G. Sell's* Conjecture for Non-autonomous Dynamical Systems

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1.- Introduction

Asymptotically autonomous systems (L. Markus -1953)

A system of differential equation

$$x' = f(x) + p(t, x) \tag{1}$$

is said to be asymptotically autonomous, if the function $p \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following condition

$$\lim_{t \to \infty} |p(t, x)| = 0 \tag{2}$$

uniformly with respect to x on every compact subset from $\mathbb{R}^n$, where $|\cdot|$ is a norm on $\mathbb{R}^n$. Autonomous system

$$x' = f(x) \tag{3}$$

is called a limiting system for (1).

Denote by $F(t, x) := f(x) + p(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ the right hand side of system (1), where $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the space of all continuous functions $F : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ equipped with the compact-open topology.

Example 1 *(***Bessel's equation***) Consider the equation*

$$t^2x'' + tx' + (t^2 - \alpha^2)x = 0, \tag{4}$$

or equivalently

$$\begin{cases} x' = y \\ y' = -\frac{1}{t}y + (\frac{\alpha^2}{t^2} - 1)x, \end{cases} \tag{5}$$

with limiting system

$$\begin{cases} x' = y \\ y' = -x \ . \end{cases} \tag{6}$$

Theorem 2 *(***L. Markus** *(1956), $n = 2$) Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $O = (0,0)$ be a critical point of limiting system (3), i.e, $f(0) = 0$. Assume that the variational system of (3) based on origin O have characteristic values with negative real parts. Then there exists a neighborhood U of O and a time T such that $\lim_{t\to\infty} |x(t)| = 0$ for any solution of equation (1) intersecting U no later than T, i.e., the origin is an attracting point for (1).*

Let $(C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}_+, \sigma)$ be the shift dynamical system on $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ (or Bebutov's dynamical system). For every function $F \in C(\mathbb{R}_+ \times$

$\mathbb{R}^n, \mathbb{R}^n)$ we denote by $H^+(F) := \overline{\{F_\tau : \ \tau \in \mathbb{R}_+\}}$ the closure of all positive translations of the function F and by Ω_F its ω-limit set, i.e., $\Omega_F := \{G :$ there exists a sequence $\tau_n \to +\infty$ such that $F_{\tau_n} \to G\}$, where F_τ is τ-shift of the function F, i.e., $F_\tau(t,x) := F(t+\tau, x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ be an arbitrary function. Consider the equation

$$x' = F(t,x). \tag{7}$$

Along with equation (7) we consider its H-class, i.e., the following family of equations

$$y' = G(t,y) \ (G \in H^+(F)). \tag{8}$$

Example 3 *1. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically autonomous, i.e., $F(t,x) = f(x) + p(t,x)$ and p satisfies condition (2). In this case $\Omega_F = \{f\}$, i.e., its ω-limit set contains a single function.*

2. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically T periodic, i.e., $F(t,x) = f(t,x) + p(t,x)$, $f(t+T,x) = f(t,x)$ and p satisfies condition (2). In

this case $\Omega_F = \{f_\tau : \ \tau \in [0,T)\}$, *i.e., its* ω-*limit set contains a continuum functions and it is isomorphic to unitary circle.*

3. If $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ *is asymptotically quasi periodic, i.e.,* $F(t,x) = f(t,x) + p(t,x)$, $f(t,x)$ *is quasi periodic with the spectrum of the frequency* $\nu_1, \nu_2, \ldots, \nu_m$ *and* p *satisfies condition (2). In this case its* ω*-limit set is isomorphic to an* m*-torus.*

Theorem 4 *(***G. Sell** *(1971)) Let* $F \in C(\mathbb{R}^n, \mathbb{R}^n)$, *be regular, asymptotically autonomous and* $O \in \mathbb{R}^n$ *be the null solution equation (7), i.e,* $F(t,0) = 0$ *for all* $t \in \mathbb{R}_+$. *Assume that the null solution of the limiting equation (3) is uniformly asymptotically stable. Then the null solution of equation (7) uniformly asymptotically stable.*

Remark 5 1. *Note that Theorem 4 generalizes Theorem of L. Markus in the following directions:*

a. n *is an arbitrary natural number (L. Markus only* $n = 2$*);*

b. *right hand side f of the limiting equation is only continuous (L. Markus, $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$);*

c. *the null solution of limiting equation (3) is only uniformly asymptotically stable (L. Markus, O is uniformly exponentially stable (In fact, $Re\lambda_i < 0$ $(i = 1, 2)$, λ_1, λ_2 are characteristic values of the origin for the variational equation for (3)).*

2. *It is easy to see that there are examples with uniformly asymptotically stable origin which is not uniformly exponentially stable. For example $x' = -x^3$ $(n = 1)$.*

2.- Problem of Global Asymptotic Stability

The aim of this talk is the study the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles).

