ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We analyze the existence of almost periodic (respectively, almost automorphic, recurrent) solutions of the following linear differential equation $u'(t)+A(\sigma(t,y))u(t) = f(\sigma(t,y))$ $(y \in Y)$, in a Banach space, with almost periodic (respectively, almost automorphic, recurrent) coefficients, where $(Y, \mathbb{R}, \sigma)$ is a dynamical system on the metric space Y. In particular, we prove that, if the operator $A(y)$ is positive and self-adjoint, then, for the equation

$$u'(t) + A(\sigma(t,y))u(t) = 0, \tag{1}$$

one of the following alternatives is fulfilled:

(i) There exists a complete trajectory of (1) with constant positive norm;

(ii) The trivial solution of equation (1) is uniformly asymptotically stable.

We investigate this problem within the framework of general linear non-autonomous dynamical systems. We apply our general results also to the cases of functional-differential equations and difference equations.

1. Introduction

In this paper we investigate the existence of almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff) solutions of some linear differential equations in a Banach space with almost periodic (respectively, almost automorphic, jointly recurrent in the sense of Birkhoff) coefficients of the form

$$u'(t) + A(\sigma(t,y))u(t) = f(\sigma(t,y)), \quad (y \in Y), \tag{2}$$

where Y is a compact metric space, and $(Y, \mathbb{R}, \sigma)$ is a dynamical system on Y. We suppose that the following conditions hold:

1. Y is a compact minimal set, and y is an almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff) point;
2. $f \in C(Y, E)$, where E is a Banach space with norm $|\cdot|$, and $C(Y, E)$ denotes the Banach space of continuous functions $f : Y \to E$ equipped with the norm $||f|| := \max_{y\in Y} |f(y)|$;
3. $A \in C(Y, [E]))$, where $[E]$ is the Banach space of all linear bounded operators acting on the space E and furnished with the operator norm
$$||A|| := \sup_{x\in E, |x|\le 1} |Ax|;$$
4. $A(y)$ is self-adjoint for all $y \in Y$;

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5. $A(y)$ is dissipative (i.e., $A(y) \geq 0$) for all $y \in Y$.

For finite-dimensional systems (i.e., $E = \mathbb{R}^n$), the problem

$$u'(t) + A(t)u(t) = f(t) \tag{3}$$

with almost periodic coefficients was studied by Cieutat and Haraux in [12], where the following results were established.

Theorem 1.1. *(See [12]) Assume that $A : \mathbb{R} \to [\mathbb{R}^n]$ is a continuous almost periodic operator-valued function, such that, for all $t \in \mathbb{R}$, $A(t)$ is symmetric and $A(t) \geq 0$. If the average matrix*

$$M(A) := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A(s)ds$$

is positive definite (i.e., $KerM(A) = \{0\}$), then, for each almost periodic forcing term $f : \mathbb{R} \to \mathbb{R}^n$, there exists a unique, exponentially stable almost periodic solution u of (3).

Recall (see [12]) that *a process* on $\mathbb{R}^n$ is a two-parameter family of maps $U(t, \tau) : \mathbb{R}^n \to \mathbb{R}^n$, defined for $(t, \tau) \in \mathbb{R} \times \mathbb{R}_+$ (notice that t represents the initial time and τ the elapsed one), and satisfying

(i) $\forall\, t \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^n$, $U(t, 0)x = x$;
(ii) $\forall\, (t, s, \tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, $\forall\, x \in \mathbb{R}^n$, $U(t, s + \tau) = U(t + \tau, s)U(t, \tau)$;
(iii) $\forall\, \tau \in \mathbb{R}_+$, the one-parameter family of maps $U(t, \tau) : \mathbb{R}^n \to \mathbb{R}^n$ with parameter $t \in \mathbb{R}$ is equicontinuous.

A process U on $\mathbb{R}^n$ is said to be *contractive* if $|U(t, \tau)x_1 - U(t, \tau)x_2| \leq |x_1 - x_2|$ $\forall (t, \tau) \in \mathbb{R} \times \mathbb{R}_+$, $\forall\, x_1, x_2 \in \mathbb{R}^n$.
A process U on $\mathbb{R}^n$ is called *almost periodic* if for any sequence $\{s'_n\} \subset \mathbb{R}$, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ such that the sequence $\{U_{s_n}(t, \tau)x\}$ converges to some $V(t, \tau)x$ in $\mathbb{R}^n$ uniformly in $t \in \mathbb{R}$ and pointwise in $(\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $U_s(t, \tau)$ is the s–translation of $U(t, \tau)$, i.e., $U_s(t, \tau) := U(t + s, \tau)$.
A *complete trajectory* through $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is a map $u : \mathbb{R} \to \mathbb{R}^n$ such that $u(t) = x$ and $u(\tau + s) = U(s, \tau)u(s)$ for all $(s, \tau) \in \mathbb{R} \times \mathbb{R}_+$.

Theorem 1.2. *(See [12]) Let $U = U(t, \tau)$ be an almost periodic linear contraction process on $\mathbb{R}^n$. Then, one of the following alternatives is fulfilled:*

(i) *There is a complete trajectory $z = z(s)$ of U with constant positive norm;*
(ii) *There are two constants $C \geq 0$, $\delta > 0$ such that*

$$||U(t, \tau)||_{[\mathbb{R}^n]} \leq Ce^{-\delta\tau},$$

for all $t \in \mathbb{R}$ and $\tau > 0$.

Denote by $\varphi(t, u_0, y)$ the unique solution of equation (2) with initial value u_0 at time $t = 0$. Then, from the general properties of linear equations, we have:

(C1) $\varphi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
(C2) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{R}$ and $(u, y) \in E \times Y$;
(C3) The mapping $\varphi : \mathbb{R} \times E \times Y \to E$ is continuous;
(C4) The mapping $\varphi(t, \cdot, y) : E \to E$ is linear for all $(t, y) \in \mathbb{R} \times Y$.

The triplet $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (shortly φ) is said to be *a cocycle* [21] over the dynamical system $(Y, \mathbb{R}, \sigma)$, if the properties (C1)–(C3) are fulfilled. If, additionally, the property (C4) holds, then the cocycle φ is called *linear*.

Remark 1.3. *Below we will also consider more general cases. Namely,*

(i) *The one-sided cocycles, i.e., when the mapping φ is defined only on $\mathbb{R}_+ \times E \times Y$;*
(ii) *The cocycles with discrete time, i.e., instead of the continuous time $\mathbb{R}$ (respectively, $\mathbb{R}_+$) we consider the discrete time $\mathbb{Z}$ (respectively, $\mathbb{Z}_+$).*

The aim of this paper is to generalize the results of Cieutat and Haraux [12] (Theorem 1.1 and Theorem 1.2 previously stated) to the case of general linear non-autonomous (cocycle) dynamical systems (Theorem 4.6 and Theorem 5.7), and to apply them to the study of ordinary linear differential equations (Theorem 6.5 and Theorem 6.11), functional-differential equations (Theorem 6.18 and Theorem 6.19) and difference equations (Theorem 6.26 and Theorem 6.27).
The paper is organized as follows.
In Section 2, some notions and facts from the theory of autonomous/non-autonomous dynamical systems are collected: almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff) motions; shift dynamical systems and almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff) functions; cocycles, skew-product dynamical systems and non-autonomous dynamical systems; global attractors, etc.
Section 3 contains a brief summary of results from [9, 10] concerning bounded motions of linear non-autonomous dynamical systems with minimal base.
In Section 4 and 5 we establish the main abstract results of our paper. Section 4 is dedicated to the study of linear contractive systems. The main result of this section is Theorem 4.6 which provides a detailed description of the asymptotic behavior of trajectories for the non-autonomous linear contractive systems with minimal base.
In Section 5, we investigate the existence of uniformly compatible (in the sense of Shcherbakov [23]-[26]) motions of linear non-homogeneous (affine) systems, if the corresponding linear homogeneous system is contractive (Theorem 5.7).
Finally, Section 6 is concerned with the applications of our general results, obtained in Section 4 and 5, to ordinary linear almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff) equations, functional-differential equations and difference equations.

2. Almost Periodic and Almost Automorphic Motions of Dynamical Systems

To make this paper as much self-contained as possible, let us collect in this section some well-known concepts and results from the theory of dynamical systems which will be necessary for our analysis.

2.1. **Recurrent, Almost Periodic and Almost Automorphic Motions.**

Let (X,ρ) be a complete metric space. By $\mathbb{S}$ we will denote either $\mathbb{R}$ or $\mathbb{Z}$ and by $\mathbb{T}$ a sub-semigroup of $\mathbb{S}$.
Let $(X,\mathbb{T},\pi)$ be *a dynamical system* on X, i.e., let $\pi : \mathbb{T}\times X\to X$ be a continuous function such that $\pi(0,x)=x$ for all $x\in X$, and $\pi(t_1+t_2,x)=\pi(t_2,\pi(t_1,x))$, for all $x\in X$, and $t_1,t_2\in\mathbb{T}$.
Given $\varepsilon>0$, a number $\tau\in\mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x, if $\rho(\pi(\tau,x),x)<\varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x))<\varepsilon$ for all $t\in\mathbb{T}$).

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$, i.e., $\sigma(t,\nu) := (\nu_1 t, \nu_2 t, \dots, \nu_m t)$ for all $t \in \mathbb{S}$ and $\nu \in \mathcal{T}^m$ and the numbers $\nu_1, \nu_2, \dots, \nu_m$ are rationally independent.
A point $x \in X$ is called *quasi-periodic* with frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ such that $\pi(t,x) := \Phi(\sigma(t,\omega))$ for all $t \in \mathbb{T}$, where $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$ and $\omega \in \mathcal{T}^m$.
A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that, in any segment of length l, there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.
If the point $x \in X$ is almost recurrent, and the set $H(x) := \overline{\{\pi(t,x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.
Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \text{ such that } \{\pi(t_n,x)\} \to x \text{ and } \{t_n\} \to \infty\}$.
A point $x \in X$ is said to be *Levitan almost periodic* (see [11, 16]) for the dynamical system $(X, \mathbb{T}, \pi)$ if there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.1. *Let $x_i \in X_i$ $(i = 1, 2, \dots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then, the point $x := (x_1, x_2, \dots, x_m)) \in X := X_1 \times X_2 \times \dots \times X_m$ is also Levitan almost periodic for the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t,x) := (\pi_1(t,x_1), \pi_2(t,x_2), \dots, \pi_m(t,x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \dots, x_m) \in X$.*

A point $x \in X$ is called *stable in the sense of Lagrange* (*st.*L), if its trajectory $\{\pi(t,x) \ : \ t \in \mathbb{T}\}$ is relatively compact.
A point $x \in X$ is called *almost automorphic* (see [16, 27]) for the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;
(ii) There exists a dynamical system $(Y, \mathbb{T}, \lambda)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \lambda)$ and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

2.2. **Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** We recall below a general method for the construction of dynamical systems on the space of continuous functions. In this way we will obtain many well-known dynamical systems on functional spaces (see, for example, [5, 23]).
Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X, Y be a complete pseudo metric space, and P be a family of pseudo metrics on Y. We denote by $C(X,Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d_K^p\}$ $(p \in P,\ K \in \mathcal{K}(X))$, where

$$d_K^p(f,g) := \sup_{x \in K} p(f(x), g(x)),$$

and $\mathcal{K}(X)$ denotes the family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X,Y) \to C(X,Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau,x))$, $x \in X$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = Id_{C(X,Y)}$;
b. $\forall \tau_1, \tau_2 \in \mathbb{T}$ $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}$;
c. $\forall \tau \in \mathbb{T}$ the mapping $\sigma_\tau : C(X,Y) \mapsto C(X,Y)$ is continuous.

