5^{th} Chaotic Modeling and Simulation
International conference
12-15 June 2012
Athens, Greece.

Chaotic Attractors of Control Systems

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1.- INTRODUCTION

Control dynamical systems with open-loop controls

Consider a control dynamical system governed by the difference equation

$$x(t+1) = f(x(t); u(t)), \qquad (1)$$

where $x \in \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$.

Let $S(\mathbb{Z}_+, \mathcal{P})$ denote the set of the sequences $u(t)$ defined on $\mathbb{Z}_+$ that assume values of the set $\mathcal{P} := \{c_1, c_2, \ldots, c_p\} \subseteq \mathbb{R}^m$. The sequences $u(t)$ in the class $S(\mathbb{Z}_+, \mathcal{P})$ are open-loop controls of system (1). Consider the set of control systems (1) with the open-loop control of the class $S(\mathbb{Z}_+, \mathcal{P})$. These systems constitute a continual set. Particularly important among all systems of this set are p systems

$$x(t+1) = f(x(t); c_i) \quad (x \in \mathbb{R}^n, \quad i = 1, 2, \ldots, p).$$

Below we will consider an examples of this type.

Let $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ be a function satisfying

$$|f(x_1) - f(x_2)| \leq \alpha |x_1 - x_2| \tag{2}$$

for all $x_1, x_2 \in \mathbb{R}^n$, where $|\cdot|$ is a norm on $\mathbb{R}^n$ and $\alpha \in (0, 1)$.

Lemma 1 *If the function f verifies the condition (2), then*

1. *the equation*

   $$x(t+1) = f(x(t)) \tag{3}$$

   generates a semigroup dynamical system $(\mathbb{R}^n$, $\mathbb{Z}_+$, $\pi)$, where $\pi(t, x)$ is a unique solution of equation (3) defined on $\mathbb{Z}_+$ with the initial condition $\pi(0, x) = x$;

2. *the following inequality holds*

   $$|\pi(t, x_1) - \pi(t, x_2)| \leq \alpha^t |u_1 - x_2|$$

   for all $x_1, x_2 \in R^n$ and $t \in \mathbb{Z}_+$.

Consider the finite set of difference equations

$$x(t+1) = f_i(x(t)) \ (i = 1, 2, \ldots, p) \tag{4}$$

with the right-hand sites $f_i \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the condition (2) with the constant $\alpha_i \in (0,1)$. Let $(\mathbb{R}^n, \mathbb{Z}_+, \pi_i)$ $(i = 1, 2, \dots, m)$ be the dynamical system, generated by (4) and (H, P_i) $(i = 1, 2, \dots, m)$ be the cascade, where $P_i(x) := \pi(1, x)$ for all $x \in \mathbb{R}^n$.

Dissipative control systems and their global attractors

Denote by $\varphi(t, x, u)$ a unique solution of equation (1) passing through the point x at the initial moment, i.e., $\varphi(0, x, u) = x$.

The discrete control system (1) is said to be dissipative, if there exists a positive number r such that

$$\lim_{t \to +\infty} |\varphi(t, x, u)| < r \tag{5}$$

for all $x \in \mathbb{R}^n$ and $u \in S(\mathbb{Z}_+, \mathcal{P})$. In this case the set $B[0, r] := \{x \in \mathbb{R}^n : \ |x| \le r\}$ is called an absorbing set for discrete control system (1).

A two-sided sequence $x : \mathbb{Z} \mapsto \mathbb{R}^n$ is said to be an entire (whole) solution of equation (1), if

$x(t+s+1) = f(x(t+s), u(s))$ for all $t \in \mathbb{Z}$ and $s \in \mathbb{Z}_+$.

Denote by $I_u := \{x \in \mathbb{R}^n :$ the solution $\varphi(t, x, u)$ of equation (1) is defined on $\mathbb{Z}$ and it is bounded $\}$.

Theorem 2 *If the discrete control system (1) is dissipative then the following statements hold:*

1. *the set $I_u \neq \emptyset$ and compact for all control $u \in S(\mathbb{Z}_+, \mathcal{P})$;*

2. *the family of compact sets $\{I_u : u \in S(\mathbb{Z}_+, \mathcal{P})\}$ is invariant, i.e., $I_{u_t} \subseteq \varphi(t, I_u, u)$ for all $t \in \mathbb{Z}_+$, where u_t is t-translation of control u (i.e., $u_t(s) := u(t+s)$ for all $s \in \mathbb{Z}_+$);*

3. *the family of compact sets $\{I_u : u \in S(\mathbb{Z}_+, \mathcal{P})\}$ is maximal, i.e., for every compact invariant family of sets $\{I'_u : u \in S(\mathbb{Z}_+, \mathcal{P})\}$, then $I'_u \subseteq I_u$ for all $u \in S(\mathbb{Z}_+, \mathcal{P})$;*

