ASYMPTOTIC STABILITY OF INFINITE-DIMENSIONAL ALMOST PERIODIC SYSTEMS

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Abstract. This paper is dedicated to the study of the problem of asymptotic stability general non-autonomous dynamical systems (both with continuous and discrete time). We study the relation between different types of attractions and asymptotic stability in the framework of general non-autonomous dynamical systems. Specially we investigate a case of almost periodic systems, i.e., when the base (driving system) is almost periodic. The obtained results we apply to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Functional Differential Equations (both with finite retard and neutral type) and Semi-Linear Parabolic Equations.

1. Introduction

The aim of this paper is the study the problem of asymptotic stability (the both local and global) of trivial solution for non-autonomous differential systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS). We formulate and prove our results for general (abstract) non-autonomous dynamical systems. The obtained results we apply to study the problem asymptotical stability for ordinary differential equations (ODEs), functional-differential equations (FDEs) and semi-linear parabolic equations (SLPEs).

Let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ be a product space of n copies of $\mathbb{R}$, W be an open subset from $\mathbb{R}^n$ containing the origin, $C(\mathbb{R}\times W, \mathbb{R}^n)$ be the space of all continuous functions $f : \mathbb{R} \times W \mapsto \mathbb{R}^n$ equipped with compact open topology.

Consider a differential equation

$$u' = f(t, u), \tag{1}$$

where $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$. Denote by $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$ the *shift dynamical system* [10, 22] on the space $C(\mathbb{R} \times W, \mathbb{R}^n)$ (*dynamical system of translations* or *Bebutov's dynamical system*), i.e., $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R}\times W, \mathbb{R}^n)$, where $f_\tau(t, x) := f(t + \tau, x)$ for all $(t, x) \in \mathbb{R} \times W$.

Below we will use the following conditions:

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(A): for all $(t_0, x_0) \in \mathbb{R}_+ \times W$ the equation (1) admits a unique solution $x(t; t_0, x_0)$ with initial data (t_0, x_0) and defined on $\mathbb{R}_+ := [0, +\infty)$, i.e., $x(t_0; t_0, x_0) = x_0$;

(B): the hand right side f *is positively compact*, if the set $\Sigma_f^+ := \{f_\tau : \tau \in \mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(C): the equation

$$v' = g(t, v), \quad g \in \Omega_f$$

is called a *limiting equation* for (1), where Ω_f is the ω-limit set of f with respect to the shift dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\} \to +\infty$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): *equation* (1) (or its hand right side f) *is regular* , if for all $p \in H^+(f)$ the equation

$$x' = p(t, x)$$

admits a unique solution $\varphi(t, x_0, p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x_0, p) = x_0$ for all $x_0 \in W$, where $H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{R}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(E): equation (1) admits a null (trivial) solution, i.e., $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$.

The null solution of equation (1) is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t, u, f_\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;

(ii) *uniformly attracting*, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, u, f_\tau)| = 0$$

uniformly with respect to $|u| \le a$ and $\tau \in \mathbb{R}_+$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

The main results are contained in the following three theorems. The firs two (Theorems A and B) are related to equation (1) and the third (Theorem C) to equation (1) with almost periodic right hand side f.

Let E be a Banach space with the norm $|\cdot|$.

Theorem 1.1. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *the cocycle φ generated by equation (1) is locally compact, i.e., for every point $u \in E$ there exists a neighborhood U of the point u and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact.*

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:

(i)

$$\lim_{t \to +\infty} \sup_{v \in K, g \in \Omega_f} |\varphi(t, v, g)| = 0$$

for every compact subset K from E;

(ii) *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (2) is relatively compact on $\mathbb{R}_+$.*

Finite-dimensional equation (1) Theorem 1.1 generalizes a statement (Theorem 2.6) established in the work [2] (see also [20, Ch.I] and the bibliography therein).

Theorem 1.2. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *the cocycle φ generated by equation (1) is completely continuous, i.e., for every bounded subset $M \in E$ there exists a positive number l such that the set $\varphi(l, M, H^+(f))$ is relatively compact.*

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:

a. *for every $g \in \Omega_f$ limiting equation (2) does not a nontrivial bounded on $\mathbb{R}$ solutions;*
b. *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (2) is bounded on $\mathbb{R}_+$.*

Recall that a function $f \in C(\mathbb{R} \times W, E)$ is called almost periodic (respectively, almost recurrent) with respect to (w.r.t.) $t \in \mathbb{R}$ uniformly w.r.t. u on every compact subset K from W, if for an arbitrary number $\varepsilon > 0$ and compact subset $K \subseteq W$ there exists a positive number $L = L(K,\varepsilon)$ such that on every segment $[a, a+L]$ ($a \in \mathbb{R}$) of the length L there exists at least one number τ such that

$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+s+\tau, u) - f(t+s)| < \varepsilon$$

(respectively,

$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+\tau, u) - f(t,u)| < \varepsilon)$$

for all $s \in \mathbb{R}$. If the function $f \in C(\mathbb{R} \times W, E)$ is almost recurrent and $H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ is compact, then f is called recurrent (w.r.t. $t \in \mathbb{R}$ uniformly w.r.t. u on every compact subset K from W).

Theorem 1.3. *Suppose that the following conditions are fulfilled:*

(i) *the function $f \in C(\mathbb{R} \times W, E)$ is recurrent with respect to $t \in \mathbb{R}$ uniformly with respect to spacial variable u on every compact subset from W;*
(ii) *$f(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
(iii) *the function f is regular;*
(iv) *the cocycle φ associated by equation (1) is asymptotically compact;*
(v) *the null solution of equation (1) is uniformly stable;*
(vi) *there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t,u,f)| = 0$$

for all $|u| \le a$.

Then the null solution of equation (1) is asymptotically stable.

Proof. This statement it follows directly from Theorem 4.5 using the same arguments as in the proof of Theorem 5.2. □

Remark 1.4. *For finite-dimensional equation (1) with almost periodic hand right side f Theorem 5.7 was established by Z. Artstein* [3] *(see also* [1]*,* [18] *and* [20, Ch.I]*).*

We establish also analogical results for the functional-differential equations and for semi-linear parabolic equations.

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, recurrence, shift dynamical systems, cocycles, non-autonomous dynamical systems etc) and facts from the theory of dynamical systems which will be necessary in this paper.

Section 3 is devoted to the analysis of different types of stabilities for non-autonomous dynamical systems (NDSs). We prove that from uniform attractiveness it follows uniform asymptotic stability. It is proved that for asymptotically compact dynamical system asymptotic stability and uniform asymptotic stability are equivalent. We formulate and proves some tests of asymptotical stability (global asymptotical stability) of infinite-dimensional NDSs (Theorem 3.6, Theorem 3.12 and Theorem 3.13).

In Section 4 we present some results about NDS with minimal base (driving system). The main result of this Section (Theorem 4.5) give a sufficient condition of global asymptotic stability of this type of systems.

Finally, Section 5 contains a series of applications of our general results from Sections 3-4 for Ordinary Differential Equations (Theorem 5.2, Theorem 5.5 and Theorem 5.7), Functional-Differential Equations (both Functional-Differential Equations with finite delay (Theorem 5.11, Theorem 5.12 and Theorem 5.13) and Neutral Functional-Differential Equations (Theorem 5.15)) and semi-linear parabolic equations (Theorem 5.17, Theorem 5.18 and Theorem 5.19).

2. Some Notions and Facts from Dynamical Systems

2.1. **Stable and asymptotically stable sets. Global attractors and Levinson center.** Let (X,ρ) be a complete metric space with the metric ρ, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of the nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ be one of the sub-semigroups $\mathbb{R}_+$ (respectively, $\mathbb{Z}_+$) or $\mathbb{R}$ (respectively, $\mathbb{Z}$).

Triplet $(X,\mathbb{T},\pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0,x) = x; \tag{2}$$

$$\pi(s,\pi(t,x)) = \pi(s+t,x); \tag{3}$$

is called a *dynamical system*. If $\mathbb{T} = \mathbb{R}$ $(\mathbb{R}_+)$ or $\mathbb{Z}$ $(\mathbb{Z}_+)$, then $(X, \mathbb{T}, \pi)$ is called a *group (semi-group) dynamical system*. In the case, when $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a *flow*, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a *cascade (discrete flow)*.

The function $\pi(\cdot, x) : \mathbb{T} \to X$ is called a *motion* passing through the point x at moment $t = 0$ and the set $\Sigma_x := \pi(\mathbb{T}, x)$ is called a *trajectory* of this motion.

A nonempty set $M \subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to dynamical system $(X, \mathbb{T}, \pi)$ or, simple, positively invariant (negatively invariant, invariant), if $\pi(t, M) \subseteq M$ $(M \supseteq \pi(t, M), \pi(t, M) = M)$ for every $t \in \mathbb{T}_+ := \{t \in \mathbb{T} :\ t \geq 0\}$.

A closed positively invariant set (respectively, invariant set), which does not contain own closed positively invariant (respectively, invariant) subset, is called *minimal*.

Let $M \subseteq X$. The set

$$\omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}$$

is called ω-limit for M. If the set M consists a single point x, i.e., $M = \{x\}$, then $\Omega(\{x\}) := \omega_x$ is called the ω-limits set of the point x.

The set $W^s(\Lambda)$, defined by equality

$$W^s(\Lambda) := \{x \in X | \lim_{t \to +\infty} \rho(\pi(t, x), \Lambda) = 0\}$$

is called a *stable manifold* of the set $\Lambda \subseteq X$.

The set M is called:

- *orbital stable*, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \geq 0$;
- *attracting*, if there exists $\gamma > 0$ such that $B(M, \gamma) \subset W^s(M)$, where $B(M, \gamma) := \{x \in X :\ \rho(x, M) < \gamma\}$;
- *asymptotic stable*, if it is orbital stable and attracting;
- *global asymptotic stable*, if it is asymptotic stable and $W^s(M) = X$;
- *uniform attracting*, if there exists $\gamma > 0$ such that

$$\lim_{t \to +\infty} \sup_{x \in B(M, \gamma)} \rho(\pi(t, x), M) = 0.$$

The system $(X, \mathbb{T}, \pi)$ is called:

– *point dissipative* if there exist a nonempty compact subset $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(\pi(t, x), K) = 0; \tag{4}$$

– *compactly dissipative* if equality (4) takes place uniformly with respect to x on the compact subsets from X;
– *locally dissipative* if for any point $p \in X$ there exist $\delta_p > 0$ such that equality (4) takes place uniformly with respect to $x \in B(p, \delta_p)$;
– *bounded dissipative* if equality (4) takes place uniformly with respect to x on every bounded subset from X;

- *local completely continuous* (*compact*) if for all point $p \in X$ there are two positive numbers δ_p and l_p such that the set $\pi(l_p, B(p, \delta_p))$ is relatively compact.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a compact set attracting every compact subset from X. Let us set

$$J = \omega(K) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, K)}. \tag{5}$$

It can be shown [10, Ch.I] that the set J defined by equality (5) doesn't depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called a *Levinson center* of the compactly dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Denote by

$$D^+(M) := \bigcap_{\varepsilon > 0} \overline{\bigcup \{\pi(t, B(M, \varepsilon)) | t \geq 0\}},$$

$$J^+(M) := \bigcap_{\varepsilon > 0} \bigcap_{t \geq 0} \overline{\bigcup \{\pi(\tau, B(M, \varepsilon)) | \tau \geq t\}},$$

$D^+_x := D^+(\{x\})$ and $J^+_x := J^+(\{x\})$.

