ON THE STRUCTURE OF THE GLOBAL ATTRACTOR FOR NON-AUTONOMOUS DIFFERENCE EQUATIONS WITH WEAK CONVERGENCE.

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Abstract. The aim of this paper is to describe the structure of global attractors for non-autonomous difference systems of equations with recurrent (in particular, almost periodic) coefficients. We consider a special class of this type of systems (the so–called weak convergent systems). We study this problem in the framework of general non-autonomous dynamical systems (cocycles). We apply the general results obtained in our early papers to study the almost periodic (almost automorphic, recurrent) and asymptotically almost periodic (asymptotically almost automorphic, asymptotically recurrent) solutions of difference equations.

Dedicated to Francisco Balibrea on occasion of his 60th birthday

1. Introduction

Denote by $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ the space of all continuous functions $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ equipped with the compact-open topology, and by $|\cdot|$ the norm in $\mathbb{R}^n$.

Consider a system of differential equations

$$x' = f(t,x), \tag{1}$$

where $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the right-hand side of (1) satisfies hypotheses ensuring the existence, uniqueness and extendability of solutions of (1), i.e., for all $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^n$ there exists a unique solution $x(t;t_0,x_0)$ of (1) with initial data t_0,x_0, and defined for all $t\geq t_0$.

Recall (see, for example, [8, 18]) that equation (1) is said to be uniformly dissipative (or uniformly ultimately bounded) if there exists a number $r_0>0$ so that, for every $r>0$, there is $L(r)>0$ such that if $|x_0|\leq r$, then $|x(t;t_0,x_0)|\leq r_0$ if $t\geq t_0+L(r)$.

In this framework, we aim to provide some results on an interesting classical question which is due to Seifert. It is described in the following way:

Date: April 11, 2011.

1991 *Mathematics Subject Classification.* primary:34A24,34A30,34B55,34C11,34C27,37C55, 37C60, 37C65,37C70,37C75,34D05,34D20,34D23,34D45.

Key words and phrases. Non-autonomous dynamical systems; skew-product systems; cocycles; global attractor; dissipative systems; convergent systems; quasi-periodic, almost periodic, almost automorphic, recurrent solutions; asymptotically almost periodic solutions; difference equations.

Problem (G. Seifert [9]): Suppose that (1) is uniformly dissipative and the function f is almost periodic (with respect to the time variable t). Does equation (1) possess an almost periodic solution?

Fink and Fredericson [9] and Zhikov [19] established that, in general, even when (1) is scalar ($n = 1$), the answer to Seifert's question is negative.

However, in view of this negative general answer, there are still several aspects which can be analyzed and provide useful information on this problem. On the one hand, it would be very interesting to investigate the existence of certain classes of dissipative differential equations for which the response to Seifert's question is affirmative. And, on the other hand, one could be interested in finding out some additional assumptions ("optimal" if possible) which guarantee the existence of at least one almost periodic solution (see [2] for a short description of already published results concerning these questions).

In our earlier articles [1, 2], we have proved some partial results on these problems. However, our main aim in the present paper is to analyze the same kind of questions but for dissipative almost periodic difference equations. To be more precise, we will investigate Seifert's problem for the following difference equation

$$u(t+1) = f(t, u(t)), \ \ (t \in \mathbb{Z}_+, \ \ u \in \mathbb{R}^n) \tag{2}$$

where $\mathbb{Z}$ (respectively, $\mathbb{Z}_+$) is the set of all entire (respectively, entire nonnegative) numbers and $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$, with almost periodic coefficients.

We show that, in general, the answer to Seifert's question for (2) is negative, even in the scalar case, i.e. when $n = 1$. We prove this claim by constructing an appropriate counterexample.

Additionally, we prove a positive answer for a special class of difference equations. To that end, we impose stronger assumptions on the right-hand side of our difference equation, and introduce the so–called equations (2) with *weak convergence*, for which we prove that the response to Seifert's question is affirmative.

We present our results within the framework of general non-autonomous dynamical systems (cocycles) and we apply our abstract theory already developed in some previous papers (see, e.g. [1, 2] and the references therein) to study some classes of difference equations.

The paper is organized as follows.

In Section 2, we briefly recall some notions (global attractor, minimal set, point/compact dissipativity, non-autonomous dynamical systems with convergence, Levitan/Bohr almost periodicity, almost automorphy, recurrence, Poisson stability, etc) and facts from the theory of dynamical systems which will be necessary in this paper. We give here also some results concerning a special class of non-autonomous dynamical systems (NAS): the so-called NAS with weak convergence.

In Section 3, we analyze Seifert's Problem for non-autonomous difference equations. We first introduce the notion of discretization of a dynamical system with continuous time (a flow), and we establish some relations between the given flow and its discretization.

Next, in Section 3.2 we construct an example of a one-dimensional almost periodic dissipative difference equation of type (2) without almost periodic solutions, what can be interpreted as a negative answer to our problem under study.

However, we establish a positive response in Section 3.3, where we prove that an almost periodic dissipative difference equation (2) with weak convergence admits a unique almost periodic solution, and this solution, in general, is not the unique solution of this equation which is bounded on $\mathbb{Z}$.

Section 3.4 is devoted to the study of uniform compatible solutions of (2) by the character of recurrence (in the sense of B. A. Shcherbakov [14, 15, 16]). In this way we obtain some tests for the existence of periodic (respectively, almost periodic, almost automorphic, recurrent) solutions of equation (2) and also asymptotically periodic (respectively, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent) solutions.

Finally, in Section 3.5, we refine these previous results of Section 3.4 for a special class of equations of type (2).

2. Nonautonomous Dynamical Systems with Convergence and/or Weak Convergence

Let us start by recalling some concepts and notation about the theory of non-autonomous dynamical systems which will be necessary for our analysis. A more detailed analysis can be found, for instance, in [1, 2].

2.1. **Compact Global Attractors of Dynamical Systems.**

Let (X, ρ) be a metric space, $\mathbb{R}$ ($\mathbb{Z}$) be the group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be the semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of the additive group $\mathbb{S}$.

A *dynamical system* is a triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0, x) = x \ (\forall x \in X);$$

$$\pi(s, \pi(t, x)) = \pi(s + t, x) \ (\forall t, \tau \in \mathbb{T} \text{ and } x \in X).$$

The function $\pi(\cdot, x) : \mathbb{T} \to X$ is called a *motion* passing through the point x at the moment $t = 0$ and the set $\Sigma_x := \pi(\mathbb{T}, x)$ is called the *trajectory* of this motion.

A nonempty set $M \subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to the dynamical system $(X, \mathbb{T}, \pi)$ or, simply, positively invariant (negatively invariant, invariant), if $\pi(t, M) \subseteq M$ ($M \subseteq \pi(t, M), \pi(t, M) = M$) for every $t \in \mathbb{T}$.

