SELL'S CONJECTURE FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. This paper is dedicated to the study of the G. Sell's conjecture for general non-autonomous dynamical systems. We give a positive answer for this conjecture and we apply this result to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Functional Differential Equations and Semi-linear Parabolic Equations.

1. Introduction

The aim of this paper is the study the problem of global asymptotic stability of trivial solution for non-autonomous dynamical systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS).

Consider a differential equation

(1) $$x' = f(t,x) \quad (f \in C(\mathbb{R} \times W, \mathbb{R}^n)),$$

where $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, W is an open subset from $\mathbb{R}^n$ containing the origin (i.e., $0 \in W$), $C(\mathbb{R} \times W, \mathbb{R}^n)$ is the space of all continuous functions $f : \mathbb{R} \times W \mapsto \mathbb{R}^n$ equipped with compact open topology. This topology is defined, for example [5, 15], by the following distance

$$\rho(f,g) := \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{\rho_k(f,g)}{1+\rho_k(f,g)},$$

where $\rho_k(f,g) := \max\{|f(t,x)-g(t,x)| : (t,x) \in [-k,k] \times W_k\}$, $\{W_k\}$ is a family of compact subsets from W with the properties: $W_k \subset W_{k+1}$ for all $k \in \mathbb{N}$, $\bigcup_{k=1}^{+\infty} W_k = W$, and $|\cdot|$ is a norm on $\mathbb{R}^n$. Denote by $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$ the *shift dynamical system* [5, 15] on the space $C(\mathbb{R} \times W, \mathbb{R}^n)$ (*dynamical system of translations* or *Bebutov's dynamical system*), i.e., $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$, where $f_\tau(t,x) := f(t+\tau, x)$ for all $(t,x) \in \mathbb{R} \times W$.

Below we will use the following conditions:

(A): for all $(t_0, x_0) \in \mathbb{R}_+ \times W$ the equation (1) admits a unique solution $x(t; t_0, x_0)$ with initial data (t_0, x_0) and defined on $\mathbb{R}_+ := [0, +\infty)$, i.e., $x(t_0; t_0, x_0) = x_0$;

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(B): the hand right side f *is positively compact*, if the set $\Sigma_f^+ := \{f_\tau : \tau \in \mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(C): the equation
$$y' = g(t, y), \quad (g \in \Omega_f)$$
is called a *limiting equation* for (1), where Ω_f is the ω-limit set of f with respect to the shift dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\} \to +\infty$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): *equation* (1) (or its hand right side f) *is regular* , if for all $p \in H^+(f)$ the equation
$$y' = p(t, y)$$
admits a unique solution $\varphi(t, x_0, p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x_0, p) = x_0$ for all $x_0 \in W$, where $H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{R}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(E): equation (1) admits a null (trivial) solution, i.e., $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;

(F): a function f satisfies to local (respectively, global) *Lipschitz condition*, if there exists a function $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ (respectively, a positive constant L) such that
$$|f(t, x_1) - f(t, x_2)| \leq L(r)|x_1 - x_2|$$
(respectively, $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$) for all $t \in \mathbb{R}_+$ and $x_1, x_2 \in W$ with $|x_1|, |x_2| \leq r$ for all $r > 0$ (respectively, for all $x_1, x_2 \in W$).

The trivial solution of equation (1) is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f_\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;

(ii) *uniformly attracting*, if there exists a positive number a
$$\lim_{t \to +\infty} |\varphi(t, x, f_\tau)| = 0$$
uniformly with respect to $|x| \leq a$ and $\tau \in \mathbb{R}_+$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

G. Sell's conjecture ([15, Ch.VIII,p.134]). Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and f be positively pre-compact. Assume that W contains the origin 0 and $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. Assume further that there exists a positive number a such that
$$\lim_{t \to +\infty} |\varphi(t, x, g)| = 0$$
takes place uniformly with respect to $|x| \leq a$ and $g \in \Omega_f$. Then the trivial solution of (1) is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Z. Artstein [1] and Bondi P. et al [2].

Remark 1.1. 1. Bondi P. et al [2] proved this conjecture under the additional assumption that the function f is local Lipschitzian.

2. Artstein Z. [1] proved this statement without Lipschitzian condition. In reality he proved a small more general statement. Namely, he supposed that only limiting equations for (1) are regular, but the function f is not obligatory regular.

3. It is well known (see, for example,[11]) that for a wide class of ordinary differential equations (ODEs) the notions of uniform asymptotic stability and the notion of stability in the sense of Duboshin (total stability or stability under the perturbation) are equivalent. There is a series of works (see, for example,[2],[8],[10],[12]-[14], [16]-[17] and the references therein), where the authors study the analog of G. Sell's problem for total stability.

In this paper we will formulate G. Sell's conjecture for the abstract NDS. We will give a positive answer to this conjecture and we will apply this result to different classes of evolution equations: infinite-dimensional differential equations, functional-differential equations and semi-linear parabolic equations .

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, point/compact dissipativity, recurrence, shift dynamical systems, etc) and facts from the theory of dynamical systems which will be necessary in this paper.

Section 3 is devoted to the analysis of G. Sell's conjecture. In this Section we formulate an analog of G. Sell's conjecture for cocycles and general NDS.

In Sections 4 we establish the relation between different types of stability of NDS. We prove that from uniform attractiveness it follows uniform asymptotic stability (Theorem 4.1). It is proved that for asymptotically compact dynamical system asymptotic stability and uniform asymptotic stability are equivalent (Theorem 4.3). The main results of this Section are Theorem 4.7 and Corollary 4.9 which contain s positive answer to G. Sell's conjecture for general NDS.

Finally, Section 5 contains some applications of our general results from Sections 2-4 for Ordinary Differential Equations (Theorem 5.1), Functional-Differential Equations (Theorem 5.5) and semi-linear parabolic equations (Theorem 5.7).

2. Compact Global Attractors of Dynamical Systems

Let X be a topological space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of the nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ be one of the sub-semigroups $\mathbb{R}_+$ (respectively, $\mathbb{Z}_+$) or $\mathbb{R}$ (respectively, $\mathbb{Z}$).

A triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0, x) = x; \tag{2}$$

$$\pi(s, \pi(t, x)) = \pi(s + t, x); \tag{3}$$

is called a *dynamical system.* If $\mathbb{T} = \mathbb{R}$ ($\mathbb{R}_+$) or $\mathbb{Z}$ ($\mathbb{Z}_+$), then $(X, \mathbb{T}, \pi)$ is called a *group (semi-group) dynamical system.* In the case, when $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a *flow*, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a *cascade (discrete flow).*

Below X will be a complete metric space with the metric ρ.

The function $\pi(\cdot,x):\mathbb{T}\to X$ is called a *motion* passing through the point x at moment $t=0$ and the set $\Sigma_x:=\pi(\mathbb{T},x)$ is called a *trajectory* of this motion.

A nonempty set $M\subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to dynamical system $(X,\mathbb{T},\pi)$ or, simple, positively invariant (negatively invariant, invariant), if $\pi(t,M)\subseteq M$ $(M\subseteq\pi(t,M),\pi(t,M)=M)$ for every $t\in\mathbb{T}$.

A closed positively invariant set (respectively, invariant set), which does not contain own closed positively invariant (respectively, invariant) subset, is called *minimal.*

Let $M\subseteq X$. The set

$$\omega(M):=\bigcap_{t\ge 0}\overline{\bigcup_{\tau\ge t}\pi(\tau,M)}$$

is called ω-limit for M.

