I := (a, b)$, where $a, b \in [-\infty, +\infty]$. For example, $I = \mathbb{R}$, $I = (0, +\infty)$, $I = (a, b)$ and $a, b \in \mathbb{R}$ etc. Consider equation (10) in the case when $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$. For example, for the equation

$$x'' + cx' + 1/x^{\alpha} = f(\sigma(t, y)) \tag{34}$$

$f(y, x, x') := -cx' - 1/x^{\alpha} + f(y)$ and $I = (0, +\infty)$, where $\alpha > 0$.

A solution $\varphi \in C(\mathbb{R}, \mathbb{R})$ of equation (10) is said to be bounded on $\mathbb{R}$ (respectively, on $\mathbb{R}_+$) if $Q := \overline{\varphi(\mathbb{R})}$ is a compact subset from I, i.e., if there exist two real numbers α and β such that $a < \alpha \leq \varphi(t) \leq \beta < b$ for all $t \in \mathbb{R}$ (respectively, $t \in \mathbb{R}_+$).

All our results about second order equations (10) (Theorems 5.4, 6.1 and Corollaries 5.5 and 6.2) remain true also when $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$. We will formulate for example the following statement.

Theorem 7.1. *Suppose that the function $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$. Then the following statements hold:*

(i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one uniformly compatible (by character of recurrence with the right-hand side) solution;*
(ii) *If $u(t)$ and $v(t)$ are two uniformly compatible solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ uniformly compatible solution.*

Proof. We omit the proof because it absolutely similar to proof of Theorem 6.1. □

Corollary 7.2. *Suppose that the function $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$ and the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then the following statements hold:*

(i) *If (10) admits a bounded on $\mathbb{R}$, together with its derivative, solution, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. This statement it follows from Theorem 7.1 and Theorem 2.13. □

Corollary 7.3. *Suppose that the following conditions are fulfilled:*

(i) *$f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ and there exists a constant $C > 0$ such that*

$$|f(y, x, x')| \le C(1 + |x'|^2) \tag{35}$$

for all $(y, x, x') \in Y \times I \times \mathbb{R}$;
(ii) *the function f is regular and monotone increasing (in the large sense) with respect to variable $x \in \mathbb{R}$;*
(iii) *the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent).*

Then the following statements hold:

(i) *If (10) admits a bounded on $\mathbb{R}$ solution, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*

(iii) *If the function f is strictly increasing with respect to variable $x \in \mathbb{R}$, then equation (10) admits at most one bounded on $\mathbb{R}$ stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. This statement it follows from Theorem 6.1 and Theorem 2.13. □

Proof. This statement it follows from Corollary 7.2. To this end it is sufficient to note that under the condition (35) if $\varphi \in C(\mathbb{R}, \mathbb{R})$ is a bounded on $\mathbb{R}$ solution of equation (10), then its derivative φ' is also bounded on $\mathbb{R}$ (see Lemma 2.1 [13] and also Lemma 5.1 from [11, Ch.XII]). □

Remark 7.4. *1. Corollary 7.3 (item (iii)) refines and generalizes some of results from* [3, 8, 10, 13] *in the case, when the function f is strict increasing with respect to second variable.*

2. We plan to study more in detail this case (f is strict increasing with respect to second variable) in one of our future publication.

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