9^{th} AIMS Conference on Dynamical Systems,
Differential Equations And Applications
July 1-5 2012
Orlando, Florida, USA.

Markus-Sell's Theorem for Infinite Dimensional Asymptotically Almost Periodic Systems

David Cheban
Department of Mathematics and Informatics
State University of Moldova
e-mail: cheban@usm.md
and

Cristiana Mammana
Department of Economic and Financial Institutions
University of Macerata
e-mail: mammana@unimc.it

1.- INTRODUCTION

This talk is dedicated to the study of asymptotic stability of asymptotically almost periodic systems. We formulate and prove the analog of Markus-Sell's theorem for asymptotically almost periodic systems (both finite and infinite dimensional cases). We study this problem in the framework of general non-autonomous dynamical systems.The obtained general results we apply to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Difference Equations, Functional Differential Equations and Semilinear Parabolic Equations.

Markus-Sell's theorem

Denote by $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, $F(t,x) := f(x) + p(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ the right hand side of system

$$x' = f(x) + p(t,x), \tag{1}$$

where $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the space of all continuous functions $F : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ equipped with the compact open topology.

A system of differential equation (1) is said to be asymptotically autonomous, if the function $p \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following condition

$$\lim_{t\to\infty} |p(t,x)| = 0 \tag{2}$$

uniformly in x on every compact subset from $\mathbb{R}^n$, where $|\cdot|$ is a norm on $\mathbb{R}^n$. Autonomous system

$$x' = f(x) \tag{3}$$

is called a limiting system for (1).

Example 1 (**Bessel's equation**) *Consider the equation*

$$t^2x'' + tx' + (t^2 - \alpha^2)x = 0,$$

or equivalently

$$\begin{cases} x' = y \\ y' = -\frac{1}{t}y + (\frac{\alpha^2}{t^2} - 1)x, \end{cases}$$

with limiting system

$$\begin{cases} x' = y \\ y' = -x \ . \end{cases}$$

Denote by $C^1(\mathbb{R}^n, \mathbb{R}^n)$ the space of all continuously differentiable functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$.

Theorem 2 *(L. Markus, 1956) Let* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $O = (0,0)$ *be a critical point of limiting system (3), i.e,* $f(0) = 0$. *Assume that the variational system of (3) based on origin* O *have characteristic values with negative real parts. Then there exists a neighborhood* U *of* O *and a time* T *such that* $\lim_{t\to\infty} |x(t)| = 0$ *for any solution of equation (1) intersecting* U *no later than* T, *i.e., the origin is an attracting point for (1).*

Theorem 3 *(G. Sell, 1971) Let* $F \in C(\mathbb{R}^n, \mathbb{R}^n)$ *be regular, asymptotically autonomous and* $O \in \mathbb{R}^n$ *be the null solution equation (4), i.e,* $F(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$. *Assume that the null solution of limiting equation (3) is uniformly asymptotically stable. Then the null solution of equation (4) is uniformly asymptotically stable.*

Let $(C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}_+, \sigma)$ be the shift dynamical system (or Bebutov's dynamical system) on $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. For every function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ we denote by $H^+(F) := \overline{\{F_\tau : \ \tau \in \mathbb{R}_+\}}$ the closure of all positive translations of the function F and by Ω_F its ω-limit set, i.e., $\Omega_F := \{G : \ \exists\ \tau_n \to +\infty$ such that $F_{\tau_n} \to G\}$, where F_τ is τ-shift of the function F.

Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ be an arbitrary function. Consider the equation

$$x' = F(t, x). \tag{4}$$

Along with equation (4) we consider its H^+-class, i.e., the following family of equations

$$y' = G(t, y) \quad (G \in H^+(F)).$$

Example 4 *1. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically autonomous, i.e., $F(t,x) = f(x) + p(t,x)$ and p satisfies condition (2). In this case $\Omega_F = \{f\}$, i.e., its ω-limit set contains a single function.*

2. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically T periodic, i.e., $F(t,x) = f(t,x) + p(t,x)$, $f(t + T, x) = f(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and p satisfies condition (2). In this case $\Omega_F = \{f_\tau : \ \tau \in [0, T)\}$, i.e., its ω-limit set contains a continuum functions and it is homeomorphic to unitary circle.