Lemma 2.1. [13] *Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $x \in X$ be a point with relatively compact semi-trajectory $\Sigma^+_x := \{\pi(t, x) : t \geq 0\}$. Then the following statements hold:*

(i) *the dynamical system $(X, \mathbb{T}, \pi)$ induces on on the $H^+(x) := \overline{\Sigma}_x$ a dynamical system $(H^+(x), \mathbb{T}_+, \pi)$, where by bar is denoted the closure of Σ^+_x in the space X;*
(ii) *the dynamical system $(H^+(x), \mathbb{T}_+, \pi)$ is compactly dissipative;*
(iii) *Levinson center $J_{H^+(x)}$ of $(H^+(x), \mathbb{T}_+, \pi)$ coincides with ω-limit set ω_x of the point x.*

2.2. **Almost periodic, almost automorphic and recurrent points (motions).** Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} :$ such that $\{\pi(t_n, x)\} \to x$ and $\{t_n\} \to \infty\}$.

A point $x \in X$ is said to be *Levitan almost periodic* (see [19]) for the dynamical system $(X, \mathbb{T}, \pi)$ if there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

A point $x \in X$ is called *stable in the sense of Lagrange* (*st.*L), if its trajectory $\{\pi(t, x) : t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called *almost automorphic* [19, 24, 11] for the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \lambda)$ and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.2. *A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is also almost automorphic (see* [4]–[7], [15],[21] *and* [24]*). In other words, a Levitan almost periodic point x is almost automorphic, if and only if its trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is relatively compact.*

2.3. **Bebutov's dynamical system.** Let X, W be two metric space. Denote by $C(\mathbb{T} \times W, X)$ the space of all continuous mappings $f : \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and σ be the mapping from $\mathbb{T} \times C(\mathbb{T} \times W, X)$ into $C(\mathbb{T} \times W, X)$ defined by the equality $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where f_τ is the τ-translation (shift) of f with respect to variable t, i.e., $f_\tau(t,x) = f(t+\tau, x)$ for all $(t,x) \in \mathbb{T} \times W$. Then [10, Ch.I] the triplet $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$ is a dynamical system on $C(\mathbb{T} \times W, X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

A function $f \in C(\mathbb{T} \times W, X)$ is said to be *almost periodic* (respectively, *almost automorphic*, *recurrent* with respect to time $t \in \mathbb{T}$ uniformly with respect to spacial variable $x \in W$ on every compact subset from W, if $f \in C(\mathbb{T} \times W, X)$ is an almost periodic (respectively, almost automorphic, recurrent) point of the Bebutov's dynamical system $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$.

2.4. **Cocycles.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $((Y, \mathbb{T}_2, \sigma)$ be a dynamical system on metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, if the the following conditions are fulfilled:

(i) $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;
(ii) $\varphi(t+\tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_1$, $u \in W$ and $y \in Y$;
(iii) the mapping $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ is continuous.

Example 2.3. Consider differential equation (1) with regular second right hand $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f), \mathbb{R}_+, \sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}_+\}}$. Let $\varphi(t, u, g)$ a unique solution of equation
$$y' = g(t,y), \quad (g \in H^+(f)),$$
then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

(i) $\varphi(0, u, g) = u$ for all $u \in W$ and $g \in H^+(f)$;
(ii) $\varphi(t+\tau, u, g) = \varphi(t, \varphi(\tau, u, g), g_\tau)$ for all $t, \tau \in \mathbb{R}_+$, $u \in W$ and $g \in H^+(f)$;
(iii) the mapping $\varphi : \mathbb{R}_+ \times W \times H^+(f) \mapsto W$ is continuous.

From above it follows that the triplet $\langle W, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle over $(H^+(f), \mathbb{R}_+, \sigma)$ with the fiber $W \subseteq \mathbb{R}^n$. Thus, every non-autonomous equation (1) with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation (1).

Suppose that $W \subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0 \in W$ (0 is the null element of E) and the cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t, 0, y) = 0$ for all $t \in \mathbb{T}_1$ and $y \in Y$.

The trivial motion/solution of cocycle φ is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for all $t \geq 0$ and $y \in Y$;

(ii) *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t\to+\infty} |\varphi(t, u, y)| = 0 \tag{6}$$

uniformly with respect to $|u| \leq a$ and $y \in Y$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

2.5. **Nonautonomous Dynamical Systems (NDS).** Recall [10] that a triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is said to be a *NDS*, where $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) is a dynamical system on X (respectively, Y) and h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$.

Below we will give some examples of nonautonomous dynamical systems which play a very important role in the study of nonautonomous differential equations.

Example 2.4. *(***NDS generated by cocyle***.)* Note that every cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ naturally generates a NDS. In fact, let $X := W \times Y$ and $(X, \mathbb{T}_1, \pi)$ be a skew-product dynamical system on X (i.e., $\pi(t, x) := (\varphi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{T}_1$ and $x := (u, y) \in X$). Then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where $h := pr_2 : X \mapsto Y$ is the second projection (i.e., $h(u, y) = y$ for all $u \in W$ and $y \in Y$), is a NDS.

Remark 2.5. *There are Examples of NDS which are not generated by cocycles (see, for example,* [13]*).*

Let (X, h, Y) be a vectorial bundle. Denote by θ_y the null element of the vectorial space $X_y := \{x \in X : \ h(x) = y\}$ and $\Theta := \{\theta_y : \ y \in Y\}$ is the null section and of (X, h, Y).

A vectorial bundle $(X, h, Y$ is said to be locally trivial with fiber F if for every point $y \in Y$ there exists a neighborhood U of the point y (U is an open subset of Y containing y) such that $h^{-1}(U)$ and $U \times F$ are homeomorphic, i.e., there exists an homeomorphism $\alpha : h^{-1}(U) \mapsto U \times F$ (trivialization).

Lemma 2.6. [13] *Let (X, h, Y) be a vectorial bundle and Θ be its null section. Suppose that the following conditions hold:*

(i) *the space Y is compact;*

(ii) *the vectorial bundle (X, h, Y) is locally trivial.*

Then the trivial section Θ *is compact.*

Consider a NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ on the vectorial bundle (X,h,Y). Everywhere in this paper we suppose that the null section Θ of (X,h,Y) is a positively invariant set, i.e., $\pi(t,\theta)\in\Theta$ for all $\theta\in\Theta$ and $t\ge 0$ $(t\in\mathbb{T}_1)$.

The *null (trivial) section* Θ of NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be:

(i) *uniformly stable*, if for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that $|x|<\delta$ implies $|\pi(t,x)|<\varepsilon$ for all $t\ge 0$ $(t\in\mathbb{T}_1)$;
(ii) *attracting*, if there exists a number $\nu>0$ such that $B(\Theta,\nu)\subseteq W^s(\Theta)$, where $B(\Theta,\nu):=\{x\in X|\ |x|<\nu\}$;
(iii) *uniformly attracting*, if there exists a number $\nu>0$ such that

$$\lim_{t\to+\infty}\sup\{|\pi(t,x)|:\ |x|\le\nu\}=0;$$

(iv) *asymptotically stable* (respectively, *uniformly asymptotically stable*), if Θ is uniformly stable and attracting (respectively, uniformly attracting);
(v) globally asymptotically (respectively, uniformly asymptotically) stable, if Θ is asymptotically (respectively, uniformly asymptotically) stable and $W^s(\Theta)=X$.

3. Some Tests of Global Asymptotical Stability of NDS

Let $(Y,\mathbb{T}_2,\sigma)$ be a compactly dissipative dynamical system, J_Y its Levinson center and $\langle (X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),\ h\rangle$ be a NDS. Denote by $\tilde{X}:=h^{-1}(J_Y)=\{x\in X:\ h(x)=y\in J_Y\}$, then evidently the following statements are fulfilled:

(i) $\tilde{X}$ is closed;
(ii) $\pi(t,\tilde{X})\subseteq\tilde{X}$ for all $t\in\mathbb{T}_1$ and, consequently, on the set $\tilde{X}$ is induced by $(X,\mathbb{T}_1,\pi)$ a dynamical system $(\tilde{X},\mathbb{T}_1,\pi)$);
(iii) the triplet $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma),\ h\rangle$ is a NDS.

A dynamical system $(X,\mathbb{T}_1,\pi)$ is said to be:

(i) *completely continuous (compact)*, if for every bounded subset $B\subseteq X$ there exists a number $l=l(B)>0$ such that the set $\pi(l,M)$ is relatively compact, where $\pi(l,M):=\{\pi(l,x):\ x\in M\}$;
(ii) *locally completely continuous* (*locally compact*), if for every point $p\in X$ there exit positive numbers $l=l(p)$ and $\delta=\delta(p)$ such that the set $\pi(l,B(p,\delta))$ is relatively compact, where $B(p,\delta):=\{x\in X:\ \rho(x,p)<\delta\}$;
(iii) *asymptotically compact*, if for any positively invariant subset $M\subseteq X$ there exists a compact subset $K\subseteq X$ such that $\lim\limits_{t\to+\infty}\beta(\pi(t,M),K)=0$.