Consider a differential equation

$$x' = f(t,x) \quad f \in C(\mathbb{R} \times W, \mathbb{R}^n), \tag{9}$$

where $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, W is an open subset from $\mathbb{R}^n$ containing the origin (i.e., $0 \in W$), $C(\mathbb{R} \times W, \mathbb{R}^n)$ is the space of all continuous functions $f : \mathbb{R} \times W \mapsto \mathbb{R}^n$ equipped with compact open topology. This topology is defined by the following distance

$$\rho(f,g) := \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{\rho_k(f,g)}{1+\rho_k(f,g)}, \tag{10}$$

where $\rho_k(f,g) := \max\{|f(t,x) - g(t,x)| : \ (t,x) \in [-k,k] \times W_k\}$, where $\{W_k\}$ is a family of compact subsets from W with the properties: $W_k \subset W_{k+1}$ for all $k \in \mathbb{N}$ and $\bigcup_{k=1}^{+\infty} W_k = W$ and $|\cdot|$ is a norm on $\mathbb{R}^n$. Denote by $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$ the

shift dynamical system on the space $C(\mathbb{R}\times W,\mathbb{R}^n)$ (dynamical system of translations or Bebutov's dynamical system), i.e., $\sigma(\tau,f):=f_\tau$ for all $\tau\in\mathbb{R}$ and $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$, where $f_\tau(t,x):=f(t+\tau,x)$ for all $(t,x)\in\mathbb{R}\times W$.

Below we will use the following conditions:

(A): for all $(t_0,x_0)\in\mathbb{R}_+\times W$ the equation (9) admits a unique solution $x(t;t_0,x_0)$ passing through point x_0 at the moment t_0 and defined on $\mathbb{R}_+$, i.e., $x(t_0;t_0,x_0)=x_0$, where $\mathbb{R}_+:=[0,+\infty)$;

(B): the hand right side is positively compact, if the set $\Sigma_f^+:=\{f_\tau:\ \tau\in\mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R}\times W,\mathbb{R}^n)$;

(C): the equation

$$y'=g(t,y)\quad(g\in\Omega_f)\tag{11}$$

is called a limiting equation for (9), where Ω_f is the ω-limit set of f with respect to shift dynamical system $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$,

i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\}$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): equation (9) is regular (or its hand right side f), if for all $g \in H^+(f)$ the equation

$$y' = g(t, y) \tag{12}$$

admits a unique solution $\varphi(t, x_0, g)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x_0, g) = x_0$ for all $x_0 \in W$, where $H^+(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}_+\}}$ and by bar is denote the closure in the space $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(E): equation (9) admits a null (trivial) solution, i.e., $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;

(F): a function f satisfies to local (respectively, global) Lipschitz condition, if there exists a function $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ (respectively, a positive constant L) such that

$$|f(t, x_1) - f(t, x_2)| \leq L(r)|x_1 - x_2| \tag{13}$$

(respectively, $|f(t, x_1) - f(t, x_2)| \leq |x_1 - x_2|$) for all $t \in \mathbb{R}_+$ and $x_1, x_2 \in W$ with $|x_1|, |x_2| \leq$

r for all $r > 0$ (respectively, for all $x_1, x_2 \in W$).

The trivial solution of equation (9) is said to be:

1. uniformly stable, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f_\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;

2. uniformly attracting, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, x, f_\tau)| = 0 \tag{14}$$

uniformly with respect to $|x| \leq a$ and $\tau \in \mathbb{R}_+$;

3. uniformly asymptotically stable, if it is uniformly stable and uniformly attracting.

