ALMOST PERIODIC MOTIONS IN SEMI-GROUP DYNAMICAL SYSTEMS AND BOHR/LEVITAN ALMOST PERIODIC SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS WITHOUT FAVARD'S SEPARATION CONDITION

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Abstract. The discrete analog of the well-known Favard Theorem states that the linear difference equation

$$x(t+1) = A(t)x(t) + f(t) \ (t \in \mathbb{Z}) \tag{1}$$

with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution. The main assumption in this theorem is the separation among bounded solutions of the homogeneous equations

$$x(t+1) = B(t)x(t), \tag{2}$$

where $B \in H(A) := \{B \mid B(t) = \lim_{n\to+\infty} A(t+t_n)\}$.

In this paper we prove that the linear difference equation (1) with Levitan almost periodic coefficients has a unique Levitan almost periodic solution, if it has at least one bounded solution, and the bounded solutions of the homogeneous equation

$$x(t+1) = A(t)x(t) \tag{3}$$

are homoclinic to zero in the positive direction (i.e., $\lim_{t\to+\infty} |\varphi(t)| = 0$ for all relatively compact solutions φ of (3)). If the coefficients of (1) are Bohr almost periodic and all relatively compact solutions of all limiting equations (2) tend to zero as $t \to +\infty$, then equation (1) admits a unique almost automorphic solution.

We study the problem of existence of Bohr/Levitan almost periodic solutions of equation (1) in the framework of general non-autonomous dynamical systems (cocycles).

Dedicated to the memory of José Real

1. Introduction

Let $\mathbb{R}$ (respectively, $\mathbb{Z}$) be the set of all real (respectively, entire) numbers and $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$. Recall (see, for example, [19],[23]-[25]) that a function φ defined on

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$\mathbb{S}$ with values in a Banach space E is called *Bohr almost periodic*, if for all $\varepsilon > 0$ there exists a positive number $l(\varepsilon)$ such that, on every interval $[a, a+l]$ $(a \in \mathbb{S})$, there exists at least one number $\tau \in \mathbb{S}$ such that

$$|\varphi(t+\tau) - \varphi(t)| < \varepsilon$$

for all $t \in \mathbb{S}$ (the number τ is called an ε *almost period* of the function φ).

A function $\varphi : \mathbb{S} \to E$ is called [15, 25] *Levitan almost periodic*, if there exists a Bohr almost periodic function $\psi : \mathbb{S} \to F$ (F is another Banach space) such that $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_\varphi$ is the family of all sequences $\{t_n\} \subset \mathbb{S}$ such that the functional sequence $\{\varphi(\cdot + t_n)\}$ converges to $\varphi(\cdot)$ uniformly on every compact subset of $\mathbb{S}$.

It is evident that every Bohr almost periodic function is Levitan almost periodic. The converse statement is not true [25].

A function $\varphi : \mathbb{S} \to E$ is called [1, 4] (see also [25, 28, 29]) *almost automorphic* (or *Bohr almost automorphic*) if for every sequence $\{t_n^{'}\}$ there exists a subsequence $\{t_n\} \subset \mathbb{S}$ for which we have local convergence (i.e., uniform convergence on every compact subset of $\mathbb{S}$)

$$\varphi(t + t_n) \to \tilde{\varphi}(t),$$

and the "returning" also holds:

$$\tilde{\varphi}(t - t_n) \to \varphi(t).$$

It is known (see, for example, [25] and also [15]) that every almost automorphic function is Levitan almost periodic. The converse is not true in general, because almost automorphic functions are bounded, but a Levitan almost periodic function may be unbounded. Recall also that any Bohr almost periodic function is almost automorphic.

This paper is dedicated to the study of linear difference equations with Bohr/Levitan almost periodic and almost automorphic coefficients. This field is called Favard's theory [25, 39], due to the fundamental contributions made by J. Favard [21, 22]. In 1927, J. Favard published his celebrated paper, where he studied the existence of almost periodic solutions of the following differential equation in $\mathbb{R}^n$:

$$x' = A(t)x + f(t) \quad (x \in \mathbb{R}^n) \tag{4}$$

with the $n \times n$ matrix $A(t)$ and the vector-function $f(t)$ almost periodic in the sense of Bohr (see, for example, [24, 25]).

Along with equation (4), consider the homogeneous equation

$$x' = A(t)x$$

and the corresponding family of *limiting equations*

$$x' = B(t)x, \tag{5}$$

where $B \in H(A)$, and $H(A)$ denotes the hull of the almost periodic matrix $A(t)$ which is composed by those functions $B(t)$ obtained as uniform limits on $\mathbb{R}$ of the type $B(t) := \lim_{n\to\infty} A(t + t_n)$, where $\{t_n\}$ is some sequence in $\mathbb{R}$.

Let us now recall Favard's result.

Theorem 1.1. *(Favard's theorem* [21]*) The linear differential equation (4) with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution, and each bounded solution $\varphi(t)$ of every limiting equation (5) $(B \in H(A))$ is separated from zero, i.e.,*

$$\inf_{t\in\mathbb{R}} |\varphi(t)| > 0.$$

Using the same arguments (namely, the Favard min-max method) as in the proof of Theorem 1.1, the following discrete analog can be established.