Remark 3.1. *1. The dynamical system* $(X,\mathbb{T}_1,\pi)$ *is completely continuous, if one of the following conditions are fulfilled:*

(i) *the space* X *possesses the property of Heine-Borel, i.e., every bounded set* $B\subseteq X$ *is relatively compact;*

(ii) *for some* $t_0 \in T_1$ *the mapping* $\pi^{t_0} : X \mapsto X$*, defined by the equality* $\pi^{t_0}(x) := \pi(t_0, x)$ *(*$\forall\, x \in X$*) is completely continuous, i.e., for any bounded subset* B *from* X *the set* $\pi^{t_0}(B)$ *is relatively compact.*

2. Every completely continuous dynamical system $(X, \mathbb{T}_1, \pi)$ *is locally completely continuous and asymptotically compact.*

3. Let $(X, \mathbb{T}, \pi)$ *be a dynamical system associated by cocycle* $\langle (W, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ *and* Y *is a compact space. Then* $(X, \mathbb{T}, \pi)$ *is asymptotically compact if and only if for every bounded sequence* $\{u_n\} \subseteq W$*,* $\{y_n\} \subseteq Y$ *and* $t_n \to +\infty$ *the sequence* $\{\varphi(t_n, u_n, y_n)\}$ *is relatively compact, if it is bounded. In this case the cocycle* φ *is called asymptotically compact.*

Theorem 3.2. [13] *Let* $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$ *be a NDS. Suppose that the following conditions are fulfilled:*

(i) Y *is compact;*
(ii) *the dynamical system* $(X, \mathbb{T}_1, \pi)$ *is locally compact;*
(iii) *the trivial section* Θ *of* (X, h, Y) *is positively invariant;*
(iv) *the trivial section* $\tilde{\Theta}$ *of NDS* $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h \rangle$ *is uniformly attracting.*

Then the trivial section Θ *of non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ *is uniformly stable.*

Remark 3.3. *Theorem 3.2 remains true:*

(i) *if we replace the condition of uniform attraction of* Θ *by the following: there exists a positive number* $\tilde{\alpha}$ *such that for all compact subset* $K \subseteq B[\tilde{\Theta}, \tilde{\alpha}]$ *we have*

$$\lim_{t \to +\infty} \sup\{|\pi(t, x)| : \ x \in K\} = 0;$$

(ii) *if we replace the condition of local compactness for* $(X, \mathbb{T}_1, \pi)$ *by the following: there are positive numbers* α *and* l *such that the set* $\pi(l, B(\Theta, \alpha))$ *is relatively compact.*

Corollary 3.4. [13] *Under the conditions of Theorem 3.2 the trivial section* Θ *of NDS* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ *is uniformly asymptotically stable.*

Theorem 3.5. *Let* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h) \rangle$ *be a NDS and the following conditions be held:*

(i) *the trivial section* Θ *of* (X, h, Y) *is positively invariant;*
(ii) Y *is compact.*

Then the following statements are equivalent:

a. $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h) \rangle$ *is compactly dissipative and its Levinson center* J_X *is included in* Θ*;*
b. *the trivial section* Θ *is globally asymptotically stable;*
c. *the equality*

$$\lim_{t \to +\infty} |\pi(t, x)| = 0 \tag{7}$$

holds for all $x \in X$ uniformly w.r.t. x on every compact subset M from X.

Proof. Suppose that condition a. is fulfilled. We will show that Θ is globally asymptotically stable. Under condition a. it is sufficient to show that Θ is stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$, $|x_n| < \delta_n$ and $t_n \to +\infty$ such that

$$|\pi(t_n, x_n)| \geq \varepsilon_0. \tag{8}$$

By Lemma 2.6 the set Θ is compact, then the sequence $\{x_n\}$ is relatively compact. Since $(X, \mathbb{T}_1, \pi)$ is compact dissipative, then the sequence $\{\pi(t_n, x_n)\}$ is relatively compact. Thus, without loss of generality we can suppose that the sequence $\{\pi(t_n, x_n\}$ is convergent. Denote by $\bar{x} := \lim\limits_{n\to\infty} \pi(t_n, x_n)$. Then $\bar{x} \in J_X \subseteq \Theta$ and, consequently, $|\bar{x}| = 0$. On the other hand passing into limit in (8) as $n \to \infty$ we obtain $|\bar{x}| \geq \varepsilon_0$. The obtained contradiction proves our statement.

Now we will prove that condition b. implies a. In fact. According to Theorem 3.6 [8] the set Θ is orbitally stable. By Theorem 1.13 [10, Ch.I] the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is included in Θ.

Suppose that condition c. is fulfilled. We will show that c. implies a. Let M be an arbitrary compact subset from X, then by condition c. we have the following equality

$$\lim_{t\to+\infty} \sup_{x\in M} \rho(\pi(t, x), \Theta) = 0. \tag{9}$$

In fact

$$\rho(\pi(t, x), \Theta) \leq \rho(\pi(t, x), \theta_{h(\pi(t,x))}) = |\pi(t, x)| \leq \max_{x\in M} |\pi(t, x)| \to 0 \tag{10}$$

as $t \to +\infty$. Since the sets M and Θ are compact, then by Lemma 1.3 [10, Ch.I] we have

(i) the set Σ^+_M is relatively compact;
(ii) the set $\Omega(M)$ is nonempty, compact and invariant;
(iii)

$$\lim_{t\to+\infty} \sup_{x\in M} \rho(\pi(t, x), \Omega(M)) = 0. \tag{11}$$

From (9) and (11) we obtain $\Omega(M) \subseteq \Theta$ for all compact subset M from X, i.e. the compact subset Θ attracts every compact subset M from X. This means that the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and, evidently, its Levinson center J_X is included in Θ, i.e., c. implies a.

Finally we will establish the implication a. $\to$ c. Suppose that it is not true, then there are a compact subset $M_0 \subseteq X$, a sequence $\{x_n\} \subseteq M_0$, $t_n \to +\infty$ and $\varepsilon_0 > 0$ such that

$$|\pi(t_n, x_n)| \geq \varepsilon_0. \tag{12}$$

Since $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and Y is compact, then without loss of generality, we can consider that the sequences $\{\pi(t_n, x_n)\}$ and $\{\sigma(t_n, y_n)\}$ are convergent, where $y_n := h(x_n)$. Denote by $\bar{y} = \lim\limits_{n\to\infty} \sigma(t_n, y_n)$ and $\bar{x} = \lim\limits_{n\to\infty} \pi(t_n, x_n)$, then $\bar{x} \in J_X$ and $h(\bar{x}) = \bar{y}$. Since $J_X \subseteq |theta$, then $|\bar{x}| = 0$. Taking into account the last equality and passing into limit in (12) as $n \to \infty$ we will have $\varepsilon_0 \leq 0$. The obtained contradiction proves our statement. Theorem is proved. □

A continuous mapping $\gamma : \mathbb{S} \mapsto X$ is called *an entire trajectory* of the semi-group dynamical system $(X, \mathbb{T}, \pi)$ passing through the point x, if $\gamma(0) = x$ and $\pi(t, \gamma(s)) = \gamma(t + s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

Denote by $\mathcal{F}_x(\pi)$ the set of all entire trajectory of $(X, \mathbb{T}, \pi)$ passing through the point x and $\mathcal{F}(\pi) := \bigcup_{x \in X} \mathcal{F}_x(\pi)$.

Theorem 3.6. *Let Y be a compact metric space and $(X, \mathbb{T}_1, \pi)$ be asymptotically compact. The following statements hold:*

- (i) *if the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h) \rangle$ is globally asymptotically stable, then:*
  - a. *every motion of $(X, \mathbb{T}_1, \pi)$ is bounded on $\mathbb{T}_1^+$, i.e., $\sup\limits_{t \in \mathbb{T}_1^+} |\pi(t, x)| < +\infty$ for all $x \in X$, where $\mathbb{T}_1^+ := \{t \in \mathbb{T}_1 : \ t \geq 0\}$;*
  - b. *the dynamical system $(X, \mathbb{T}_1, \pi)$ does not have nontrivial entire bounded on $\mathbb{S}$ motions.*
- (ii) *if $(X, \mathbb{T}_1, \pi)$ is locally compact, then under conditions a. and b. the trivial section Θ of NDS $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h) \rangle$ is globally asymptotically stable.*

Proof. Let Y be compact, $(X, \mathbb{T}_1, \pi)$ be asymptotically compact and the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h) \rangle$ be globally asymptotically stable. According to Theorem 3.5 the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is included in Θ. Hence, every positively semi-trajectory $\Sigma_x^+ := \{\pi(t, x) : \ t \geq 0\}$ is relatively compact and, in particular, it is bounded. Let now $\gamma \in \mathcal{F}(\pi)$ be an arbitrary entire trajectory of dynamical system $(X, \mathbb{T}_1, \pi)$ bounded on $\mathbb{S}$. Since the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact, then $\gamma(\mathbb{S})$ is relatively compact. Taking into account that Levinson center J_X is a maximal compact invariant set of dynamical system $(X, \mathbb{T}_1, \pi)$, then $\gamma(\mathbb{S}) \subseteq J_X \subseteq \Theta$. Thus the first statement of Theorem is proved.

Now we will establish the second statement of Theorem. From condition a. and asymptotically compactness of $(X, \mathbb{T}_1, \pi)$ it follows that every semi-trajectory Σ_x^+ is relatively compact and, consequently, every ω–limit set ω_x $(x \in X)$ is non-empty, compact and invariant. Note that $\omega_x \subseteq \Theta$. In fact, let $x \in X$ and $p \in \omega_x$ be an arbitrary point from ω_x. Since the set ω_x is compact and invariant, then there exists an entire trajectory $\gamma \in \mathcal{F}_x$ such that $\gamma(\mathbb{S}) \subseteq \omega_x$. According to condition b. we have $\gamma(0) = p \in \gamma(\mathbb{S}) \subseteq \Theta$. Thus we established the inclusion $\Omega_X := \overline{\bigcup \{\omega_x : \ x \in X\}} \subseteq \Theta$. This means that the dynamical system $(X, \mathbb{T}_1, \pi)$ is point dissipative. By Theorem 1.10 [10, Ch.I] it is also compactly dissipative. Let J_X be its Levinson center and $x \in J_X$. Since J_X is a compact invariant set of dynamical system $(X, \mathbb{T}_1, \pi)$, then there exists an entire motion $\gamma \in \mathcal{F}_x$ such that $\gamma(\mathbb{S}) \subseteq J_X$. According to condition b. we obtain $x \in \gamma(\mathbb{S}) \subseteq \Theta$ and, consequently, $J_X \subseteq \Theta$. Now to finish the proof of Theorem it sufficient to apply Theorem 3.5. □

Remark 3.7. *Under the conditions of Theorem 3.6 condition a. is equivalent to the following one:* $\lim_{t\to+\infty}|\pi(t,x)|=0$ *for all* $x\in X$*.-*

Remark 3.8. *It is not difficult to check that Theorem 3.6 remains true if we replace condition b. by the following one:*

> b_1. *the dynamical system* $(\tilde{X},\mathbb{T}_1,\pi)$ *does not have nontrivial entire bounded on* $\mathbb{S}$ *motions.*

This statement directly follows from Theorem 3.6. In fact, if $\gamma\in\mathcal{F}(\pi)$ is a bounded on $\mathbb{S}$ motion of $(X,\mathbb{T}_1,\pi)$, then under the conditions of Theorem 3.6 the set $\gamma(S)$ is relatively compact and, consequently, $\nu:=h\circ\gamma$ (i.e., $\nu(s):=h(\gamma(s))\ \forall\ s\in\mathbb{S}$) is an entire trajectory with relatively compact rank $\nu(\mathbb{S})$. This means that $\nu(\mathbb{S})\subseteq J_Y$ and, consequently, $\gamma(\mathbb{S})\subseteq\tilde{X}$.