3. If $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ is asymptotically quasi periodic, i.e., $F(t,x) = f(t,x) + p(t,x)$, where

$f(t, x)$ is a quasi periodic function with the spectrum of frequency $\nu_1, \nu_2, \ldots, \nu_m$ and p satisfies condition (2). In this case its ω-limit set is homeomorphic to an m-torus.

Remark 5 *Note that Theorem 3 generalizes Theorem of L. Markus in the following directions:*

a. *right hand side f of the limiting equation is only continuous (L. Markus, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$);*

b. *the null solution of limiting equation (3) is only uniformly asymptotically stable (L. Markus, O is uniformly exponentially stable (In fact, $Re\lambda_i < 0$ $(i = 1, \ldots, n)$, $\lambda_1, \ldots, \lambda_n$ are characteristic values of the origin for the variational equation for (3)).*

In connection with Theorems 2 and 3 by G. Sell was formulated the following problem.

G. Sell's conjecture (1971)

Let $F \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and F be positively pre-compact. Assume that W contains the origin 0 and $F(t,0) = 0$ for all $t \in \mathbb{R}_+$. Assume further that there exists a positive number a such that the equality

$$\lim_{t \to +\infty} |\varphi(t,x,G)| = 0$$

takes place uniformly in $|x| \leq a$ and $G \in \Omega_F$. Then the trivial solution of equation $x' = F(t,x)$ is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Z. Artstein (JDEs, 1978) and Bondi P. et al. (NA, 1977).

Remark 6 *1. Bondi P., Moaurd V. and Visentin F. (Nonlinear Analysis, 1977) proved this conjecture under the additional assumption that the function F is locally Lipschitzian.*

2. Artstein Z. (Journal of Differential Equations, 1978) proved this statement without Lipschitzian condition. In reality he proved a more general statement. Namely, he supposed that *only limiting equations for (**??**) are regular, but the function F is not obligatory regular.*

3. By D. Cheban (Nonlinear Analysis, 2012) was formulated G. Sell's conjecture for abstract NDSs (the both with continuous and discrete time). By D. Cheban is given a positive answer to this conjecture and also are presented some applications of this result to different classes of evolution equations: infinite-dimensional differential equations, functional-differential equations and semilinear parabolic equations.

In 1973 (S. M. Shamim Imdadi and M. Rama Mohama Rao, Proceeding of The American Mathematical Society) it was published the following remarkable (but **false** in general case) statement.

Theorem 7 *Let F be a regular function with $F(t,0) = 0$ for all $t \geq 0$. If there exits a function $G \in \Omega_F$ such that the null solution of equation $y' = G(t,y)$ is uniformly asymptotically stable, then the null solution of equation $x' = F(t,x)$ is uniformly asymptotically stable.*

Bondi P. at al. (Nonlinear Analysis, 1977) give the following counterexample to Theorem 7.

$$ax'' + bx' + cx = x \sin \sqrt{t} \tag{5}$$

$(x \in \mathbb{R},\ t \in \mathbb{R}_+,\ a, b > 0,\ c \in (0,1))$.

For every $\mu \in [-1, 1]$

$$ax'' + bx' + cx = \mu x \tag{6}$$

is a limiting equation for (5). For $\mu \in [-1, c)$ the null solution of equation (6) is uniformly asymptotically stable, but the null solution of equation (5) is not uniformly stable.

From the main result of this talk it follows that Theorem 7 is true if the function f is asymptotically recurrent (in particularly, asymptotically almost periodic) in $t \in \mathbb{R}$ uniformly in u on every compact subset Q from $\mathbb{R}^n$.

