G. Sell's Conjecture for Non-autonomous Dynamical Systems

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The aim of this talk is the study the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles).

Consider a differential equation

$$x' = f(t,x) \quad f \in C(\mathbb{R}\times W, \mathbb{R}^n), \tag{1}$$

where $\mathbb{R} := (-\infty,+\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, W is an open subset from $\mathbb{R}^n$ containing the origin (i.e., $0\in W$), $C(\mathbb{R}\times W,\mathbb{R}^n)$ is the space of all continuous functions $f:\mathbb{R}\times W\mapsto \mathbb{R}^n$ equipped with compact open topology. This topology is defined by the following distance

$$\rho(f,g) := \sum_{k=1}^{+\infty}\frac{1}{2^k}\frac{\rho_k(f,g)}{1+\rho_k(f,g)}, \tag{2}$$

where $\rho_k(f,g) := \max\{|f(t,x)-g(t,x)| :\ (t,x)\in[-k,k]\times W_k\}$, where $\{W_k\}$ is a family of compact subsets from W with the properties: $W_k\subset W_{k+1}$ for all $k\in\mathbb{N}$ and $\bigcup_{k=1}^{+\infty} W_k = W$ and $|\cdot|$ is a norm on $\mathbb{R}^n$. Denote by $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$ the shift dynamical system on the space $C(\mathbb{R}\times W,\mathbb{R}^n)$ (dynamical system of translations or Bebutov's dynamical system), i.e., $\sigma(\tau,f) := f_\tau$ for all $\tau\in\mathbb{R}$ and $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$, where $f_\tau(t,x) := f(t+\tau,x)$ for all $(t,x)\in\mathbb{R}\times W$.

Below we will use the following conditions:

(A): for all $(t_0,x_0)\in\mathbb{R}_+\times W$ the equation (1) admits a unique solution $x(t;t_0,x_0)$ passing through point x_0 at the moment t_0 and defined on $\mathbb{R}_+$, i.e., $x(t_0;t_0,x_0)=x_0$, where $\mathbb{R}_+ := [0,+\infty)$;

(B): the hand right side is positively compact, ifthe set $\Sigma_f^+ := \{f_\tau :\ \tau\in\mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R}\times W,\mathbb{R}^n)$;

(C): the equation
$$y' = g(t,y) \quad g\in\Omega_f \tag{3}$$
is called a limiting equation for (1), where Ω_f is the ω-limit set of f with respect to shift dynamical system $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$, i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\}$ such that $f_{\tau_k}\to g$ as $k\to\infty\}$;

(D): equation (1) is regular (or its hand right side f), if for all $p\in H^+(f)$ the equation
$$y' = p(t,y) \tag{4}$$
admits a unique solution $\varphi(t,x_0,p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0,x_0,p)=x_0$ for all $x_0\in W$, where $H^+(f) := \overline{\{f_\tau :\ \tau\in\mathbb{R}_+\}}$ and by bar is denote the closure in the space $C(\mathbb{R}\times W,\mathbb{R}^n)$;

(E): equation (1) admits a null (trivial) solution, i.e., $f(t,0)=0$ for all $t\in\mathbb{R}_+$;

(F): a function f satisfies to local (respectively, global) Lipschitz condition, if there exists a function $L:\mathbb{R}_+\mapsto\mathbb{R}_+$ (respectively, a positive constant L) such that

$$|f(t,x_1)-f(t,x_2)|\le L(r)|x_1-x_2| \tag{5}$$

(respectively, $|f(t,x_1)-f(t,x_2)|\le|x_1-x_2|$) for all $t\in\mathbb{R}_+$ and $x_1,x_2\in W$ with $|x_1|,|x_2|\le r$ for all $r>0$ (respectively, for all $x_1,x_2\in W$).

The trivial solution of equation (1) is said to be:

1. uniformly stable, if for all positive number ε there exists a number $\delta=\delta(\varepsilon)$ ($\delta\in(0,\varepsilon)$) such that $|x|<\delta$ implies $|\varphi(t,x,f_\tau)|<\varepsilon$ for all $t,\tau\in\mathbb{R}_+$;

2. uniformly attracting, if there exists a positive number a

$$\lim_{t\to+\infty}|\varphi(t,x,f_\tau)|=0 \tag{6}$$

uniformly with respect to $|x|\le a$ and $\tau\in\mathbb{R}_+$;

3. uniformly asymptotically stable, if it is uniformly stable and uniformly attracting.

Theorem 1. (***Sell, 1971***) *Let $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$ be a regular function and f is positively pre-compact. If the trivial solution of (1) is uniformly uniformly asymptotically stable, then the following statements hold:*

1. for every $\varepsilon>0$ there exists a $\delta(\varepsilon)\in(0,\varepsilon)$ such that $|x|<\delta$ implies $|\varphi(t,x,g)|<\varepsilon$ for all $t\in\mathbb{R}_+$ and $g\in H^+(f)$;

2. there exists a positive number a such that

$$\lim_{t\to+\infty}|\varphi(t,x,g)|=0 \tag{7}$$

uniformly with respect to $|x|\le a$ and $g\in H^+(f)$.

Theorem 2. (***Sell, 1971***) *Let $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$ be a regular function and f is positively pre-compact. Assume further that*

1. the trivial solution of (1) is uniformly stable;

2. there exists a positive constant a such that equality (7) takes place uniformly with respect to $|x|\le a$ and $g\in\Omega_f$.

Then the trivial solution of equation (1) is uniformly asymptotically stable.

G. Sell's conjecture (***G. Sell, Ch. VIII,p.134***). Let $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$ be a regular function and f be positively pre-compact. Assume that W contains the origin 0 and $f(t,0)=0$ for all $t\in\mathbb{R}_+$. Assume further that there exists a positive number a such that the equality (7) takes place uniformly with respect to $|x|\le a$ and $g\in\Omega_f$. Then the trivial solution of (1) is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Z. Artstein (1978) and Bondi P. et all (1977).

Remark 1. Bondi P. et all (1977) proved this conjecture under the additional assumption that the function f is local Lipschitzian.

2. Artstein Z. (1978) proved this statement without Lipschitzian condition. In reality he proved a small more general affirmation. Namely, he supposed that only limiting equations for (1) are regular, but the function f is not obligatory regular.

In this paper we will formulate G. Sell's conjecture for the abstract non-autonomous dynamical systems (the both with continuous and discrete time). We will give a positive answer to this conjecture and we will apply this result to different classes of evolution equations: infinite-dimensional differential equations, functional-differential equations and difference equations.

References:

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3. Bondi P., Moaurd V. and Visentin F., *Limiting equations in the stability problem.* Nonlinear Analysis, **1 (2)** (1977), p.123–128.