A closed positively invariant set, which does not contain any own closed positively invariant subset, is called *minimal*.

It is easy to see that every positively invariant minimal set is invariant.

Let $M \subseteq X$. The set

$$\omega(M) := \bigcap_{t \ge 0} \overline{\bigcup_{\tau \ge t} \pi(\tau, M)}$$

is called the *ω-limit* of M.

The dynamical system $(X, \mathbb{T}, \pi)$ is called:

– *point dissipative* if there exists a nonempty compact subset $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(\pi(t, x), K) = 0; \tag{3}$$

– *compact dissipative* if the equality (3) takes place uniformly with respect to x in any compact subset of X.

Let $(X, \mathbb{T}, \pi)$ be compact dissipative and K be a compact set attracting every compact subset from X. Let us set

$$J_X := \omega(K) := \bigcap_{t \ge 0} \overline{\bigcup_{\tau \ge t} \pi(\tau, K)}. \tag{4}$$

It can be shown [3, Ch.I] that the set J_X defined by equality (4) does not depend on the choice of the attracting set K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J_X is called the *Levinson center* of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

More generally, a compact invariant set $J \subset X$ is called the Levinson center of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$ if J attracts every compact subset of X, which means that

$$\lim_{t \to +\infty} \rho(\pi(t, x), J) = 0,$$

uniformly with respect to $x \in M$, and for all compact subset M of X. It is worth noticing that this concept does not coincide, in general, with that of global attractor (since the latter attracts the bounded subsets of X). For a more detailed analysis on the relationship between these two concepts, see Cheban [3].

2.2. **Global attractor of cocycles.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}_1$).

A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ (i.e., h is continuous and $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$ and $x \in X$), is called a *non-autonomous dynamical system.*

Let $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system, W a complete metric space, and φ a continuous mapping from $\mathbb{T}_1 \times W \times Y$ into W, possessing the following properties:

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then, the triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ (or shortly φ) is called [17] *a cocycle* on $(Y, \mathbb{T}_2, \sigma)$ with fiber W.

Let $X := W \times Y$ and let us define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as follows: $\pi((u,y),t) := (\varphi(t,u,y), \sigma(t,y))$ (i.e., $\pi = (\varphi, \sigma)$). Then, it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a *skew-product dynamical system* [17] and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, hence, $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system. Note that pr_1 and pr_2 denote the projection mappings with respect to the first and second variables, i.e., $pr_1(w,y) = w$, $pr_2(w,y) = y$ for $(w,y) \in X$.

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma), h \rangle$ $(X := W \times Y)$, called non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential/difference equations. Under appropriate assumptions, every non-autonomous differential/difference equation generates a cocycle (a non-autonomous dynamical system). Several examples can be found, for instance, in [1].

A family $\{I_y \mid y \in Y\}$ $(I_y \subset W)$ of nonempty compact subsets of W is called (see, for example, [3]) *a compact pullback attractor* (*uniform pullback attractor*) of the cocycle φ, if the following conditions hold:

(i) the set $I := \bigcup \{I_y \mid y \in Y\}$ is relatively compact;
(ii) the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for all $t \in \mathbb{T}_+$ and $y \in Y$;
(iii) for all $y \in Y$ (uniformly in $y \in Y$) and $K \in \mathcal{C}(W)$

$$\lim_{t \to +\infty} \beta(\varphi(t, K, \sigma(-t, y)), I_y) = 0,$$

where $\beta(A, B) := \sup\{\rho(a, B) : a \in A\}$ is the Hausdorff semi-distance, and $\mathcal{C}(W)$ denotes de family of compact subsets of W.

Below in this subsection we suppose that $\mathbb{T}_2 = \mathbb{S}$.

A family $\{I_y \mid y \in Y\}(I_y \subset W)$ of nonempty compact subsets is called a *compact global attractor* of the cocycle φ, if the following conditions are fulfilled:

(i) the set $I := \bigcup \{I_y \mid y \in Y\}$ is relatively compact;
(ii) the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ;
(iii) the equality

$$\lim_{t \to +\infty} \sup_{y \in Y} \beta(\varphi(t, K, y), I) = 0$$

holds for every $K \in \mathcal{C}(W)$.

Let $M \subseteq W$ and

$$\omega_y(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M, \sigma(-\tau, y))}$$

for all $y \in Y$.

A cocycle φ over $(Y, \mathbb{S}, \sigma)$ with fiber W is said to be *compact dissipative*, if there exits a nonempty compact $K \subseteq W$ such that

$$\lim_{t \to +\infty} \sup\{\beta(\varphi(t, M, y), K) \mid y \in Y\} = 0 \tag{5}$$

for any $M \in \mathcal{C}(W)$.

Recall that a function $F \in C(\mathbb{T}, \mathbb{R})$ is said *to possess the* (S)*-property (for example, periodicity, almost periodicity, recurrence, asymptotically almost periodicity and so on)*, if the motion $\sigma(\tau, F)$, generated by the function F in the shift dynamical system $(C(\mathbb{T}, \mathbb{R}), \mathbb{T}, \sigma)$, possesses this property (see [3, Ch. II] for more details).

Then, we can now establish the following result which will be used in the proof of our main results in this paper.

Theorem 2.1. [3, ChII] *Let Y be compact, $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ compact dissipative, and K the nonempty compact subset of W appearing in (5). Then:*

1. $I_y = \omega_y(K) \neq \emptyset$, *is compact,* $I_y \subseteq K$ *and*
$$\lim_{t \to +\infty} \beta(\varphi(t, K, \sigma(-t, y)), I_y) = 0$$
for every $y \in Y$*;*
2. $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ *for all* $y \in Y$ *and* $t \in \mathbb{S}_+$*;*
3. $$\lim_{t \to +\infty} \beta(\varphi(t, M, \sigma(-t, y)), I_y) = 0$$
for all $M \in \mathcal{C}(W)$ *and* $y \in Y$ *;*
4. $$\lim_{t \to +\infty} \sup\{\beta(\varphi(t, M, \sigma(-t, y)), I) \mid y \in Y\} = 0$$
for any $M \in \mathcal{C}(W)$*, where* $I := \cup\{I_y \mid y \in Y\}$*;*
5. $I_y = pr_1 J_y$ *for all* $y \in Y$*, where* J *is the Levinson center of* $(X, \mathbb{T}_+, \pi)$*, and hence* $I = pr_1 J$*;*
6. *the set* I *is compact;*
7. *the set* I *is connected if one of the next two conditions is fulfilled:*
   (a) $\mathbb{S}_+ = \mathbb{R}_+$ *and the spaces* W *and* Y *are connected;*
   (b) $\mathbb{S}_+ = \mathbb{Z}_+$ *and the space* $W \times Y$ *possesses the* (S)*-property or it is connected and locally connected.*

2.3. **Non-Autonomous Dynamical Systems with Convergence and Weak Convergence.** First, let us recall (see [3]) that a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is said to be *convergent* if the following conditions are fulfilled:

(i) the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative;
(ii) the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$, where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$).