The aim of this paper is investigation the problem of asymptotic stability of trivial solution for asymptotically almost periodic (respectively, asymptotically recurrent) systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS). We formulate and prove the analog of Theorem 7 for abstract non-autonomous dynamical systems. The obtained result we apply to different classes of evolution equations: Ordinary Differential Equations (both finite and infinite-dimensional cases), Difference Equations, Functional Differential Equations, SemiLinear Parabolic Equations .

G. Sell's conjecture for non-autonomous dynamical systems

Let $\mathbb{TS}$ ($\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$) be a sub-semigroups of $\mathbb{S}$ and $(Y, \mathbb{T}, \sigma)$ be a dynamical system on the metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T} \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}, \sigma)$ with the fiber W, if the following conditions are fulfilled:

1. $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;

2. $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau \in \mathbb{T}$, $u \in W$ and $y \in Y$;

3. the mapping $\varphi : \mathbb{T} \times W \times Y \mapsto W$ is continuous.

Example 8 Consider differential equation $x' = f(t,x)$ with regular second right hand side $f \in C(\mathbb{R}\times W, \mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f)$ $\mathbb{R}_+$, $\sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R}\times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $H^+(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}_+\}}$. Let $\varphi(t,u,g)$ a unique solution of equation

$$y' = g(t,y), \quad (g \in H^+(f)),$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

1. $\varphi(0,u,g) = u$ for all $u \in W$ and $g \in H^+(f)$;

2. $\varphi(t+\tau,u,g) = \varphi(t,\varphi(\tau,u,g),g_\tau)$ for all $t,\tau \in \mathbb{R}_+$, $u \in W$ and $g \in H^+(f)$;

3. the mapping $\varphi : \mathbb{R}_+ \times W \times H^+(f) \mapsto W$ is continuous.

From above it follows that the triplet $\langle W, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle over $(H^+(f), \mathbb{R}_+, \sigma)$ with the fiber $W \subseteq \mathbb{R}^n$. Thus, every non-autonomous equation $x' = f(t,x)$ with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation $x' = f(t,x)$.

Suppose that $W \subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0 \in W$ (0 is the null element of E) and the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t, 0, y) = 0$ for all $t \in \mathbb{T}$ and $y \in Y$.

The trivial motion/solution of cocycle φ is said to be:

1. *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for all $t \geq 0$ and $y \in Y$;

2. *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0 \tag{7}$$

uniformly with respect to $|u| \leq a$ and $y \in Y$;

3. *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

Asymptotic Stability of NDS with Asymptotically Recurrent Base (driving system)

Below we study the asymptotic stability of NDS with asymptotically recurrent base. The main result of this talk are Theorem 9 which contains some test of asymptotic stability for asymptotically recurrent (in particularly, asymptotically almost periodic) NDS.

Theorem 9 *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ (or simply φ) with fiber E. Suppose that the following conditions are fulfilled:*

1. *the space Y is compact;*

2. *the cocycle φ is locally compact;*

3. *Levinson center J (global attractor) of the dynamical system $(Y, \mathbb{T}, \sigma)$ is minimal (i.e., every trajectory is dense in Y);*

4. $\varphi(t, 0, y) = 0$ *for all* $t \in \mathbb{T}$ *and* $y \in Y$*;*

5. *there null motion of the cocycle $\langle E, \varphi, (J, \mathbb{T}, \sigma)\rangle$ is uniformly asymptotically stable.*

Then the trivial motion of the cocycle $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is uniformly asymptotically stable.

Some applications

Below we give a series of applications of our general results for Ordinary Differential Equations, Functional-Differential Equations with finite delay and Semilinear Parabolic Equations.

Ordinary differential equations

Denote by $C(\mathbb{S}\times W,E)$ the space of all continuous mappings $f:\mathbb{S}\times W\mapsto E$ equipped with the compact open topology. On the space $C(\mathbb{S}\times W,E)$ it is defined a shift dynamical system (dynamical system of translations or Bebutov's dynamical system) $(C(\mathbb{S}\times W,E),\mathbb{S},\sigma)$, where σ is a mapping from $\mathbb{S}\times C(\mathbb{S}\times W,E)$ onto $C(\mathbb{S}\times W,E)$ defined as follow $\sigma(\tau,f):=f_\tau$ for all $(\tau,f)\in\mathbb{S}\times C(\mathbb{S}\times W,E)$, where f_τ is the τ-translation of f in t, i.e., $f_\tau(t,x):=f(t+\tau,x)$ for all $(t,x)\in\mathbb{S}\times W$. Consider a differential equation

$$u'=f(t,u), \tag{8}$$

where $f\in C(\mathbb{R}\times W,E)$.