Remark 3.9. *Note that the completely continuous dynamical system* $(X,\mathbb{T},\pi)$ *is asymptotically compact and locally compact.*

From Theorem 3.6 and Remark 3.9 directly it follows the following statement.

Corollary 3.10. *Let* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ *and the following conditions are fulfilled:*

(i) Y *is compact;*
(ii) *the dynamical system* $(X,\mathbb{T}_1,\pi)$ *is completely continuous.*

Then the trivial section Θ *is globally asymptotically stable if and only if conditions a. and b. of Theorem 3.6 hold.*

Remark 3.11. *Corollary 3.10 was established in* [5] *in the case particular, when* (X,h,Y) *is finite-dimensional and* Y *is a compact and invariant set.*

Theorem 3.12. *Let* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ *be a NDS and* Y *be compact. The trivial section* Θ *of* $\langle(X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),h\rangle$ *is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section* $\tilde{\Theta}$ *of* $\langle(\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$ *is globally asymptotically stable;*
2. *for all compact subset* $K\subseteq X$ *the set* $\Sigma_K^+:=\{\pi(t,x):\ t\geq 0,\ x\in K\}$ *is relatively compact.*

Proof. Necessity. Suppose that the trivial section Θ of $\langle(X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),\ h\rangle$ is globally asymptotically stable, then by Theorem 3.5 the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative and its Levinson center J_X is contained in Θ. Since Levinson center J_Y of $(Y,\mathbb{T}_2,\sigma)$ is its maximal compact invariant set, then the set $\tilde{\Theta}$ is also invariant and, consequently, $J_X=\tilde{\Theta}$. Taking into account that $\Theta\supseteq\tilde{\Theta}=J_X$, then it is easy to check that $\tilde{\Theta}$ is globally asymptotically stable set of NDS $\langle(\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$. To finish the proof of the first statement of Theorem it is sufficient to note that since the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative, then by Theorem 1.5 [10, Ch.I] for every compact subset $K\subseteq X$ the set Σ_K^+ is relatively compact.

Sufficient. Let the trivial section $\tilde{\Theta}$ of NDS $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma),h\rangle$ be globally asymptotically stable. By Theorem 3.5 the dynamical system $(\tilde{X},\mathbb{T}_1,\pi)$ is compactly dissipative and its Levinson center $J_{\tilde{X}}$ is included in $\tilde{\Theta}$. Reasoning as well as in the proof of the first statement of Theorem and taking into account the invariance of the set J_Y we conclude that $J_{\tilde{X}}=\tilde{\Theta}$. Now we will establish that the dynamical system $(X,\mathbb{T}_1,\pi)$ is also compactly dissipative. To prove this statement, according to Theorem 1.15 [10, Ch.I], it is sufficient to establish that $(X,\mathbb{T}_1,\pi)$ is point dissipative. Let x be an arbitrary point of X, since its positive semi-trajectory Σ_x^+ is relatively compact, then its ω–limit set ω_x is a non-empty, compact, invariant set, and

$$\lim_{t\to+\infty}\rho(\pi(t,x),\omega_x)=0.$$

Note that $h(\omega_x)\subseteq J_Y$, since J_Y is a maximal compact invariant set of $(Y,\mathbb{T}_2,\sigma)$), and, consequently, $\omega_x\subseteq\tilde{X}$. On the other hand $\tilde{\Theta}$ is a maximal compact invariant set of $(\tilde{x},\mathbb{T}_2,\sigma)$, hence $\omega_x\subseteq\tilde{\Theta}$. Thus we have $\Omega_X:=\overline{\{\omega_x:\ x\in X\}}$ is a compact set, i.e., the dynamical system system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative. Let now J_X be its Levinson center, then $h(J_X)\subseteq J_Y$ and, consequently, $J_X\subseteq\tilde{X}$. On the other hand $J_{\tilde{X}}=\tilde{\Theta}$ is a maximal compact set of $(\tilde{X},\mathbb{T}_1,\pi)$ and, consequently, $J_X\subseteq\tilde{\Theta}$. Now we will prove that the set Θ is uniformly stable. Suppose that it is not true, then there are $\delta_n\to 0$ $(\delta_n>0)$, $\{x_n\}\subseteq X$ and $t_n\to+\infty$ such that

$$|x_n|<\delta_n\ \text{ and }\ |\pi(t_n,x_n)|\geq\varepsilon_0 \tag{13}$$

for all $n\in\mathbb{N}$. By Lemma 2.6 Θ is a compact set and the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative, then without loss of generality, we can suppose that the sequences $\{x_n\}$ and $\{\pi(t_n,x_n\}$ are convergent. Denote by x_0 (respectively, by $\bar{x}_0$) the limit of $\{x_n\}$ (respectively, $\{\pi(t_n,x_n)\}$). Then by (13) we have $x_0\in\Theta$ and $|\bar{x}|\geq\varepsilon_0>0$. On the other hand $\bar{x}\in J_X\subseteq\tilde{\Theta}$ and, consequently, $|\bar{x}|=0$. The obtained contradiction prove our statement. Let now x be an arbitrary point from X, then $\lim\limits_{t\to+\infty}|\pi(t,x)|=0$. In fact, if we suppose the contrary, then there exist $x_0\in X$, $\varepsilon_0>0$, and $t_n\to+\infty$ such that

$$|\pi(t_n,x_0)|\geq\varepsilon_0 \tag{14}$$

for all $n\in\mathbb{N}$. Since the semi-trajectory $\Sigma_{x_0}^+$ of point x_0 is relatively compact, then we can suppose that the sequence $\{\pi(t_n,x_0)\}$ is convergent. Let $\bar{x}_0$ be its limit, then from (14) we have $|\bar{x}_0|\geq\varepsilon_0>0$. On the other hand $\bar{x}_0\in\omega_{x_0}\subseteq J_X\subseteq\tilde{\Theta}$ and, consequently, $|\bar{x}_0|=0$. The obtained contradiction complete the proof of the global asymptotic stability of trivial section Θ. Theorem is proved. □

Theorem 3.13. *Let $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS, Y be compact and $(X,\mathbb{T}_1,\pi)$ be locally compact. The trivial section Θ of $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section $\tilde{\Theta}$ of $\langle(\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable;*
2. *for all $x\in X$ the set $\Sigma_x^+:=\{\pi(t,x):\ t\geq 0\}$ is relatively compact.*

Proof. The necessity of Theorem it follows from Theorem 3.12. To prove the sufficiency, according to Theorem 3.12, it is sufficient to show that the set Σ_K^+ is relatively compact for all compact subset $K \subseteq X$. To this end we note (reasoning as well as in the proof of Theorem 3.12) that the dynamical system $(X, \mathbb{T}_1, \pi)$ is point dissipative. Since dynamical system $(X, \mathbb{T}_1, \pi)$ is locally compact, then by Theorem 1.10 [10, Ch.I] this system is also compactly dissipative. Conform to Theorem 1.15 [10, Ch.I] for all compact subset $K \subseteq X$ the set Σ_K^+ is relatively compact. □

Corollary 3.14. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a NDS, Y be compact and $(X, \mathbb{T}_1, \pi)$ be completely continuous. The trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section $\tilde{\Theta}$ of $\langle (\tilde{X}, \mathbb{T}_1, \pi), (J_Y, \mathbb{T}_2, \sigma), h \rangle$ is globally asymptotically stable;*
2. *for all $x \in X$ the set $\Sigma_x^+ := \{\pi(t, x) : \ t \geq 0\}$ is bounded.*

Proof. This statement follows directly from Theorem 3.13. To this end it is sufficient to note that every completely continuous dynamical system is locally compact and every bounded semi-trajectory Σ_x is relatively compact, if $(X, \mathbb{T}_1, \pi)$ is completely continuous. □

Lemma 3.15. *Suppose that the following conditions hold:*

(i) *$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a NDS;*
(ii) *Y is compact;*
(iii) *the trivial section Θ of (X, h, Y) is positively invariant.*

Then the following two statements are equivalent:

(i) *Θ is uniformly stable;*
(ii) *Θ is orbitally stable with respect to $(X, \mathbb{T}_1, \pi)$.*

Proof. Let Θ be uniformly stable, then it is orbitally stable with respect to $(X, \mathbb{T}_1, \pi)$. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$, $\{x_n\}$ and $t_n \to +\infty$ such that

$$\rho(x_n, \Theta) < \delta_n \quad \text{and} \quad \rho(\pi(t_n, x_n), \Theta) \geq \varepsilon_0. \tag{15}$$

Since Θ is compact then, without loss of generality, we can suppose that the sequence $\{x_n\}$ is convergent. Denote its limit by x_0, then $y_0 = \lim_{n\to\infty} y_n$, where $y_n := h(x_n)$. Denote by $\delta_0 = \delta(\varepsilon_0/2)$ a positive number chosen for $\varepsilon_0/2$ from the uniform stability of Θ, i.e., $|x| < \delta_0$ implies $|\pi(t, x)| < \varepsilon_0/2$ for all $t \geq 0$ ($t \in \mathbb{T}_1$). Since $|x_n| = \rho(x_n, \theta_{y_n}) \leq \rho(x_n, \theta_{y_0}) + \rho(\theta_{y_0}, \theta_{y_n}) \to 0$ as $n \to \infty$. Thus, there exists a number $n_0 \in \mathbb{N}$ such that $|x_n| < \delta_0$ for all $n \geq n_0$ and, consequently, we obtain

$$|\pi(t_n, x_n)| < \varepsilon_0/2. \tag{16}$$

On the other hand from (15) we receive

$$|\pi(t_n, x_n)| \geq \rho(\pi(t_n, x_n), \Theta) \geq \varepsilon_0. \tag{17}$$

The inequalities (16) and (17) are contradictory. The obtained contradiction proves our statement.