It is well-known (see [10]) that the triple $(C(X,Y),\mathbb{T},\sigma)$, where σ is defined by the equality $\sigma(\tau,f) := \sigma_\tau f \ \ (f \in C(X,Y),\ \tau \in \mathbb{T})$, is a dynamical system.
Consider now a dynamical system given in the form of $(C(X,Y),\mathbb{T},\sigma)$, and which is useful in the applications.

Example 2.2. *Let $X = \mathbb{T}$, and denote by $(X,\mathbb{T},\pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t,x) := x + t$. The dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$ is called* Bebutov's dynamical system *(a dynamical system of translations, or shifts dynamical system) (see* [23]*). For example, the equality*

$$d(f,g) := \sup_{L>0} \max\{d_L(f,g), L^{-1}\},$$

*where $d_L(f,g) := \max_{|t|\le L} \rho(f(t),g(t))$, defines a complete metric (*Bebutov's metric*) on the space $C(\mathbb{T},Y)$ which is compatible with the compact-open topology on $C(\mathbb{T},Y)$.*

The function $\varphi \in C(\mathbb{T},Y)$ is said *to possess a property* (P), if the motion $\sigma(\cdot,\varphi) : \mathbb{T} \to C(\mathbb{T},Y)$ possesses this property in the Bebutov's dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$, generated by the function φ. As property (P) we can take periodicity, quasi-periodicity, almost periodicity, almost automorphy, recurrence, etc.

2.3. **Cocycles, Skew-Product Dynamical Systems and Non-autonomous Dynamical Systems.** Let $\mathbb{T}_i$ $(i = 1,2)$ be a sub-semigroup of the group $\mathbb{S}$, and $\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$. Consider now two dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$. A triplet $\langle (X,\mathbb{T}_1,\pi),\, (Y,\mathbb{T}_2,\sigma),\, h\rangle$ is called *a non-autonomous dynamical system* if h is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$.
Let $(Y,\mathbb{T}_2,\sigma)$ be a dynamical system on Y, W a complete metric space, and φ a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in W, possessing the following properties:

a. $\varphi(0,u,y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t+\tau,u,y) = \varphi(\tau,\varphi(t,u,y),\sigma(t,y))$ $(t,\tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then, the triplet $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ (or shortly φ) is called *a cocycle* on $(Y,\mathbb{T}_2,\sigma)$ with fiber W (see [21]).
Given a cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$, let us set $X := W \times Y$, and define a mapping $\pi : \mathbb{T}_1 \times X \to X$ as follows: $\pi(t,(u,y)) := (\varphi(t,u,y),\sigma(t,y))$ (i.e., $\pi = (\varphi,\sigma)$). Then, it is easy to see that $(X,\mathbb{T}_1,\pi)$ is a dynamical system on X, which is called *a skew-product* dynamical system [21] and $h = pr_2 : X \to Y$ is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$ and, hence, $\langle (X,\mathbb{T}_1,\pi),\, (Y,\mathbb{T}_2,\sigma), h\rangle$ is a non-autonomous dynamical system.
Thus, if we have a cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ on the dynamical system $(Y,\mathbb{T}_2,\sigma)$ with fiber W, then it generates a non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\sigma), h\rangle$ $(X := W \times Y)$, called the non-autonomous dynamical system generated by the cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ on $(Y,\mathbb{T}_2,\sigma)$.
It is well known that non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions, every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system).

2.4. **Global attractors of dynamical systems.** Let $M \subseteq X$. The set

$$\omega(M) := \bigcap_{t\ge 0} \overline{\bigcup_{\tau \ge t} \pi(\tau, M)}$$

is called the *ω-limit* set of M.
The set M is called *orbitally stable*, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \geq 0$.
The dynamical system $(X, \mathbb{T}, \pi)$ is called:

– *point dissipative* if there exists a nonempty compact subset $K \subseteq X$ such that for every $x \in X$

$$\lim_{t\to+\infty} \rho(\pi(t, x), K) = 0; \tag{4}$$

– *compact dissipative* if the equality (4) takes place uniformly with respect to x on every compact subset of X;
– *locally complete (compact)* if for any point $p \in X$, there exist $\delta_p > 0$ and $l_p > 0$ such that the set $\pi(l_p, B(p, \delta_p))$ is relatively compact, where $B(x, \delta) := \{x \in X \mid \rho(x, p) < \delta\}$.

Let $(X, \mathbb{T}, \pi)$ be compact dissipative and K be a compact set attracting every compact subset of X. Let us set

$$J_X := \omega(K) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi(\tau, K)}. \tag{5}$$

It can be shown [10, Ch.I] that the set J_X defined by equality (5) does not depend on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J_X is called a *Levinson center* of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.
The next theorem characterizes the compact dissipative dynamical systems.

Theorem 2.3. *(See* [10, ChI]*) Let $(X, \mathbb{T}, \pi)$ be point dissipative. For $(X, \mathbb{T}, \pi)$ to be compact dissipative it is necessary and sufficient that there exists a nonempty compact set M possessing the following properties:*

(i) $\Omega_X := \overline{\cup\{\omega(x) : x \in X\}} \subseteq M$;
(ii) *M is orbitally stable.*

In this case, $J_X \subseteq M$ where J_X is the center of Levinson of $(X, \mathbb{T}, \pi)$.

Recall that a dynamical system $(X, \mathbb{S}, \pi)$ is said to be *asymptotically compact* if for any bounded positively invariant set $B \subseteq X$, there is a non-empty compact set $K \subseteq X$ such that

$$\lim_{t\to+\infty} \sup\{\rho(\pi(t, x), K) \mid x \in B\} = 0.$$

A continuous mapping $\gamma : \mathbb{S} \to X$ is said to be *an entire (full) trajectory* of the dynamical system $(X, \mathbb{T}, \pi)$, if $\gamma(t + \tau) = \pi(t, \gamma(\tau))$ for all $t \in \mathbb{T}$ and $\tau \in \mathbb{S}$.
Denote by Φ_x the set of all entire trajectories of $(X, \mathbb{T}, \pi)$ with $\gamma(0) = x$ and $\Phi := \bigcup\{\Phi_x : x \in X\}$.

Remark 2.4. *(i) Assume that $x \in X$ is such that $\Sigma_x^+ := \{\pi(t, x) \mid t \in \mathbb{S}_+\}$ is bounded, and $(X, \mathbb{T}, \pi)$ is asymptotically compact. Then, Σ_x^+ is relatively compact.*
(ii) Let $M \subseteq X$ be bounded and invariant. Then, M is relatively compact if the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact. In particular, if $x \in X$ and $\gamma \in \Phi_x$ is such that $\gamma(\mathbb{S})$ is bounded, then $\gamma(\mathbb{S})$ is relatively compact.

3. Bounded motions of linear systems

Let (X, h, Y) be a locally trivial Banach fiber bundle (see [4]), and recall that a non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is said to be *linear* (see [6],[19, 20]) if the map $\pi(t, \cdot) : X_y \to X_{\sigma(t,y)}$ is linear for every $t \in \mathbb{T}$ and $y \in Y$.
The non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is said to be *distal* on $\mathbb{S}_+$ in the fiber $X_y := \{x \in X \mid h(x) = y\}$ if $\inf\{\rho(\pi(t, x_1), \pi(t, x_2)) \mid t \in \mathbb{S}_+\} > 0$ for all $x_1, x_2 \in X_y, x_1 \neq x_2$ (see [5, 16]).
In the case of a group non-autonomous dynamical system, the concept of distalness on $\mathbb{S}_+$ and $\mathbb{S}$, on the fiber X_y can be defined likewise.
A non-autonomous system is said to be *distal* on $\mathbb{S}_+$ ($\mathbb{S}_-$, $\mathbb{S}$), if it is distal in every fiber X_y, $y \in Y$.
The following lemma provides more information on distal non-autonomous dynamical systems.

Lemma 3.1. *(See* [5, 16]*) The following assertions hold.*

(i) *Assume that X is compact, and $(Y, \mathbb{S}, \sigma)$ is minimal. If the group non-autonomous dynamical system $\langle (X, \mathbb{S}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is distal on $\mathbb{S}_+(\mathbb{S}_-)$, then it is distal on $\mathbb{S}$.*

(ii) *Assume that X is compact, $(Y, \mathbb{S}, \sigma)$ is minimal, and $y \in Y$. Then, the following conditions are equivalent:*
   (a) *The group non-autonomous system $\langle (X, \mathbb{S}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is distal on $\mathbb{S}$ in the fiber X_y;*
   (b) *For any points $x_1, ..., x_k \in X$, where $k \geq 2$ is an integer, the point $(x_1, ..., x_k) \in X^k$ is recurrent in $(X^k, \mathbb{S}, \pi)$.*

Assume that $(X_i, \mathbb{T}, \pi_i)$ is a dynamical system on $X_i, i = 1, \ldots, k$; let $X := X_1 \times ... \times X_k$, and let $\pi := (\pi_1, ..., \pi_k) : X \times \mathbb{T} \to X$ be defined by the formula

$$\pi(x, t) := (\pi_1(x_1, t), ..., \pi_k(x_k, t))$$

for all $t \in \mathbb{T}$ and $x := (x_1, ..., x_k) \in X$.
The dynamical system $(X, \mathbb{T}, \pi)$, where $X := X_1 \times ... \times X_k$ and $\pi := (\pi_1, ..., \pi_k)$, is called *the direct product* of the dynamical systems $(X_i, \mathbb{T}, \pi_i), i = 1, ..., k$ and denoted by $(X_1, \mathbb{T}, \pi_1)\times, \ldots, (X_k, \mathbb{T}, \pi_k)$.
If $X_i = X, i = 1, ..., k$, and $\pi_i = \pi, i = 1, ..., k$, then

$$(X, \mathbb{T}, \pi) \times (X, \mathbb{T}, \pi) \times ... \times (X, \mathbb{T}, \pi) := (X^k, \mathbb{T}, \pi).$$