The set $W^s(\Lambda)$, defined by equality

$$W^s(\Lambda):=\{x\in X|\lim_{t\to+\infty}\rho(\pi(t,x),\Lambda)=0\}$$

is called a *stable manifold* of the set $\Lambda\subseteq X$.

The set M is called:

- *orbital stable*, if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(x,M)<\delta$ implies $\rho(\pi(t,x),M)<\varepsilon$ for all $t\ge 0$;
- *attracting*, if there exists $\gamma>0$ such that $B(M,\gamma)\subset W^s(M)$, where $B(M,\gamma):=\{x\in X:\ \rho(x,M)<\gamma\}$;
- *asymptotic stable*, if it is orbital stable and attracting;
- *global asymptotic stable*, if it is asymptotic stable and $W^s(M)=X$;
- *uniform attracting*, if there exists $\gamma>0$ such that

$$\lim_{t\to+\infty}\sup_{x\in B(M,\gamma)}\rho(\pi(t,x),M)=0.$$

The system $(X,\mathbb{T},\pi)$ is called:

– *point dissipative* if there exist a nonempty compact subset $K\subseteq X$ such that for every $x\in X$

$$\lim_{t\to+\infty}\rho(\pi(t,x),K)=0; \tag{4}$$

– *compactly dissipative* if equality (4) takes place uniformly with respect to x on the compact subsets from X;
– *locally dissipative* if for any point $p\in X$ there exist $\delta_p>0$ such that equality (4) takes place uniformly with respect to $x\in B(p,\delta_p)$;
– *bounded dissipative* if equality (4) takes place uniformly with respect to x on every bounded subset from X;
– *local completely continuous* (*compact*) if for all point $p\in X$ there are two positive numbers δ_p and l_p such that the set $\pi(l_p,B(p,\delta_p))$ is relatively compact.

Let $(X,\mathbb{T},\pi)$ be compactly dissipative and K be a compact set attracting every compact subset from X. Let us set

$$J=\omega(K):=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi(\tau,K)}. \tag{5}$$

It can be shown [5, Ch.I] that the set J defined by equality (5) doesn't depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X,\mathbb{T},\pi)$ itself. The set J is called a *Levinson center* of the compactly dissipative dynamical system $(X,\mathbb{T},\pi)$.

Denote by

$$D^{+}(M):=\bigcap_{\varepsilon>0}\overline{\bigcup\{\pi(t,B(M,\varepsilon))|t\geq 0\}},$$

$$J^{+}(M):=\bigcap_{\varepsilon>0}\bigcap_{t\geq 0}\overline{\bigcup\{\pi(\tau,B(M,\varepsilon))|\tau\geq t\}},$$

$D^{+}_{x}:=D^{+}(\{x\})$ and $J^{+}_{x}:=J^{+}(\{x\})$.

Lemma 2.1. *Let $(X,\mathbb{T},\pi)$ be a dynamical system and $x\in X$ be a point with relatively compact semi-trajectory $\Sigma^{+}_{x}:=\{\pi(t,x):\ t\geq 0\}$. Then the following statements hold:*

(i) *the dynamical system $(X,\mathbb{T},\pi)$ induces on the $H^{+}(x):=\overline{\Sigma}^{+}_{x}$ a dynamical system $(H^{+}(x),\mathbb{T}_{+},\pi)$, where by bar is denoted the closure of Σ^{+}_{x} in the space X;*
(ii) *the dynamical system $(H^{+}(x),\mathbb{T}_{+},\pi)$ is compactly dissipative;*
(iii) *Levinson center $J_{H^{+}(x)}$ of $(H^{+}(x),\mathbb{T}_{+},\pi)$ coincides with ω-limit set ω_{x} of the point x.*

Proof. The first and second statements are evident. Now we will establish the equality $J_{H^{+}(x)}=\omega_{x}$. It is clear that $\omega_{x}\subseteq J_{H^{+}(x)}$ because $\omega_{x}\subseteq H^{+}(x)$ is a compact and invariant subset of $(H^{+}(x),\mathbb{T}_{+},\pi)$. To finish the proof of Lemma it is sufficient to show that $J_{H^{+}(x)}\subseteq\omega_{x}$. Let $y\in\omega_{x}$, then there are sequences $\{x_n\}\subseteq H^{+}(x)$ and $\{t_n\}\subseteq\mathbb{T}_{+}$ such that $t_n\to\infty$ and $\{\pi(t_n,x_n)\}\to y$ as $n\to\infty$. Logically two cases are possible:

(i) there exists a subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ such that $\{x_{n_k}\}\subseteq\Sigma^{+}_{x}$. This means that there is a sequence $\{s_k\}\subseteq\mathbb{T}_{+}$ such that $x_{n_k}=\pi(s_k,x))$ and, consequently, we have

$$y=\lim_{k\to\infty}\pi(t_{n_k},x_{n_k})=\lim_{k\to\infty}\pi(t_{n_k},\pi(s_k,x))=\lim_{k\to\infty}\pi(t_{n_k}+s_k,x).$$

Since $s_k+t_{n_k}\to\infty$ as $k\to\infty$, then $y\in\omega_{x}$.
(ii) there exists a subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ such that $\{x_{n_k}\}\subseteq\omega_{x}$ and, consequently, $\pi(t_{n_k},x_{n_k})\in\omega_{x}$ for all $k\in\mathbb{N}$. Since the set ω_{x} is invariant and closed, then y also belongs to ω_{x}.

Lemma is completely proved. □

3. G. Sell's conjecture for non-autonomous dynamical systems

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{T}$ be two sub-semigroups of $\mathbb{S}$ and $((Y, \mathbb{T}_2, \sigma)$ be a dynamical system on metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, if the following conditions are fulfilled:

(i) $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;
(ii) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_1$, $u \in W$ and $y \in Y$;
(iii) the mapping $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ is continuous.

Example 3.1. (**Bebutov's dynamical system**) Let X, W be two metric space. Denote by $C(\mathbb{T} \times W, X)$ the space of all continuous mappings $f : \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and σ be the mapping from $\mathbb{T} \times C(\mathbb{T} \times W, X)$ into $C(\mathbb{T} \times W, X)$ defined by the equality $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where f_τ is the τ-translation (shift) of f with respect to variable t, i.e., $f_\tau(t, x) = f(t+\tau, x)$ for all $(t, x) \in \mathbb{T} \times W$. Then [5, Ch.I] the triplet $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$ is a dynamical system on $C(\mathbb{T} \times W, X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

A function $f \in C(\mathbb{T} \times W, X)$ is said to be *recurrent* with respect to time $t \in \mathbb{T}$ uniformly with respect to spacial variable $x \in W$ on every compact subset from W, if $f \in C(\mathbb{T} \times W, X)$ is a recurrent point of the Bebutov's dynamical system $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$.

Example 3.2. Consider differential equation (1) with regular second right hand $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f), \mathbb{R}_+, \sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $H^+(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}_+\}}$. Let $\varphi(t, u, g)$ a unique solution of equation

$$y' = g(t, y), \quad (g \in H^+(f)),$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

(i) $\varphi(0, u, g) = u$ for all $u \in W$ and $g \in H^+(f)$;
(ii) $\varphi(t + \tau, u, g) = \varphi(t, \varphi(\tau, u, g), g_\tau)$ for all $t, \tau \in \mathbb{R}_+$, $u \in W$ and $g \in H^+(f)$;
(iii) the mapping $\varphi : \mathbb{R}_+ \times W \times H^+(f) \mapsto W$ is continuous.