4. *$I_u \subseteq B[0,r]$ for all $u \in S(\mathbb{Z}_+,\mathcal{P})$ and, consequently, the set $\mathbf{I} := \bigcup\{I_u : \ u \in S(\mathbb{Z}_+,\mathcal{P})\}$ is compact;*

5. I *is a uniform (with respect to control $u(t)$) attracting set for discrete control system (1), i.e.,*

$$\lim_{t\to+\infty} \sup_{u\in S(\mathbb{Z}_+,\mathcal{P})} \rho(\varphi(t,x,u),\mathbf{I}) = 0, \qquad (6)$$

where $\rho(x,A) := \inf_{p\in A} |x-p|$ and $A \subseteq \mathbb{R}^n$;

6. *the mapping $u \mapsto I_u$ is upper semi-continuous, i.e.,*

$$\lim_{u\to u_0} \beta(I_u, I_{u_0}) = 0, \qquad (7)$$

where $\beta(A,B) := \sup_{a\in A} \beta(a,B)$.

We will say that the family of compact subsets $\{I_u : \ u \in S(\mathbb{Z}_+,\mathcal{P})\}$ is a global attractor for the discrete control system (1) if the following conditions are fulfilled:

1. $\mathbf{I} = \bigcup\{I_u : \ u \in S(\mathbb{Z}_+,\mathcal{P})\}$ is a compact set;

2. $\{I_u : \ u \in S(\mathbb{Z}_+, \mathcal{P})\}$ is a maximal invariant set;

3. $\mathbf{I}$ is an uniform attracting set for (1).

Thus Theorem 2 states that every dissipative discrete control system (1) admits a compact global attractor.

Discrete inclusions, ensemble of dynamical systems (collages) and cocycles

Let us consider the set-valued function $\mathcal{F} : W \to C(\mathbb{R}^n)$ defined by the equality $\mathcal{F}(x) := \{f(x; c_i) \,|i = 1, 2, \dots, p\}$. Then the discrete control system (1) is equivalent to the difference inclusion

$$x_{t+1} \in \mathcal{F}(x_t). \tag{8}$$

Denote by $\mathcal{F}_{x_0}$ the set of all trajectories of discrete inclusion (8) issuing from the point $x_0 \in W$ and $\mathcal{F} := \bigcup\{\mathcal{F}_{x_0} \mid x_0 \in W\}$.

Below we will give a new approach concerning the study of discrete control system (1) or difference inclusion (8). Denote by $C(\mathbb{Z}_+, \mathcal{P})$ the space of all continuous mappings $f : \mathbb{Z}_+ \to \mathcal{P}$ equipped with the compact-open topology. Denote by $(C(\mathbb{Z}_+, \mathcal{P}), \mathbb{Z}_+, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov on $C(\mathbb{Z}_+, \mathcal{P})$, i.e., $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (i.e., $f_k(n) := f(n + k)$ for all $n \in \mathbb{Z}_+$).

We may now rewrite inclusion (8) in the following way:

$$x_{t+1} = \omega(t)x_t, \ (\omega \in \Omega := C(\mathbb{Z}_+, \mathcal{M})) \qquad (9)$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(t) := f(\cdot, c_{i_{t+1}})$ for all $t \in \mathbb{Z}_+$. We denote by $\varphi(t, x_0, \omega)$ the solution of equation (9) issuing from the point $x_0 \in \mathbb{R}^n$ at the initial moment $t = 0$. Note that $\mathcal{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathcal{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in \mathbb{R}^n, \omega \in \Omega\}$, i.e., $DI(\mathcal{M})$ (or inclusion (8)) is equivalent to the family of non-autonomous equations (9) ($\omega \in \Omega$).

From the general properties of difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ satisfies the following conditions:

1. $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in \mathbb{R}^n \times \Omega$;

2. $\varphi(t+\tau, x_0, \omega) = \varphi(t, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in \mathbb{R}^n \times \Omega$;

3. the mapping φ is continuous;

4. for any $t, \tau \in \mathbb{Z}_+$ and $\omega_1, \omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that
$$U(t, \omega_2)U(\tau, \omega_1) = U(t + \tau, \omega_3), \qquad (10)$$
where $\omega \in \Omega$, $U(t, \omega) := \varphi(t, \cdot, \omega) = \prod_{k=0}^{t} \omega(k)$, $\omega(k) := f_{i_k}$ $(k = 0, 1, \ldots, n)$ and $f_{i_0} := Id_{\mathbb{R}^n}$.