Theorem 1. (**Sell, 1971**) *Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and f is positively precompact. If the trivial solution of (9) is uniformly asymptotically stable, then the following statements hold:*

1. *for every* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) \in (0, \varepsilon)$ *such that* $|x| < \delta$ *implies* $|\varphi(t, x, g)| < \varepsilon$ *for all* $t \in \mathbb{R}_+$ *and* $g \in H^+(f)$*;*

2. *there exists a positive number* a *such that*

$$\lim_{t \to +\infty} |\varphi(t, x, g)| = 0 \tag{15}$$

uniformly with respect to $|x| \leq a$ *and* $g \in H^+(f)$*.*

Theorem 2. (**Sell, 1971**) *Let* $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ *be a regular function and* f *is positively precompact. Assume further that*

1. *the trivial solution of (9) is uniformly stable;*

2. *there exists a positive constant* a *such that equality (15) takes place uniformly with respect to* $|x| \leq a$ *and* $g \in \Omega_f$*.*

Then the trivial solution of equation (9) is uniformly asymptotically stable.

3.- G. Sell's Conjecture for Non-Autonomous ODEs

G. Sell's conjecture (**G. Sell, Ch.VIII,p.134**). Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and f be positively pre-compact. Assume that W contains the origin 0 and $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. Assume further that there exists a positive number a such that the equality (15) takes place uniformly with respect to $|x| \leq a$ and $g \in \Omega_f$. Then the trivial solution of (9) is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Z. Artstein (1978) and Bondi P. et all (1977).

Remark 1. Bondi P. et all (1977) proved this conjecture under the additional assumption that the function f is local Lipschitzian.

2. Artstein Z. (1978) proved this statement without Lipschitzian condition. In reality he proved a small more general affirmation. Namely, he supposed that only limiting equations for (9) are regular, but the function f is not obligatory regular.

4.- G. Sell's Conjecture for Non-Autonomous Dynamical Systems

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, if the following conditions are fulfilled:

1. $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;

2. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_1$, $u \in W$ and $y \in Y$;

3. the mapping $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ is continuous.

Example 6 Consider differential equation (9) with regular second right hand $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f), \mathbb{R}_+, \sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R} \times$

$W, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $H^+(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}_+\}}$. Let $\varphi(t, u, g)$ a unique solution of equation

$$y' = g(t, y), \quad (g \in H^+(f)),$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

1. $\varphi(0, u, g) = u$ for all $u \in W$ and $g \in H^+(f)$;

2. $\varphi(t + \tau, u, g) = \varphi(t, \varphi(\tau, u, g), g_\tau)$ for all $t, \tau \in \mathbb{R}_+$, $u \in W$ and $g \in H^+(f)$;

3. the mapping $\varphi : \mathbb{R}_+ \times W \times H^+(f) \mapsto W$ is continuous.

From above it follows that the triplet $\langle W, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle over $(H^+(f), \mathbb{R}_+, \sigma)$ with the fiber $W \subseteq \mathbb{R}^n$. Thus, every non-autonomous equation (9) with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation (9).

Suppose that $W \subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0 \in W$ (0 is the null element of E) and the cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t, 0, y) = 0$ for all $t \in \mathbb{T}_1$ and $y \in Y$.

The trivial motion/solution of cocycle φ is said to be:

1. *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for all $t \geq 0$ and $y \in Y$;

2. *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0 \tag{16}$$

   uniformly with respect to $|u| \leq a$ and $y \in Y$;

3. *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

G.Sell's conjecture for cocycle. Suppose that $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{T}_2, \sigma)$ with the fiber W and the following conditions are fulfilled:

1. the cocycle φ admits a trivial motion/solution;

2. the space Y is compact;

3. there exists a positive constant a such that (16) takes place uniformly with respect to $|u| \leq a$ and $y \in J_Y$, where J_Y is Levinson center (maximal compact invariant set) of compactly dissipative dynamical system $(Y, \mathbb{T}_2, \sigma)$;

Recall [5] that a triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is said to be a *NDS*, where $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) is a dynamical system on X (respectively, Y) and h is an homomorphism from (X, $\mathbb{T}_1$, π) onto (Y, $\mathbb{T}_2$, σ).