Now we show that from the orbital stability of Θ it follows that it is uniformly stable. This statement may be proved using the same reasoning as in the proof of Theorem 3.5. □

Theorem 3.16. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical systems, Y be a compact metric space, (X,h,Y) be a finite-dimensional fiber space and Θ be its null section. If Θ is uniformly stable, then the following properties are equivalent:*

1. *for every $\varepsilon>0$ and $x\in X$ there exists a number $\tau=\tau(\varepsilon,x)>0$ such that $|\pi(\tau,x)|<\varepsilon$;*
2. *for every $\varepsilon>0$ and $x\in X$ there exists a number $l=l(\varepsilon,x)>0$ such that $|\pi(t,x)|<\varepsilon$ for all $t\ge l$;*
3. *the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative and $\Omega_X\subseteq\Theta$;*
4. *$\omega_x\bigcap\Theta\neq\emptyset$ for all $x\in X$;*
5. *for all $\varepsilon>0$ and $r>0$ there exists $L=L(\varepsilon,r)>0$ such that*

$$|\pi(t,x)|<\varepsilon \text{ for all } t\ge L(\varepsilon,r) \quad\text{and}\quad |x|\le r. \tag{18}$$

Proof. It easy to check that, under the conditions of Theorem, the following implications 2. $\Longleftrightarrow$ 3. $\Rightarrow$ 4. $\Longleftrightarrow$ 1. hold. Now we will establish the implication 4. $\Rightarrow$ 3. To this end we note that by Lemma 3.15 the set Θ is orbitally stable and, consequently, $D^+(\Theta)=\Theta$. According to Theorem 1.13 [10, Ch.I] the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative and its Levinson center J_X is included in $D^+(\Theta)$. Thus we obtain $J_X\subseteq\Theta$. Since $(X,\mathbb{T}_1,\pi)$ is point dissipative and $\Omega_X\subseteq J_X$ we obtain the necessary statement.

To finish the proof of Theorem it is sufficient, for example, to show that 3. $\Longleftrightarrow$ 5. The implication 5. $\rightarrow$ 3. is evident. According to condition 3. the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative and $\Omega_X\subseteq\Theta$. By Theorem 1.10 [10, Ch.I] the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative and $J_X=D^+(\Omega_X)\subseteq\Theta$, since the set Θ is uniformly stable. Since the Levinson center J_X attracts every compact subset from J_X we have (18). Indeed, if we suppose that it is not true, then there are $\varepsilon_0>0$, $r_0>0$, $\{x_n\}$ and $t_n\to+\infty$ such that

$$|x_n|\le r_0 \quad\text{and}\quad |\pi(t_n,x_n)|\ge\varepsilon_0 \tag{19}$$

for all $n\in\mathbb{N}$. Since Y compact, (X,h,Y) is finite-dimensional and $(X,\mathbb{T}_1,\pi)$ is compact dissipative, then we can suppose that the sequence $\{\pi(t_n,x_n)\}$ is convergent. Denote by $\bar{x}$ its limit, then passing into limit in (19) we obtain $|\bar{x}|\ge\varepsilon_0>0$. On the other hand $\bar{x}\in J_X\subseteq\Theta$ and, consequently, $|\bar{x}|=0$. The obtained contradiction complete the proof of Theorem. □

Remark 3.17. *1. Note that Theorem 3.16 remains true also for the infinite-dimensional case too (i.e., (X,h,Y) is infinite-dimensional), if we suppose that the dynamical system $(X,\mathbb{T}_1,\pi)$ is completely continuous.*

2. Theorem 3.16 remains true if we replace the uniform stability of- the set Θ by uniform stability of $\Theta'=h^{-1}(J_Y)\bigcap\Theta$.

4. Asymptotical Stability of NDS with Minimal Base

In this section we suppose that the complete metric space Y is compact and the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal, i.e., every trajectory $\Sigma_y := \{\sigma(t, y) : t \in \mathbb{T}_2\}$ is dense in Y (this means that $H(y) = Y$ for all $y \in Y$, where $H(y) := \overline{\Sigma}_y$).

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

(i) *the trivial section Θ is uniformly stable with respect to NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$;*
(ii) *$L^+(X) = X$, where $L^+(X) := \{x \in X : \Sigma_x^+$ is relatively compact $\}$;*
(iii) *there exists a point $y_0 \in Y$ such that $X^s_{y_0} = X_{y_0}$, where $X_y := \{x \in X : h(x) = y\}$ and $X^s_y := \{x \in X_y : \lim\limits_{t\to+\infty} |\pi(t, x)| = 0\}$.*

Then $X^s_y = X_y$ for all $y \in Y$.

Proof. Suppose that there exists $\tilde{y} \in Y$ such that $X^s_{\tilde{y}} \neq X_{\tilde{y}}$ and let $\tilde{x} \in X_{\tilde{x}} \setminus X^s_{\tilde{y}}$. Since $\Sigma^+_{\tilde{x}}$ is relatively compact, then the omega limit set $\omega_{\tilde{x}}$ of the point $\tilde{x}$ is a nonempty compact and invariant set. According to choose of the point $\tilde{x}$ there exists at least one point $\bar{x} \in \omega_{\tilde{x}}$ such that $|\bar{x}| \neq 0$. Let $\gamma \in \mathcal{F}_{\bar{x}}(\pi)$ be an entire trajectory of $(X, \mathbb{T}_1, \pi)$ passing through the point $\bar{x}$ at initial moment with the condition $\gamma(\mathbb{S}) \subseteq \omega_{\tilde{x}}$. We will show that

$$\alpha := \inf_{s\le 0} |\gamma(s)| > 0. \tag{20}$$

If we suppose that (20) is not true, then there exists a sequence $s_n \to -\infty$ such that $|\gamma(s_n)| \to 0$ as $n \to \infty$. Since Θ is uniformly stable then for all $0 < \varepsilon < |\bar{x}|/2$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $|x| < \delta$ implies the inequality $|\pi(t, x)| < \varepsilon$ for all $t \geq 0$. Let $n_0 \in \mathbb{N}$ be a sufficiently large number (such that $|\gamma(s_n)| < \delta$ for all $n \geq n_0$), then we have $|\bar{x}| = |\pi(-s_{n_0}, \gamma(s_{n_0}))| < \varepsilon < |\bar{x}|/2$. The obtained contradiction proves our statement. Denote by ν the entire trajectory of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ defined by equality $\nu := h\circ\gamma$, i.e., $\nu(s) = h(\gamma(s))$ for all $s \in \mathbb{S}$, then $\nu \in \mathcal{F}_{\bar{y}}(\sigma)$, where $\bar{y} := h(\bar{x})$. Since Y is minimal, then there exists a sequence $\{\tau_n\}$ from $\mathbb{S}$ such that $\tau_n \to -\infty$ and $\nu(\tau_n) \to y_0$. Under the conditions of Theorem, without loss of generality, we may suppose that the functional sequences $\{\gamma(t + \tau_n)\}_{t\in\mathbb{S}}$ and $\{\nu(t + \tau_n)\}_{t\in\mathbb{S}}$ are convergent (uniformly with respect to t on every compact subset from $\mathbb{S}$). Let $\tilde{\gamma}$ (respectively, $\tilde{\nu}$) be the limit of the sequence $\{\gamma(t+\tau_n)\}_{t\in\mathbb{S}}$ (respectively, $\{\nu(t+\tau_n)\}_{t\in\mathbb{S}}$). Then it is clear that $\tilde{\gamma} \in \mathcal{F}_{\tilde{\gamma}(0)}(\pi)$, $\tilde{\gamma}(\mathbb{S}) \subseteq \alpha_\gamma := \{z :$ there exists a sequence $s_n \to -\infty$ such that $\gamma(s_n) \to z\}$ and $|\tilde{\gamma}(s)| \geq \alpha$ for all $s \in \mathbb{S}$. On the other hand $\tilde{\gamma}(t) = \pi(t, \tilde{\gamma}(0))$ for all $t \geq 0$, $\tilde{\gamma}(0) \in X_{y_0}$ and, consequently, $\lim\limits_{t\to+\infty} |\pi(t, \tilde{\gamma}(0))| = 0$. The obtained contradiction complete the proof of Theorem. □

Lemma 4.2. *Suppose that the trivial section Θ is uniformly stable with respect to NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$. Let $y_0 \in Y$ be an arbitrary point, then the following conditions are equivalent:*

1. *$X^s_{y_0} = X_{y_0}$;*
2. *for every $x \in X_{y_0}$ the semi-trajectory $\Sigma^+_x := \{\pi(t, x) : t \geq 0\}$ is relatively compact and $\omega_x \subseteq \Theta$;*

3. $\omega_x \bigcap \Theta \neq \emptyset$ *for all* $x \in X_{y_0}$*;*
4. *for arbitrary* $\varepsilon > 0$ *and* $x \in X_{y_0}$ *there exists a positive number* $\tau = \tau(x, \varepsilon)$ *such that* $|\pi(\tau, x)| < \varepsilon$*.*

Proof. Note that the implications 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 4. are evident. To finish the proof of Lemma it is sufficient that 4. implies 1.. Indeed, let $\varepsilon > 0$ be an arbitrary positive number, $x \in X$, $\varepsilon_k := 1/k$ $(k \in \mathbb{N})$, and τ_k be a positive number such that $|\pi(\tau_k, x)| < 1/k$. Denote by $\delta(\varepsilon)$ the positive number from uniform stability of Θ for ε (i.e., $|x| < \delta$ implies $|\pi(t, x)| < \varepsilon$ for all $t \geq 0$), then for the sufficiently large k $(1/k < \delta)$ we have $|\pi(t + \tau_k, x)| < \varepsilon$ for all $t \geq 0$. Thus for $\varepsilon > 0$ there exists $l(\varepsilon, x) > 0$ such that $|\pi(t, x)| < \varepsilon$ for all $t \geq l(\varepsilon, x)$, i.e., $x \in X^s_{y_0}$. □

Remark 4.3. *1. The implications* 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 4. *are true without assumption of uniform stability of* Θ*.*

2. Lemma 4.2 remains true without assumption of compactness and minimality of Y*.*

3. If (X, h, Y) *is finite-dimensional, then Lemma 4.2 follows from Theorem 3.16.*

From Theorem 4.1 and Lemma 4.2 we have the following statement.

Corollary 4.4. *Suppose that the following conditions are fulfilled:*

(i) *the trivial section* Θ *is uniformly stable with respect to NDS* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$*;*
(ii) $L^+(X) = X$*, where* $L^+(X) := \{x \in X : \ \Sigma^+_x$ *is relatively compact* $\}$*;*
(iii) *there exists a point* $y_0 \in Y$ *such that one of the conditions 1.–4. of Lemma 4.2 is fulfilled.*

Then $X^s_y = X_y$ *for all* $y \in Y$*.*

Below we give a local version of Theorem 4.1.