If the function f is regular, then equation (8) naturally defines a cocycle $\langle W,\ \varphi,\ (H^+(f),\ \mathbb{R}_+,\ \sigma)\rangle$, where $(H^+(f),\mathbb{R}_+,\sigma)$ is a (semi-group) dynamical system on $H^+(f)$ induced by Bebutov's dynamical system.

Denote by $\Omega_f:=\{g\in H^+(f):$ there exists a sequence $\tau_n\to+\infty$ such that $g=\lim\limits_{n\to\infty}f_{\tau_n}\}$ the ω-limit set of f.

Theorem 10 *Assume that the following conditions are fulfilled:*

1. *the function f is regular;*

2. *the set $H^+(f)$ is compact;*

3. *the ω-limit set Ω_f of function f is a compact minimal set of Bebutov's dynamical system $(C(\mathbb{R} \times W, E), \mathbb{R}, \sigma)$;*

4. *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*

5. *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*

6. *there exists a function $P \in \Omega_f$ such that the trivial solution of equation*

$$x' = P(t, x) \tag{9}$$

is uniformly attraction, i.e., there exists a positive number a such that

$$\lim_{t\to+\infty}\sup_{|v|\le a,\tau\ge 0}|\varphi(t,v,P_\tau)|=0. \qquad (10)$$

Then the null solution of equation (8) is uniformly asymptotically stable.

Theorem 11 *Assume that the following conditions are fulfilled:*

1. *the function f is regular;*

2. *f is asymptotically recurrent in t uniformly in x on every compact subset from W;*

3. *$f(t,0)=0$ for all $t\in\mathbb{R}_+$;*

4. *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l,U,H^+(f))$ is relatively compact;*

5. *the trivial solution of equation (9) is uniformly attracting, i.e., there exists a positive number a such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, \tau\ge 0} |\varphi(t,v,P_\tau)| = 0.$$

Then the null solution of equation (8) is uniformly asymptotically stable.

Difference equations

Consider a difference equation

$$u(t+1) = f(t, u(t)), \qquad (11)$$

where $f \in C(\mathbb{Z} \times W, E)$.

Along with equation (11) we consider the family of equations

$$v(t+1) = g(t, v(t)), \qquad (12)$$

where $g \in H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{Z}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z} \times W, E)$. Let $\varphi(t,v,g)$ be a unique solution of equation (12) with initial data $\varphi(0,v,g) = v$. Denote by $(H^+(f), \mathbb{Z}_+, \sigma)$ the shift dynamical system on

$H^+(f)$, then the triplet $\langle W, \varphi, (H^+(f), \mathbb{Z}_+, \sigma)\rangle$ is a cocycle (with discrete time) over $(H^+(f), \mathbb{Z}_+, \sigma)$ with the fibre W.

Theorem 12 *Assume that the following conditions are fulfilled:*

1. *the function $f \in C(\mathbb{Z} \times W, E)$ is asymptotically recurrent in t uniformly in x on every compact subset from W;*

2. *$f(t, 0) = 0$ for all $t \in \mathbb{Z}_+$;*

3. *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*

4. *the trivial solution of equation*

$$x(t+1) = P(t, x)$$

is uniformly attracting, i.e., there exists a positive number a such that

$$\lim_{t\to+\infty} \sup_{|v|\le a, \tau\ge 0} |\varphi(t, v, P_\tau)| = 0.$$

Then the null solution of equation (11) is uniformly asymptotically stable.