Note that every cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ naturally generates a NDS. In fact. Let $X := W \times Y$

and $(X, \mathbb{T}_1, \pi)$ be a skew-product dynamical system on X (i.e., $\pi(t,x) := (\varphi(t,u,y), \sigma(t,y))$ for all $t \in \mathbb{T}_1$ and $x := (u,y) \in X$), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where $h := pr_2 : X \mapsto Y$ is the second projection (i.e., $h(u,y) = y$ for all $u \in W$ and $y \in Y$), is a NDS.

Let (X, h, Y) be a vectorial bundle. Denote by θ_y the null element of the vectorial space $X_y := \{x \in X : \ h(x) = y\}$ and $\Theta := \{\theta_y : \ y \in Y\}$ is the null section of (X, h, Y).

Consider a NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ on the vectorial bundle (X, h, Y). Everywhere in this talk we suppose that the null section Θ of (X, h, Y) is a positively invariant set, i.e., $\pi(t, \theta) \in \Theta$ for all $\theta \in \Theta$ and $t \geq 0$ $(t \in \mathbb{T}_1)$.

The *null (trivial) section* Θ of NDS $\langle (X, \mathbb{T}_1, \pi)$, $(Y, \mathbb{T}_2, \sigma)$, $h \rangle$ is said to be:

1. *uniformly stable*, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(t,x)| < \varepsilon$ for all $t \geq 0$ $(t \in \mathbb{T}_1)$;

2. *attracting*, if there exists a number $\nu > 0$ such that $B(\Theta, \nu) \subseteq W^s(\Theta)$, where $B(\Theta, \nu) := \{x \in X | \ |x| < \nu\}$;

3. *uniform attracting*, if there exists a number $\nu > 0$ such that

$$\lim_{t \to +\infty} \sup\{|\pi(t, x)| : \ |x| \leq \nu\} = 0;$$

4. *asymptotically stable* (respectively, *uniformly asymptotically stable*) if, Θ is uniformly stable and attracting (respectively, uniformly attracting).

Let $(Y, \mathbb{T}_2, \sigma)$ be a compactly dissipative dynamical system, J_Y its Levinson center and $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a NDS. Denote by $\tilde{X} := h^{-1}(J_Y) = \{x \in X : \ h(x) = y \in J_Y\}$, then evidently the following statements are fulfilled:

1. $\tilde{X}$ is closed;

2. $\pi(t, \tilde{X}) \subseteq \tilde{X}$ for all $t \in \mathbb{T}_1$ and, consequently, on the set $\tilde{X}$ is induced by $(X, \mathbb{T}_1, \pi)$ a dynamical system $(\tilde{X}, \mathbb{T}_1, \pi)$);

3. the triplet $\langle(\tilde{X}, \mathbb{T}_1, \pi), (J_Y, \mathbb{T}_2, \sigma), h\rangle$ is a NDS.

G.Sell's conjecture for NDS. Suppose that $\langle(X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a NDS and the following conditions are fulfilled:

1. the null/trivial section Θ of (X, h, Y) is a positively invariant set;

2. the space Y is compact;

3. the null section $\tilde{\Theta}$ of the NDS $\langle(\tilde{X}, \mathbb{T}_1, \pi)$, $(J_Y, \mathbb{T}_2, \sigma)$, $h\rangle$ is uniformly attracting.

Then the trivial section Θ of NDS $\langle(X, \mathbb{T}_1, \pi)$, $(Y, \mathbb{T}_2, \sigma)$, $h\rangle$ is uniformly asymptotically stable.

The main goal of this talk is to give a positive answer to G.Sell's conjecture for general NDS

(the both cases: with continuous and discrete time) and applications of this result to different classes of differential/difference equations (ODEs in Banach space, FDEs, DEs, some classes of PDES).

5.- Some Applications

Applying the general results obtained in this talk we will obtain a series of results for equation (9). Below we formulate some of them.

ODEs in Banach space

Let E be a Banach space. Consider differential equation (9) on the space E.

Theorem 7 *Assume that the following conditions are fulfilled:*

1. *the function f is regular;*

2. *the set $H^+(f)$ is compact;*

3. $f(t,0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

4. *There exists a neighborhood $U \subset E$ of the origin* 0 *and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*

5. *there exists a positive number* a *such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0. \qquad (17)$$

Then the null solution of equation (9) is uniformly asymptotically stable.