From above it follows that the triplet $\langle W, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle over $(H^+(f), \mathbb{R}_+, \sigma)$ with the fiber $W \subseteq \mathbb{R}^n$. Thus, every non-autonomous equation (1) with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation (1).

Suppose that $W \subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0 \in W$ (0 is the null element of E) and the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t, 0, y) = 0$ for all $t \in \mathbb{T}_1$ and $y \in Y$.

The trivial motion/solution of cocycle φ is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for all $t \geq 0$ and $y \in Y$;

(ii) *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t\to+\infty} |\varphi(t,u,y)| = 0 \tag{6}$$

uniformly with respect to $|u| \le a$ and $y \in Y$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

G.Sell's conjecture for cocycle. Suppose that $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{T}_2, \sigma)$ with the fiber W and the following conditions are fulfilled:

(i) the cocycle φ admits a trivial motion/solution;
(ii) the space Y is compact;
(iii) there exists a positive constant a such that (6) takes place uniformly with respect to $|u| \le a$ and $y \in J_Y$, where J_Y is Levinson center (maximal compact invariant set) of compactly dissipative dynamical system $(Y, \mathbb{T}_2, \sigma)$;

Recall [5] that a triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is said to be a *NDS*, where $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) is a dynamical system on X (respectively, Y) and h is an homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$.

Below we will give some examples of nonautonomous dynamical systems which play a very important role in the study of nonautonomous differential equations.

Example 3.3. *(***NDS generated by cocyle***.)* Note that every cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ naturally generates a NDS. In fact. Let $X := W \times Y$ and $(X, \mathbb{T}_1, \pi)$ be a skew-product dynamical system on X (i.e., $\pi(t, x) := (\varphi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{T}_1$ and $x := (u, y) \in X$), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where $h := pr_2 : X \mapsto Y$ is the second projection (i.e., $h(u, y) = y$ for all $u \in W$ and $y \in Y$), is a NDS.

Example 3.4. *(***NDS generated by tangent flow***.)* Let M be a compact differentiable manifold and $(M, \mathbb{R}, \sigma)$ be a differentiable flow on M. Denote by TM the tangent bundle of M and by $D\sigma : \mathbb{R} \times TM \mapsto TM$ the derivative flow on TM: if $x := (u, y) \in TM$, i.e., $u \in T_yM$ (T_yM is the tangent space for M at the point $y \in M$), then $\pi(t, (u, y)) := (D\sigma^t(y)u, \sigma^t(y))$ (where $\sigma^t(y) := \sigma(t, y)$ for all $t \in \mathbb{R}$ and $y \in Y$). The triplet $\langle (TM, \mathbb{R}, \pi), (M, \mathbb{R}, \sigma), h\rangle$, where $h : TM \mapsto M$ is the natural projection ($h(u, y) := y$ for all $y \in Y$ and $u \in T_yM$), is a nonautonomous dynamical system.

Example 3.5. *(***NDS generated by equation (1) with non-regular right hand side.***)* Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a local Lipschitzian function and consider a differential equation (1). Let $Y = H(f) := \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ and by bar is denoted a closure in the space $C(\mathbb{R} \times W, \mathbb{R}^n)$ and $(Y, \mathbb{R}, \sigma)$ be a shift dynamical system on Y. Denote by X the set of all points $x := (u, g) \in W \times Y$ such that the equation

$$y' = g(t, y) \tag{7}$$

admits a (unique) solution $\varphi(t, u, g)$ defined on $\mathbb{R}_+$ with initial data $\varphi(0, u, g) = u$. Now we define a mapping $\pi : \mathbb{R}_+ \times X \mapsto X$ by the following way: $\pi(\tau, (u, g)) := (\varphi(\tau, u, g), g_\tau)$ for all $\tau \in \mathbb{R}_+$ and $(u, g) = x \in X$. From the general properties of solutions of ODEs it follows that the triplet $(X, \mathbb{R}_+, \pi)$ is a semi-group dynamical system. Note that a triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $h : X \mapsto Y$ is the second projection, is a nonautonomous dynamical system generated by equation (1).

Remark 3.6. 1. Note that the NDS in Example 3.4 (respectively, Example 3.5) is not generated by a cocycle.

2. Equation (1) from Example 3.5, generally speaking, is not regular because we suppose the uniqueness of Cauchy problem for (1) (the function f is local Lipschitzian), but we do not suppose the existence on the all space W. As a consequence we obtain that the bundle space (X, h, Y), figuring in Example 3.5, is not trivial.

Let (X, h, Y) be a vectorial bundle. Denote by θ_y the null element of the vectorial space $X_y := \{x \in X : \ h(x) = y\}$ and $\Theta := \{\theta_y : \ y \in Y\}$ is the null section and of (X, h, Y).

A vectorial bundle $(X, h, Y$ is said to be local trivial with fiber F if for every point $y \in Y$ there exists a neighborhood U of the point y (U is an open subset of Y containing y) such that $h^{-1}(U)$ and $U \times F$ are homeomorphic, i.e., there exists an homeomorphism $\alpha : h^{-1}(U) \mapsto U \times F$ (trivialization).

Lemma 3.7. *Let (X, h, Y) be a vectorial bundle and Θ be its null section. Suppose that the following conditions hold:*

(i) *the space Y is compact;*
(ii) *the vectorial bundle (X, h, Y) is local trivial.*

Then the trivial section Θ is compact.

Proof. Let Y be compact and $y \in Y$. Since (X, h, Y) is trivial, then there exists a neighborhood U_y of the point y such that $h^{-1}(U_y)$ and $U_y \times F$ are homeomorphic. Let $\{\theta_{y_k}\}$ be an arbitrary sequence from Θ. We will prove that from $\{\theta_{y_k}\}$ can be extracted a convergent subsequence. In fact, since the space Y is compact then without loss of generality we may suppose that the sequence $\{y_k\}$ converges to $y_0 \in Y$ and, consequently, there exists a natural number k_0 such that $y_k \in U_{y_0}$ for all $k \geq k_0$. Denote by α the homeomorphism (trivialization) between $h^{-1}(U_{y_0})$ and $U_{y_0} \times F$, then $\theta_{y_k} = \alpha^{-1}(y_k, 0)$ (0 is the null element of the vectorial space F) for all $k \geq k_0$ and, consequently, $\theta_{y_k} \to \theta_{y_0}$ as $k \to \infty$. □

Consider a NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ on the vectorial bundle (X, h, Y). Everywhere in this paper we suppose that the null section Θ of (X, h, Y) is a positively invariant set, i.e., $\pi(t, \theta) \in \Theta$ for all $\theta \in \Theta$ and $t \geq 0$ ($t \in \mathbb{T}_1$).

The *null (trivial) section* Θ of NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is said to be:

(i) *uniformly stable*, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(t, x)| < \varepsilon$ for all $t \geq 0$ ($t \in \mathbb{T}_1$);
(ii) *attracting*, if there exists a number $\nu > 0$ such that $B(\Theta, \nu) \subseteq W^s(\Theta)$, where $B(\Theta, \nu) := \{x \in X | \ |x| < \nu\}$;
(iii) *uniform attracting*, if there exists a number $\nu > 0$ such that
$$\lim_{t \to +\infty} \sup\{|\pi(t, x)| : \ |x| \leq \nu\} = 0;$$
(iv) *asymptotically stable* (respectively, *uniformly asymptotically stable*) if, Θ is uniformly stable and attracting (respectively, uniformly attracting).