Theorem 1.2. *(Discrete version of Favard's theorem) The linear difference equation*

$$x(t+1) = A(t)x(t) + f(t) \quad (x \in \mathbb{R}^n), \tag{6}$$

with Bohr almost periodic coefficients, admits at least one Bohr almost periodic solution if it has a bounded solution, and each bounded solution $\varphi(t)$ of every limiting equation

$$y(t+1) = B(t)y(t) \quad (B \in H(A)) \tag{7}$$

is separated from zero, i.e.,

$$\inf_{t\in\mathbb{Z}} |\varphi(t)| > 0. \tag{8}$$

Remark 1.3. *In this paper we study the problem of existence of Levitan/Bohr almost periodic and almost automorphic solutions of linear difference equations in a more general framework. Namely, we study the problem of existence of Poisson stable solutions (in particular, periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable) of linear difference equations with Poisson stable coefficients. The powerful tool to study this problem is the notion of comparability and uniform comparability of motions by the character of recurrence introduced by B. Shcherbakov* [32]–[34].

In our previous paper [8] some generalizations of Theorem 1.2 without Favard's separation condition (8) were proved. More precisely, denote by $[E]$ the Banach space of linear and bounded mappings from E into itself equipped with the operator-norm.

Theorem 1.4. *(*[8, Theorem 4.16]*) Let $(A, f) \in C(\mathbb{T}, [E]) \times C(\mathbb{T}, E)$ and suppose that the following conditions hold:*

(i) *Eq. (6) admits a solution $\varphi(t, u_0, (A, f))$ which is relatively compact on $\mathbb{Z}$;*
(ii) *for all $B \in H(A)$ the solutions of equation (7), which are relatively compact on $\mathbb{Z}$, tend to zero as the time t tends to ∞, i.e.,*

$$\lim_{|t|\to+\infty} |\varphi(t, u, B)| = 0, \tag{9}$$

if $\varphi(t, u, B)$ is relatively compact on $\mathbb{Z}$.

If $(A, f) \in C(\mathbb{Z}, [E]) \times C(\mathbb{Z}, E)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (6) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.

It is worth noticing that difference equations which appear in the applications are often defined only on the (discrete) semi-axis $\mathbb{Z}_+$. Therefore, it is desirable to generalize Theorem 1.4 in the sense that we can also handle equation (6) defined only on $\mathbb{Z}_+$. The main aim of this paper is to study this problem.

More precisely, we generalize Theorem 1.4, in this paper, in the following two directions:

(i) To replace condition (9) by a weaker (one-sided) one which reads as
$$\lim_{t\to+\infty} |\varphi(t,u,B)| = 0;$$
(ii) To consider the difference equation (6) defined also on $\mathbb{Z}_+$.

One of the main result that we will prove in our paper is the following.

Theorem 1.5. *Let $(A,f)\in C(\mathbb{T},[E])\times C(\mathbb{T},E)$, where $\mathbb{T}=\mathbb{Z}$ or $\mathbb{Z}_+$, and suppose that the following conditions hold:*

(i) *equation (6) admits a solution $\varphi(t,u_0,(A,f))$ which is relatively compact on $\mathbb{Z}_+$;*
(ii) *for all $B\in H(A)$ the solutions of equation (7), which are relatively compact on $\mathbb{Z}$, tend to zero as the time variable t tends to $+\infty$, i.e.,*

$$\lim_{t\to+\infty} |\varphi(t,u,B)| = 0, \tag{10}$$

if $\varphi(t,u,B)$ is defined on $\mathbb{Z}$ and is relatively compact.

Then, if $(A,f)\in C(\mathbb{Z},[E])\times C(\mathbb{Z},E)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), equation (6) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.

Remark 1.6. ***In our opinion, condition (10) seems much more natural than condition (9) when the equation (6) is defined on $\mathbb{Z}_+$. In addition, condition (10) is simpler to verify than condition (9). Moreover, condition (10) takes place, for example, if $\lim\limits_{t\to+\infty}|\varphi(t,u,B)| = 0$ for all solution $\varphi(t,u,B)$ of equation (7) defined on $\mathbb{Z}_+$ with relatively compact rank ($\varphi(\mathbb{Z}_+,u,B)$ is relatively compact). In particular, this condition is fulfilled if the trivial solution of equation (7) is attracting, i.e., condition (10) holds true for all solution of equation (7).***

This paper is organized as follows.

In Section 2 we study the almost periodic motions of semi-group dynamical systems. In the mathematical literature there are two definitions of almost periodicity (in the sense of Bohr) of motions. The first one was introduced by Bhatia and Chow [7] (see also [30, Ch. III]) and the second was introduced by Seifert [31]. We establish the equivalence of both notions of almost periodicity for semi-group dynamical systems (Theorem 2.9).