Remark 8 *If the space* E *is finite-dimensional, then Theorem 7 coincides with the result of Artstein Z. and Bondi P. et all because in this case the cocycle* φ *associated by equation (9) is local compact.*

Theorem 9 *Let* $f \in C(\mathbb{R}\times E, E)$. *Under the conditions of Theorem 7 the null solution of equation (9) is globally asymptotically stable if and only if the following conditions hold:*

1.

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0$$

for every $a > 0$;

2. *for every* $v \in E$ *and* $g \in H^+(f)$ *the solution* $\varphi(t, v, g)$ *of equation (12) is bounded on* $\mathbb{R}_+$.

Theorem 10 *Suppose that the following conditions are fulfilled:*

1. *the function* $f \in C(\mathbb{R} \times W, E)$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact from* W*;*

2. $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

3. *the function* f *is regular;*

4. *the cocycle* φ *associated by equation (9) is asymptotically compact, i.e., the sequence* $\{\varphi(t_n, v_n, g_n)\}$ *is relatively compact,if it is bounded, for all* $t_n \to +\infty$ *and* $(v_n, g_n) \in W \times H^+(f)$ *with bounded sequence* $\{v_n\} \subseteq W$*;*

5. *the null solution of equation (9) is uniformly stable;*

6. there exists a positive number a such that

$$\lim_{t\to+\infty} \sup_{|u|\le a} |\varphi(t,u,f)| = 0.$$

Then the null solution of equation (9) is asymptotically stable.

Difference equations

Consider a difference equation

$$u(t+1) = f(t,u(t)), \tag{18}$$

where $f \in C(\mathbb{Z}\times W, E)$.

Along with equation (18) we consider the family of equations

$$v(t+1) = g(t,v(t)), \tag{19}$$

where $g \in H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{Z}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z}\times W, E)$. Let $\varphi(t,v,g)$ be the unique solution of equation (19) with initial data $\varphi(0,v,g) = v$. Denote by $(H^+(f), \mathbb{Z}_+, \sigma)$ the shift dynamical system on $H^+(f)$, then the triplet $\langle W, \varphi, (H^+(f), \mathbb{Z}_+, \sigma)\rangle$ is

a cocycle (with discrete time) over $(H^+(f), \mathbb{Z}_+, \sigma)$ with the fiber W.

Theorem 11 *Let $f \in C(\mathbb{Z} \times W, E)$. Assume that the following conditions are fulfilled:*

1. *the set $H^+(f)$ is compact;*

2. *$f(t, 0) = 0$ for all $t \in \mathbb{Z}_+$;*

3. *there exists a neighborhood U of 0 and a positive number l such that $\varphi(l, U, H^+(f))$ is relatively compact;*

4. *there exists a positive number a such that*

$$\lim_{t \to +\infty} \sup_{|v| \leq a, g \in \Omega_f} |\varphi(t, v, g)| = 0.$$

Then the null solution of equation (18) is uniformly asymptotically stable.

Functional-differential equations (FDEs) with finite delay

Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a,b] = [-r,0]$, then we set $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $u \in C([\sigma - r, \sigma + A],\mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for all $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t+\theta)$, $-r \leq \theta \leq 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{20}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is defined a shift dynamical system $(C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $\sigma(\tau, f) := f_\tau$ for all $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f_τ is τ-translation of f, i.e., $f_\tau(t,\phi) := f(t+\tau,\phi)$ for all $(t,\phi) \in \mathbb{R} \times \mathcal{C}$.

Let us set $H^+(f) := \overline{\{f_s : s \in \mathbb{R}_+\}}$, where by bar we denote the closure in $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$.

Along with the equation (20) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{21}$$

where $g \in H^+(f)$.

A function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ (respectively, equation (20)) is called *regular*, if for $v \in \mathcal{C}$ and $g \in H^+(f)$ equation (21) admits a unique solution passing through v at the initial moment $t = 0$.

Below, in this subsection, we suppose that equation (20) is regular.

It is well known that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H^+(f) \mapsto \mathbb{R}^n$ possesses the following properties:

1. $\varphi(0, v, g) = u$ for all $v \in \mathcal{C}$ and $g \in H^+(f)$;

2. $\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), \sigma(\tau, g))$ for all $t, \tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H^+(f)$;

3. the mapping φ is continuous.