The direct product of group dynamical systems is defined likewise.
The points $x_1, ..., x_k \in X$ are said to be *jointly recurrent* if the point $(x_1, ..., x_k) \in X^k$ is recurrent in the dynamical system $(X^k, \mathbb{T}, \pi)$.
If $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is a skew product over $(Y, \mathbb{S}, \sigma)$ with the fiber E, then it is *linear* if and only if E is a Banach space and the map $\varphi(t, \cdot, y) : E \to E$ is linear for every $y \in Y$ and $t \in \mathbb{T}$.
Throughout the rest of this section we assume that Y is compact, the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, $X = E \times Y, E$ is a Banach space with norm $|\cdot|$, the non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is linear, $\pi = (\varphi, \sigma)$, and $h = pr_2$.
Let $F \subseteq E \times Y$ be a closed vectorial subset of the trivial fiber bundle $(E \times Y, pr_2, Y)$ that is positively invariant relative to $(X, \mathbb{T}, \pi)$. We set

$$\mathbb{B}^+ = \{(x, y) \in F \mid \sup\{|\varphi(t, x, y)| \ : \ t \in \mathbb{S}_+\} < +\infty\}.$$

The set $\mathbb{B}^-$ is defined likewise. If $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is a semigroup non-autonomous dynamical system, then $\mathbb{B}$ is the set of all points of F with the following property: there is an entire trajectory of the dynamical system $(F,\mathbb{S}_+,\pi)$ which is bounded on $\mathbb{S}$, and which passes through this point. We put $\mathbb{B}^+_y := \mathbb{B}^+\bigcap X_y$ and $\mathbb{B}_y := \mathbb{B}\bigcap X_y, y\in Y$.
We can now establish the following result.

Theorem 3.2. *(See* [9, 10]*) Assume that $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is a linear non-autonomous dynamical system, $(X,\mathbb{S}_+,\pi)$ is asymptotically compact, and there is an $M>0$ such that the inequality*

$$|\varphi(t,x,y)|\le M|x|$$

is fulfilled for all $\gamma\in\Phi_{(x,y)}, (x,y)\in\mathbb{B}$, and $t\in\mathbb{S}$. Then, the following assertions hold:

(i) *Any two different entire trajectories γ_1 and γ_2 ($h(\gamma_1(0))=h(\gamma_2(0))$) are jointly recurrent;*
(ii) *For any $(x,y)\in\mathbb{B}$, the set $\Phi_{(x,y)}$ consists of a single entire recurrent trajectory;*
(iii) *$\mathbb{B}$ is closed in F;*
(iv) *$(X,\mathbb{S}_+,\pi)$ induces a group dynamical system $(\mathbb{B},\mathbb{S},\pi)$ on $\mathbb{B}$;*
(v) *For any $y\in Y$, the set $\mathbb{B}_y$ is finite-dimensional and dim $\mathbb{B}_y$ does not depend on $y\in Y$.*

We conclude this section with a sufficient condition ensuring that a linear non-autonomous system is asymptotically compact.
Let $P:X\to X$ be a projection of the vector bundle, that is, $P_y := P|_{X_y}$ projection in X_y for every $y\in Y$. Then, P is said to be *completely continuous* if $P(M)$ is relatively compact for any bounded set $M\subseteq X$.
Recall that a dynamical system $(X,\mathbb{S}_+,\pi)$ is said to be *conditionally β-condensing* (see [15]) if there exists $t_0>0$ such that $\beta(\pi(t_0,B))<\beta(B)$ for all bounded sets B in X with $\beta(B)>0$. The dynamical system $(X,\mathbb{S}_+,\pi)$ is said to be *β-condensing* if it is conditionally β-condensing and the set $\pi(t_0,B)$ is bounded for all bounded sets $B\subseteq X$.
According to Lemma 2.3.5 in [15, p.15] and Lemma 3.3 in [8], a conditionally condensing dynamical system $(X,\mathbb{S}_+,\pi)$ is asymptotically compact.
A cocycle $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is called *conditionally α-condensing* if there exists $t_0>0$ such that, for any bounded set $B\subseteq E$, the inequality $\alpha(\varphi(t_0,B,Y))<\alpha(B)$ holds if $\alpha(B)>0$. The cocycle φ is called *α-condensing* if it is a conditionally α-condensing cocycle, and the set $\varphi(t_0,B,Y)=\cup\{\varphi(t_0,u,Y)|u\in B,y\in Y\}$ is bounded for all bounded set $B\subseteq E$.
A cocycle φ is said to be *a conditionally α-contraction* of order $k\in[0,1)$, if there exists $t_0>0$ such that for any bounded set $B\subseteq E$ for which $\varphi(t_0,B,Y)=\cup\{\varphi(t_0,u,Y)|u\in B,y\in Y\}$ is bounded, the inequality $\alpha(\varphi(t_0,B,Y))\le k\alpha(B)$ holds. The cocycle φ is called *α-contraction* if it is a conditionally α-contraction cocycle, and the set $\varphi(t_0,B,Y)=\cup\{\varphi(t_0,u,Y)|u\in B,y\in Y\}$ is bounded for all bounded sets $B\subseteq E$.
Let us now recall a helpful result to determine when a cocycle is an α-contraction.

Theorem 3.3. *(See* [9, 10]*) Let E be a Banach space, and φ a cocycle on $(Y,\mathbb{S},\sigma)$ with fiber E, which satisfies the following conditions:*

(i) *$\varphi(t,u,y)=\psi(t,u,y)+\gamma(t,u,y)$ for all $t\in\mathbb{S}_+, u\in E$ and $y\in Y$.*
(ii) *There exists a function $m:\mathbb{R}^2_+\to\mathbb{R}_+$ satisfying the condition $m(t,r)\to 0$ as $t\to+\infty$ (for every $r>0$) such that $|\psi(t,u_1,y)-\psi(t,u_2,y)|\le m(t,r)|u_1-u_2|$ for all $t\in\mathbb{S}_+, u_1,u_2\in B[0,r]$ and $y\in Y$.*
(iii) *$\gamma(t,A,Y)$ is compact for all bounded $A\subset X$ and $t>0$.*

Then, the cocycle φ is an α-contraction.

Lemma 3.4. *(See* [10, ChXIII]*) Let Y be compact and the cocycle φ be α–condensing. Then, the skew-product dynamical system $(X,\mathbb{T},\pi)$ is also α–condensing.*

4. Linear contractive systems

We will analyze in this section the asymptotic properties of linear contractive non-autonomous dynamical systems.

Let X be a Banach space, let $C(\mathbb{S},X)$ be the space of all continuous maps $\varphi:\mathbb{S}\to X$ equipped with the compact-open topology, and let $(C(\mathbb{S},X),\mathbb{S},\sigma)$ be the dynamical system of translations (shifts) on $C(\mathbb{S},X)$. Let d be a metric on $C(\mathbb{S},X)$, which is consistent with its topology (for example, the Bebutov metric).

A linear non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is said to be *contractive*, if

$$|\pi(t,x)|\le |x|, \tag{6}$$

for all $t\in\mathbb{S}_+$ and $x\in X$.

A subset $M\subseteq Y$ is said to be *minimal* if $H(y)=M$ for all $y\in M$, where $H(y):=\overline{\{\sigma(t,y):\ t\in\mathbb{S}\}}$.

We can now prove the following result.

Lemma 4.1. *Suppose that the following conditions hold:*

(i) *$\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is a linear non-autonomous contractive dynamical system;*
(ii) *$\gamma:\mathbb{S}\to X$ is an entire relatively compact trajectory (i.e. the set $\overline{\gamma(\mathbb{S})}$ is compact).*
(iii) *The dynamical system $(Y,\mathbb{S},\sigma)$ is compact and minimal.*

Then, the following statements take place:

(i)
$$|\gamma(t)|\ge|\gamma(0)| \tag{7}$$
for all $t\in\mathbb{S}_-$;
(ii) *The full trajectory γ is recurrent, i.e., the function $\gamma:\mathbb{S}\to X$ is recurrent in the sense of Birkhoff w.r.t. the shift dynamical system (the Bebutov dynamical system) $(C(\mathbb{S},X),\mathbb{S},\sigma)$;*
(iii) *$|\gamma(t)|=|\gamma(0)|$ for all $t\in\mathbb{S}$.*

Proof. Let γ be an entire trajectory of $(X,\mathbb{S}_+,\pi)$. Then,

$$\gamma(t+\tau)=\pi(t,\gamma(\tau)) \tag{8}$$

for all $t\in\mathbb{S}_+$ and $t\in\mathbb{S}$. From (6) and (8) we obtain (7).

To prove the second statement we consider the non-autonomous dynamical system $\langle (H(\gamma),\mathbb{S},\sigma),(Y,\mathbb{S},\lambda),\tilde{H}\rangle$, where $H(\gamma):=\overline{\{\sigma(\tau,\gamma):\ \tau\in\mathbb{S}\}}$ and $\sigma(\tau,\gamma)(t):=\gamma(\tau+t)$ for all $\tau,t\in\mathbb{S}$, noting that by bar it is denoted the closure in the space

$C(\mathbb{S}, X)$, $(H(\gamma), \mathbb{S}, \sigma)$ is the shift dynamical system on $H(\gamma)$ induced by the Bebutov dynamical system $(C(\mathbb{S}, X), \mathbb{S}, \sigma)$, and $\tilde{H}$ is the mapping from $H(\gamma)$ onto Y defined by the equality $\tilde{H}(\psi) := h(\psi(0))$ for all $\psi \in H(\gamma)$. Since γ is relatively compact, then the set $H(\gamma)$ is compact in the space $C(\mathbb{S}, X)$. From the inequality (7) it follows that the non-autonomous dynamical system $\langle (H(\gamma), \mathbb{S}, \sigma), (Y, \mathbb{S}, \lambda), \tilde{H} \rangle$ is distal in the negative direction. Since the system $(Y, \mathbb{S}, \lambda)$ is minimal, then by Lemma 3.1 it is also distal in the positive direction, and the function $\gamma \in C(\mathbb{S}, X)$ is recurrent in the sense of Birkhoff.
Now, we will prove the third statement of Lemma. Denote by $\varphi(t) := |\gamma(t)|$. It is evident that $\varphi \in C(\mathbb{S}, \mathbb{R})$ is a recurrent function and $\varphi(t_2) = |\pi(t_2 - t_1, \gamma(t_1))| \leq |\gamma(t_1)| = \varphi(t_1)$ for all $t_2 \geq t_1$ $(t_1, t_2 \in \mathbb{S})$, i.e. the function φ is non-increasing. From the last property of the function φ and its recurrence, it follows that φ is a constant. In fact, if we suppose that there exists a $t_0 \in \mathbb{S}$ such that $\varphi(t_0) \neq \varphi(0)$, then, without loss of generality, we may suppose that $\varphi(t_0) < \varphi(0)$. Since the function is recurrent in the sense of Birkhoff, then there exists a sequence $\{t_n\} \subset \mathbb{S}$ such that $t_n \to +\infty$ and $\gamma(t + t_n) \to \gamma(t)$ uniformly w.r.t. t on every compact from $\mathbb{S}$. In particular we have $\varphi(t_n) \leq \varphi(t_0) < \varphi(0)$ for sufficiently large n. Thus we will have $\varphi(0) = \lim_{n \to +\infty} \varphi(t_n) \leq \varphi(t_0) < \varphi(0)$. This contradiction proves our statement. □

Remark 4.2. *Notice that the first statement holds without assuming any compactness and minimality of* $(Y, \mathbb{S}, \lambda)$.