Thus, a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent, if the systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively, and J_X has "trivial" sections, i.e., $J_X \bigcap X_y$ consists of a single point for all $y \in J_Y$. In this case, the Levinson center J_X of the dynamical system $(X, \mathbb{T}_1, \pi)$ is a copy (an homeomorphic image) of the Levinson center J_Y of the dynamical system $(Y, \mathbb{T}_2, \sigma)$. Thus, the dynamics on J_X is the same as on J_Y.

Before introducing the class of non-autonomous dynamical systems with weak convergence, let us recall some definition concerning the different recurrence properties of points and motions (see [5] for more details).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$*shift* (respectively, an $\varepsilon-$*almost period*) of the point $x \in X$, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where by bar we denote the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \text{ such that } \{\pi(t_n, x)\} \to x \text{ and } \{t_n\} \to \infty\}$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called *Levitan almost periodic* [12], if there exists a dynamical system $(Y, \mathbb{T}, \lambda)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

A point $x \in X$ is called *stable in the sense of Lagrange* (*st.*L for short), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called *almost automorphic* [12] for the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

- (i) x is st.L;
- (ii) there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \lambda)$, and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.2. *1. Every almost automorphic point $x \in X$ is also Levitan almost periodic.*

2. A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t, x) : t \in T\}$ is also almost automorphic. In other words, a Levitan almost periodic point x is almost automorphic, if and only if its trajectory $\{\pi(t, x) : \ t \in T\}$ is relatively compact.

3. Let (X, T, π) and (Y, T, λ) be two dynamical systems, $x \in X$ and the following conditions be fulfilled:

- (i) *a point $y \in Y$ is Levitan almost periodic;*
- (ii) $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Then, the point x is also Levitan almost periodic.

4. Let $x \in X$ be a st.L point, $y \in Y$ be an almost automorphic point and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then, the point x is almost automorphic too.

Recall [5] that a point $x \in X$ is called *asymptotically τ–periodic* (respectively, *asymptotically Bohr almost periodic*, *asymptotically recurrent*), if there exists a τ-periodic (respectively, Bohr almost periodic, recurrent) point $p \in X$ such that $\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,p)) = 0$.

Now, we introduce the concept of non-autonomous dynamical systems with weak convergence, which is very close to convergent systems, but possessing a non-trivial global attractor. This means that this class of non-autonomous systems will conserve almost all properties of convergent systems, but will have a "nontrivial" global attractor J_X, i.e., there exists at least one point $y \in J_Y$ such that the set $J_X \bigcap X_y$ contains more than one point.

A non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *weak convergent*, if the following conditions hold:

(i) the dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively;

(ii) it follows that
$$\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0,$$
for all $x_1, x_2 \in J_X$ with $h(x_1) = h(x_2)$.

Remark 2.3. It is clear that every convergent non-autonomous dynamical system is weak convergent. The inverse statement, generally speaking, is not true. See [2] for a counterexample confirming this statement.

3. Analysis of Seifert's problem for dissipative difference equations

We can now analyze the questions related to Seifert's problem mentioned in the Introduction of this paper. To be more precise, consider again the initial differential equation (1)
$$x' = f(t,x),$$
where $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the right-hand side of (1) satisfies hypotheses ensuring the existence, uniqueness and extendability of solutions of (1), i.e., for all $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^n$ there exists a unique solution $x(t;t_0,x_0)$ of (1) with initial data t_0, x_0, and defined for all $t \geq t_0$.

Equation (1) (respectively, the function f) is called *regular*, if for all $x_0 \in \mathbb{R}^n$ and $g \in H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ (where the bar denotes the closure in the space $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $f_\tau(t,x) := f(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$) the equation
$$x' = g(t,x)$$
has a unique solution $\varphi(t,x,g)$ passing through the point x_0 at the initial moment $t = 0$, and defined on $\mathbb{R}_+ := \{t \in \mathbb{R}|\ t \geq 0\}$.

Then, the following result holds.

Theorem 3.1. [3, ChII] *Suppose that $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is regular and $H(f)$ is a compact subset of $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Then, the following statements are equivalent:*

(i) *equation (1) is uniformly dissipative;*

(ii) *there exists a positive number R_0 such that*

$$\limsup_{t\to+\infty} |\varphi(t,x,g)| \le R_0 \tag{6}$$

for all $(x,g) \in \mathbb{R}^n \times H(f)$.

At light of Theorem 3.1, it is said that equation (1) is *dissipative* (in fact the family of equations (6) is collectively dissipative, but we use this shorter terminology) if (6) holds.

Remark 3.2. *Let $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{R},\sigma)\rangle$ ($Y = H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ and $(Y,\mathbb{R},\sigma)$ is the shift dynamical system on Y) be the cocycle generated by the differential equation (1). If $H(f)$ is a compact subset of $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$), then (1) is dissipative if and only if the cocycle φ, generated by (1), is compact dissipative.*

Now, before establishing and proving our main results about Seifert's problem in the case of difference equations (or discrete non-autonomous dynamical systems), we need some results on discretization which we will consider in the next subsection.

3.1. Discretization of NAS with continuous time.

Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$. Consider a non-autonomous dynamical system $\langle (X,\mathbb{T},\tilde{\pi}),\ (Y,\mathbb{T},\tilde{\sigma}),\ h\rangle$ with continuous time $\mathbb{T}$ and denote by $\mathcal{T} := \mathbb{T}\bigcap\mathbb{Z}$.

The non-autonomous dynamical system $\langle (X,\mathcal{T},\pi),(Y,\mathcal{T},\sigma),h\rangle$ with discrete time $\mathcal{T}$ is called the discretization of $\langle (X,\mathbb{T},\tilde{\pi}),(Y,\mathbb{T},\tilde{\sigma}),h\rangle$, if the following conditions are fulfilled:

(i) $\pi(t,x) = \tilde{\pi}(t,x)$ for all $(t,x) \in \mathcal{T}\times X$;
(ii) $\sigma(t,x) = \tilde{\sigma}(t,x)$ for all $(t,x) \in \mathcal{T}\times Y$.