Let $(Y, \mathbb{T}_2, \sigma)$ be a compactly dissipative dynamical system, J_Y its Levinson center and $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h\rangle$ be a NDS. Denote by $\tilde{X} := h^{-1}(J_Y) = \{x \in X : h(x) = y \in J_Y\}$, then evidently the following statements are fulfilled:

(i) $\tilde{X}$ is closed;
(ii) $\pi(t, \tilde{X}) \subseteq \tilde{X}$ for all $t \in \mathbb{T}_1$ and, consequently, on the set $\tilde{X}$ is induced by $(X, \mathbb{T}_1, \pi)$ a dynamical system $(\tilde{X}, \mathbb{T}_1, \pi)$);
(iii) the triplet $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h\rangle$ is a NDS.

G.Sell's conjecture for NDS. Suppose that $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a NDS and the following conditions are fulfilled:

(i) the null/trivial section Θ of (X, h, Y) is a positively invariant set;
(ii) the space Y is compact;
(iii) the null section $\tilde{\Theta}$ of the NDS $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h\rangle$ is uniformly attracting.

Then the trivial section Θ of NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is uniformly asymptotically stable.

One of the main goal of this paper is a positive answer to G.Sell's conjecture for general NDS (Corollary 4.9).

4. Relation between different types of stability for NDS

In this Section we establish the relation between different types of stability of NDS. In particular, it is proved that for asymptotically compact dynamical system asymptotic stability and uniform asymptotic stability are equivalent (Theorem 4.3). The main results of this Section are Theorem 4.7 and Corollary 4.9 which contain s positive answer to G. Sell's conjecture for general NDS.

Theorem 4.1. *Let Y be compact. If the null section Θ of non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h\rangle$ is uniformly attracting, then it is uniformly asymptotically stable.*

Proof. To prove this statement it is sufficient that under the condition of Theorem, the null section Θ is uniformly stable. Let r be a positive number such that

$$\lim_{t\to+\infty} \sup_{|x|\le r} |\pi(t, x)| = 0. \tag{8}$$

If we suppose that under the condition of Theorem the set Θ is not uniformly stable, then there are $\varepsilon_0 > 0$, $|x_n| \le 1/n$ and $t_n \ge n$ such that

$$|\pi(t_n, x_n)| \ge \varepsilon_0 \tag{9}$$

for all $n \in \mathbb{N}$. On the other hand from (8) we have for any $0 < \varepsilon < \varepsilon_0/2$ a positive number $L = L(\varepsilon)$ such that

$$|\pi(t, x)| < \varepsilon \tag{10}$$

for all $|x| \le r$ and $t \ge L(\varepsilon)$. For sufficiently large $n \in \mathbb{N}$ ($n > \max\{L,\ r^{-1}\}$) we have $|x_n| < r$ and, consequently, from (10) we obtain

$$|\pi(t_n, x_n)| < \varepsilon < \varepsilon_0/2. \tag{11}$$

Inequalities (9) and (11) are contradictory. The obtained contradiction completes the proof of Theorem. □

Remark 4.2. *1. Note that Theorem 4.1 remains true without assumption of the compactness of Y.*

2. By Theorem 4.1, if Θ is uniformly attracting, then it is uniformly asymptotically stable.

3. It is evident that from uniform asymptotic stability of Θ it follows its asymptotic stability. The converse statement, generally speaking, is not true (see, for example, [5, Ch.I] *Example 1.8).*

Theorem 4.3. *Suppose that the following conditions are fulfilled:*

- (i) *the space Y is compact;*
- (ii) *the null section Θ is asymptotically stable;*
- (iii) *the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.*

Then the null section Θ is uniformly asymptotically stable.

Proof. By Lemma 3.7 the trivial section Θ is a compact positively invariant set. Since the null section Θ is asymptotically stable, then there exists a positive number δ such that $B(\Theta, \delta) \subset W^s(\Theta)$. Let μ be a positive number such that $|\pi(t, x)| \le \delta$ for all $x \in X$ with $|x| \le \mu$ and $t \ge 0$. Consider the set $M := \{\pi(t, x) : \ |x| \le \mu, \ t \ge 0\}$. It is cleat that M is bounded and positively invariant. Since the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact, then there exists a nonempty compact set K from X such that

$$\lim_{t \to +\infty} \beta(\pi(t, M), K) = 0,$$

where $\beta(A, B) := \sup_{a \in A} \rho(a, B)$ and $\rho(a, B) := \inf_{b \in B} \rho(a, b)$. According to Lemma 1.3 [5, Ch.I] the set $\Omega(M)$ is compact, invariant and

$$\lim_{t \to +\infty} \beta(\pi(t, M), \Omega(M)) = 0, \tag{12}$$

where $\pi(s, M) := \{\pi(s, x) : \ x \in M\}$. It is clear that $\Omega(M) \subseteq \overline{M} \subseteq W^s(\Theta)$. According to Theorem 1.37 [5, Ch.I] the set $\Omega(\Theta) \subseteq \Theta$ is a maximal compact invariant set of dynamical system $(X, \mathbb{T}_1, \pi)$ in $W^s(\Theta)$ and, consequently,

$$\Omega(M) \subseteq \Omega(\Theta) \subseteq \Theta. \tag{13}$$

From (12) and (13) it follows that

$$\lim_{t \to +\infty} \sup_{|x| \le \mu} |\pi(t, x)| = 0.$$

Indeed, if we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\{x_n\}$ with $|x_n| \le \mu$ and $t_n \to +\infty$ such that

$$|\pi(t_n, x_n)| \ge \varepsilon_0 \tag{14}$$

for all $n \in \mathbb{N}$. According to Lemma 1.3 [5, Ch.I] the sequence $\{\pi(t_n, x_n)\}$ is relatively compact and, consequently, without loss of generality we can suppose that $\{\pi(t_n, x_n)\}$ is convergent. We denote by $\bar{x} := \lim_{n \to \infty} \pi(t_n, x_n)$, then from (14) we obtain $|\bar{x}| \ge \varepsilon_0 > 0$. On the other hand $\bar{x} \in \Omega(M) \subseteq \Theta$ and, consequently, $|\bar{x}| = 0$. The obtained contradiction proves our statement. The theorem is proved. □

Directly from Theorem 4.3 it follows the following statement.

Corollary 4.4. *Suppose that the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *the null section Θ is globally asymptotically stable;*
(iii) *the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.*

Then the null section Θ is globally uniformly asymptotically stable.

Corollary 4.5. *Suppose that the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *the null section Θ is asymptotically stable;*
(iii) *one of the following two conditions hold:*
   a. *the dynamical system $(X, \mathbb{T}_1, \pi)$ is completely continuous;*
   b. *the fiber bundle (X, h, Y) is finite-dimensional.*

Then the null section Θ is uniformly asymptotically stable.

Proof. This statement it follows from Theorem 4.3. Indeed, condition a. (or b.) implies the asymptotically compactness of the dynamical system $(X, \mathbb{T}_1, \pi)$ and now it is sufficient to apply Theorem 4.3. □

From Theorem 4.1 and Theorem 4.3 we have.