Theorem 4.3. *Let* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ *be a non-autonomous dynamical system, and* $M \neq \emptyset$ *be a compact positively invariant set. Suppose that the following conditions are fulfilled:*

(i) $h(M) = Y$;
(ii) $M_y := M \bigcap X_y$ *contains a single point (i.e.,* $M_y = \{m_y\}$*) for all* $y \in Y$;
(iii) M *is uniformly stable, i.e., for all* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon) > 0$ *such that* $\rho(x, m_y) < \delta$ $(x \in X_y)$ *implies* $\rho(\pi(t, x), m_{\sigma(t,y)}) < \varepsilon$ *for all* $t \geq 0$.

Then, M *is orbitally stable.*

Proof. We will show that set M is orbitally stable in $(X, \mathbb{T}_1, \pi)$. Suppose that it is not true, then there exist $\varepsilon_0 > 0, \delta_n \to 0, x_n \in B(M, \delta_n) := \cup\{B(x, \delta_n) : x \in M\}$ and $t_n \to +\infty$ such that

$$\rho(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{9}$$

Since M is compact, then we may suppose that the sequence $\{x_n\}$ is convergent. Let $x_0 := \lim_{n \to +\infty} x_n$, with $x_{y_n} \in M_{y_n}, \rho(x_n, M) = \rho(x_n, x_{y_n})$ and $y_0 = h(x_0)$. Then, $x_0 = \lim_{n \to +\infty} x_{y_n}$ and $x_0 \in M_{y_0}$. Let $q_n = h(x_n)$ and note that

$$\rho(x_n, x_{q_n}) \leq \rho(x_n, x_{y_n}) + \rho(x_{y_n}, x_{q_n}) \to 0 \tag{10}$$

as $n \to +\infty$, because $q_n \to y_0$ and $x_{q_n} \to x_0$. Taking into account (10) and the asymptotic stability of the set M, we have

$$\rho(\pi(t_n, x_n), \pi(t_n, x_{q_n})) \to 0. \tag{11}$$

But the equalities (9) and (11) are contradictory. Hence, the set M is orbitally stable in $(X, \mathbb{T}_1, \pi)$. □

Theorem 4.4. *Suppose that $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is a linear non-autonomous dynamical system satisfying the following conditions:*

(i) *Y is compact;*
(ii) *The dynamical system $(X,\mathbb{S}_+,\pi)$ is asymptotically compact;*
(iii) *There exists a positive number M such that $|\pi(t,x)| \le M|x|$ for all $t\in\mathbb{S}_+$ and $x\in X$.*

Then, the following assertions are equivalent:

(i) *The non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is point dissipative and $\mathbb{B}\subseteq\Theta$, where Θ is the zero section of the vector fibering (X,h,Y) defined by $\Theta := \{\theta_y : y\in Y, \theta_y\in X_y, |\theta_y| = 0\}$;*
(ii) *There exist positive number $\mathcal{N}$, ν such that $|\pi(t,x)| \le \mathcal{N}e^{-\nu t}|x|$ for all $t\in\mathbb{S}_+$ and $x\in X$.*

Proof. Denote by $\Theta = \{\theta_y : y\in Y, \theta_y\in X_y, |\theta_y| = 0\}$ the zero section of the vector fibering (X,h,Y). Since (X,h,Y) is locally trivial and Y is compact, then Θ is a compact positively invariant set of the dynamical system $(X,\mathbb{S}_+,\pi)$, $\Theta\bigcap X_y$ contains a single point θ_y, and Θ is uniformly stable, i.e., for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that $|x|<\delta$ implies $|\pi(t,x)|<\varepsilon$ for all $t\in\mathbb{S}_+$ and $x\in X$. By Theorem 4.3 the set Θ is orbitally stable. Since $(X,\mathbb{S}_+,\pi)$ is point dissipative, then $\Omega_X\subseteq\Theta$, where $\Omega_X := \overline{\bigcup\{\omega(x) :\ x\in X\}}$. According to Theorem 2.3 the dynamical system $(X,\mathbb{S}_+,\pi)$ is compact dissipative and its Levinson center J_X coincides with $\tilde{\Theta} := \Theta\bigcap h^{-1}(J_Y)$. Since $(X,\mathbb{S}_+,\pi)$ is asymptotically compact, according to Theorem 1.6.4 in [10, Ch.I] , $(X,\mathbb{S}_+,\pi)$ is local dissipative. It follows from Theorem 2.11.3 in [10, Ch.II] that $(X,\mathbb{S}_+,\pi)$ is uniformly exponentially stable. Assume now that the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi), (Y, \mathbb{S}, \sigma), h\rangle$ is uniformly exponentially stable. Then, according to Theorem 2.11.3 in [10, Ch.II], it is locally dissipative. Let J_X be its Levinson centre (i.e., the maximal compact invariant set of the dynamical system $(X,\mathbb{S}_+,\pi)$) . We note that, according to the linearity of the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$, we have $J_X = \tilde{\Theta} := \Theta\bigcap h^{-1}(J_Y)$. Let φ be an entire bounded trajectory of the dynamical system $(X,\mathbb{S}_+,\pi)$. Since the non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is asymptotically compact, and the set $M=\varphi(\mathbb{S})$ is relatively compact, we note that $\varphi(\mathbb{S})\subseteq J_X=\tilde{\Theta}$ because J_X is the maximal compact invariant set of $(X,\mathbb{S}_+,\pi)$. The theorem is therefore proved. □

Theorem 4.5. *Suppose that the following conditions are satisfied:*

(i) *The dynamical system $(Y,\mathbb{S},\sigma)$ is compact and minimal.*
(ii) *The linear non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is generated by the cocycle φ (i.e., $X=E\times Y$, $\pi=(\varphi,\sigma)$ and $h=pr_2 : X\to Y$).*
(iii) *The dynamical system $(X,\mathbb{S}_+,\pi)$ is asymptotically compact.*
(iv) *There exists a positive number μ such that $|\varphi(t,u,y)| \le \mu|u|$ for all $t\in Y$ and $t\in\mathbb{S}_+$.*

Then, there are two vectorial positively invariant sub-fiberings (X^0,h,Y) and (X^s,h,Y) of (X,h,Y) such that:

a. *$X_y = X^0_y + X^s_y$ and $X^0_y\cap X^s_y = \theta_y$ for all $y\in Y$, where $\theta_y=(0,y)\in X = E\times Y$ and 0 is the zero in the Banach space E.*

b. *The vector sub-fibering* (X^0,h,Y) *is finite dimensional, invariant (i.e.,* $\pi(t,X^0)=X^0$ *for all* $t\in\mathbb{S}_+$*) and every trajectory of the dynamical system* $(X,\mathbb{S}_+,\pi)$ *belonging to* X^0 *is recurrent.*

c. *There exist two positive numbers* N *and* ν *such that* $|\varphi(t,u,y)|\le Ne^{-\nu t}|u|$ *for all* $(u,y)\in X^s$ *and* $t\in\mathbb{S}_+$.

Proof. Let $X^0:=\mathbb{B}$. Then, according to Theorem 3.2, statement b. holds. Denote by P_y the projection of $X_y:=h^{-1}(y)$ to $\mathbb{B}_y:=\mathbb{B}\cap h^{-1}(y)$, then $P_y(u,y)=(\mathcal{P}(y)u,y)$ for all $u\in E, \mathcal{P}^2(y)=\mathcal{P}(y)$ and the mapping $\mathcal{P}:Y\to[E]$ ($y\mapsto\mathcal{P}(y)$) is continuous, where we recall that $[E]$ denotes the set of all linear continuous operators acting on E. Now we set $X^s_y:=\mathcal{Q}(y)X_y$ and $X^s:=\cup\{X^s_y : y\in Y\}$, where $\mathcal{Q}(y):=Id_E-\mathcal{P}(y)$. We will show that X^s is closed in X. In fact, let $\{x_n\}=\{(u_n,y_n)\}\subseteq X^s$ and $x_0=(u_0,y_0)=\lim\limits_{n\to\infty}x_n$. Note that $P_{y_0}(x_0)=(\mathcal{P}(y_0)u_0,y_0)=(\lim\limits_{n\to\infty}\mathcal{P}(y_n)u_n,y_0)=(0,y_0)=\theta_{y_0}$ and, consequently, $x_0\in X^s_{y_0}\subseteq X^s$.
Let $(X^s,\mathbb{S}_+,\pi)$ be the dynamical system induced by $(X,\mathbb{S}_+,\pi)$. It is clear that, under the conditions of Theorem 4.5, the dynamical system $(X^s,\mathbb{S}_+,\pi)$ is asymptotically compact and every positive semi-trajectory is relatively compact and, consequently, $\lim\limits_{t\to\infty}|\pi(t,x)|=0$ for all $x\in X^s$, because the dynamical system $(X^s,\mathbb{S}_+,\pi)$ does not possess any non-trivial entire trajectory bounded on $\mathbb{S}$. Indeed, if we suppose that it is not true, then there exist $x_0=(u_0,y_0)$ and $t_n\to+\infty$ such that $|u_0|\ne 0$, $\lim\limits_{n\to+\infty}\pi(t_n,x)=x_0$, and there exists a non-trivial entire trajectory, which is bounded on $\mathbb{S}$, passing through the point x_0. This contradiction proves the necessary assertion. Thus, by applying Theorem 4.4, there exist two positive constants N and ν such that $|\varphi(t,u,y)|\le Ne^{-\nu t}|u|$ for all $(u,y)\in X^s$ and $t\in\mathbb{S}_+$. The theorem is proved. □

Theorem 4.6. *Suppose that the following conditions are satisfied:*

(i) *The dynamical system* $(Y,\mathbb{S},\sigma)$*is compact and minimal.*

(ii) *The linear non-autonomous dynamical system* $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ *is generated by the cocycle* φ *(i.e.* $X=E\times Y$, $\pi=(\varphi,\sigma)$ *and* $h=pr_2:X\to Y$*).*

(iii) *The dynamical system* $(X,\mathbb{S}_+,\pi)$ *is asymptotically compact.*

(iv) *The cocycle* φ *is non-expanding, i.e.,* $|\varphi(t,u,y)|\le|u|$ *for all* $(u,y)\in E\times Y$ *and* $t\in\mathbb{S}_+$.