Lemma 3.3. *Let $(X,\mathbb{R}_+,\tilde{\pi})$ be an autonomous dynamical system with continuous time $\mathbb{R}_+$ and $(X,\mathbb{Z}_+,\pi)$ be its discretization. Then, the following statements hold:*

(i) *If $\tilde{\gamma} : \mathbb{R} \mapsto X$ is an entire trajectory of $(X,\mathbb{R}_+,\tilde{\pi})$, then the mapping $\gamma : \mathbb{Z} \to X$ defined by the equality $\gamma(n) := \tilde{\gamma}(n)$ (for all $n\in\mathbb{Z}$) is an entire trajectory of $(X,\mathbb{Z}_+,\pi)$;*
(ii) *If $\gamma : \mathbb{Z} \mapsto X$ is an entire trajectory of $(X,\mathbb{Z}_+,\pi)$, then the mapping $\tilde{\gamma} : \mathbb{R} \mapsto X$ defined by the equality*

$$\tilde{\gamma}(t) := \tilde{\pi}(\{t\},\gamma([t]))\ (\forall\ \ t\in\mathbb{R}) \tag{7}$$

is an entire trajectory of $(X,\mathbb{R}_+,\tilde{\pi})$, where $[t]\in\mathbb{Z}$ is the entire part of the number t and $\{t\}\in[0,1)$ is its fractional part.

Proof. The first statement of Lemma is trivial. To prove the second statement we need to verify that the equality

$$\tilde{\gamma}(t+\tau) = \tilde{\pi}(t,\tilde{\gamma}(\tau)) \tag{8}$$

holds for all $t\in\mathbb{R}_+$ and $\tau\in\mathbb{R}$. Let now $t\in\mathbb{R}_+$, $\tau\in\mathbb{R}$ and $\tilde{\gamma} : \mathbb{R}\to X$ be the mapping defined by (7), then we have

$$\tilde{\pi}(t,\tilde{\gamma}(\tau)) = \tilde{\pi}([t]+\{t\},\tilde{\gamma}([\tau]+\{\tau\})) = \tilde{\pi}(\{t\}+\{\tau\},\gamma([t]+[\tau]). \tag{9}$$

Logically, two cases are possible:

1. $\{t\}+\{\tau\}\in[0,1)$: Then, from (9) it follows (8).

2. $\{t\}+\{\tau\}=1+r$, where $r\in[0,1)$ and, consequently, $r=\{t+\tau\}$ and $[t+\tau]=[t]+[\tau]+1$: Then, from (9) we have

$$
\begin{aligned}
\tilde{\pi}(t,\tilde{\gamma}(\tau)) &= \tilde{\pi}(\{t\}+\{\tau\},\gamma([t]+[\tau]) \\
&= \tilde{\pi}(r+1,\gamma([t]+[\tau])) \\
&= \tilde{\pi}(r,\gamma(1+[t]+[\tau])) \\
&= \tilde{\gamma}(t+\tau).
\end{aligned} \tag{10}
$$

Lemma is proved. □

Let $(X,\mathbb{R}_+,\tilde{\pi})$ be a semi-flow and $\tilde{\gamma}$ be an entire trajectory, then $\gamma:\mathbb{Z}\mapsto X$, defined by the equality $\gamma(n):=\tilde{\gamma}(n)$ for all $n\in\mathbb{Z}$ is called *discretization* of $\tilde{\gamma}$.

Lemma 3.4. *Suppose that the following conditions are fulfilled:*

(i) *$(X,\mathbb{R}_+,\tilde{\pi})$ is an autonomous dynamical system with continuous time $\mathbb{R}_+$ and $(X,\mathbb{Z}_+,\pi)$ is its discretization;*
(ii) *$\tilde{\gamma}:\mathbb{R}\mapsto X$ is an entire trajectory of $(X,\mathbb{R}_+,\tilde{\pi})$ and γ is its discretization.*

Then the trajectory $\tilde{\gamma}$ is almost periodic with respect to $(X,\mathbb{R}_+,\tilde{\pi})$ if and only if γ is almost periodic with respect to $(X,\mathbb{Z}_+,\pi)$.

Proof. Let $\tilde{\gamma}$ be an almost periodic motion of $(X,\mathbb{R}_+,\tilde{\pi}))$ and $\{n_k^{'}\}\subseteq\mathbb{Z}$ be an arbitrary sequence of entire numbers. Consider the sequence $\{\gamma(n+n_k^{'})\}_{n\in\mathbb{Z}}$. Since the trajectory $\tilde{\gamma}$ is almost periodic with respect to $(X,\mathbb{R}_+,\tilde{\pi})$, then from the functional sequence $\{\tilde{\gamma}(t+n_k^{'})\}$ $(t\in\mathbb{R})$ we can extract a subsequence $\{\tilde{\gamma}(t+n_k)\}$ $(t\in\mathbb{R})$ which is uniformly convergent with respect to $t\in\mathbb{R}$. In particular we have

$$
\sup_{t\in\mathbb{R}}\rho(\tilde{\gamma}(t+n_k),\tilde{\gamma}(t+n_l))\to 0 \tag{11}
$$

as $k,l\to+\infty$. Taking into account (11) we have

$$
\begin{aligned}
\sup_{t\in\mathbb{Z}}\rho(\gamma(t+n_k),\gamma(t+n_l)) &= \sup_{t\in\mathbb{Z}}\rho(\tilde{\gamma}(t+n_k),\tilde{\gamma}(t+n_l)) \\
&\le \sup_{t\in\mathbb{R}}\rho(\tilde{\gamma}(t+n_k),\tilde{\gamma}(t+n_l))\to 0
\end{aligned} \tag{12}
$$

as $k,l\to+\infty$. Since the space X is complete, then the sequence $\{\gamma(n+n_k^{'})\}_{n\in\mathbb{Z}}$ converges uniformly with respect to $n\in\mathbb{Z}$ and, consequently, $\gamma:\mathbb{Z}\mapsto X$ is almost periodic.

As for the converse statement, let $\gamma:\mathbb{Z}\mapsto X$ be an almost periodic motion of $(X,\mathbb{Z}_+,\pi)$ and $\varepsilon>0$ an arbitrary positive number. We choose a number $\delta=\delta(\varepsilon)>0$ from the integral continuity of the dynamical system $(X,\mathbb{R}_+,\tilde{\pi})$, i.e., such that $\rho(x_1,x_2)<\delta$ implies $\rho(\tilde{\pi}(t,x_1),\tilde{\pi}(t,x_2))<\varepsilon$, for all $t\in[0,1]$. For the number δ, we can choose a relatively dense subset $\mathcal{P}_\delta\subseteq\mathbb{Z}$ such that

$$
\rho(\gamma(n+\tau),\gamma(n))<\delta \tag{13}
$$

for all $n\in\mathbb{Z}$ and $\tau\in\mathcal{P}_\delta$. Then we have

$$
\rho(\tilde{\gamma}(t+\tau),\tilde{\gamma}(t))=\rho(\tilde{\pi}(\{t\},\gamma([t]+\tau)),\tilde{\pi}(\{t\},\gamma([t]))<\varepsilon \tag{14}
$$

because $\rho(\gamma([t]+\tau),\gamma([t])) < \delta$ for all $t \in \mathbb{R}$ and $\tau \in \mathcal{P}_\delta \subseteq \mathbb{Z} \subseteq \mathbb{R}$. Thereby, for arbitrary $\varepsilon > 0$ we find a relatively dense subset $\mathcal{P}_{\delta(\varepsilon)} \subseteq \mathbb{R}$ of ε–almost periods of the motion $\tilde{\gamma}$. This means its almost periodicity, and the lemma is proved. □