Thus, a triplet $\langle \mathcal{C}, \varphi, (H^+(f), \mathbb{R}_+, \sigma) \rangle$ is a cocycle which is associated to equation (20).

Lemma 12 *Suppose that the following conditions hold:*

1. *the function* $f \in C(\mathbb{R} \times W, \mathcal{C})$ *is regular;*

2. *the set* $H^+(f)$ *is compact;*

3. *the function* f *is completely continuous, i.e., the set* $f(\mathbb{R}_+ \times A)$ *is bounded for all bounded subset* $A \subseteq \mathcal{C}$*.*

Then the cocycle φ *associated by (20) is completely continuous, i.e., for all bounded subset* $A \subseteq W$ *there exists a positive number* $l = l(A)$ *such that the set* $\varphi(l, A, H^+(f))$ *is relatively compact in* $\mathcal{C}$*.*

Theorem 13 *Assume that the following conditions are fulfilled:*

1. *the function* f *is regular;*

2. *the set* $H^+(f)$ *is compact;*

3. $f(t,0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

4. *there exists a positive number* a *such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0. \qquad (22)$$

Then the null solution of equation (20) is uniformly asymptotically stable.

The null solution of equation (20) with $f \in C(\mathbb{R}\times\mathcal{C},\mathcal{C})$ is said to be *globally asymptotically stable* if it is asymptotically stable and

$$\lim_{t\to+\infty} |\varphi(t,v,g)| = 0$$

for all $(v,g) \in \mathcal{C}\times H^+(f)$.

Theorem 14 *Let* $f \in C(\mathbb{R}\times\mathcal{C},\mathcal{C})$*. Under the conditions of Theorem 13 the null solution of equation (20) is globally asymptotically stable if and only if the following conditions hold:*

1.

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0$$

for every $a > 0$*;*

2. *for every* $v \in E$ *and* $g \in H^+(f)$ *the solution* $\varphi(t,v,g)$ *of equation (21) is bounded on* $\mathbb{R}_+$*.*

Theorem 15 *Suppose that the following conditions are fulfilled:*

1. *the function* $f \in C(\mathbb{R}\times\mathcal{C},\mathcal{C})$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact from* $\mathcal{C}$*;*

2. $f(t,0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

3. *the function* f *is regular;*

4. *the null solution of equation (20) is uniformly stable;*

5. there exists a positive number a such that

$$\lim_{t\to+\infty} \sup_{|u|\le a} |\varphi(t,u,f)| = 0.$$

Then the null solution of equation (20) is asymptotically stable.

Neutral functional-differential equations

Now consider the neutral functional-differential equation

$$\frac{d}{dt}Du_t = f(t,u_t), \tag{23}$$

where $f \in C(\mathbb{R}\times\mathcal{C},\mathcal{C})$ and the operator $D : \mathcal{C} \mapsto \mathbb{R}^n$ is atomic at zero. Like (20), equation (23) generates a NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$, where $X := \mathcal{C}\times Y,\ Y := H^+(f)$, and $\pi := (\varphi,\sigma)$.

An operator D is said to be stable, if the zero solution of difference equation $Dy_t = 0$ is uniformly asymptotically stable.

Lemma 16 *Let $H^+(f)$ be compact. Then the NDS $(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (23) is asymptotically compact.*

Theorem 17 *Suppose that the following conditions are fulfilled:*

1. *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact subset from* $\mathcal{C}$*;*

2. $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

3. *the function* f *is regular;*

4. *the null solution of equation (23) is uniformly stable;*

5. *there exists a positive number* a *such that*

$$\lim_{t \to +\infty} \sup_{|u| \leq a} |\varphi(t, u, f)| = 0. \qquad (24)$$

Then the null solution of equation (23) is asymptotically stable, i.e., there exists a positive

number δ *such that* $\lim_{t\to+\infty} |\varphi(t,v,g)| = 0$ *for all* $|v| < \delta$ *and* $g \in H^+(f)$.

Semi-linear parabolic equations

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|^2 := \langle\cdot,\cdot\rangle$, and A be a self-adjoint operator with domain $D(A)$.