Then, there are two vectorial positively invariant sub-fiberings (X^0,h,Y) *and* (X^s,h,Y) *of* (X,h,Y) *such that:*

a. $X_y=X^0_y+X^s_y$ *and* $X^0_y\cap X^s_y=\theta_y$ *for all* $y\in Y$*, where* $\theta_y=(0,y)\in X=E\times Y$ *and* 0 *is the zero in the Banach space* E.

b. *The vector sub-fibering* (X^0,h,Y) *is finite dimensional, invariant (i.e.,* $\pi^tX^0=X^0$ *for all* $t\in\mathbb{S}_+$*).*

(i) *For every* $x:=(u,y)\in X^0_y$ *there exists a unique entire trajectory of the dynamical system* $(X,\mathbb{S}_+,\pi)$ *belonging to* X^0.

(ii) *Every trajectory of the dynamical system* $(X,\mathbb{S}_+,\pi)$ *belonging to* X^0 *is recurrent, and its norm is constant, i.e.* $|\varphi(t,u,y)|=|u|$ *for all* $x=(u,y)\in X^s_y$ *and* $t\in\mathbb{S}$.

c. *There exists two positive numbers* N *and* ν *such that* $|\varphi(t,u,y)|\le Ne^{-\nu t}|u|$ *for all* $(u,y)\in X^s$ *and* $t\in\mathbb{S}_+$.

Proof. This theorem follows directly from Theorem 4.5 and Lemma 4.1. □

Corollary 4.7. *Suppose that the following conditions are satisfied:*

(i) *The dynamical system $(Y,\mathbb{S},\sigma)$ is compact and minimal.*
(ii) *The linear non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is generated by the cocycle φ.*
(iii) *The dynamical system $(X,\mathbb{S}_+,\pi)$ is asymptotically compact.*
(iv) *The cocycle φ is non-expanding.*

Then, one of the following alternatives is fulfilled:

a. *There is an entire trajectory γ of $(X,\mathbb{S}_+,\pi)$ with constant positive norm.*
b. *There exist two positive numbers N and ν such that $|\varphi(t,u,y)|\le Ne^{-\nu t}|u|$ for all $(u,y)\in X$ and $t\in\mathbb{S}_+$.*

Remark 4.8. *Theorems 4.4–4.6 generalize some results from* [9, 10]*, where these facts were proved for conditionally α–condensing systems.*

5. Comparability and Uniform Comparability of Motions by the Character of Recurrence in the Sense of Shcherbakov

Let $(Y,\mathbb{T},\sigma)$ be a dynamical system. A point $y\in Y$ is said to be *positively* (respectively, *negatively*) *stable in the sense of Poisson* (see, for example, [26] and [28]), if there exists a sequence $t_n\to+\infty$ (respectively, $t_n\to-\infty$) such that $\sigma(t_n,y)\to y$ If the point y is Poisson stable in both directions, it is called *Poisson stable.*
Denote by $\mathfrak{N}_y=\{\{t_n\}\subset\mathbb{T}\mid \sigma(t_n,y)\to y,\quad\text{as}\quad n\to+\infty\}$.
A point $x\in X$ is called *comparable with $y\in Y$ by the character of recurrence* if $\mathfrak{N}_y\subseteq\mathfrak{N}_x$ (see [23]–[26]).

Remark 5.1. *If a point $x\in X$ is comparable with $y\in Y$ by the character of recurrence, and y is stationary (respectively, τ-periodic, recurrent, Poisson stable), then so is the point x (see* [26]*).*

Denote by $\mathfrak{M}_y:=\{\{t_n\}\subset\mathbb{T}\mid\text{the sequence }\{\sigma(t_n,y)\}\text{ is convergent}\}$.
A point $x\in X$ is called *uniformly comparable with $y\in Y$ by the character of recurrence* if $\mathfrak{M}_y\subseteq\mathfrak{M}_x$ (see [23]–[26]).

Remark 5.2. *1. If a point $x\in X$ is uniformly comparable with $y\in Y$ by the character of recurrence, and y is stationary (respectively, τ-periodic, almost periodic, almost automorphic, recurrent, Poisson stable), then so is the point x (see* [23]*–*[26]*).*
2. Every almost periodic point is recurrent.

Let $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system. Recall that a mapping $\zeta: Y\to X$ is called a *section (selector)* of the homomorphism h, if $h(\gamma(y))=y$ for all $y\in Y$. The section ζ of the homomorphism h is *invariant* if $\zeta(\sigma(t,y))=\pi(t,\zeta(y))$ for all $y\in Y$ and $t\in\mathbb{S}_+$.
Let Y be a compact metric space. Consider a non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ and denote by $\Gamma(Y,X)$ the family of all continuous sections of the homomorphism h. By the equality

$$d(\varphi_1,\varphi_2):=\max_{y\in Y}\rho(\varphi_1(y),\varphi_2(y)), \tag{12}$$

it is defined a metric on $\Gamma(Y,X)$.

Remark 5.3. *A continuous section $\zeta \in \Gamma(Y,X)$ is invariant if and only if $\zeta \in \Gamma(Y,X)$ is a stationary point of the semigroup $\{S^t \mid t \in \mathbb{S}_+\}$, where $S^t : \Gamma(Y,X) \to \Gamma(Y,X)$ is defined by $(S^t\zeta)(y) := \pi(t,\zeta(\sigma(-t,y)))$ for all $y \in Y$ and $t \in \mathbb{S}_+$.*

Let $\langle (X,\mathbb{T},\pi), (Y,\mathbb{T},\sigma), h\rangle$ be a linear non-autonomous dynamical system. A non-autonomous dynamical system $\langle (W, \mathbb{T}, \mu), (Z, \mathbb{T}, \lambda), \varrho\rangle$ is said to be *linear non-homogeneous*, generated by the linear (homogeneous) dynamical system $\langle (X,\mathbb{T},\pi), (Y,\mathbb{T},\sigma), h\rangle$, if the following conditions hold:

1. There exits a homomorphism q of the dynamical system $(Z,\mathbb{T},\lambda)$ onto $(Y,\mathbb{T},\sigma)$;
2. The space $W_y := (q\circ \rho)^{-1}(y)$ is affine for all $y \in (q\circ\varrho)(W) \subseteq Y$ and the vectorial space $X_y = h^{-1}(y)$ is an associated space to W_y ([22, p.175]). The mapping $\mu(t,\cdot) : W_y \to W_{\sigma(t,y)}$ is affine and $\pi(t,\cdot) : X_y \to X_{\sigma(t,y)}$ is its linear associated function ([22, p.179]), i.e. $X_y = \{w_1 - w_2 \mid w_1, w_2 \in W_y\}$ and $\mu(t,w_1) - \mu(t,w_2) = \pi(t,(w_1 - w_2))$ for all $w_1, w_2 \in W_y$ and $t \in T$.

Example 5.4. Let $\langle E,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a linear cocycle (shortly, φ), and $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$ be the non-autonomous dynamical system generated by the cocycle φ. For each $f \in C(Y,E)$ we consider a mapping $\phi_f : \mathbb{S}_+ \times E \times Y \mapsto E$ defined by equality

$$\phi_f(t,u,y) := \varphi(t,u,y) + \int_0^t \varphi(t-s, f(\sigma(s,y)), \sigma(s,y))ds. \tag{13}$$

Note that ϕ_f is a cocycle. Denote by $\langle (X,\mathbb{S}_+,\pi_f), (Y,\mathbb{S},\lambda), h\rangle$ the non-autonomous dynamical system generated by the cocycle ϕ_f (i.e., $X := E\times Y$, $\pi_f := (\phi_f,\sigma)$ and $h := pr_2 : X \mapsto Y$). It is easy to check that $\langle (X,\mathbb{S}_+,\pi_f), (Y,\mathbb{S},\sigma), h\rangle$ is a linear non-homogeneous dynamical system, generated by the linear dynamical system $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$ (by the linear cocycle φ) and the function $f \in C(Y,E)$.

Remark 5.5. *1. If the time is discrete, i.e. $\mathbb{S} = \mathbb{Z}$, then it is necessary to write a sum in equality (13) instead of an integral .*
2. Note that $\zeta \in \Gamma(Y,X)$ is a continuous section of the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$, generated by cocycle $\langle E,\varphi,(Y,\mathbb{S},\sigma)\rangle$, if and only if $\zeta = (\nu, Id_Y)$ and $\nu(\sigma(t,y)) = \varphi(t,\nu(y),y)$ for all $t \in \mathbb{S}$ and $y \in Y$, where $\nu \in C(Y,E)$. In this case, the function ν is called a continuous invariant section of the cocycle φ.

The next theorems establish conditions which ensure the existence of invariant sections.

Theorem 5.6. *(See* [10, ChII]*) Assume that the following conditions hold:*

(i) $\Gamma(Z,W) \neq \emptyset$;
(ii) $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$ *is a linear homogeneous dynamical system;*
(iii) $\langle (W,\mathbb{S}_+,\mu), (Z,\mathbb{S},\lambda), \varrho\rangle$ *is a linear non-homogeneous dynamical system generated by* $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$;
(iv) q *is a homomorphism from* $(Z,\mathbb{S},\lambda)$ *onto* $(Y,\mathbb{S},\sigma)$;
(v) *The spaces Y and Z are compact, and (X,h,Y) is a normed fiber bundle;*
(vi) *There are two positive numbers N and ν such that*

$$|\pi(t,x)| \leq Ne^{-\nu t}|x|$$

for all $x \in X$ and $t \in \mathbb{S}_+$, i.e., the linear non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$ is uniformly exponentially stable.

Then, the dynamical system $\langle (W, \mathbb{S}_+, \mu), (Z, \mathbb{S}, \lambda), \varrho\rangle$ *admits a unique invariant section* $\zeta \in \Gamma(Z, W)$.

Theorem 5.7. *Suppose that the following conditions are satisfied:*

(i) *The dynamical system* $(Y, \mathbb{S}, \sigma)$ *is compact and minimal.*
(ii) *The cocycle* φ *is non-expanding.*
(iii) *The linear non-autonomous dynamical system* $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ *is generated by cocycle* φ.
(iv) *The dynamical system* $(X, \mathbb{S}_+, \pi)$ *is asymptotically compact.*

Then, one of the following alternatives is fulfilled:

a. *There is at least one entire recurrent trajectory* ζ *of* $(X, \mathbb{S}_+, \pi)$ *with constant positive norm.*
b. *For every* $f \in C(Y, E)$ *the non-homogeneous linear cocycle* φ_f *(see, Example 5.4) admits a unique continuous invariant section* $\zeta_f \in C(Y, E)$.