Remark 3.5. *Note that Lemma 3.4 is close to Theorem 1.27* [7, ChI,p.47]*, but does not follow from this statement.*

Corollary 3.6. *Let $(X, \mathbb{R}_+, \tilde{\pi})$ be a semi-flow without almost periodic motions. Then, its discretization $(X, \mathbb{Z}_+, \pi)$ does not possess any almost periodic motion.*

3.2. **A negative answer: One-dimensional almost periodic dissipative difference equation without almost periodic solutions.** Now we investigate the following interesting question.

Problem.(*Seifert's problem for almost periodic dynamical systems*) Suppose that the following conditions are fulfilled:

- (i) $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is a compact dissipative cocycle with continuous ($\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$) or discrete ($\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{Z}$) time;
- (ii) $(Y, \mathbb{T}, \pi)$ is an almost periodic minimal set.

Does the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$, generated by the cocycle φ ($X = W \times Y$, $\pi = (\varphi, \sigma)$ and $\mathbb{T}_+ := \{t \in \mathbb{T} : \ t \geq 0\}$) possess an almost periodic motion?

Fink and Fredericson [9] and Zhikov [19] established that, in general, even when (1) is scalar, the answer to Seifert's question is negative. Namely, in these works they constructed a differential equation

$$x' = f(\tilde{\sigma}(t, y), x), \quad (x \in \mathbb{R},\ y \in \mathcal{T}^2) \tag{15}$$

where $\mathcal{T}^2$ is a two-dimensional torus, $(\mathcal{T}^2, \mathbb{R}, \tilde{\sigma})$ is an almost periodic minimal dynamical system (in fact this is an irrational winding of a two-dimensional torus $\mathcal{T}^2$) and $f \in C(\mathcal{T}^2 \times \mathbb{R}, \mathbb{R})$ with the following properties:

- (i) the cocycle generated by (15) is dissipative;
- (ii) the skew-product dynamical system generated by (15) does not possess any almost periodic motions.

Below we will prove that there exists a one-dimensional ($W = \mathbb{R}$) almost periodic compact dissipative cocycle φ with discrete time for which the corresponding skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ does not have any almost periodic motions. To this end we denote by $\langle \mathbb{R}, \tilde{\varphi}, (\mathcal{T}^2, \mathbb{R}, \tilde{\sigma})\rangle$ the cocycle generated by (15). Then we have

$$\tilde{\varphi}(t, u, y) = u + \int_0^t f(\tilde{\sigma}(\tau, y), \tilde{\varphi}(\tau, u, y))d\tau. \tag{16}$$

Denote by $(\mathcal{T}^2, \mathbb{Z}, \sigma)$ the discretization of the dynamical system $(\mathcal{T}^2, \mathbb{Z}, \sigma)$, and let $y_0 \in \mathcal{T}^2$ be an arbitrary point. Then, by Lemma 3.4, the trajectory of the point y_0 is almost periodic and, consequently, the set $Y = H(y_0) := \overline{\{\sigma(n, y_0) : \ n \in \mathbb{Z}\}}$ is a compact minimal set of the dynamical system $(\mathcal{T}^2, \mathbb{Z}, \sigma))$ consisting of almost periodic motions. Now, we define a mapping $\varphi : \mathbb{Z} \times \mathbb{R} \times Y \to \mathbb{R}$ by the equality $\varphi(n,u,y) := \tilde{\varphi}(n,u,y)$ and, consequently, we obtain

$$\varphi(n,u,y) = u + \int_0^1 f(\tilde{\sigma}(\tau,\sigma(n-1,y)),\tilde{\varphi}(\tau,\varphi(n-1,u,y),\sigma(n-1,y)))d\tau. \tag{17}$$

Thus, $\varphi(n,u,y)$ is a solution of the difference equation

$$y(t+1) = F(\sigma(t,y),y(t)), \quad (t \in \mathbb{Z},\ y \in Y = H(y_0)) \tag{18}$$

where $F \in C(Y \times \mathbb{R}, \mathbb{R})$ is defined by

$$F(y,u) := u + \int_0^1 f(\tilde{\sigma}(\tau,y),\tilde{\varphi}(\tau,u,y))d\tau. \tag{19}$$

Since the cocycle $\tilde{\varphi}$, generated by equation (15), is dissipative by its construction, then the cocycle $\langle \mathbb{R}, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ is also dissipative. Now, consider the non-autonomous dynamical system $\langle (X,\mathbb{Z},\pi),(Y,\mathbb{Z},\sigma),h\rangle$ generated by the cocycle φ (i.e., $X := \mathbb{R} \times Y$, $\pi := (\varphi,\sigma)$ and $h = pr_2 : X \to Y$). Note that the dynamical system $(X,\mathbb{Z},\pi)$ does not possess any almost periodic motion. In fact, if we suppose that it is not true, then there exists a point $(\bar{u},\bar{y}) \in X = \mathbb{R} \times Y$ such that the motion $(\varphi(n,\bar{u},\bar{y}),\sigma(n,\bar{y}))$ is almost periodic with respect to the dynamical system $(X,\mathbb{Z},\pi)$ and, consequently, with respect to the dynamical system $(\tilde{X},\mathbb{Z},\sigma)$ (where $\tilde{X} := \mathbb{R} \times \mathcal{T}^2$) because $(X,\mathbb{Z},\pi)$ is a subsystem of $(\tilde{X},\mathbb{Z},\sigma)$. Finally, we note that $(\tilde{X},\mathbb{Z},\sigma))$ is the discretization of the skew-product dynamical system generated by (15). The obtained contradiction proves our statement. Thus (18) is a one-dimensional almost periodic dissipative difference equation without almost periodic solutions.

3.3. **A positive answer: Almost periodic solutions of almost periodic dissipative difference equations.** Let $(Y,\mathbb{Z}_+,\sigma)$ be a dynamical system on the metric space Y. In this subsection we prove a positive answer to Seifert's Problem assuming additional assumptions on our difference equations. First, we suppose that Y is a compact space. Consider a difference equation

$$u(t+1) = f(\sigma(t,y),u(t)) \quad (t \in \mathbb{Z}_+,\ y \in Y), \tag{20}$$

where $f \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$.