An operator is said to have a *discrete spectrum* if in the space H, there exists an ortho-normal basis $\{e_k\}$ of eigenvectors, such that $\langle e_k, e_j\rangle = \delta_{kj}$, $Ae_k = \lambda_k e_k$ $(k,j = 1,2,\ldots)$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_k \leq \ldots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

One can define an operator $f(A)$ for a wide class of functions f defined on the positive semi-axis as follows:

$$D(f(A)) := \{h = \textstyle\sum_{k=1}^{\infty} c_k e_k \in H : \sum_{k=1}^{\infty} c_k [f(\lambda_k)]^2 < +\infty\}, \tag{25}$$

$$f(A)h := \textstyle\sum_{k=1}^{\infty} c_k f(\lambda_k) e_k, \ h \in D(f(A)). \tag{26}$$

In particular, we can define operators A^α for all $\alpha \in \mathbb{R}$. For $\alpha = -\beta < 0$ this operator is bounded.

The space $D(A^{-\beta})$ can be regarded as the completion of the space H with respect to the norm $|\cdot|_\beta := |A^{-\beta}\cdot|$.

The following statements hold:

1. The space $\mathcal{F}_{-\beta} := D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum_{k=1}^{\infty} c_k e_k$ such that
$$\sum_{k=1}^{\infty} c_k \lambda_k^{-2\beta} < +\infty;$$

2. For any $\beta \in \mathbb{R}$, the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded operator mapping $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that
$$A^\beta D(A^\alpha) = D(A^{\alpha-\beta}),\ A^{\beta_1+\beta_2} = A^{\beta_1}A^{\beta_2}.$$

3. For all $\alpha \in \mathbb{R}$, the space $\mathcal{F} := D(A^\alpha)$ is a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\alpha := \langle A^\alpha\cdot, A^\alpha\cdot\rangle$ and the norm $|\cdot|_\alpha := |A^\alpha\cdot|$.

4. The operator A with the domain $\mathcal{F}_{1+\alpha}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\alpha$.

5. The embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$ is continuous, i.e., $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and there exists a positive constant $C = C(\alpha, \beta)$ such that $|\cdot|_\beta \leq C|\cdot|_\alpha$.

6. $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.

7. Let $\alpha_1 > \alpha_2$, then the space $\mathcal{F}_{\alpha_1}$ is compactly embedded into $\mathcal{F}_{\alpha_2}$, i.e., every sequence bounded in $\mathcal{F}_{\alpha_1}$ is relatively compact in $\mathcal{F}_{\alpha_2}$.

8. The resolvent $\mathcal{R}_\lambda(A) := (A - \lambda I)^{-1}$, $\lambda \neq \lambda_k$ is a compact operator in each space $\mathcal{F}_\alpha$, where I is the identity operator.

According to (25) we can define an exponential operator e^{-tA}, $t \geq 0$, in the scale spaces $\{\mathcal{F}_\alpha\}$. Note some of its properties:

a. For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator e^{-tA} maps $\mathcal{F}_\alpha$ into $\bigcap_{\beta \geq 0} \mathcal{F}_\beta$ and

$$|e^{-tA}x|_\alpha \leq e^{-\lambda_1 t}|x|_\alpha$$

for all $x \in \mathcal{F}_\alpha$.

b. $e^{-t_1 A}e^{-t_2 A} = e^{-(t_1+t_2)A}$ for all $t_1, t_2 \in \mathbb{R}_+$;

c.

$$|e^{-tA}x - e^{-\tau A}x|_\beta \to 0$$

as $t \to \tau$ for every $x \in \mathcal{F}_\beta$ and $\beta \in \mathbb{R}$;

d. For any $\beta \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\beta, e^{-tA})$;

e.

$$|A^\alpha e^{-tA}h| \leq \left[\left(\frac{\alpha-\beta}{t}\right)^{\alpha-\beta} + \lambda_1^{\alpha-\beta}\right]e^{-t\lambda_1}|A^\beta h|, \ \alpha \geq \beta \tag{27}$$

$$||A^\alpha e^{-tA}|| \leq \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}, \ t > 0, \ \alpha > 0.$$

Consider an evolutionary differential equation

$$u' + Au = F(t, u) \tag{28}$$

in the separable Hilbert space H, where A is a linear (generally speaking unbounded) positive operator with discrete spectrum, and F is a nonlinear continuous mapping acting from $\mathbb{R} \times \mathcal{F}_\theta$ into H, $0 \leq \theta < 1$, possessing the property

$$|F(t, u_1) - F(t, u_2)| \leq L(r)|A^\theta(u_1 - u_2)| \tag{29}$$

for all $u_1, u_2 \in B_\theta(0, r) := \{u \in \mathcal{F}_\theta : \ |u|_\theta \leq r\}$. Here $L(r)$ denotes the Lipschitz constant of F on the set $B_\theta(0, r)$.