Corollary 4.6. *Suppose that the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *the null section Θ is asymptotically stable;*
(iii) *the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.*

Then the null section Θ is uniformly asymptotically stable if and only if it is a uniform attracting set.

Theorem 4.7. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a NDS. Suppose that the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *the dynamical system $(X, \mathbb{T}_1, \pi)$ is locally compact;*
(iii) *the trivial section Θ of (X, h, Y) is positively invariant;*
(iv) *the trivial section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{T}_1, \pi), (J_Y, \mathbb{T}_2, \sigma), h \rangle$ is uniformly attracting.*

Then the trivial section Θ of non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is uniformly stable.

Proof. Since $\tilde{\Theta}$ is uniformly attracting with respect to NDS $\langle(\tilde{X},\mathbb{T}_1,\pi)$, $(J_Y,\mathbb{T}_2,\sigma)$, $h\rangle$, then there exists a positive number $\tilde{\alpha}$ such that

$$\lim_{t\to+\infty}\sup\{|\pi(t,x)|:\ |x|\le\tilde{\alpha},\ x\in\tilde{X}\}=0. \tag{15}$$

According to local compactness of the dynamical system $(X,\mathbb{T}_1,\pi)$ for all $y\in Y$ there exists a positive number l_y such that the set $\pi(l_y,B[\theta_y,\tilde{\alpha}])$ is relatively compact, where $B[x,\delta]:=\{x\in X:\ |x|\le\delta\}$. By Lemma 3.7 the trivial section Θ is compact, then from the covering $\{B[\theta_y,\tilde{\alpha}]:\ y\in Y\}$ of Θ we can extract a finite sub-covering $\{B[\theta_{y_i},\tilde{\alpha}]:\ i=1,2,\dots,m\}$. Put $l:=\max\{l_{y_i}:\ i=1,2,\dots,m\}$, then the set $\pi(l,B(\Theta,\alpha))$ is relatively compact, where $\alpha\in(0,\tilde{\alpha})$ such that $B(\Theta,\alpha)\subset\bigcup_{i=1}^m B[\theta_{y_i},\tilde{\alpha}]$.

Now we will show that for arbitrary $\varepsilon\in(0,\alpha)$ there exists a number $\delta(\varepsilon)\in(0,\varepsilon)$ such that $|x|<\delta$ implies $|\pi(t,x)|<\varepsilon$ for all $t\ge0$ $(t\in\mathbb{T}_1)$. If we suppose that it is not true, then there are $\varepsilon_0\in(0,\alpha)$, a sequence $0<\delta_n\to0$ as $n\to+\infty$ and $t_n\to+\infty$ such that

$$|x_n|\le\delta_n,\ |\pi(t_n,x_n|=\varepsilon_0 \text{ (if the time } \mathbb{T}_1 \text{ is continuous, i.e., } \mathbb{R}_+\subseteq\mathbb{T}_1)$$

or

$$|\pi(t_n+1,x_n)|>\varepsilon_0 \text{ (if the time } \mathbb{T}_1 \text{ is discrete, i.e., } \mathbb{T}_1\subseteq\mathbb{Z})$$

and

$$|\pi(t,x_n)|\le\varepsilon_0 \ \text{ for all } t\in[0,t_n].$$

Denote by $\bar{x}_n:=\pi(t_n,x_n)$ and $\tilde{x}_n:=\pi(l,\bar{x}_n)$. According to choose of the number l, the sequence $\{\tilde{x}_n\}$ is relatively compact. Let γ_n be a continuous mapping defined on $\mathbb{S}$ with the values from X, defined by equality

$$\gamma_n(s):=\begin{cases}\pi(s+l+t_n,x_n)=\pi(s,\tilde{x}_n), & s\ge-l-t_n\\ x_n, & s\le-l-t_n.\end{cases}$$

The sequence $\{\gamma_n\}$ is relatively compact with respect to compact-open topology. In fact, to establish this fact it is sufficient to show that for all $r>0$ the sequence $\{\varphi_n\}$, where $\varphi_n(t):=\gamma_n(t)$ for all $t\in[-r,r]$, satisfies the conditions of Arcela-Ascoly theorem. This means that there exists a compact $K_r\subset X$ that $\varphi_n([-r,r])\subseteq K_r$ for all $n\in\mathbb{N}$ and $\{\varphi_n\}$ is equi-continuous on $[-r,r]$. Since $t_n\to+\infty$ as $n\to\infty$, then for the given $r>0$ there exists a number $n_0\in\mathbb{N}$ such that $[-r,r]\subset[-r-l-t_n,+\infty)$ for all $n\ge n_0$. Since $\gamma_n([-r,r])\subseteq\pi(l,B[\Theta,\tilde{\alpha}])$, then in the quality of K_r we can take the set $\overline{\pi(l,B[\Theta,\tilde{\alpha}])}\bigcup M$, where $M:=\bigcup_{i=1}^{n_0}\gamma_i([-r,r])$. Now we will show that for arbitrary $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that $|s_1-s_2|<\delta$ implies $\rho(\varphi_n(s_1),\varphi_n(s_2))<\varepsilon$. If we suppose that it is not true, then there are $\varepsilon_0>0$, $0<\delta_n\to0$ and $s_n^i\in[-r,r]$ $(i=1,2)$ such that

$$\rho(\varphi_n(s_n^1),\varphi_n(s_n^2))\ge\varepsilon_0. \tag{16}$$

From (16) we have

$$\rho(\pi(s_n^1+l+t_n,x_n),\pi(s_n^2+l+t_n,x_n))\ge\varepsilon_0 \tag{17}$$

for all $n\ge n_0$. Without loss of generality we can suppose that the sequence $\{x_n\}$ is convergent. Denote by $\tilde{x}:=\lim_{n\to\infty}\tilde{x}_n$ and $s^i:=\lim_{n\to\infty}s_n^i$ $(i=1,2)$. Since $|s_n^1-s_n^2|<\delta_n$, then $s^1=s^2$. Passing into limit in (17) as $n\to\infty$ we obtain

$$0=\rho(\pi(s^1+r,\tilde{x}),\pi(s^2+r,\tilde{x})\ge\varepsilon_0.$$

The obtained contradiction proves our statement. Thus the sequence $\{\gamma_n\}$ is relatively compact in the compact-open topology. Without loss of generality we can suppose that the sequence $\{\gamma_n\}$ converges. Denote by γ the limit of $\{\gamma_n\}$, then according to Theorem 1.3.5 [6] γ is an entire trajectory of $(X, \mathbb{T}_1, \pi)$ with $\gamma(0) = \tilde{x}$. On the other hand we have

$$|\gamma(-l)| = \lim_{n\to\infty} |\gamma_n(-l)| = \lim_{n\to\infty} |\pi(t_n, x_n)| = \varepsilon_0 > 0, \tag{18}$$

if the time $\mathbb{T}_1$ is continuous and

$$|\gamma(-l)| = \lim_{n\to\infty} |\gamma_n(-l)| = \lim_{n\to\infty} |\pi(t_n + 1, x_n)| \geq \varepsilon_0 > 0, \tag{19}$$

if the time $\mathbb{T}_1$ is discrete. Thus we have $\gamma(-l)| \geq \varepsilon_0 > 0$, i.e., the entire trajectory γ of $(X, \mathbb{T}_1, \pi)$ is not trivial. It easy to see that under the conditions of Theorem

$$|\gamma(s)| \leq \tilde{\alpha} \tag{20}$$

for all $s \leq -l$. On the other hand from (15) it follows that for $\varepsilon_0 > 0$ there exists a positive number $L = L(\varepsilon_0)$ such that

$$|\pi(t, x)| < \varepsilon_0/2 \tag{21}$$

for all $t \geq L$ and $|x| \leq \tilde{\alpha}$. Let $t_0 \geq L$, then from (18)-(21) we obtain

$$0 < \varepsilon_0 \leq |\gamma(-l)| = |\pi(t_0, \gamma(-t_0 - l))| < \varepsilon_0/2.$$