Theorem 4.5. *Suppose that the following conditions are fulfilled:*

(i) *the dynamical system* $(X, \mathbb{T}_1, \pi)$ *is asymptotically compact;*
(ii) *the trivial section* Θ *is uniformly stable with respect to NDS* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$*;*
(iii) *there exist positive number* δ_0 *and point* $y_0 \in Y$ *such that* $B(\theta_{y_0}, \delta_0) \subset X^s_{y_0}$*, where* $B(\theta_y, r) := \{x \in X_y : \ |x| < r\}$*.*

Then the trivial section Θ *is asymptotically stable, i.e., there exists a positive number* β *such that* $B(\Theta, \beta) \subset X^s$*, where* $B(\Theta, \beta) := \bigcup\{B(\theta_y, \beta) : \ y \in Y\}$ *and* $X^s := \bigcup\{X^s_y : \ y \in Y\}$*.*

Proof. Since Θ is uniformly stable, then there exists a positive number δ_1 such that $|\pi(t, x)| \leq \delta_0$ for all $t \geq 0$ and $x \in X$ with $|x| \leq \delta_1$. Let now $\beta := \min\{\delta_0, \delta_1\}$. We will show that $B(\Theta, \beta) \subset X^s$. If we suppose that it is not so, then using the same reasoning as in the proof of Theorem 4.1 we obtain a contradiction which proves our statement. □

Remark 4.6. *All results of Sections 3–4 remain true if:*

1. we replace the positive invariance of the trivial section Θ by the following condition: there exists a compact positively invariant set $M \subseteq X$ such that $M_y := \{x \in M : \ h(x) = y\}$ consists a single point for all $y \in Y$;

2. the compact metric space Y we replace by an arbitrary compact regular topological space.

5. Some Applications

5.1. **Ordinary differential equations.** Let E be a Banach space with the norm $|\cdot|$, W be an open subset of E and $0 \in W$. Denote by $C(\mathbb{S} \times W, E)$ the space of all continuous mappings $f : \mathbb{S} \times W \mapsto E$ equipped with the compact open topology. On the space $C(\mathbb{S} \times W, E)$ it is defined a shift dynamical system [10, ChI] (dynamical system of translations or Bebutov's dynamical system) $(C(\mathbb{S} \times W, E), \mathbb{S}, \sigma)$, where σ is a mapping from $\mathbb{S} \times C(\mathbb{S} \times W, E)$ onto $C(\mathbb{S} \times W, E)$ defined as follow $\sigma(\tau, f) := f_\tau$ for all $(\tau, f) \in \mathbb{S} \times C(\mathbb{S} \times W, E)$, where f_τ is the τ-translation of f with respect to time t, i.e., $f_\tau(t, x) := f(t + \tau, x)$ for all $(t, x) \in \mathbb{S} \times W$. Consider a differential equation

$$u' = f(t, u), \tag{21}$$

where $f \in C(\mathbb{R} \times W, E)$.

The function f (respectively, equation (21) is said to be *regular*, if for all $v \in W$ and $g \in H^+(f) := \overline{\{f_\tau : \ \tau \in \mathbb{R}_+\}}$, where by bar is denoted the closure in $C(\mathbb{R} \times W, E)$, the equation

$$v' = g(t, v) \tag{22}$$

admits a unique solution $\varphi(t, v, g)$ passing through the point $v \in W$ at moment $t = 0$ and defined on $\mathbb{R}_+$.

If the function f is regular, then the equation (21) naturally defines a cocycle $\langle W, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$, where $(H^+(f), \mathbb{R}_+, \sigma)$ is a (semi-group) dynamical system on $H^+(f)$ induced by Bebutov's dynamical system.

Applying the general results from Sections 3-4 we will obtain a series of results for equation (21). Below we formulate some of them.

Denote by $\Omega_f := \{g \in H^+(f) : \ \text{there exists a sequence} \tau_n \to +\infty \text{ such that } g = \lim_{n\to\infty} f_{\tau_n}\}$ the ω-limit set of f.

The null solution of equation (21) with $f \in C(\mathbb{R} \times E, E)$ is said to be *globally asymptotically stable* if it is asymptotically stable and

$$\lim_{t \to +\infty} |\varphi(t, v, g)| = 0$$

for all $(v, g) \in E \times H^+(f)$.

A trivial solution of equation (21) is called uniformly attracting (respectively, eventual uniform attracting [2]), if for every compact subset $K \subset E$ and for every

$\varepsilon > 0$ there exists $T = T(K, \varepsilon) > 0$ (respectively, there exist $\gamma = \gamma(K) > 0$ and $T = T(K, \varepsilon) > 0$) such that

$$x_0 \in K, \ t \geq t_0 + T \ \text{ implies } \ |x_f(t; t_0, x_0)| < \varepsilon$$

(respectively,

$$x_0 \in K, \ t_0 \geq \gamma, \ \ t \geq t_0 + T \ \text{ implies } \ |x_f(t; t_0, x_0)| < \varepsilon),$$

where by $x_f(t; t_0, x_0)$ is denoted a unique solution $x(t)$ of equation (21) with initial data $x(t_0) = x_0$.

The solutions of equation (21) are said to be uniformly bounded [2] if for all $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that

$$|x_0| \leq \alpha, \ \ t_0 \in \mathbb{R}_+, \ t \geq t_0 \ \rightarrow \ |x_f(t; t_0, x_0)| \leq \beta.$$

Lemma 5.1. *Suppose that the following conditions are fulfilled:*

(i) $f \in C(\mathbb{R} \times E, E)$*;*
(ii) *the function* f *is regular;*
(iii) *the set* $H^+(f)$ *is compact;*
(iv) $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$.

Let φ *be a cocle, generated by equation (21), then the following statements hold:*

(i) *if the trivial solution of equation (21) is uniformly attraction, then the trivial solution/motion of the cocycle* φ *is uniformly attracting;*
(ii) *if the trivial solution of equation (21) is eventual uniform attracting, then the trivial solution/motion of the cocycle* φ *possesses the following property:*

$$\lim_{t \to +\infty} \max_{x \in K, g \in \Omega_f} |\varphi(t, x, g)| = 0 \tag{23}$$

for all compact subset K *from* E*;*
(iii) *if the solutions of equation (21) are uniformly bounded, then the solutions/motions of the cocycle* φ *uniformly bounded, i.e., for all* $\alpha > 0$ *there exists* $\beta = \beta(\alpha) > 0$ *such that* $|x| \leq \alpha$ *implies* $|\varphi(t, x, g)| \leq \beta$ *for all* $t \in \mathbb{R}_+$ *and* $g \in H^+(f)$.

Proof. The first statement of Lemma is well known [22, Ch.VIII].

To prove the second statement we note that

$$\varphi(t, x, f_{t_0}) = x(t + t_0; t_0, x) \tag{24}$$

for all $t, t_0 \in \mathbb{R}_+$ and $x \in E$. Let now K be an arbitrary compact subset from E and $\varepsilon > 0$ be an arbitrary positive number. Denote by $\gamma = \gamma(K)$ and $T = T(K, \varepsilon)$ positive numbers from eventual uniform attractivity of null solution of equation (21). Let now $x \in K$ and $g \in \Omega_f$, then there exists a sequence $t_n \to +\infty$ such that $f_{t_n} \to g$ (in the space $C(\mathbb{R} \times E, E)$) and, consequently, $t_n \geq \gamma$ for sufficiently large n. Note that

$$|\varphi(t, x, g)| = \lim_{n \to +\infty} |\varphi(t, x, f_{t_n})| = \lim_{n \to +\infty} |x_f(t + t_n; t_n, x)| \leq \varepsilon \tag{25}$$

$$\text{for all } \ t \geq T(K, \varepsilon).$$

From (23) evidently it follows (23).

Finally we will prove the third statement. Let $\alpha > 0$ and $\beta = \beta(\alpha) > 0$ is taken from the uniformly boundedness of the solutions of (21). Let $|x| \le \alpha$, $g \in H^+(f)$ and $t \in \mathbb{R}_+$, then there exists a sequence $\{t_n\} \subseteq \mathbb{R}_+$ such that $g = \lim\limits_{t\to+\infty} f_{t_n}$. Note that

$$|\varphi(t,x,g)| = \lim_{n\to\infty} |\varphi(t,x,f_{t_n})| = \lim_{n\to\infty} |x_f(t+t_n;t_n,x)| \le \beta(\alpha).$$

Lemma is completely proved. □

Theorem 5.2. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *the cocycle φ generated by equation (21) is locally compact, i.e., for every point $u \in E$ there exists a neighborhood U of the point u and a positive number l such that the set $\varphi(l,U,H^+(f))$ is relatively compact.*

Then the null solution of equation (21) is globally asymptotically stable if and only if the following conditions hold:

(i)

$$\lim_{t\to+\infty} \sup_{v\in K, g\in\Omega_f} |\varphi(t,v,g)| = 0 \tag{26}$$

for every compact subset K from E;

(ii) *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (22) is relatively compact on $\mathbb{R}_+$.*

Proof. Consider the dynamical system $(H^+(f),\mathbb{R}_+,\sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center (maximal compact invariant set) $J_{H^+(f)}$ evidently coincides with ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X := W \times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generates by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ, i.e., $\pi(t,(v,g)) := (\varphi(t,v,g),\sigma(t,g))$ for all $t \in \mathbb{R}_+$ and $(v,g) \in X$. Now consider a non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ ($h := pr_2$) associated by equation (21). It easy to verify that this NDS possesses the following properties:

(i) by Lemma 2.1 the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
(iv) according to (26) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X},\mathbb{R}_+,\pi),(J_Y,\mathbb{R}_+,\sigma),h\rangle$ is uniformly attracting because $|\pi(t,x)| = |\varphi(t,v,g)|$ for all $t \in \mathbb{R}_+$ and $x := (v,g) \in X$;
(v) every trajectory $\Sigma^+_{(u,g)}$ ($(u,g) \in E\times H^+(f)$) of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$, generated by equation (21), is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 3.5 and Theorem 3.13. □

Corollary 5.3. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *the cocycle φ generated by equation (21) is locally compact;*
(v) *the null solution of equation (21) is eventual uniform attracting;*
(vi) *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (22) is relatively compact on $\mathbb{R}_+$.*

Then the null solution of equation (21) is globally asymptotically stable.

Proof. This statement follows from Theorem 5.2. In fact. According to Lemma 5.1 from uniform eventual attraction of the null solution of equation (21 it follows condition (26). Now to finish the proof of this statement it is sufficient to apply Theorem 5.2. □

Remark 5.4. *1. For finite-dimensional equation (21) Corollary 5.3 generalizes a statement (Theorem 2.6) established in the work* [2] *(see also* [20, Ch.I] *and the bibliography therein).*

2. If the cocycle φ associated by equation (21) is asymptotically compact (in particularly, if it is completely continuous), then Theorem 5.2 remains true if we replace condition (ii) by the following: for all $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ is bounded on $\mathbb{R}_+$.