Proof. The statements follow directly from Theorem 4.6 and Theorem 5.6. □

6. Applications

We will now apply the abstract theory developed in the previous sections to some interesting applications: linear differential equations in Banach spaces, functional-differential equations, and difference equations.

6.1. **Ordinary Linear Differential Equations in a Banach Space.**

Let X be a real Banach space with the norm $|\cdot|$, and let X^* denote its dual space with the dual norm also denoted by $|\cdot|$. The value of $f \in X^*$ at the point $x \in X$ will be denoted by $\langle x, f\rangle$. Let $J : X \to X^*$ be the duality mapping between X and X^* (see [29]), i.e., for $x \in X$, $J(x) := \{f \in X^* \mid \langle x, f\rangle = |x|^2 = |f|^2\}$.

Definition 6.1. *The mapping* $F : X \to X$ *is called dissipative, if for any* $x, y \in X$,

$$\langle F(x) - F(y), f\rangle \le 0 \tag{14}$$

for all $f \in J(x - y)$.

If X is a Hilbert space, then for any $x \in X$, $J(x) = x$, hence (14) becomes

$$\langle F(x) - F(y), x - y\rangle \le 0$$

for $x, y \in X$.

Lemma 6.2. *(See* [14]*) Let* $(Y, \mathbb{R}, \sigma)$ *be a dynamical system, and let* $F : Y \times X \to X$ *be a continuous function, and for each* $y \in Y$ *the partial mapping* $F(y, \cdot) : X \to X$ *is dissipative in* x. *If* $x_1(t)$ *and* $x_2(t)$ *are two solutions on the interval* $(a, b) \subseteq \mathbb{R}$ *of the equation*

$$x' = F(\sigma(t, y), x) \ (y \in Y),$$

then

$$|x_1(t) - x_2(t)| \le |x_1(s) - x_2(s)|,$$

for $a \le s \le t \le b$.

Let Y be a compact metric space and $(Y, \mathbb{R}, \sigma)$ be a minimal dynamical system (i.e., every trajectory of $(Y, \mathbb{R}, \sigma)$ is dense in Y). Consider the equation

$$u' = A(\sigma(t, y))u, \quad (y \in Y) \tag{15}$$

where $A \in C(Y,[E])$. Denote by $\varphi(t,u,y)$ the unique solution of equation (15) passing through the point $u \in E$ at the initial moment $t = 0$. It is well known (see, for example, [10, 5, 21]) that $\langle E, \varphi, (Y, \mathbb{R}, \sigma)$ is a linear cocycle.
As a straightforward consequence of Lemma 6.2 we have the following result.

Corollary 6.3. *Let E be a Hilbert space and*

$$\langle A(y)u, u\rangle \le 0$$

for all $u \in E$ and $y \in Y$ (i.e., the operator-function $A \in C(Y,[E])$ is dissipative). Then, the linear cocycle φ, generated by equation (15), is non-expanding.

Applying our general results, obtained in Sections 4 and 5 (namely, Corollary 4.7 and Theorem 5.7), to the linear cocycle φ, we will obtain the following results.

Theorem 6.4. *Suppose that the following conditions are satisfied:*

(i) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is compact and minimal.*
(ii) *The operator-function $A \in C(Y,[E])$ is dissipative.*
(iii) *The linear cocycle φ, generated by equation (15), is asymptotically compact.*

Then, one of the following alternatives is fulfilled:

a. *For every $y \in Y$, equation (15) admits at least one recurrent solution with constant positive norm.*
b. *The trivial solution of equation (15) is uniformly exponentially stable, i.e., there exist two positive numbers N and ν such that $|\varphi(t,u,y)| \le Ne^{-\nu t}|u|$ for all $(u,y) \in X$ and $t \in \mathbb{R}_+$.*

Theorem 6.5. *Assume the following hypotheses:*

(i) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is compact and minimal.*
(ii) *The operator-function $A \in C(Y,[E])$ is dissipative.*
(iii) *The linear cocycle φ, generated by equation (15), is asymptotically compact.*

Then, one of the following alternatives is fulfilled:

a. *For every $y \in Y$, equation (15) admits at least one recurrent solution with constant positive norm.*
b. *For every $f \in C(Y,E)$, there exists a unique function $\nu_f \in C(Y,E)$ such that*

$$\phi_f(t, \nu_f(y), y) = \nu_f(\sigma(t,y))$$

for all $t \in \mathbb{R}$ and $y \in Y$, where $\phi_f(t,u,y)$ is a unique solution of the equation

$$v' = A(\sigma(t,y))v + f(\sigma(t,y)) \quad (y \in Y) \tag{16}$$

passing through the point $u \in E$ at the initial moment $t = 0$.

Corollary 6.6. *Under the conditions of Theorem 6.5 if the statement b. holds and the point $y \in Y$ is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent), then the solution $\varphi(t, \nu(y), y) = \nu(\sigma(t,y))$ of equation (16) is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent).*

Remark 6.7. *1. When the space E is finite dimensional ,and the dynamical system $(Y, \mathbb{R}, \sigma)$ is almost periodic, Theorem 6.5 generalizes and precises Theorem 3.4 from* [12].

2. Note that, under the conditions of Theorem 6.5, if $(Y,\mathbb{R},\sigma)$ is an almost periodic minimal set we cannot state (in item a.) that there exists at least one almost periodic solution with constant positive norm. This fact is confirmed by the following example.

Example 6.8. Consider the almost periodic function

$$f(t) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}} \sin \frac{t}{2k+1}$$

and its primitive

$$F(t) := \int_0^t a(s)ds = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)}.$$

Note that $F(t)$ is unbounded. Indeed, using the inequality $|\sin t| \geq \frac{1}{2}|t|$ with $|t| \leq 1$, we obtain that

$$F(t) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)} \geq \sum_{k \geq \frac{1}{2}(\frac{|t|}{2}-1)} \frac{t^2}{8} \frac{1}{(2k+1)^{5/2}}$$

$$\geq \frac{t^2}{8} \int_{|s| \geq \frac{1}{2}(\frac{|t|}{2}-1)} \frac{ds}{(2s+1)^{5/2}} = \frac{t^2 2^{3/2}}{24|t|^{3/2}} = \frac{1}{6\sqrt{2}}|t|^{1/2} \to +\infty$$

as $|t| \to +\infty$.
Denote by $Y := H(f)$, where $H(f)$ is the closure in $C(\mathbb{R},\mathbb{R})$ of the family of all shifts $\{f_\tau : \ \tau \in \mathbb{R}\}$ ($f_\tau(t) := f(t+\tau)$ for all $t \in \mathbb{R}$). Let $(Y,\mathbb{R},\sigma)$ be a shift dynamical system on $H(f)$ as a restriction of the Bebutov dynamical system $(C(\mathbb{R},\mathbb{R}),\mathbb{R},\sigma)$ on the closed invariant subset $H(f) \subset C(\mathbb{R},\mathbb{R})$. Now, consider the equation

$$u' = \mathcal{F}(\sigma(t,g))u, \quad (g \in H(f), \ u \in \mathbb{R}^2) \tag{17}$$

where $\mathcal{F} \in C(H(f),[\mathbb{R}^2])$ is the function $\mathcal{F}$ defined as

$$\mathcal{F}(g) = \begin{pmatrix} 0 & -g(0) \\ g(0) & 0 \end{pmatrix}.$$

Then, the equation can be rewritten as follows

$$\begin{cases} x' = -g(t)y \\ y' = g(t)x \ , \end{cases} \tag{18}$$

where $g \in H(f)$. Note that $\langle \mathcal{F}(g)u,u\rangle = -g(0)xy + g(0)xy = 0$ for all $u=(x,y) \in \mathbb{R}^2$ and $g \in H(f)$ and, consequently, Theorem 6.5 can be applied to equation (17). Let $\varphi(t,x_0,y_0,g)$ be the unique solution of equation (17) with initial condition $\varphi(0,x_0,y_0,g) = (x_0,y_0) \in \mathbb{R}^2$. Since

$$\frac{d(x^2(t)+y^2(t))}{dt} = 2x'(t)x(t) + 2y'(t)y(t) = 2(-g(t))y(t) + 2g(t)x(t)y(t) = 0$$

for all $t \in \mathbb{R}$ and, consequently, for every non-trivial solution $\varphi(t,x_0,y_0,g)$ of equation (17) (respectively, of system (18)) we have $|\varphi(t,x_0,y_0,g)|^2 = x_0^2 + y_0^2$ for all $t \in \mathbb{R}$. These solutions are recurrent thanks to Theorem 4.6.
Consider the two-dimensional system of differential equations

$$\begin{cases} x' = -f(t)y \\ y' = f(t)x \end{cases} \tag{19}$$

which is included in the family (18) because $f \in H(f)$. The system (19) has no non-trivial almost periodic solutions [5, Ch.IV,pp.269-270].

Let $(Y, \mathbb{R}, \sigma)$ be a compact minimal dynamical system.
We will say that the dynamical system $(Y, \mathbb{R}, \sigma)$ *satisfies the condition* (C), if for any Banach space F and any continuous function $\mathcal{F} \in C(Y, F)$ there exists

$$M(y) := \lim_{T \to +\infty} \frac{1}{T} \int_0^T F(\sigma(t, y))dt. \tag{20}$$

The dynamical system $(Y, \mathbb{R}, \sigma)$ is said to be *ergodic* (with respect to an invariant measure μ on Y) or μ*–ergodic* (see [30]) if for every μ–measurable invariant set $A \subset Y$ one has $\mu(A) \cdot \mu(Y \setminus A) = 0$, that is, either A or its complement has μ–measure zero.

Remark 6.9. *1. If the dynamical system $(Y, \mathbb{R}, \sigma)$ is uniquely ergodic (i.e. there exists a unique normalized invariant measure μ on the space Y and the dynamical system $(Y, \mathbb{R}, \sigma)$ is μ–ergodic), then (see [30])*

(i) *The dynamical system $(Y, \mathbb{R}, \sigma)$ satisfies the condition (C);*
(ii) *The function $M : Y \to F$ appearing in (20) is constant;*
(iii) *The convergence in (20) is uniformly with respect to $y \in Y$.*

2. If the dynamical system $(Y, \mathbb{R}, \sigma)$ is almost periodic, then, there exists a unique invariant measure μ on the space Y, and $(Y, \mathbb{R}, \sigma)$ is μ–ergodic.