Similarly as for differential equations, the non-autonomous difference equation (20) is said to be [3] *dissipative*, if there exists a positive number r such that

$$\limsup_{t\to+\infty} |\varphi(t,u,y)| < r$$

for all $u \in \mathbb{R}^n$ and $y \in Y$, where $|\cdot|$ is a norm on $\mathbb{R}^n$, and $\varphi(t,u,y)$ is the unique solution of (20) passing through $u \in \mathbb{R}^n$ at the initial moment $t = 0$.

Below we provide a simple geometric condition which guarantees existence of a unique almost periodic solution, and this solution, generally speaking, is not the unique solution of (20) which is bounded on $\mathbb{Z}$.

Theorem 3.7. *Suppose that the following conditions are fulfilled:*

(i) *equation (20) is dissipative;*
(ii) *the space Y is compact, and the dynamical system $(Y,\mathbb{Z}_+,\sigma)$ is minimal;*

(iii) *for all* $y \in Y$

$$\lim_{t\to+\infty} |\varphi(t,u_1,y) - \varphi(t,u_2,y)| = 0, \tag{21}$$

where $\varphi(t,u_i,y)$ $(i = 1,2)$ *is the solution of equation (20) passing through* u_i *at the initial moment* $t = 0$*, which is bounded on* $\mathbb{Z}$*.*

Then,

(i) *if the point* y *is* τ*–periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (20) admits a unique* τ*–periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution* $\varphi(t,u_y,y)$ $(u_y \in \mathbb{R}^n)$*;*
(ii) *every solution* $\varphi(t,x,y)$ *is asymptotically* τ*–periodic (respectively, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent)*

Proof. Let $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{Z}_+,\sigma)\rangle$ be the cocycle associated to equation (20). Denote by $(X,\mathbb{Z}_+,\pi)$ the skew-product dynamical system, where $X := \mathbb{R}^n \times Y$ and $\pi := (\varphi,\sigma)$ (i.e., $\pi(t,(u,y)) := (\varphi(t,u,y),\sigma(t,y))$ for all $x := (u,y) \in \mathbb{R}^n \times Y$ and $t \in \mathbb{Z}_+$). Consider the non-autonomous dynamical system $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z}_+,\sigma),h\rangle$ generated by the cocycle φ (respectively, by (20), where $h := pr_2 : X \to Y$). Since Y is compact, it is evident that the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compact dissipative and its Levinson center J_Y coincides with Y. By Theorem 2.23 in [3], the skew-product dynamical system $(X,\mathbb{Z}_+,\pi)$ is compact dissipative. Denote by J_X its Levinson center and by $I_y := pr_1(J_X \bigcap X_y)$ for all $y \in Y$, where $X_y := \{x \in X :\ h(x) = y\}$. According to the definition of the set $I_y \subseteq \mathbb{R}^n$ and by Theorem 2.1, $u \in I_y$ if and only if the solution $\varphi(t,u,y)$ is defined on $\mathbb{Z}$ and bounded (i.e., the set $\overline{\varphi(\mathbb{Z},u,y)} \subseteq \mathbb{R}^n$ is compact). Thus, $I_y = \{u \in \mathbb{R}^n : \text{such that } (u,y) \in J_X\}$. It is easy to see that condition (21) means that the non-autonomous dynamical system $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z}_+,\sigma),h\rangle$ is weak convergent. To finish the proof, it is sufficient to apply Lemma 3.2 and Corollary 3.10 in [2] for the non-autonomous system $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z}_+,\sigma),h\rangle$ generated by (20). □

Remark 3.8. Under the assumptions of Theorem 3.7, there exists a unique almost periodic solution of (20), but (20) has, in general, more than one solution defined and bounded on $\mathbb{Z}$. Below we will give an example which confirms this statement.

Example 3.9. In our paper [1] we proved that the following system of almost periodic differential equations

$$\begin{cases} u' = \frac{(u-\sin t)^2(2v-u+2\sin\sqrt{2}t-\sin t)+2(v-\sin\sqrt{2}t)^5}{((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)[1+((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)^2]} + \cos t \\ \\ v' = \frac{8(v-\sin\sqrt{2}t)^2(v-u+\sin\sqrt{2}t-\sin t)}{((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)[1+((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)^2]} + \sqrt{2}\cos\sqrt{2}t \end{cases} \tag{22}$$

possesses the following properties:

(i) system (22) is dissipative;
(ii) the non-autonomous dynamical system $\langle (X,\mathbb{R},\tilde{\pi}),(\tilde{Y},\mathbb{R},\tilde{\sigma}),h\rangle$, generated by (22), is weak convergent;
(iii) system (22) has a unique almost periodic solution;

(iv) system (22) has more than one solution defined and bounded on $\mathbb{R}$.

Now using this example and arguing as in Section 3.1 we can prove that the discretization $\langle (X,\mathbb{Z},\pi),(Y,\mathbb{Z},\sigma),h\rangle$ of the dynamical system $\langle (X,\mathbb{R},\tilde{\pi}),(\tilde{Y},\mathbb{R},\tilde{\sigma}),h\rangle$ (more exactly the subsystem $\langle (X,\mathbb{R},\tilde{\pi}),(Y,\mathbb{R},\tilde{\sigma}),h\rangle$ of this system, where $Y := H(y_0)$ and y_0 is some point from $\tilde{Y}$) possesses the necessary properties. Namely,

(i) the dynamical system $(Y,\mathbb{Z},\sigma)$ is almost periodic;
(ii) $\langle (X,\mathbb{Z},\pi),(Y,\mathbb{Z},\sigma),h\rangle$ is compact dissipative and weak convergent;
(iii) the Levinson center J_X of the dynamical system $(X,\mathbb{Z},\pi)$ contains a unique almost periodic minimal set $M \subseteq J_X$;
(iv) $J_X \neq M$.

3.4. **Uniform compatible solutions of strict dissipative equations.** In this subsection we consider our difference equation (20) when the driving system $(Y,\mathbb{Z},\sigma)$ is recurrent, and the function $f \in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ is *strictly contracting* with respect to its second variable $x \in \mathbb{R}^n$, i.e.,

$$|f(y,u_1)-f(y,u_2)| < |u_1-u_2| \tag{23}$$

for all $u_1,u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$) and $y \in Y$.

Lemma 3.10. *Suppose that the function $f \in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ is strictly contracting with respect to $x \in \mathbb{R}^n$, then*

$$|\varphi(t,u_1,y)-\varphi(t,u_2,y)| < |u_1-u_2| \tag{24}$$

for all $u_1,u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$), $t \geq 1$ and $y \in Y$.