Functional differential-equations (FDEs) with finite delay

We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a,b] = [-r,0]$, then we set $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $u \in C([\sigma - r, \sigma + A],\mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for all $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t+\theta)$, $-r \leq \theta \leq 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{13}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R} \times$

$\mathcal{C}, \mathbb{R}^n)$ is defined a shift dynamical system $(C(\mathbb{R}\times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $\sigma(\tau, f) := f_\tau$ for all $f \in C(\mathbb{R}\times \mathcal{C}, \mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f_τ is τ-translation of f, i.e., $f_\tau(t, \phi) := f(t+\tau, \phi)$ for all $(t, \phi) \in \mathbb{R}\times\mathcal{C}$.

Let us set $H^+(f) := \overline{\{f_s : s \in \mathbb{R}_+\}}$, where by bar we denote the closure in $C(\mathbb{R}\times\mathcal{C}, \mathbb{R}^n)$.

Along with the equation (13) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{14}$$

where $g \in H^+(f)$.

Below, in this subsection, we suppose that equation (13) is regular.

Remark 13 1. Denote by $\tilde{\varphi}(t, u, f)$ the solution of equation (13) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0, u, f) = u \in \mathcal{C}$, i.e., $\varphi(s, u, f) = u(s)$ for all $s \in [-r, 0]$. By $\varphi(t, u, f)$ we will denote below the trajectory of equation (13), corresponding to the solution $\tilde{\varphi}(t, u, f)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t, u, f)(s) := \tilde{\varphi}(t+s, u, f)$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r, 0]$.

2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (13) as synonym concepts.

It is well known that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H^+(f) \mapsto \mathbb{R}^n$ possesses the following properties:

1. $\varphi(0, v, g) = u$ for all $v \in \mathcal{C}$ and $g \in H^+(f)$;

2. $\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), \sigma(\tau, g))$ for all $t, \tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H^+(f)$;

3. the mapping φ is continuous.

Thus, a triplet $\langle \mathcal{C}, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle which is associated to equation (13).

A function $f \in C(\mathbb{R} \times W, \mathcal{C})$ is said to be completely continuous, if the set $f(\mathbb{R}_+ \times A)$ is bounded for all bounded subset $A \subseteq \mathcal{C}$.

Theorem 14 *Assume that the following conditions are fulfilled:*

a. *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is regular and completely continuous;*

b. f *is asymptotically recurrent in* t *uniformly in* x *on every compact subset from* $\mathcal{C}$*;*

c. $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*

d. *the trivial solution of equation*

$$x' = P(t, x_t)$$

is uniformly attraction.

Then the null solution of equation (13) is uniformly asymptotically stable.

Semilinear parabolic equations

Let E be a Banach space, and let $A : D(A) \to E$ be a linear closed operator with the dense domain $D(A) \subseteq E$.

An operator A is called sectorial if for some $\varphi \in (0, \pi/2)$, some $M \geq 1$, and some real a, the sector

$$S_{a,\varphi} := \{\lambda : \varphi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$.

If A is a sectorial operator, then there exists an $a_1 \geq 0$ such that *Re* $\sigma(A + a_1 I) > 0$ ($\sigma(A) := \mathbb{C} \setminus \rho(A)$). Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator

$$A_1^{-\alpha} := \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha := D(A_1^\alpha)$, and let us equip the space E^α with the graph norm $|x|_\alpha := |A_1^\alpha x|$ $(x \in E)$, $E^0 := E$, and $E^1 := D(A)$. Then E^α is a Banach space with the norm $|\cdot|_\alpha$ and is densely and continuously embedded in E.

Consider differential equation

$$x' + Ax = f(t, x), \tag{15}$$

where $f \in C(\mathbb{R} \times E^{\alpha}, E)$ and $C(\mathbb{R} \times E^{\alpha}, E)$ is the space of all the continuous functions equipped with compact open topology.

Along with equation (15), consider family of equations

$$y' + Ay = g(t, y), \tag{16}$$

where $g \in H^{+}(f) := \overline{\{f_{\tau} : \tau \in \mathbb{R}_{+}\}}$.