We can now prove the following results.

Lemma 6.10. *Let E be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Assume the following conditions:*

(i) *$A \in C(Y, [E])$;*
(ii) *The operator $A(y)$ is self-adjoint for all $y \in Y$;*
(iii) *$A(y) \geq 0$ for all $y \in Y$, i.e., $\langle A(y)u, u \rangle \geq 0$;*
(iv) *The operator $\int_0^h A(\sigma(t, y))dt$ is invertible.*

Then, $|\varphi(h, u, y)| < |u|$ for all $u \in E$ with $|u| \neq 0$ and $y \in Y$, where $\varphi(t, u, y)$ is the unique solution of equation (15) with initial condition $\varphi(0, u, y) = u$.

Proof. We argue by contradiction. Suppose that the statement is not true. Then, there exist $y_0 \in Y$ and $u_0 \in E$ with $|u_0| \neq 0$ such that $|\varphi(h, u_0, y_0)| = |u_0|$. Since

$$\begin{aligned} \frac{d}{dt}|\varphi(t, u_0, y_0)|^2 &= 2\langle \varphi'(t, u_0, y_0), \varphi(t, u_0, y_0) \rangle \\ &= 2\langle A(\sigma(t, y_0))\varphi(t, u_0, y_0), \varphi(t, u_0, y_0) \rangle \\ &\leq 0, \end{aligned}$$

for all $t \in [0, h]$ and, it follows that $\langle A(\sigma(t, y_0))\varphi(t, u_0, y_0), \varphi(t, u_0, y_0) \rangle \equiv 0$ on $[0, h]$. Since $A^*(y) = A(y) \geq 0$ for all $y \in Y$, we have $A(\sigma(t, y_0))\varphi(t, u_0, y_0) \equiv 0$ which implies the equality $\varphi'(t, u_0, y_0) = 0$ for all $t \in [0, h]$. Thus $\varphi(t, u_0, y_0) \equiv u_0$ on $[0, h]$ and consequently, $\{\int_0^h A(\sigma(t, y_0))dt\}u_0 = 0$. Since the operator $\int_0^h A(\sigma(t, y))dt$ is invertible we obtain $u_0 = 0$. This contradiction proves our statement. $\square$

Theorem 6.11. *Let E be a Hilbert space. Assume that the following conditions are satisfied:*

(i) *$A \in C(Y, [E])$;*
(ii) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is uniquely ergodic;*

(iii) *The operator*

$$M(y) := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A(\sigma(t,y))dt$$

is invertible;

(iv) *The operator $A(y)$ is self-adjoint for all $y \in Y$;*

(v) *$A(y) \geq 0$ for all $y \in Y$;*

(vi) *The linear cocycle φ, generated by equation (15), is asymptotically compact.*

Then, the linear cocycle φ is uniformly asymptotically stable.

Proof. Thanks to Theorem 4.6, in order to prove this statement, it is sufficient to show that the sub-bundle X^0 is trivial. Let $x \in X^0$, then there exist $u_0 \in E$ and $y_0 \in Y$ such that $x = (u_0, y_0)$ and $|\varphi(t, u_0, y_0)| = |u_0|$ for all $t \in \mathbb{R}$. As the dynamical system $(Y, \mathbb{R}, \sigma)$ is almost periodic, and the operator $M \in [E]$ is invertible, the operator $\int_0^h A(\sigma(t, y_0))dt$ is also invertible for all sufficiently large $h > 0$. From Lemma 6.10 it follows that the equality $|\varphi(t, u_0, y_0)| = |u_0|$ (for all $t \in \mathbb{R}$) takes place only if $u_0 = 0$. The theorem is proved. □

Corollary 6.12. *Let E be a Hilbert space. Assume that the following conditions hold:*

(i) *$A \in C(Y, [E])$;*

(ii) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is uniquely ergodic;*

(iii) *The operator*

$$M(y) := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A(\sigma(t,y))dt \tag{21}$$

is invertible;

(iv) *The operator $A(y)$ is self-adjoint for all $y \in Y$;*

(v) *$A(y) \geq 0$ for all $y \in Y$;*

(vi) *The linear cocycle φ, generated by equation (15), is asymptotically compact.*

Then, for all $f \in C(Y, E)$ there exists a unique function $\nu_f \in C(Y, E)$ such that $\phi_f(t, \nu_f(y), y) = \nu_f(\sigma(t, y))$ for all $t \in \mathbb{R}$ and $y \in Y$, where ϕ_f denotes the cocycle generated by the linear non-homogeneous equation (16).

Proof. This statement follows from Theorem 6.11 and 6.5. □

Corollary 6.13. *Under the conditions of Corollary 6.12, if the point $y \in Y$ is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent), then the solution $\phi_f(t, \nu_f(y), y) = \nu_f(\sigma(t, y))$ of equation (16) is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent).*

6.2. **Linear Functional Differential Equations with Finite Delay.** Let $r > 0, C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ with the sup-norm. For $[a, b] := [-r, 0]$, we set $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$. Let $c \in \mathbb{R}, a \geq 0$, and $u \in C([c - r, c + a], \mathbb{R}^n)$. We define $u_t \in \mathcal{C}$, for any $t \in [c, c + a]$ by the relation $u_t(\theta) := u(t + \theta), -r \geq \theta \geq 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C}, \mathbb{R}^n)$ be the Banach space of all linear operators from $\mathcal{C}$ to $\mathbb{R}^n$ equipped with the operator norm. Let $C(\mathbb{R}, \mathfrak{A})$ be the space of all operator-valued functions $A : \mathbb{R} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{R}, A)$. Let

$H(A) := \overline{\{A_\tau \mid \tau \in \mathbb{R}\}}$, where $A_\tau$is the shift of the operator-valued function A by τ, and the bar denotes closure in $C(\mathbb{R}, \mathfrak{A})$.

Example 6.14. Consider the linear functional-differential equation with delay

$$u' = A(\sigma(t,y))u_t, \ (y \in Y) \tag{22}$$

where $A \in C(Y, \mathfrak{A})$.

Remark 6.15. *1. Denote by $\tilde{\varphi}(t,u,y)$ the solution of equation (22) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0,u,y) = u \in \mathcal{C}$, i.e., $\varphi(s,u,y) = u(s)$ for all $s \in [-r,0]$. By $\varphi(t,u,y)$ we will denote below the trajectory of equation (22), corresponding to the solution $\tilde{\varphi}(t,u,y)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t,u,f)(s) := \tilde{\varphi}(t+s,u,f)$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r,0]$.*
2. Taking into account item 1. of this remark, we use the notions "solution" and "trajectory" for equation (22) as synonyms.

Let $\varphi(t,u,y)$ be the solution of equation (22) satisfying the condition $\varphi(0,u,y) = u$ and defined for all $t \geq 0$. Let $X := \mathcal{C} \times Y$ and let $\pi := (\varphi, \sigma)$ be the dynamical system on X defined by the equality $\pi(\tau,(u,y)) := (\varphi(\tau,u,y), \sigma(\tau,y))$. It is easy to see that the non-autonomous system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle (h := pr_2 : X \to Y)$ is linear.

Lemma 6.16. *(See [10, Ch.12]) Let Y be compact. Then, the linear non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (22) is completely continuous, that is, for any bounded set $\mathcal{A} \subseteq X$ there is an $l = l(\mathcal{A}) > 0$ such that $\pi(l,\mathcal{A})$ is relatively compact.*

Lemma 6.17. *Suppose that the following conditions are fulfilled:*

(i) *The space Y is compact;*
(ii) *$A \in C(Y, \mathfrak{A})$;*
(iii) *The operator function $A : Y \to \mathfrak{A}$ is dissipative, i.e.,*

$$\langle A(y)\phi, \phi(0)\rangle \leq 0 \tag{23}$$

for all $y \in Y$, and $\phi \in \mathcal{C}$.

Then, the linear cocycle, generated by equation (22), is non-expanding, i.e., $||\varphi(t,u,y)|| \leq ||u||$ for all $t \geq 0$, $u \in \mathcal{C}$ and $y \in Y$.

Proof. Let $\tilde{\varphi}(t,u,y)$ be the solution of equation (22) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$), and denote by $\tilde{\alpha}(t) := |\tilde{\varphi}(t,u,y)|^2$ for all $t \in \mathbb{R}_+$ (respectively, for all $t \in \mathbb{R}$). Then, from (23) we have

$$\frac{d\tilde{\alpha}(t)}{dt} = 2\langle A(\sigma(t,y))\varphi(t,u,y)), \tilde{\varphi}(t,u,y)\rangle \leq 0,$$

for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and, consequently, we obtain

$$\tilde{\alpha}(t_2) \leq \tilde{\alpha}(t_1), \tag{24}$$

for all $t_1, t_2 \in \mathbb{R}_+$ (respectively, $t_1, t_2 \in \mathbb{R}$) with $t_2 \geq t_1$. Note that from (24) we have

$$||\varphi(t,u,y)|| = \max_{-r \leq s \leq 0} |\tilde{\varphi}(t+s,u,y)| =$$
$$= |\tilde{\varphi}(t+s_t,u,y)| \leq |\tilde{\varphi}(s_t,u,y)| \leq ||u||,$$

for all $t \geq 0$, $y \in Y$ and $u \in \mathcal{C}$, where s_t is some number (depending on t) in the segment $[-r,0]$. The proof is now complete. □

Applying our general results (namely, Theorem 4.6, Corollary 4.7 and Lemma 6.17) to the linear cocycle φ, generated by equation (22), we will obtain the following results.

Theorem 6.18. *Assume the following conditions:*

(i) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is compact and minimal.*
(ii) *The operator-function $A \in C(Y, \mathfrak{A})$ is dissipative.*

Then, one of the following alternatives is fulfilled:

a. *For every $y \in Y$, equation (22) admits at least one recurrent solution with constant positive norm.*
b. *The trivial solution of equation (22) is uniformly exponentially stable, i.e., there exist two positive numbers N and ν such that $|\varphi(t,u,y)| \leq Ne^{-\nu t}|u|$ for all $(u,y) \in X$ and $t \in \mathbb{R}_+$.*

Proof. Let φ be the cocycle generated by equation (22), and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ the linear non-autonomous dynamical system associated to φ (see Example 6.14). According to Lemma 6.16, the dynamical system $(X, \mathbb{R}_+, \pi)$ is completely continuous and, in particular, the cocycle is asymptotically compact. Now, to finish the proof, it is sufficient to apply Theorem 4.6 and Corollary 4.7. □

Theorem 6.19. *Assume the following conditions:*

(i) *The dynamical system $(Y, \mathbb{R}, \sigma)$ is compact and minimal.*
(ii) *The operator-function $A \in C(Y, \mathfrak{A})$ is dissipative.*

Then, one of the following alternatives is fulfilled:

a. *For every $y \in Y$, equation (22) admits, at least, one recurrent solution with constant positive norm.*
b. *For every $f \in C(Y, \mathfrak{A})$, there exists a unique function $\nu_f \in C(Y, \mathcal{C})$ such that*
$$\phi_f(t, \nu_f(y), y) = \nu_f(\sigma(t,y))$$
for all $t \in \mathbb{R}$ and $y \in Y$, where $\phi_f(t,u,y)$ is the unique solution of the equation
$$v' = A(\sigma(t,y))v_t + f(\sigma(t,y)) \quad (y \in Y) \tag{25}$$
passing through the point $u \in \mathcal{C}$ at the initial moment $t = 0$.