Theorem 5.5. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *the cocycle φ generated by equation (21) is completely continuous, i.e., for every bounded subset $M \in E$ there exists a positive number l such that the set $\varphi(l, M, H^+(f))$ is relatively compact.*

Then the null solution of equation (21) is globally asymptotically stable if and only if the following conditions hold:

a. *for every $g \in \Omega_f$ limiting equation (22) does not a nontrivial bounded on $\mathbb{R}$ solutions;*
b. *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (22) is bounded on $\mathbb{R}_+$.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 5.2 plus application Corollary 3.10. □

Remark 5.6. *Theorem 5.5 remains true if we replace the completely continuity by the following two conditions:*

(i) *the cocycle* φ *is asymptotically compact:*
(ii) *the cocycle* φ *is locally completely continuous.*

Theorem 5.7. *Suppose that the following conditions are fulfilled:*

(i) *the function* $f \in C(\mathbb{R} \times W, E)$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact subset from* W*;*
(ii) $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*
(iii) *the function* f *is regular;*
(iv) *the cocycle* φ *associated by equation (21) is asymptotically compact;*
(v) *the null solution of equation (21) is uniformly stable;*
(vi) *there exists a positive number* a *such that*

$$\lim_{t \to +\infty} |\varphi(t, u, f)| = 0$$

for all $|u| \le a$*.*

Then the null solution of equation (21) is asymptotically stable.

Proof. This statement it follows directly from Theorem 4.5 using the same arguments as in the proof of Theorem 5.2. □

Remark 5.8. *For finite-dimensional equation (21) with almost periodic hand right side* f *Theorem 5.7 was established by Z. Artstein* [3] *(see also* [1]*,* [18] *and* [20, Ch.I]*).*

5.2. **Functional differential-equations.** We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

5.2.1. *Functional-differential equations (FDEs) with finite delay.* Let us first recall some notions and notations from [16]. Let $r > 0$, $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a, b] = [-r, 0]$, then we set $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \ge 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for all $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t + \theta)$, $-r \le \theta \le 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{27}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is defined (see, for example, [10, ChI]) a shift dynamical system $(C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $\sigma(\tau, f) := f_\tau$ for all $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f_τ is τ-translation of f, i.e., $f_\tau(t, \phi) := f(t + \tau, \phi)$ for all $(t, \phi) \in \mathbb{R} \times \mathcal{C}$.

Let us set $H^+(f) := \overline{\{f_s : s \in \mathbb{R}_+\}}$, where by bar we denote the closure in $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$.

Along with the equation (27) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{28}$$

where $g \in H^+(f)$.

A function $f \in C(\mathbb{R} \times C_r, \mathbb{R}^n)$ (respectively, equation (27)) is called [22] *regular*, if for $v \in C_r$ and $g \in H^+(f)$ equation (28) admits a unique solution passing through v at the initial moment $t = 0$.

Below, in this subsection, we suppose that equation (27) is regular.

Remark 5.9. 1. Denote by $\tilde{\varphi}(t, u, f)$ the solution of equation (27) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0, u, f) = u \in C_r$, i.e., $\varphi(s, u, f) = u(s)$ for all $s \in [-r, 0]$. By $\varphi(t, u, f)$ we will denote below the trajectory of equation (27), corresponding to the solution $\tilde{\varphi}(t, u, f)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into C_r, defined by $\varphi(t, u, f)(s) := \tilde{\varphi}(t + s, u, f)$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r, 0]$.

2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (27) as synonym concepts.

It is well known [7, 22] that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H^+(f) \mapsto \mathbb{R}^n$ possesses the following properties:

(i) $\varphi(0, v, g) = u$ for all $v \in \mathcal{C}$ and $g \in H^+(f)$;
(ii) $\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), \sigma(\tau, g))$ for all $t, \tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H^+(f)$;
(iii) the mapping φ is continuous.

Thus, a triplet $\langle \mathcal{C}, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle which is associated to equation (27). Applying the results from Sections 3-4 we will obtain a series of results for functional differential equation (27). Below we formulate some of them.

Lemma 5.10. [13] *Suppose that the following conditions hold:*

(i) *the function* $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ *is regular;*
(ii) *the set* $H^+(f)$ *is compact;*
(iii) *the function* f *is completely continuous, i.e., the set* $f(\mathbb{R}_+ \times A)$ *is bounded for all bounded subset* $A \subseteq \mathcal{C}$.

Then the cocycle φ *associated by (27) is completely continuous, i.e., for all bounded subset* $A \subseteq W$ *there exists a positive number* $l = l(A)$ *such that the set* $\varphi(l, A, H^+(f))$ *is relatively compact in* $\mathcal{C}$.

The null solution of equation (27) with $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is said to be *globally asymptotically stable* if it is asymptotically stable and $\lim_{t \to +\infty} |\varphi(t, v, g)| = 0$ for all $(v, g) \in \mathcal{C} \times H^+(f)$.

Theorem 5.11. *Let* $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$. *Assume that the following conditions are fulfilled:*

(i) *the function* f *is regular;*
(ii) *the set* $H^+(f)$ *is compact;*
(iii) $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$;
(iv) *the function* f *is completely continuous.*

Then the null solution of equation (27) is globally asymptotically stable if and only if the following conditions hold:

(i)
$$\lim_{t\to+\infty}\sup_{|v|\le a, g\in\Omega_f}|\varphi(t,v,g)|=0$$
for every $a>0$;

(ii) *for every $v\in\mathcal{C}$ and $g\in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (28) is bounded on $\mathbb{R}_+$.*

Proof. Consider the dynamical system $(H^+(f),\mathbb{R}_+,\sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and by Lemma 2.1 its Levinson center $J_{H^+(f)}$ coincides with ω-limit set Ω_f of f. Let $Y:=H^+(f)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X:=\mathcal{C}\times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generates by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ. Now consider a NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ ($h:=pr_2$) associated by equation (27). It easy to verify this NDS posses the following properties:

(i) the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compact dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;
(iii) Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
(iv) according to (29) the null section $\tilde{\Theta}$ of NDS $\langle(\tilde{X},\mathbb{R}_+,\pi),(J_Y,\mathbb{R}_+,\sigma),h\rangle$ is uniformly attracting because $|\pi(t,x)|=|\varphi(t,v,g)|$ for all $t\in\mathbb{R}_+$ and $x:=(v,g)\in X$;
(v) according to Lemma 5.10 the dynamical system $(X,\mathbb{R}_+,\pi)$ is completely continuous;
(vi) every positive semi-trajectory Σ_x^+ of skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ is relatively compact.

Now to finish the proof it is sufficient to apply Corollary 3.14. □

Theorem 5.12. *Let $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *$f(t,0)=0$ for all $t\in\mathbb{R}_+$;*
(iv) *the function f is completely continuous.*

Then the null solution of equation (27) is globally asymptotically stable if and only if the following conditions hold:

a. *for every $g\in\Omega_f$ limiting equation (28) does not a nontrivial bounded on $\mathbb{R}$ solutions;*
b. *for every $v\in\mathcal{C}$ and $g\in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (28) is bounded on $\mathbb{R}_+$.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 5.11 plus application Corollary 3.10. □

Theorem 5.13. *Suppose that the following conditions are fulfilled:*

(i) *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact from* $\mathcal{C}$*;*
(ii) $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*
(iii) *the function* f *is regular;*
(iv) *the function* f *is completely continuous;*
(v) *the null solution of equation (27) is uniformly stable;*
(vi) *there exists a positive number* a *such that*

$$\lim_{t\to+\infty} \sup_{|u|\le a} |\varphi(t, u, f)| = 0.$$

Then the null solution of equation (27) is asymptotically stable.

Proof. This statement follows directly from Theorem 4.5 using the same arguments as in the proof of Theorem 5.11. □

5.2.2. *Neutral functional-differential equations.* Now consider the neutral functional-differential equation

$$\frac{d}{dt} Du_t = f(t, u_t), \tag{29}$$

where $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ and the operator $D : \mathcal{C} \mapsto \mathbb{R}^n$ is atomic at zero [16, p.67]. Like (27), equation (29) generates a NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$, where $X := \mathcal{C} \times Y$, $Y := H^+(f)$, and $\pi := (\varphi, \sigma)$.

An operator D is said to be stable, if the zero solution of difference equation $Dy_t = 0$ is uniformly asymptotically stable (see, for example, [16, p.337]).

Lemma 5.14. *Let* $H^+(f)$ *be compact. If the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ *is completely continuous, then the NDS* $(X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ *generated by equation (29) is asymptotically compact.*

Proof. This statement can be proved by slight modification of the proof of Theorem 12.6.3 and Lemma 12.3.2 from [16, Ch.XII] and taking into account that $Y = H^+(A)$ is compact. □

Theorem 5.15. *Suppose that the following conditions are fulfilled:*

(i) *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is recurrent with respect to* $t \in \mathbb{R}$ *uniformly with respect to spacial variable* u *on every compact subset from* $\mathcal{C}$*;*
(ii) $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$*;*
(iii) *the function* f *is regular;*
(iv) *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ *is completely continuous;*
(v) *the null solution of equation (29) is uniformly stable;*
(vi) *there exists a positive number* a *such that*

$$\lim_{t\to+\infty} |\varphi(t, u, f)| = 0 \tag{30}$$

for all $|u| \le a$.

Then the null solution of equation (29) is asymptotically stable, i.e., there exists a positive number δ such that $\lim_{t\to+\infty}|\varphi(t,v,g)|=0$ for all $|v|<\delta$ and $g\in H^+(f)$.

Proof. Let $(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a NDS generated by equation (29). It is easy to check that under the conditions of Theorem 5.15 the following conditions hold:

(i) the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compact dissipative and its Levinson center J_Y coincides with $Y=H^+(f)=\Omega_f$;
(ii) the null section Θ of $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;
(iii) Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
(iv) according to (30) we have $(0_f,a)\subset X^s_f$, where $0_f:=(0,f)$ and 0 is the null element of E;
(v) according to Lemma 5.14 the dynamical system $(X,\mathbb{R}_+,\pi)$ is asymptotically compact.

Now to finish the proof of Theorem it is sufficient to apply Theorem 4.5. □

5.3. **Semi-linear parabolic equations.** Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|^2:=\langle\cdot,\cdot\rangle$, and A be a self-adjoint operator with domain $D(A)$.

An operator is said (see, for example, [14]) to have a *discrete spectrum* if in the space H, there exists an ortho-normal basis $\{e_k\}$ of eigenvectors, such that $\langle e_k,e_j\rangle=\delta_{kj}$, $Ae_k=\lambda_k e_k$ $(k,j=1,2,\dots)$ and $0<\lambda_1\le\lambda_2\le\dots,\lambda_k\le\dots$, and $\lambda_k\to+\infty$ as $k\to+\infty$.