Proof. We will prove inequality (24) by the method of mathematical induction. It is easy to check inequality (24) for $n = 1$. Indeed, from (23) we have

$$|\varphi(1,u_1,y)-\varphi(1,u_2,y)| = |f(y,u_1)-f(y,u_2)| < |u_1-u_2| \tag{25}$$

for all $u_1,u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$) and $y \in Y$. Suppose now that (24) is true for all $1 \leq n \leq m$ and we will show that then it is also true for $n = m+1$. Indeed, by (25) and the inductive assumption we have

$$\begin{aligned} &|\varphi(m+1,u_1,y)-\varphi(m+1,u_2,y)| \\ &\quad = |\varphi(1,\varphi(m,u_1,y),\sigma(1,y))-\varphi(1,\varphi(m,u_2,y),\sigma(1,y))| \\ &\quad < |\varphi(m,u_1,y)-\varphi(m,u_2,y)| < |u_1-u_2| \end{aligned} \tag{26}$$

for all $u_1,u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$) and $y \in Y$. The proof is therefore complete. □

Recall [14, 15, 16] that the point $x \in X$ is called *comparable (respectively, uniformly comparable) by the character of recurrence* with the point $y \in Y$ if $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_x$), where, as defined previously, $\mathfrak{N}_x := \{\{t_n\} \subseteq \mathbb{T}$: such that $\pi(t_n,x) \to x$ as $n \to +\infty$ $\}$ (respectively, $\mathfrak{M}_x := \{\{t_n\} \subseteq \mathbb{T}$: such that the sequence $\{\pi(t_n,x)\}$ converges $\}$).

Now, we can prove a result which will be helpful for our analysis.

Theorem 3.11. [14, 16] *Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$ be two dynamical systems, $x \in X$ and $y \in Y$. Then, the following statements hold:*

(i) *If x is comparable by the character of recurrence with y, and y is τ–periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point x.*

(ii) *If x is uniformly comparable by the character of recurrence with y, and y is τ–periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then so is the point x.*

Following B. A. Shcherbakov [14, 16], a solution $\varphi(t,u,y)$ of (20) is said to be *compatible (respectively, uniformly compatible) by the character of recurrence with the right hand-side* if $\mathfrak{N}_y \subseteq \mathfrak{N}_{\varphi(\cdot,u,y)}$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot,u,y)}$), where $\mathfrak{N}_{\varphi(\cdot,u,y)}$ (respectively, $\mathfrak{M}_{\varphi(\cdot,u,y)}$) is the family of all subsequences $\{t_n\} \subseteq \mathbb{T}$ such that the sequence $\{\varphi(t+t_n,u,y)\}$ converges to $\varphi(t,u,y)$ (respectively, converges to some function $\psi : \mathbb{T} \mapsto \mathbb{R}^n$) uniformly with respect to t on every compact from $\mathbb{T}$.

Theorem 3.12. *Let $(Y,\mathbb{Z},\sigma)$ be pseudo recurrent, $f \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ be strict contracting with respect to the variable x, and there exists at least one solution $\varphi(t,u_0,y)$ of (20) which is bounded on $\mathbb{Z}_+$.*

Then,

(i) *system (20) is convergent, i.e., the cocycle φ associated to (20) is convergent;*

(ii) *for all $y \in Y$, (20) admits a unique solution $\varphi(t,x_y,y)$ which is bounded on $\mathbb{Z}$ and uniformly compatible, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot,u_y,y)}$;*

(iii) *if the point y is τ–periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then*
   (a) *equation (20) has a unique τ–periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution;*
   (b) *every solution $\varphi(t,u,y)$ is asymptotically τ–periodic (respectively, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent);*
   (c) $\lim\limits_{t\to\infty} |\varphi(t,u,y) - \varphi(t,u_y,y)| = 0$ *for all $x \in \mathbb{R}^n$ and $y \in Y$.*

Proof. Let $V : X\dot{\times}X \to \mathbb{R}_+$ be the mapping defined by the equality $V((u_1,y),(u_2,y)) := |u_1 - u_2|$ for all $u_1,u_2 \in \mathbb{R}^n$ and $y \in Y$, where $|\cdot|^2 := \langle\cdot,\cdot\rangle$ and $\langle\xi,\eta\rangle := \sum\limits_{i=1}^{n} \xi_i^1\xi_i^2$ ($\xi^i := (\xi_1^i,\xi_2^i,\ldots,\xi_n^i) \in \mathbb{R}^n$ $(i=1,2)$). Let $(X,\mathbb{Z}_+,\pi)$ be a skew-product dynamical system associated to the cocycle φ. By Lemma 3.10 we have $V(\pi(t,(u_1,y)),\pi(t,(u_2,y)) < V((u_1,y),(u_2,y))$ for all $u_1,u_2 \in \mathbb{R}^n$ $(u_1 \neq u_2)$ and $y \in Y$. Now to prove the first statement it is sufficient to apply Theorem 3.11 in [2].

According to the first statement of the theorem, the skew-product dynamical system $(X,\mathbb{Z}_+,\pi)$ is compact dissipative and, if J_X is its Levinson center, then $J_X \bigcap X_y$ consists of a single point $x_y = (u_y,y)$ and $\varphi(t,u_y,y)$ is the unique solution of (20) defined and bounded on $\mathbb{Z}$. Now, we will prove that the solution $\varphi(\cdot,u_y,y)$ is uniformly compatible by the character of recurrence, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot,u_y,y)}$. It easy to see that the last statement is equivalent to the following inclusion $\mathfrak{M}_y \subseteq \mathfrak{M}_{x_y}$. Let $\{t_k\} \in \mathfrak{M}_y$. Then, there exists a point $q \in Y$ such that $\sigma(t_n,y) \to q$ as $n \to \infty$. Consider the sequence $\{\pi(t_k,x_y)\}$. Since $x_y \in J_X$, the sequence

$\{\pi(t_k, x_y)\}$ is relatively compact. Let p_1 and p_2 be two points of accumulation of this sequence. Then, there exist two subsequences $\{t_k^{(i)}\} \subseteq \{t_k\}$ $(i = 1, 2)$ such that $p_i = \lim_{k\to\infty} \pi(t_k^i, x_y)$ $(i = 1, 2)$. Since J_X is a compact invariant set, then $p_i \in J_X$ $(i = 1, 2)$. On the other hand, $\pi(t_k^i, x_y) = (\varphi(t_k^i, u_y, y), \sigma(t_k^i, y)) \to (\bar{u}_i, q) = p_i$ and, consequently, $p_i \in X_q$. Thus $p_i \in J_X \bigcap X_q$ $(i = 1, 2)$ and, consequently, $p_1 = p_2$. This means that the sequence $\{\pi(t_k, x_y)\}$ is convergent. The second statement is therefore proved.