The obtained contradiction complete the proof of Theorem. □

Remark 4.8. *1. Note that the assumption of the compactness of Y plays a very important role in the proof of Theorem 4.3 and Theorem 4.7. Probably these Theorems remain true if we replace this condition by the weaker one: the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is compactly dissipative. But it is an open problem.*

2. Theorem 4.7 remains true:

(i) *if we replace the condition of uniform attraction of Θ by the following: there exists a positive number $\tilde{\alpha}$ such that for all compact subset $K \subseteq B[\tilde{\Theta}, \tilde{\alpha}]$ we have*

$$\lim_{t\to+\infty} \sup\{|\pi(t, x)| : \; x \in K\} = 0;$$

(ii) *if we replace the condition of local compactness for $(X, \mathbb{T}_1, \pi)$ by the following: there are positive numbers α and l such that the set $\pi(l, B(\Theta, \alpha))$ is relatively compact.*

Corollary 4.9. *Under the conditions of Theorem 4.7 the trivial section Θ of NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is uniformly asymptotically stable.*

Proof. To prove this statement it is sufficient to show that Θ is attracting. According to Theorem 4.7 there are positive constants α and l such that the set $\pi^l B(\Theta, \alpha)$ is relatively compact. Let $\varepsilon_0 \in (0, \alpha]$ and $a := \delta(\varepsilon_0) > 0$ (without loss of generality we can suppose that $a \leq \min\{\alpha, \tilde{\alpha}\}$, where $\tilde{\alpha}$ is a positive number figuring in Corollary 4.9) be a positive number from uniform stability of Θ, then $|\pi(t, x)| \leq a$ for all $t \geq 0$ and $|x \geq a$. Thus the positive semi-trajectory Σ_x^+ with $|x| \leq a$ is relatively compact. Now it is easy to see that $\lim_{t\to+\infty}|\pi(t,x)|=0$. In fact. If we suppose that it is not true, then there exist $\tilde{\varepsilon_0}>0$, $0<|x_0|\le a$ and $t_n\to+\infty$ such that

$$|\pi(t_n,x_0)|\ge\tilde{\varepsilon_0}. \tag{22}$$

Since the sequence $\{\pi(t_n,x_0)\}$ is relatively compact, then we can suppose that it converges. Denote by $\bar{x}_0$ its limit. Passing into limit in (22) as $n\to+\infty$ we obtain

$$|\bar{x}_0|\ge\tilde{\varepsilon_0}>0. \tag{23}$$

Note that $\bar{x_0}$ is an ω-limit point for x_0. Taking into account that the ω-limit set ω_{x_0} is a nonempty, compact invariant set, then there exists an entire motion $\gamma\in\Phi_{\bar{x}_0}(\pi)$ such that $|\gamma(s)|\le a\le\tilde{\alpha}$ for all $s\in\mathbb{S}$ and we will have

$$\tilde{\varepsilon_0}\le|\bar{x}_0|=|\pi(t,\gamma(-t))|\le\sup_{x\in H(\gamma)}|\pi(t,x) \tag{24}$$

for all $t\ge0$, where $H(\gamma):=\overline{\{\gamma(s):\in\mathbb{S}\}}$ is a compact subset of ω_{x_0}. Passing into limit in (24) and taking into account Remark 4.8 we receive $\tilde{\varepsilon_0}\le0$. The last inequality contradicts to (23). The obtained contradiction proves our statement. □

Remark 4.10. *1. Note that Corollary 4.9 gives a positive answer to G.Sell's conjecture for local-compact NDS.*

2. Application this result (Corollary 4.9) to ODEs (classical G. Sell's conjecture [15, Ch.VIII,p.134]*) we give below (Theorem 5.1 and its proof).*

5. Some applications

5.1. **Ordinary differential equations.** Let E be a Banach space with the norm $|\cdot|$, W be an open subset of E and $0\in W$. Denote by $C(\mathbb{S}\times W,E)$ the space of all continuous mappings $f:\mathbb{S}\times W\mapsto E$ equipped with the compact open topology. On the space $C(\mathbb{S}\times W,E)$ it is defined a shift dynamical system [5, ChI] (dynamical system of translations or Bebutov's dynamical system) $(C(\mathbb{S}\times W,E),\mathbb{S},\sigma)$, where σ is a mapping from $\mathbb{S}\times C(\mathbb{S}\times W,E)$ onto $C(\mathbb{S}\times W,E)$ defined as follow $\sigma(\tau,f):=f_\tau$ for all $(\tau,f)\in\mathbb{S}\times C(\mathbb{S}\times W,E)$, where f_τ is the τ-translation of f with respect to time t, i.e., $f_\tau(t,x):=f(t+\tau,x)$ for all $(t,x)\in\mathbb{S}\times W$. Consider a differential equation

$$u'=f(t,u), \tag{25}$$

where $f\in C(\mathbb{R}\times W,E)$.

If the function f is regular, then the equation (25) naturally defines a cocycle $\langle W,\varphi,(H^+(f),\mathbb{R}_+,\sigma)\rangle$, where $(H^+(f),\mathbb{R}_+,\sigma)$ is a (semi-group) dynamical system on $H^+(f)$ induced by Bebutov's dynamical system.

Applying the general results from Sections 2-4 we will obtain a series of results for equation (25). Below we formulate some of them.

Denote by $\Omega_f:=\{g\in H^+(f):$ there exists a sequence$\tau_n\to+\infty$ such that $g=\lim_{n\to\infty}f_{\tau_n}\}$ the ω-limit set of f.

Theorem 5.1. *Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t,0)=0$ for all $t\in\mathbb{R}_+$;*
(iv) *There exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l,U,H^+(f))$ is relatively compact;*
(v) *there exists a positive number a such that*

$$\lim_{t\to+\infty}\sup_{|v|\le a, g\in\Omega_f}|\varphi(t,v,g)|=0. \tag{26}$$

Then the null solution of equation (25) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(f),\mathbb{R}_+,\sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center (maximal compact invariant set) $J_{H^+(f)}$ evidently coincides with ω-limit set Ω_f of f. Let $Y:=H^+(f)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X:=W\times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generates by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ, i.e., $\pi(t,(v,g)):=(\varphi(t,v,g),\sigma(t,g))$ for all $t\in\mathbb{R}_+$ and $(v,g)\in X$. Now consider a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ ($h:=pr_2$) associated by equation (25). It easy to verify this NDS posses the following properties:

(i) by Lemma 2.1 the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;
(iii) Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
(iv) according to (26) the null section $\tilde{\Theta}$ of NDS $\langle(\tilde{X},\mathbb{R}_+,\pi),(J_Y,\mathbb{R}_+,\sigma),h\rangle$ is uniformly attracting because $|\pi(t,x)|=|\varphi(t,v,g)|$ for all $t\in\mathbb{R}_+$ and $x:=(v,g)\in X$.