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of group $\mathbb{S}$, X, Y be two metric (or topological) spaces and $(X, \mathbb{T}_1, \pi)$ (respectively $(Y, \mathbb{T}_2, \sigma)$) be a semigroup dynamical system on X (respectively on Y). A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called a non-autonomous dynamical system, where $h : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $x \in X$ and $t \in \mathbb{T}_1$.

Let W, Ω be two topological spaces and $(\Omega, \mathbb{T}_2, \sigma)$ be a semi-group dynamical system on Ω.

Recall that a triplet $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ (or briefly φ) is called a cocycle over $(\Omega, \mathbb{T}_2, \sigma)$ with the fiber W, if φ is a mapping from $\mathbb{T}_1 \times W \times \Omega$ to W satisfying the following conditions:

1. $\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;
2. $\varphi(t + \tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{T}_1$ and $(x, \omega) \in W \times \Omega$;
3. the mapping φ is continuous.

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T}_1 \to X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e., $\pi = (\varphi, \sigma)$). Then it is easy to check that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system; but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(\Omega, \mathbb{T}_2, \sigma)$ and hence $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.

From the presented above it follows that every $DI(\mathcal{M})$ (respectively, inclusion (8)) in a natural way generates a cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(t, x, \omega)$ is the solution of equation (9) issuing from the point $x \in W$ at the initial moment $t = 0$. Thus, we can study inclusion (8) (respectively, $DI(\mathcal{M})$) in the framework of the theory of cocycles with discrete time.

Below we need the following result.

Theorem 3 *[?] Let $\mathcal{M}$ be a compact subset of $C(W, W)$ and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. Then*

1. $\Omega = \overline{Per(\sigma)}$, *where* $Per(\sigma)$ *is the set of all periodic points of* $(\Omega, \mathbb{Z}_+, \sigma)$ *(i.e.* $\omega \in Per(\sigma)$*, if there exists* $\tau \in \mathbb{N}$ *such that* $\sigma(\tau, \omega) = \omega$*);*

2. *the set* Ω *is compact;*

3. Ω *is invariant, i.e.,* $\sigma^t\Omega = \Omega$ *for all* $t \in \mathbb{Z}_+$;

4. φ *satisfies the condition (10).*

Chaotic attractors of discrete control systems

In this Section we give the conditions of existence of chaotic attractor for discrete control systems.

Denote by $\mathcal{A}$ the set of all mapping $\psi : \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties:

(G1) ψ is continuous;

(G2) there exists a positive number t_0 such that:

(a) $\psi(t_0, r) < r$ for all $r > 0$;

(b) the mapping $\psi(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotone increasing.

(G3) $\psi(t+\tau, r) \leq \psi(t, \psi(\tau, r))$ for all $t, \tau \in \mathbb{Z}_+$ and $r \in \mathbb{R}_+$.

Remark 4 *1. Note that the functions $\psi(t,r) = \mathcal{N} q^t r$ ($\mathcal{N} > 0$ and $q \in (0,1)$) and $\psi(t,r) = \frac{r}{1+rt}$ belong to $\mathcal{A}$, where $(t,r) \in \mathbb{Z}_+ \times \mathbb{R}_+$.*

2. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function satisfying the conditions:

1. *$f(r) < r$ for all $r > 0$;*

2. *f is monotone increasing.*

Then the mapping $\psi : \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by equality

$$\psi(t, r) = x(t)$$

for all $(t,r) \in \mathbb{Z}_+ \times \mathbb{R}_+$*, where* $x(t)$ *is a unique solution of difference equation* $x_{t+1} = f(x_t)$ *with initial data* $x_0 = r$*, belongs to* $\mathcal{A}$*.*

Let $\psi \in \mathcal{A}$. A set $\mathcal{M}$ of operators from $C(W,W)$ is said to be ψ-contracting, if

$$\rho(f_{i_t} f_{i_{t-1}} \dots f_{i_1}(x_1), f_{i_t} f_{i_{t-1}} \dots f_{i_1}(x_2)) \leq \quad (11)$$

$$\psi(t, \rho(x_1, x_2))$$

for all $x_1, x_2 \in W$ and $t \in \mathbb{N}$, where $f_{i_1}, f_{i_2}, \dots, f_{i_t} \in C(W)$ and $i_1, i_2, \dots, i_t \in \mathbb{N}$.

The set $\mathcal{S} \subset W$ is

1. nowhere dense, provided the interior of the closure of $\mathcal{S}$ is empty set, $int(cl(\mathcal{S})) = \emptyset$;

2. totally disconnected, provided the connected components are single points;

3. perfect, provided it is closed and every point $p \in \mathcal{S}$ is the limit of points $q_n \in \mathcal{S}$ with $q_n \neq p$.