One can define an operator $f(A)$ for a wide class of functions f defined on the positive semi-axis as follows:

$$D(f(A)):=\{h=\textstyle\sum_{k=1}^{\infty}c_ke_k\in H:\ \sum_{k=1}^{\infty}c_k[f(\lambda_k)]^2<+\infty\},$$
$$f(A)h:=\textstyle\sum_{k=1}^{\infty}c_kf(\lambda_k)e_k,\quad h\in D(f(A)). \tag{31}$$

In particular, we can define operators A^α for all $\alpha\in\mathbb{R}$. For $\alpha=-\beta<0$ this operator is bounded. The space $D(A^{-\beta})$ can be regarded as the completion of the space H with respect to the norm $|\cdot|_\beta:=|A^{-\beta}\cdot|$.

The following statements hold [14]:

(i) The space $\mathcal{F}_{-\beta}:=D(A^{-\beta})$ with $\beta>0$ can be identified with the space of formal series $\sum_{k=1}^{\infty}c_ke_k$ such that
$$\sum_{k=1}^{\infty}c_k\lambda_k^{-2\beta}<+\infty;$$
(ii) For any $\beta\in\mathbb{R}$, the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded operator mapping $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that
$$A^\beta D(A^\alpha)=D(A^{\alpha-\beta}),\ A^{\beta_1+\beta_2}=A^{\beta_1}A^{\beta_2}.$$
(iii) For all $\alpha\in\mathbb{R}$, the space $\mathcal{F}_\alpha:=D(A^\alpha)$ is a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\alpha:=\langle A^\alpha\cdot,A^\alpha\cdot\rangle$ and the norm $|\cdot|_\alpha:=|A^\alpha\cdot|$.
(iv) The operator A with the domain $\mathcal{F}_{1+\alpha}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\alpha$.

(v) The embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$ is continuous, i.e., $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and there exists a positive constant $C = C(\alpha, \beta)$ such that $|\cdot|_\beta \le C|\cdot|_\alpha$.
(vi) $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.
(vii) Let $\alpha_1 > \alpha_2$, then the space $\mathcal{F}_{\alpha_1}$ is compactly embedded into $\mathcal{F}_{\alpha_2}$, i.e., every sequence bounded in $\mathcal{F}_{\alpha_1}$ is relatively compact in $\mathcal{F}_{\alpha_2}$.
(viii) The resolvent $\mathcal{R}_\lambda(A) := (A - \lambda I)^{-1}$, $\lambda \neq \lambda_k$ is a compact operator in each space $\mathcal{F}_\alpha$, where I is the identity operator.

According to (31) we can define an exponential operator e^{-tA}, $t \ge 0$, in the scale spaces $\{\mathcal{F}_\alpha\}$. Note some of its properties [14]:

a. For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator e^{-tA} maps $\mathcal{F}_\alpha$ into $\bigcap\limits_{\beta \ge 0} \mathcal{F}_\beta$ and
$$|e^{-tA}x|_\alpha \le e^{-\lambda_1 t}|x|_\alpha$$
for all $x \in \mathcal{F}_\alpha$.
b. $e^{-t_1A}e^{-t_2A} = e^{-(t_1+t_2)A}$ for all $t_1, t_2 \in \mathbb{R}_+$;
c.
$$|e^{-tA}x - e^{-\tau A}x|_\beta \to 0$$
as $t \to \tau$ for every $x \in \mathcal{F}_\beta$ and $\beta \in \mathbb{R}$;
d. For any $\beta \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\beta, e^{-tA})$;
e.
$$|A^\alpha e^{-tA}h| \le \Big[\big(\tfrac{\alpha-\beta}{t}\big)^{\alpha-\beta} + \lambda_1^{\alpha-\beta}\Big]e^{-t\lambda_1}|A^\beta h|,\ \alpha \ge \beta$$
$$||A^\alpha e^{-tA}|| \le \big(\tfrac{\alpha}{t}\big)^\alpha e^{-\alpha},\ t > 0,\ \alpha > 0.$$

Consider an evolutionary differential equation

$$u' + Au = F(t, u) \tag{32}$$

in the separable Hilbert space H, where A is a linear (generally speaking unbounded) positive operator with discrete spectrum, and F is a nonlinear continuous mapping acting from $\mathbb{R} \times \mathcal{F}_\theta$ into H, $0 \le \theta < 1$, possessing the property

$$|F(t, u_1) - F(t, u_2)| \le L(r)|A^\theta(u_1 - u_2)| \tag{33}$$

for all $u_1, u_2 \in B_\theta(0, r) := \{u \in \mathcal{F}_\theta :\ |u|_\theta \le r\}$. Here $L(r)$ denotes the Lipschitz constant of F on the set $B_\theta(0, r)$.

A function $u : [0, a) \mapsto \mathcal{F}_\theta$ is said to be a *weak solution* (in $\mathcal{F}_\theta$) of equation (32) passing through the point $x \in \mathcal{F}_\theta$ at moment $t = 0$ (notation $\varphi(t, x, F)$) if $u \in C([0, T], \mathcal{F}_\theta)$ and satisfies the integral equation

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A}F(\tau, u(\tau))d\tau$$

for all $t \in [0, T]$ and $0 < T < a$.

In the book [14], it is proved that, under the conditions listed above, there exists a unique solution $\varphi(t, x, F)$ of equation (33) passing through the point x at moment $t = 0$, and it is defined on a maximal interval $[0, a)$, where a is some positive number depending on (x, F).

Denote by $C(\mathcal{R}\times\mathcal{F}_\theta, H)$ the space of all continuous mappings equipped with the compact open topology and by $(C(\mathcal{R}\times\mathcal{F}_\theta, H),\mathbb{R},\sigma)$ the shift dynamical system on $C(\mathcal{R}\times\mathcal{F}_\theta, H)$.

A function $F\in C(\mathbb{R}\times\mathcal{F}_\theta, H)$ is said to be *regular* if for all $v\in\mathcal{F}_\theta$ and $G\in H^+(F):=\overline{\{\sigma(\tau,F):\ \tau\in\mathbb{R}_+\}}$, where by bar is denote the closure in the space $C(\mathbb{R}\times\mathcal{F}_\theta, H)$, there exists a unique (weak) solution $\varphi(t,v,G)$ of equation

$$u'+Au=G(t,u) \tag{34}$$

defined on $\mathbb{R}_+$ and passing through the point v at moment. Denote by $(H^+(F),\mathbb{R}_+,\sigma)$ a shift dynamical system on $H^+(F)$ induced by $(C(\mathcal{R}\times\mathcal{F}_\theta, H),\mathbb{R},\sigma)$. From general properties of solutions of evolution equation (32) and Theorem 5.1 [9] it follows that the triplet $\langle\mathcal{F}_\theta,\varphi,(H^+(F),\mathbb{R}_+,\sigma)\rangle$ is a cocycle over $(H^+(F),\mathbb{R}_+,\sigma)$ with the fiber $\mathcal{F}_\theta$.

Applying results from Sections 3-4 we obtain a series of results for evolution equation (32). Now we will formulate some of them.

Lemma 5.16. *Under the conditions listed above, if the function F is regular and the set $H^+(F)$ is compact, then the cocycle φ associated by equation (32) is completely continuous.*

Proof. This statement can be proved with the slight modification of the proof of Lemma 5.3 [9]. □

Theorem 5.17. *Let $F\in C(\mathbb{R}\times\mathcal{F}_\theta, H)$. Assume that the following conditions are fulfilled:*

(i) *the function F is regular;*
(ii) *the set $H^+(F)$ is compact;*
(iii) *$F(t,0)=0$ for all $t\in\mathbb{R}_+$.*

Then the null solution of equation (32) is globally asymptotically stable if and only if the following conditions hold:

(i)
$$\lim_{t\to+\infty}\sup_{|v|\le a,g\in\Omega_f}|\varphi(t,v,G)|=0$$
for every $a>0$;

(ii) *for every $v\in\mathcal{F}_\theta$ and $G\in H^+(F)$ the solution $\varphi(t,v,G)$ of equation (34) is bounded on $\mathbb{R}_+$.*

Proof. Consider the dynamical system $(H^+(F),\mathbb{R}_+,\sigma)$. Since the space $H^+(F)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center $J_{H^+(F)}$ coincides with ω-limit set Ω_F of F. Let $Y:=H^+(F)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X:=\mathcal{F}_\theta\times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generates by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ. Consider a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ ($h:=pr_2$) associated by equation (32). It easy to verify that this NDS posses the following properties:

(i) the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compact dissipative and by Lemma 2.1 its Levinson center J_Y coincides with Ω_F;
(ii) the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
(iv) according to (35) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h\rangle$ is uniformly attracting because $|\pi(t, x)| = |\varphi(t, v, g)|$ for all $t \in \mathbb{R}_+$ and $x := (v, g) \in X$;
(v) by Lemma 5.16 the cocycle φ (and, consequently, the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ too) is completely continuous;
(vi) every positive semi-trajectory Σ_x^+ of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 3.13. □

Theorem 5.18. *Let $F \in C(\mathbb{R} \times \mathcal{F}_\theta, H)$. Assume that the following conditions are fulfilled:*

(i) *the function F is regular;*
(ii) *the set $H^+(F)$ is compact;*
(iii) *$F(t, 0) = 0$ for all $t \in \mathbb{R}_+$.*

Then the null solution of equation (32) is globally asymptotically stable if and only if the following conditions hold:

a. *for every $G \in \Omega_F$ limiting equation (34) does not a nontrivial bounded on $\mathbb{R}$ solutions;*
b. *for every $v \in \mathcal{C}$ and $G \in H^+(F)$ the solution $\varphi(t, v, g)$ of equation (34) is bounded on $\mathbb{R}_+$.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 5.17 plus application Corollary 3.10. □

Theorem 5.19. *Suppose that the following conditions are fulfilled:*

(i) *the function $F \in C(\mathbb{R} \times \mathcal{F}_\theta, H)$ is recurrent with respect to $t \in \mathbb{R}$ uniformly with respect to spacial variable u on every compact subset from $W \subseteq \mathcal{F}_\theta$;*
(ii) *$F(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
(iii) *the function F is regular;*
(iv) *the null solution of equation (32) is uniformly stable;*
(v) *there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t, u, F)| = 0$$

for all $|u| \leq a$.

Then the null solution of equation (32) is asymptotically stable.

Proof. This statement follows directly from Theorem 4.5 using the same arguments as in the proof of Theorem 5.17. □

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