Now to finish the proof it is sufficient to apply Corollary 4.9. □

Remark 5.2. *If the space E is finite-dimensional, then Theorem 5.1 coincides with the result of Artstein Z.* [1] *and Bondi P. et all* [2] *because in this case the cocycle φ associated by equation (25) is local compact.*

5.2. **Functional differential-equations.** We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

5.2.1. *Functional-differential equations (FDEs) with finite delay.* Let us first recall some notions and notations from [9]. Let $r>0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi:[a,b]\to\mathbb{R}^n$ equipped with the sup–norm. If $[a,b]=[-r,0]$, then we set $C_r:=C([-r,0],\mathbb{R}^n)$. Let $\sigma\in\mathbb{R}$, $A\ge 0$ and $u\in C([\sigma-r,\sigma+A],\mathbb{R}^n)$. We will define $u_t\in C_r$ for all $t\in[\sigma,\sigma+A]$ by the equality $u_t(\theta):=u(t+\theta)$, $-r\le\theta\le 0$. Consider a functional differential equation

$$\dot{u}=f(t,u_t), \tag{27}$$

where $f:\mathbb{R}\times C_r\to\mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ the space of all continuous mappings $f:\mathbb{R}\times\mathcal{C}\mapsto\mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ is defined (see, for example, [5, ChI]) a shift dynamical system $(C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n),\mathbb{R},\sigma)$, where $\sigma(\tau,f):=f_\tau$ for all $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ and $\tau\in\mathbb{R}$ and f_τ is τ-translation of f, i.e., $f_\tau(t,\phi):=f(t+\tau,\phi)$ for all $(t,\phi)\in\mathbb{R}\times\mathcal{C}$.

Let us set $H^+(f):=\overline{\{f_s : s\in\mathbb{R}_+\}}$, where by bar we denote the closure in $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$.

Along with the equation (27) let us consider the family of equations

$$\dot{v}=g(t,v_t), \tag{28}$$

where $g\in H^+(f)$.

Below, in this subsection, we suppose that equation (27) is regular.

Remark 5.3. 1. Denote by $\tilde{\varphi}(t,u,f)$ the solution of equation (27) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0,u,f)=u\in C_r$, i.e., $\varphi(s,u,\ f)=u(s)$ for all $s\in[-r,0]$. By $\varphi(t,u,f)$ we will denote below the trajectory of equation (27), corresponding to the solution $\tilde{\varphi}(t,u,f)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into C_r, defined by $\varphi(t,u,f)(s):=\tilde{\varphi}(t+s,u,f)$ for all $t\in\mathbb{R}_+$ (respectively, $t\in\mathbb{R}$) and $s\in[-r,0]$.

2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (27) as synonym concepts.

It is well known [3, 15] that the mapping $\varphi:\mathbb{R}_+\times C_r\times H^+(f)\mapsto\mathbb{R}^n$ possesses the following properties:

(i) $\varphi(0,v,g)=u$ for all $v\in C_r$ and $g\in H^+(f)$;
(ii) $\varphi(t+\tau,v,g)=\varphi(t,\varphi(\tau,v,g),\sigma(\tau,g))$ for all $t,\tau\in\mathbb{R}_+$, $v\in C_r$ and $g\in H^+(f)$;
(iii) the mapping φ is continuous.

Thus, a triplet $\langle C_r,\varphi,(H^+(f),\mathbb{R}_+,\sigma)\rangle$ is a cocycle which is associated to equation (27). Applying the results from Sections 2-4 we will obtain a series of results for functional differential equation (27). Below we formulate some of them.

Lemma 5.4. *Suppose that the following conditions hold:*

(i) *the function $f\in C(\mathbb{R}\times W,\mathcal{C})$ is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *the function f is completely continuous, i.e., the set $f(\mathbb{R}_+\times A)$ is bounded for all bounded subset $A\subseteq\mathcal{C}$.*

Then the cocycle φ associated by (27) is completely continuous, i.e., for all bounded subset $A\subseteq W$ there exists a positive number $l=l(A)$ such that the set $\varphi(l,A,H^+(f))$ is relatively compact in $\mathcal{C}$.

Proof. This statement follows from the general properties of solutions of equation (27) (see, for example [9], Lemma 2.2.3 and Lemma 3.3.1) because the set $H^+(f)$ is compact. □

Theorem 5.5. *Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t,0)=0$ for all $t\in\mathbb{R}_+$;*
(iv) *there exists a positive number a such that*

$$\lim_{t\to+\infty}\sup_{|v|\le a,g\in\Omega_f}|\varphi(t,v,g)|=0. \tag{29}$$

Then the null solution of equation (27) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(f),\mathbb{R}_+,\sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and by Lemma 2.1 its Levinson center $J_{H^+(f)}$ coincides with ω-limit set Ω_f of f. Let $Y:=H^+(f)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X:=\mathcal{C}\times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generates by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ. Now consider a NDS $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ $(h:=pr_2)$ associated by equation (27). It easy to verify this NDS posses the following properties:

(i) the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compact dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;
(iii) Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
(iv) according to (29) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X},\mathbb{R}_+,\pi),(J_Y,\mathbb{R}_+,\sigma),h\rangle$ is uniformly attracting because $|\pi(t,x)|=|\varphi(t,v,g)|$ for all $t\in\mathbb{R}_+$ and $x:=(v,g)\in X$;
(v) according to Lemma 5.4 the dynamical system $(X,\mathbb{R}_+,\pi)$ is completely continuous.

Now to finish the proof it is sufficient to apply Corollary 4.9. □

5.3. **Semi-linear parabolic equations.** Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|^2:=\langle\cdot,\cdot\rangle$, and A be a self-adjoint operator with domain $D(A)$.

An operator is said (see, for example, [7]) to have a *discrete spectrum* if in the space H, there exists an ortho-normal basis $\{e_k\}$ of eigenvectors, such that $\langle e_k,e_j\rangle=\delta_{kj}$, $Ae_k=\lambda_k e_k$ $(k,j=1,2,\dots)$ and $0<\lambda_1\le\lambda_2\le\dots,\lambda_k\le\dots$, and $\lambda_k\to+\infty$ as $k\to+\infty$.

One can define an operator $f(A)$ for a wide class of functions f defined on the positive semi-axis as follows:

$$\begin{aligned} D(f(A)) &:= \{h=\textstyle\sum_{k=1}^{\infty}c_ke_k\in H:\ \sum_{k=1}^{\infty}c_k[f(\lambda_k)]^2<+\infty\},\\ f(A)h &:= \textstyle\sum_{k=1}^{\infty}c_kf(\lambda_k)e_k,\quad h\in D(f(A)). \end{aligned} \tag{30}$$

In particular, we can define operators A^α for all $\alpha\in\mathbb{R}$. For $\alpha=-\beta<0$ this operator is bounded. The space $D(A^{-\beta})$ can be regarded as the completion of the space H with respect to the norm $|\cdot|_\beta:=|A^{-\beta}\cdot|$.