Taking into account the first and second statements, to finish the proof of the third statement it is sufficient to apply Theorem 3.11. □

Remark 3.13. If we replace condition (23) by a stronger condition, then Theorem 3.12 is also true without the requirement that there exists at least one solution which is bounded on $\mathbb{Z}_+$. Namely, if equation (20) is uniformly contracting, which means that there exists a number $\alpha \in (0, 1)$ such that

$$|f(y, u_1) - f(y, u_2)| \leq \alpha |u_1 - u_2| \tag{27}$$

for all $u_1, u_2 \in \mathbb{R}^n$ and $y \in Y$. The proof of this statement will be given in the next section.

3.5. **Uniformly contracting difference equations.** In this final subsection, we will prove additional results in the case in which we assume that our difference equation is uniformly contracting.

Denote by $||\gamma|| := \max_{y\in Y} |\gamma(y)|$ the norm in the Banach space $C(Y, \mathbb{R}^n)$.

Lemma 3.14. *If there exists a positive number α such that $|f(y, u_1) - f(y, u_2)| \leq \alpha|u_1 - u_2|$ (for all $y \in Y$ and $u_1, u_2 \in \mathbb{R}^n$), then*

$$|\varphi(n, u_1, y) - \varphi(n, u_2, y)| \leq \alpha^n |u_1 - u_2| \tag{28}$$

$\forall\ y \in Y$ and $u_1, u_2 \in \mathbb{R}^n$.

Proof. We omit the proof of this statement because it is a slight modification of the proof of Lemma 3.10. □

Theorem 3.15. *Suppose that the following conditions are fulfilled:*

- (i) *the dynamical system $(Y, \mathbb{Z}, \sigma))$ is pseudo recurrent;*
- (ii) *equation (20) is uniformly contracting, i.e., there exists a number $\alpha \in (0, 1)$ such that inequality (27) takes place.*

Then, the following statements hold:

- (i) *there exists a unique mapping $\gamma \in C(Y, \mathbb{R}^n)$ such that $\gamma(\sigma(t, y)) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{Z}_+$;*
- (ii) *the equality*

$$\lim_{t\to+\infty} |\varphi(t, u, y) - \varphi(t, \gamma(y), y)| = 0 \tag{29}$$

holds for all $y \in Y$ and $u \in \mathbb{R}^n$.

Proof. Consider the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ generated by (20), where $\varphi(t,u,y)$ is the unique solution of (20) passing through $u \in \mathbb{R}^n$ at the initial moment $t = 0$. For each $k \in \mathbb{Z}_+$, we define a mapping $S^k : C(Y, \mathbb{R}^n) \mapsto C(Y, \mathbb{R}^n)$ as

$$(S^k\eta)(y) := \varphi(k, \eta(y), \sigma(-k, y)) \tag{30}$$

for all $\eta \in C(Y, \mathbb{R}^n)$ and $y \in Y$. Thanks to Lemma 3.14,

$$\begin{aligned} d(S^k\eta_1, S^k\eta_2) &= \max_{y\in Y} |\varphi(k, \eta_1(y), \sigma(-k, y)) - \varphi(k, \eta_2(y), \sigma(-k, y))| \\ &\leq \alpha^k \max_{y\in Y} |\eta_1(y) - \eta_2(y)| \\ &= \alpha^k d(\eta_1, \eta_2) \end{aligned} \tag{31}$$

for all $\eta_1, \eta_2 \in C(Y, \mathbb{R}^n)$. It follows from (31) that $Lip(S^k) \leq \alpha^k$ ($Lip(F)$ is the Lipschitz constant of F) and, consequently, for $k \geq 1$, the mapping S^k is a contraction. Since the semigroup $\{S^k\}_{k\in\mathbb{Z}_+}$ is commutative, then it admits a unique fixed point γ, i.e., $\gamma(\sigma(t, y)) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{Z}_+$. Thus, the first statement of our theorem is proved.

The second statement follows from (28). Indeed, we have

$$|\varphi(k, u, y) - \varphi(k, \gamma(y), y)| \leq \alpha^k |u - \gamma(y)| \tag{32}$$

for all $y \in Y$, $k \in \mathbb{Z}_+$ and $u \in \mathbb{R}^n$. Passing to the limit in (32) we obtain the necessary statement and the result is completely proved. ☐

From Theorem 3.15 it follows the following result.

Corollary 3.16. *Under the assumptions of Theorem 3.15, the following statements hold:*

(i) *equation (20) is convergent;*
(ii) *if the point $y \in Y$ is τ–periodic (respectively, almost periodic, almost automorphic, recurrent), the equation (20) admits a unique τ–periodic (respectively, almost periodic, almost automorphic, recurrent) solution and every solution of (20) is asymptotically τ–periodic (respectively, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

Remark 3.17. Notice that Theorem 3.15 remains true if we replace (27) by a more general condition: there exist positive numbers $\mathcal{N}$, $\nu \in (0, 1)$ and a function $\omega : Y \to (0, +\infty)$ such that

$$|f(y, u_1) - f(y, u_2)| \leq \omega(y, |u_1 - u_2|) \ (\forall y \in Y, \ u_1, u_2 \in \mathbb{R}^n) \tag{33}$$

and

$$\prod_{k=0}^{n-1} \omega(\sigma(k, y)) \leq \mathcal{N}\nu^n \tag{34}$$

for all $y \in Y$ and $n \in \mathbb{Z}_+$.

This statement can be proved by using slight modifications of the proof of Theorem 3.15.

Remark 3.18. 1. If the dynamical system $(Y, \mathbb{Z}, \sigma)$ is almost periodic (in particular, it is uniquely ergodic) and the mapping $\omega : Y \mapsto (0, +\infty)$ is continuous, then we have (see, for example, [13, ChIV])

(i) there exists the limit

$$\mu := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \omega(\sigma(k, y)); \tag{35}$$

(ii) this limit exists uniformly with respect to $y \in Y$;
(iii) the limit μ in (35) does not depend on $y \in Y$.

2. Note that condition (34) is fulfilled if, for example, the dynamical system $(Y, \mathbb{Z}, \sigma)$ is almost periodic, $\omega \in C(Y, \mathbb{R}_+)$ and the number μ in (35) is negative.

Acknowledgements. We would like to thank the referees for their interesting comments and helpful suggestions which allowed us to improve the presentation of this paper.

This paper was written while the second author was visiting the University of Sevilla (February–September 2010) under the Programa de Movilidad de Profesores Universitarios y Extranjeros (Ministerio de Educación, Spain) grant SAB2009-0078. He would like to thank people of this university for their very kind hospitality. He also gratefully acknowledges the financial support of the Ministerio de Educación (Spain). The first author is partially supported by grant MTM2008-00088 (Ministerio de Ciencia e Innovación, Spain) and Proyecto de Excelencia P07-FQM02468 (Junta de Andalucía, Spain).

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