The set $\mathcal{S} \subset W$ is called a Cantor set, provided it is totally disconnected, perfect and compact.

The subset M of $(X, \mathbb{T}, \pi)$ is called chaotic, if the following conditions hold:

1. the set M is transitive, i.e. there exists a point $x_0 \in X$ such that $M = H(x_0) := \overline{\{\pi(t, x_0) \ : t \in \mathbb{T}\}}$;

2. $M = \overline{Per(\pi)}$, where $Per(\pi)$ is the set of all periodic points of $(X, \mathbb{T}, \pi)$.

Recall that a point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called Poisson's stable, if x belongs to its ω-limit set $\omega_x := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, x)}$.

Theorem 5 *Suppose that the following conditions are fulfilled:*

a. *$\mathcal{M}$ is a finite subset of $C(W, W)$, i.e., $\mathcal{M} := \{f_1, f_2, \dots, f_m\}$ and $m \geq 2$;*

b. $\mathcal{M}$ is ψ-contracting for some $\psi \in \mathcal{A}$.

Then the following statement hold:

1. the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ ($\Omega := C(\mathbb{Z}_+, \mathcal{M})$) generated by $DI(\mathcal{M})$ is compactly dissipative;

2. the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by $DI(\mathcal{M})$ is compactly dissipative;

3. $I = \overline{Per(\varphi)}$, where $Per(\varphi) := \{u \in W \ : \exists \ \tau \in \mathbb{N}$ and $\omega \in \Omega$ such that $\sigma(\tau, \omega) = \omega$ and $\varphi(\tau, u, \omega) = u\}$;

4. if every map $f \in \mathcal{M}$ is invertible, then

    1. Levinson's center J of the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is a chaotic Cantor set;

2. *there exists a residual subset $J_0 \subseteq J$ (large in the sense of Baire category), consisting from Poisson's stable points, such that the positive semi-trajectory of every point $x_0 \in J_0$ is dense on J;*

3. *$I = pr_1(J)$ ($pr_1 : X \to \Omega$, where I is the Levinson's center of cocycle φ and $X := W \times \Omega$), i.e., I is a continuous image of the Cantor set J.*

Let $\mathcal{M} \subset C(W)$, $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \pi)\rangle$ (respectively $(X, \mathbb{Z}_+, \pi)$) be a cocycle (a skew-product dynamical system) generated by $DI(\mathcal{M})$ and let I (J) be Levinson center of the cocycle φ (respectively, skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$

The set I is said to be a chaotic attractor of $DI(\mathcal{M})$, if

1. the set J is chaotic, i.e. J is transitive and $J = \overline{Per(\sigma)}$, where J is the Levinson center of the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by $DI(\mathcal{M})$;

2. $I = pr_1(J)$.

Remark 6 *1. Theorem 5 it was proved in [**?**] for the special case, when* $\psi(t,r) = \mathcal{N}q^t$ $((t,r) \in \mathbb{Z}_+ \times \mathbb{R}_+$, $\mathcal{N} > 0$ *and* $q \in (0,1))$.

2. The problem of the existence of compact global attractors for $DI(\mathcal{M})$ *with finite* $\mathcal{M}$ *(collage or iterated function system (IFS)) was studied before in works [**?**, **?**] (see also the bibliography therein). In [**?**, **?**] the statement close to Theorem 5 was proved. Namely:*

1. *in [**?**] it was announced the first and proved the second statement of Theorem 5, if* $\psi(t,r) = q^t r$ $(t \in \mathbb{Z}_+$ *and* $q \in (0,1))$;

2. *in [**?**] they considered the case when* W *is a compact metric space and every map* $f \in \mathcal{M} = \{f_1, f_2, \dots, f_p\}$ $(i = 1, \dots, p)$ *is contracting (not obligatory invertible). For this type of* $DI(\mathcal{M})$ *it was proved the existence of a compact global attractor* $\mathcal{A}$ *such that for all* $u \in \mathcal{A}$ *and almost all* $\omega \in \Omega$ *(with*

respect to certain measure on Ω) the trajectory $\varphi(t,u,\omega) = U(t,\omega)u$ *(*$U(n,\omega) := \prod_{k=0}^{t} f_{i_k}$, $(i_k$ $\{1,\ldots,m\})$ *and* $f_{i_0} := Id_W$*) was dense in* $\mathcal{A}$.

The problem of description of the structure of the attractor I of $DI(\mathcal{M})$ in general case (when the maps $f \in \mathcal{M}$ are not invertible) is more complicated. We plan to study this problem in one of our next publication.

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