The following statements hold [7]:

(i) The space $\mathcal{F}_{-\beta} := D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum_{k=1}^{\infty} c_k e_k$ such that

$$\sum_{k=1}^{\infty} c_k \lambda_k^{-2\beta} < +\infty;$$

(ii) For any $\beta \in \mathbb{R}$, the operator A^{β} can be defined on every space $D(A^{\alpha})$ as a bounded operator mapping $D(A^{\alpha})$ into $D(A^{\alpha-\beta})$ such that

$$A^{\beta} D(A^{\alpha}) = D(A^{\alpha-\beta}),\ A^{\beta_1+\beta_2} = A^{\beta_1} A^{\beta_2}.$$

(iii) For all $\alpha \in \mathbb{R}$, the space $\mathcal{F} := D(A^{\alpha})$ is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\alpha} := \langle A^{\alpha} \cdot, A^{\alpha} \cdot \rangle$ and the norm $|\cdot|_{\alpha} := |A^{\alpha} \cdot|$.

(iv) The operator A with the domain $\mathcal{F}_{1+\alpha}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_{\alpha}$.

(v) The embedding of the space $\mathcal{F}_{\alpha}$ into $\mathcal{F}_{\beta}$ for $\alpha > \beta$ is continuous, i.e., $\mathcal{F}_{\alpha} \subset \mathcal{F}_{\beta}$ and there exists a positive constant $C = C(\alpha, \beta)$ such that $|\cdot|_{\beta} \le C |\cdot|_{\alpha}$.

(vi) $\mathcal{F}_{\alpha}$ is dense in $\mathcal{F}_{\beta}$ for any $\alpha > \beta$.

(vii) Let $\alpha_1 > \alpha_2$, then the space $\mathcal{F}_{\alpha_1}$ is compactly embedded into $\mathcal{F}_{\alpha_2}$, i.e., every sequence bounded in $\mathcal{F}_{\alpha_1}$ is relatively compact in $\mathcal{F}_{\alpha_2}$.

(viii) The resolvent $\mathcal{R}_{\lambda}(A) := (A - \lambda I)^{-1}$, $\lambda \ne \lambda_k$ is a compact operator in each space $\mathcal{F}_{\alpha}$, where I is the identity operator.

According to (30) we can define an exponential operator e^{-tA}, $t \ge 0$, in the scale spaces $\{\mathcal{F}_{\alpha}\}$. Note some of its properties [7]:

a. For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator e^{-tA} maps $\mathcal{F}_{\alpha}$ into $\bigcap_{\beta \ge 0} \mathcal{F}_{\beta}$ and

$$|e^{-tA} x|_{\alpha} \le e^{-\lambda_1 t} |x|_{\alpha}$$

for all $x \in \mathcal{F}_{\alpha}$.

b. $e^{-t_1 A} e^{-t_2 A} = e^{-(t_1+t_2)A}$ for all $t_1, t_2 \in \mathbb{R}_+$;

c.

$$|e^{-tA} x - e^{-\tau A} x|_{\beta} \to 0$$

as $t \to \tau$ for every $x \in \mathcal{F}_{\beta}$ and $\beta \in \mathbb{R}$;

d. For any $\beta \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_{\beta}), e^{-tA})$;

e.

$$|A^{\alpha} e^{-tA} h| \le \Big[\big(\tfrac{\alpha-\beta}{t}\big)^{\alpha-\beta} + \lambda_1^{\alpha-\beta} \Big] e^{-t\lambda_1} |A^{\beta} h|,\ \alpha \ge \beta$$
$$||A^{\alpha} e^{-tA}|| \le \big(\tfrac{\alpha}{t}\big)^{\alpha} e^{-\alpha},\ t > 0,\ \alpha > 0.$$

Consider an evolutionary differential equation

$$u' + Au = F(t, u) \tag{31}$$

in the separable Hilbert space H, where A is a linear (generally speaking unbounded) positive operator with discrete spectrum, and F is a nonlinear continuous mapping acting from $\mathbb{R} \times \mathcal{F}_{\theta}$ into H, $0 \le \theta < 1$, possessing the property

$$|F(t, u_1) - F(t, u_2)| \le L(r) |A^{\theta}(u_1 - u_2)| \tag{32}$$

for all $u_1, u_2 \in B_\theta(0,r) := \{u \in \mathcal{F}_\theta : |u|_\theta \le r\}$. Here $L(r)$ denotes the Lipschitz constant of F on the set $B_\theta(0,r)$.

A function $u : [0,a) \mapsto \mathcal{F}_\theta$ is said to be a *weak solution* (in $\mathcal{F}_\theta$) of equation (31) passing through the point $x \in \mathcal{F}_\theta$ at moment $t = 0$ (notation $\varphi(t,x,F)$) if $u \in C([0,T],\mathcal{F}_\theta)$ and satisfies the integral equation

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A}F(\tau,u(\tau))d\tau$$

for all $t \in [0,T]$ and $0 < T < a$.

In the book [7], it is proved that, under the conditions listed above, there exists a unique solution $\varphi(t,x,F)$ of equation (32) passing through the point x at moment $t = 0$, and it is defined on a maximal interval $[0,a)$, where a is some positive number depending on (x,F).

Denote by $C(\mathcal{R} \times \mathcal{F}_\theta, H)$ the space of all continuous mappings equipped with the compact open topology and by $(C(\mathcal{R} \times \mathcal{F}_\theta, H), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathcal{R} \times \mathcal{F}_\theta, H)$.

Denote by $(H^+(F), \mathbb{R}_+, \sigma)$ a shift dynamical system on $H^+(F)$ induced by $(C(\mathcal{R} \times \mathcal{F}_\theta, H), \mathbb{R}, \sigma)$. From general properties of solutions of evolution equation (31) and Theorem 5.1 [4] it follows that the triplet $\langle \mathcal{F}_\theta, \varphi, (H^+(F), \mathbb{R}_+, \sigma)\rangle$ is a cocycle over $(H^+(F), \mathbb{R}_+, \sigma)$ with the fiber $\mathcal{F}_\theta$.

Applying results from Sections 2-4 we obtain a series of results for evolution equation (31). Now we will formulate some of them.

Lemma 5.6. *Under the conditions listed above, if the function F is regular and the set $H^+(F)$ is compact, then the cocycle φ associated by equation (31) is completely continuous.*

Proof. This statement can be proved with the slight modification of the proof of Lemma 5.3 [4]. □

Theorem 5.7. *Assume that the following conditions are fulfilled:*

(i) *the function F is regular;*
(ii) *the set $H^+(F)$ is compact;*
(iii) *$F(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *there exists a positive number a such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, G\in\Omega_F} |\varphi(t,v,G)| = 0. \tag{33}$$

Then the null solution of equation (31) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(F), \mathbb{R}_+, \sigma)$. Since the space $H^+(F)$ is compact, then $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center $J_{H^+(F)}$ coincides with ω-limit set Ω_F of F. Let $Y := H^+(F)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote by $X := \mathcal{F}_\theta \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ. Consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ $(h := pr_2)$ associated by equation (31). It easy to verify that this NDS posses the following properties:

(i) the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compact dissipative and by Lemma 2.1 its Levinson center J_Y coincides with Ω_F;
(ii) the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
(iv) according to (33) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h\rangle$ is uniformly attracting because $|\pi(t,x)| = |\varphi(t, v, g)|$ for all $t \in \mathbb{R}_+$ and $x := (v, g) \in X$;
(v) by Lemma 5.6 the cocycle φ and, consequently, the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ too, is completely continuous.

Now to finish the proof it is sufficient to apply Corollary 4.9. $\square$

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