Proof. This statement can be proved in the same way as Theorem 6.18, but instead of Theorem 4.6 and Corollary 4.7, we need to apply Theorem 5.7. □

Corollary 6.20. *Under the conditions of Theorem 6.19, if the statement b. holds and the point $y \in Y$ is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent), then the solution $\phi_f(t, \nu_f(y), y) = \nu_f(\sigma(t,y))$ of equation (25) is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent).*

6.3. **Difference Equations.** Let us now apply our general theory to some applications from the theory of difference equations.
Let $(Y, \mathbb{Z}, \sigma)$ be a dynamical system with discrete time $\mathbb{Z}$ on the compact metric space Y. Consider the difference equation
$$x(n+1) = A(\sigma(n,y))x(n), \ (y \in Y) \tag{26}$$
where $A \in C(Y, [E])$. Denote by $\varphi(n,u,y)$ the unique solution of equation (26) with initial data $\varphi(0,u,y) = u$. It is well known (see, for example, [5, 10, 21]) that

the triplet $\langle E, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ is a linear cocycle over $(Y, \mathbb{Z}, \sigma)$ with discrete time. Let $(X, \mathbb{Z}_+, \pi)$ be the corresponding skew-product dynamical system (i.e., $X := E \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$ be the non-autonomous dynamical system generated by the cocycle φ, where $h := pr_2$.

Remark 6.21. *1. Consider the difference equation*

$$x(n+1) = A(n)x(n), \tag{27}$$

where $A \in C(\mathbb{Z}, [E])$. Along with equation (27), we consider a family of equations

$$y(n+1) = B(n)y(n), \tag{28}$$

where $B \in H(A) := \overline{\{A_m |\ m \in \mathbb{Z}\}}$, A_m is m shift of operator A (i.e., $A_m(n) := A(n+m)$ for all $n \in \mathbb{Z}$) and by bar we denote the closure on the space $C(\mathbb{Z}, [E])$) which is endowed with the compact-open topology. The family of equations (28) can be written in the form (26). Indeed, let $Y = H(A)$ and by $(Y, \mathbb{Z}, \sigma)$ we denote the shift dynamical system on $H(A)$. Now we can rewrite the family of equations (27) as follow

$$x(n+1) = \tilde{A}(\sigma(n, y))x(n),\ \ (y \in Y)$$

where $\tilde{A} : H(A) \to [E]$ is the mapping defined by the equality $\tilde{A}(B) := B(0)$, for all $B \in H(A)$.
2. Note that $H(A)$ is a compact minimal set (of the Bebutov dynamical system $(C(\mathbb{Z}, [E]), \mathbb{Z}, \sigma))$) if and only if the operator-function $A \in C(\mathbb{Z}, [E])$ is recurrent in the sense of Birkhoff with respect to $n \in \mathbb{Z}$ (in particular, almost periodic or almost automorphic).

Applying our general results from Sections 4 and 5, we will be able to obtain some new and interesting results for the difference equation (26).
Namely, the following results hold.
Recall that an operator $A \in C(Y, [E])$ is said to be *completely continuous*, if, for any bounded subset $B \subset E$, the set $\bigcup\{A(y)B :\ y \in Y\}$ is relatively compact.

Remark 6.22. *It is clear that if the space E is finite-dimensional and Y is compact, then, every operator $A \in C(Y, [E])$ is completely continuous.*

Definition 6.23. *An operator $A \in C(Y, [E])$ is said to be asymptotically compact, if there are $A', A'' \in C(Y, [E])$ such that*

(i) *$A(y) = A'(y) + A''(y)$ for all $y \in Y$;*
(ii) *For all $y \in Y$, the operator $A'(y)$ is a contraction, i.e., $||A'(y)|| < 1$;*
(iii) *For every $y \in Y$, the operator $A''(y)$ is compact (completely continuous).*

Lemma 6.24. *Suppose that the following conditions are fulfilled:*

(i) *The space Y is compact;*
(ii) *The operator $A \in C(Y, [E])$ is asymptotically compact.*

Then, the cocycle φ, generated by equation (26), is asymptotically compact.

Proof. Let φ be the cocycle generated by equation (26). Then,

$$\varphi(n, u, y) = A(\sigma(n, y))A(\sigma(n-1, y)) \dots A(\sigma(1, y))A(y)u$$

for all $y \in Y$, $n \in \mathbb{Z}_+$ and $u \in E$. Denote by

$$\varphi_1(n, u, y) := A'(\sigma(n, y))A'(\sigma(n-1, y)) \dots A'(\sigma(1, y))A'(y)u$$

and $\varphi_2(n, u, y) := \varphi(n, u, y) - \varphi_1(n, u, y)$. Thanks to our assumptions, φ_1 and φ_2 satisfy the following conditions:

(i) $|\varphi_1(n,u,y)| \le \alpha^n |u|$ for all $n \in \mathbb{T}_+$ and $(u,y) \in E \times Y$, where $\alpha := \min\{||A(y)|| : y \in Y\}$;
(ii) The set $\varphi_2(n,A,Y)$ is relatively compact for all $n \in \mathbb{N}$, and any bounded subset A from E.

Now, to finish the proof, it is sufficient to apply Theorem 3.3 and Lemma 3.4. □

Lemma 6.25. *Suppose that the following conditions are fulfilled:*
(i) *The space Y is compact;*
(ii) *$A \in C(Y,[E])$ is asymptotically compact;*
(iii)

$$||A(y)|| \le 1 \tag{29}$$

for all $y \in Y$, where $||\cdot||$ is the operator norm;
(iv) *A is asymptotically compact.*

Then,
(i) *The linear cocycle, generated by equation (26), is non-expanding, i.e., $|\varphi(n,\ u,\ y)| \le |u|$ for all $n \in \mathbb{Z}_+$, $u \in E$ and $y \in Y$;*
(ii) *The cocycle φ, generated by equation (26), is asymptotically compact.*

Proof. Let $\varphi(n,u,y)$ be the solution of equation (26). Then,

$$\varphi(n,u,y) = A(\sigma(n,y))A(\sigma(n-1,y))\dots A(\sigma(1,y))A(y)u \tag{30}$$

for all $u \in E$ and $y \in Y$. From equalities (29) and (30) it follows that $|\varphi(n,u,y)| \le |u|$ for all $n \in \mathbb{Z}_+$ and $(u,y) \in E \times Y$. Thus the first statement of Lemma is proved. The second statement follows from Lemma 6.24. □

Theorem 6.26. *Assume the following conditions:*
(i) *The dynamical system $(Y,\mathbb{Z},\sigma)$ is compact and minimal.*
(ii) *The operator $A \in C(Y,[E])$ is asymptotically compact;*
(iii) *$||A(y)|| \le 1$ for all $y \in Y$.*

Then, one of the following alternatives is fulfilled:
a. *For every $y \in Y$, equation (26) admits at least one recurrent solution with constant positive norm.*
b. *The trivial solution of equation (26) is uniformly exponentially stable, i.e., there exist two positive numbers N and ν such that $|\varphi(n,u,y)| \le Ne^{-\nu n}|u|$ for all $(u,y) \in X$ and $t \in \mathbb{Z}_+$.*

Proof. Let φ be the cocycle generated by equation (26), and $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z},\sigma),h\rangle$ the linear non-autonomous dynamical system associated to φ. Since the cocycle φ is asymptotically compact, then the dynamical system $(X,\mathbb{Z}_+,\pi)$ is also asymptotically compact. Now, to finish the proof, it is sufficient to apply Theorem 4.6 and Corollary 4.7. □

Theorem 6.27. *Suppose that the following conditions are fulfilled:*
(i) *The dynamical system $(Y,\mathbb{Z},\sigma)$ is compact and minimal.*
(ii) *The operator $A \in C(Y,[E])$ is asymptotically compact;*
(iii) *$||A(y)|| \le 1$ for all $y \in Y$.*

Then, one of the following alternatives is fulfilled:
a. *For every $y \in Y$, equation (26) admits, at least, one recurrent solution with constant positive norm.*

b. *For every $f \in C(Y,E)$, there exists a unique function $\nu_f \in C(Y,E)$ such that*

$$\phi_f(n,\nu_f(y),y) = \nu_f(\sigma(n,y))$$

for all $t \in \mathbb{Z}$ and $y \in Y$, where $\phi_f(n,u,y)$ is the unique solution of equation

$$v(n+1) = A(\sigma(n,y))v(n) + f(\sigma(n,y)), \quad (y \in Y), \tag{31}$$

passing through the point $u \in E$ at the initial moment $n = 0$.

Proof. This statement can be proved as Theorem 6.26, but instead of Theorem 4.6 and Corollary 4.7 we need to apply Theorem 5.7. □

Corollary 6.28. *Under the conditions of Theorem 6.27, if statement b. holds, and the point $y \in Y$ is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent), then the solution $\phi_f(n,\nu_f(y),y) = \nu_f(\sigma(n,y))$ of equation (31) is quasi-periodic (respectively, almost periodic, almost automorphic, recurrent).*

Remark 6.29. 1. If the dynamical system $(Y,\mathbb{Z},\sigma)$ is almost periodic (in particular, it is uniquely ergodic), then we have (see, for example, [18, ChIV]):

(i) There exists the limit

$$\mu := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln ||A(\sigma(k,y))||; \tag{32}$$

(ii) This limit exists uniformly with respect to $y \in Y$;

(iii) The limit μ in (32) does not depend on $y \in Y$.

2. If, under the conditions of Theorem 6.27, we replace (iii) by the condition $\mu < 0$, then, for every $f \in C(Y,E)$, there exists a unique function $\gamma_f \in C(Y,E)$ such that

$$\phi_f(n,\gamma_f(y),y) = \gamma_f(\sigma(n,y))$$

for all $t \in \mathbb{Z}$ and $y \in Y$, where $\phi_f(n,u,y)$ is the unique solution of equation (31), passing through the point $u \in E$ at the initial moment $n = 0$. This fact is a particular case of one result established in our paper [7].

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