A function $u : [0, a) \mapsto \mathcal{F}_\theta$ is said to be a *mild solution* (in $\mathcal{F}_\theta$) of equation (28) passing through the point $x \in \mathcal{F}_\theta$ at moment $t = 0$ (notation $\varphi(t, x, F)$) if $u \in C([0, T], \mathcal{F}_\theta)$ and satisfies the integral equation

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A} F(\tau, u(\tau)) d\tau$$

for all $t \in [0, T]$ and $0 < T < a$.

Under the conditions listed above, there exists a unique solution $\varphi(t, x, F)$ of equation (29) passing through the point x at moment $t = 0$, and

it is defined on a maximal interval $[0, a)$, where a is some positive number depending on (x, F).

Denote by $C(\mathcal{R} \times \mathcal{F}_\theta, H)$ the space of all continuous mappings equipped with the compact open topology and by $(C(\mathcal{R} \times \mathcal{F}_\theta, H), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathcal{R} \times \mathcal{F}_\theta, H)$.

A function $F \in C(\mathbb{R} \times \mathcal{F}_\theta, H)$ is said to be *regular* if for all $v \in \mathcal{F}_\theta$ and $G \in H^+(F)$, where by bar is denote the closure in the space $C(\mathbb{R} \times \mathcal{F}_\theta, H)$, there exists a unique (mild) solution $\varphi(t, v, G)$ of equation

$$u' + Au = G(t, u) \tag{30}$$

defined on $\mathbb{R}_+$ and passing through the point v at moment. Denote by $(H^+(F), \mathbb{R}_+, \sigma)$ a shift dynamical system on $H^+(F)$ induced by $(C(\mathcal{R} \times \mathcal{F}_\theta, H), \mathbb{R}, \sigma)$. From general properties of solutions of evolution equation (28) it follows that the triplet $\langle \mathcal{F}_\theta,\ \varphi,\ (H^+(F),\ \mathbb{R}_+,\ \sigma) \rangle$ is a cocycle over $(H^+(F), \mathbb{R}_+, \sigma)$ with the fiber $\mathcal{F}_\theta$.

Lemma 18 *Under the conditions listed above, if the function F is regular and the set $H^+(F)$*

is compact, then the cocycle φ associated by equation (28) is completely continuous.

Theorem 19 *Assume that the following conditions are fulfilled:*

1. *the function F is regular;*

2. *the set $H^+(F)$ is compact;*

3. *$F(t,0) = 0$ for all $t \in \mathbb{R}_+$;*

4. *there exists a positive number a such that*

$$\lim_{t \to +\infty} \sup_{|v| \leq a, G \in \Omega_F} |\varphi(t, v, G)| = 0. \qquad (31)$$

Then the null solution of equation (28) is uniformly asymptotically stable.

Theorem 20 *Let $F \in C(\mathbb{R} \times \mathcal{F}_\theta, H)$. Under the conditions of Theorem 19 the null solution of equation (28) is globally asymptotically stable if and only if the following conditions hold:*

1.

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0$$

for every $a > 0$*;*

2. *for every* $v \in \mathcal{F}_\theta$ *and* $G \in H^+(F)$ *the solution* $\varphi(t,v,G)$ *of equation (30) is bounded on* $\mathbb{R}_+$*.*

Theorem 21 *Suppose that the following conditions are fulfilled:*

1. *the function* $F \in C(\mathbb{R}\times\mathcal{F}_\theta, H)$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact subset from* $W \subseteq \mathcal{F}_\theta$*;*

2. $F(t,0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

3. *the function* F *is regular;*

4. *the null solution of equation (28) is uniformly stable;*

5. there exists a positive number a such that

$$\lim_{t \to +\infty} \sup_{|u| \leq a} |\varphi(t, u, F)| = 0.$$

Then the null solution of equation (28) is asymptotically stable.

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