Recall that a function f is said to be regular, if for every $(v, g) \in E^{\alpha} \times H^{+}(f)$ equation (16) admits a unique solution $\varphi(t, v, g)$ with initial data $\varphi(0, v, g) = v$ and the mapping $\varphi : \mathbb{R}_{+} \times E^{\alpha} \times H^{+}(F) \mapsto E^{\alpha}$ is continuous.

Assuming that f is regular, a non-autonomous dynamical system can be associated in a natural way with equation (15). Namely, we set $Y := H^{+}(f)$ and by $(Y, \mathbb{R}_{+}, \sigma)$ denote the dynamical system of translations on Y. Further, let $X := E^{\alpha} \times Y$, and let $(X, \mathbb{R}_{+}, \pi)$ be the dynamical system on X defined by the relation $\pi^{\tau}(v, g) = \langle \varphi(\tau, v, g), g_{\tau} \rangle$. Finally, by setting $h = pr_2 : X \to Y$, we obtain the non-autonomous system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$ determined by equation (15).

Recall that a function $f \in C(\mathbb{R}\times E^{\alpha}, E)$ is said to be locally Hölder continuous in t and locally Lipschitz in x, if for every $(t_0, x_0) \in \mathbb{R}\times E^{\alpha}$ there exists a neighborhood V ($(t_0,x_0)\in V$) and positive numbers L and θ such that

$$|f(t_1,x_1)-f(t_2,x_2)| \le L(|t_1-t_2|^{\theta}+|x_1-x_2|_{\alpha})$$

for all $(t_i,x_i)\in V$ $(i=1,2)$.

Lemma 15 *Suppose that the following conditions are fulfilled:*

1. *A is a sectorial operator;*

2. *the resolvent of operator A is compact;*

3. *$0 \le \alpha < 1$ and $f \in C(\mathbb{R}\times E^{\alpha}, E)$;*

4. *the function f is locally Hölder continuous in t and locally Lipschitz in x;*

5. *the set* $f(\mathbb{R}_+ \times B)$ *is bounded in* E *for all bounded subset* B *from* E^α.

Under the conditions listed above, if the function f *is regular and the set* $H^+(f)$ *is compact, then the cocycle* φ *associated by equation (15) is completely continuous.*

Theorem 16 *Assume that the following conditions hold:*

a. *the function* $f \in C(\mathbb{R} \times E^\alpha, E)$ *is asymptotically recurrent in* t *uniformly in* x *on every compact subset from* E^α;

b. $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$;

c. *the set* $f(\mathbb{R}_+ \times B)$ *is bounded in* E *for all bounded subset* B *from* E^α;

d. *the trivial solution of equation*

$$x' = Ax + P(t, x)$$

is uniformly attracting.

Then the null solution of equation (15) is uniformly asymptotically stable.

Remark 17 *Theorem 16 for asymptotically autonomous equations it was established in the book of D. Henry (Chapter IV, Theorem 4.3.7).*

REFERENCES

1. Artstein Zvi, Uniform asymptotic stability via the limiting equations. Journal of Differential Equations, 27(2), 172-189, 1978.

2. Bondi P., Moaurd V. and Visentin F., Limiting equations in the stability problem. Nonlinear Analisys, v.1, no.2, 123–128, 1977.

3. Bondi P., Moaurd V. and Visentin F., Addendum to the Paper "Limiting equations in the stability problem". Nonlinear Analysis, v.1, no.6, 1977. pp.701.

4. Cheban D. N., Sell's conjecture for Non-Autonomous Dynamical Systems. Nonlinear Analysis: TMA, Vol.75, no.7, 2012, pp.3393-3406.

5. Henry D., Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, No. 840, Springer-Verlag, New York 1981.

6. Markus L., Contributions to Nonlinear Oscilations, Vol.3 Princeton, Univ. Press, Princeton, N. J., 1956, pp.17-29.

7. Sell G. R., Topological Dynamics and Ordinary Differential Equations. Van Nostrand-Reinhold, London, 1971.