Section 3 is dedicated to the study of comparability for the motions of dynamical systems by the character of their recurrence. We also prove some generalizations of the well-known B. A. Shcherbakov principle of comparison for motions of dynamical systems by the character of their recurrence. Our main abstract result is Theorem 3.24 which guarantees the existence of a unique uniformly compatible solution of some abstract evolution equation, if the complete compact trajectories tend to zero as the time goes to infinity.

In Section 4 we analyze the compatible (respectively, uniformly compatible) solutions of linear difference and functional-difference equations with finite delay in a Banach space. Here we present a test for the existence of Bohr (respectively, Levitan) almost periodic and almost automorphic solutions of non-homogeneous linear difference equations with Bohr (respectively, Levitan) almost periodic and almost automorphic coefficients.

2. Almost periodic motions in semi-group dynamical systems

Although we could refer to other references for some of the preliminaries below, we prefer to include them here for the sake of completeness and convenience of the readers.

2.1. **Poisson Stable Motions.**

Let us collect in this subsection some well-known concepts and results from the theory of dynamical systems which will be necessary for our analysis in this paper.

Let (X,ρ) be a complete metric space. By $\mathbb{S}$ we will denote either $\mathbb{R}$ or $\mathbb{Z}$ and by $\mathbb{T}=\mathbb{S}$ or $\mathbb{S}_+ := \{s\in\mathbb{S}|\ s\ge 0\}$).

Unlike the definitions established in [8], the ones below are also valid when we are working with a semigroup instead of a group dynamical system.

Let $(X,\mathbb{T},\pi)$ be *a dynamical system* on X, i.e., let $\pi : \mathbb{T}\times X\to X$ be a continuous function such that $\pi(0,x)=x$ for all $x\in X$, and $\pi(t_1+t_2,x)=\pi(t_2,\pi(t_1,x))$, for all $x\in X$, and $t_1,t_2\in\mathbb{T}$.

Let $\tau\in\mathbb{T}$ be a positive number. A point $x\in X$ is called *$\tau-$periodic*, if $\pi(t+\tau,x)=\pi(t,x)$ for all $t\in\mathbb{T}$. If the point $x\in X$ is $\tau-$periodic for all $\tau>0$, then it is called a *stationary point.*

Given $\varepsilon>0$, a number $\tau\in\mathbb{T}$ is called an *$\varepsilon-$shift* (respectively, an *$\varepsilon-$ almost period*) of x, if $\rho(\pi(\tau,x),x)<\varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x))<\varepsilon$ for all $t\in\mathbb{T}$).

A point $x\in X$ is called [37] *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon>0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x\in X$.

If the point $x\in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t,x)\mid t\in\mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\}\subset\mathbb{T}\ :\ \text{such that } \{\pi(t_n,x)\}\to x\}$, $\mathfrak{N}_x^\infty := \{\{t_n\}\in\mathfrak{N}_x\ :\ \text{such that } t_n\to\infty \text{ as } n\to\infty\}$.

A point $x \in X$ is said to be *Levitan almost periodic* (see [15] and also [25]) for the dynamical system $(X, \mathbb{T}, \pi)$ if there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.1. *Let $x_i \in X_i$ $(i = 1, 2, \ldots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then the point $x := (x_1, x_2, \ldots, x_m)) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic for the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t, x) := (\pi_1(t, x_1), \pi_2(t, x_2), \ldots, \pi_m(t, x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \ldots, x_m) \in X$.*

A point $x \in X$ is called *stable in the sense of Lagrange* (*st*.L), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called *almost automorphic* (see [15, 25, 36]) for the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \lambda)$ and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.2. *Let $x \in X$ be a st.L point, $y \in Y$ an almost automorphic point, and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then, the point x is almost automorphic too.*

Denote by ω_x the ω-limit set of the point $x \in X$, i.e., $\omega_x := \{p \in X :$ there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $t_n \to +\infty$ and $\{\pi(t_n, x)\} \to p$ as $n \to \infty\}$.

A point $x \in X$ is said to be (positively) *Poisson stable* if $x \in \omega_x$.

A point $x \in X$ is called *uniformly* (respectively, uniformly positively) *Poisson stable* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x^{\infty}$ (respectively, $\{t_n\} \in \mathfrak{N}_x^{+\infty}$) such that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{T}} \rho(\pi(t + t_n, x), \pi(t, x)) = 0. \tag{11}$$

Remark 2.3. *Every almost periodic point is uniformly Poisson stable.*

2.2. **Two definitions of almost periodicity for semi-group dynamical systems.** Let (X, ρ) be a complete metric space and $(X, \mathbb{T}, \pi)$ be a dynamical system on X.

A subset $\mathcal{P} \subseteq \mathbb{T}$ is said to be *relatively dense* in $\mathbb{T}$ if there exists a positive number $l \in \mathbb{T}$ such that $[t, t+l] \bigcap \mathcal{P} \neq \emptyset$ for all $t \in \mathbb{T}$, where $[t, t+l] := \{s \in \mathbb{T} : \ t \leq s \leq t+l\}$.

Bhatia & Chow's definition. A point $x \in X$ (respectively, a motion $\pi(t, x)$) is called *almost periodic* [7] (see also [16, Ch.I]), if for any positive number ε there exits a relatively dense subset $\mathcal{P}_{\varepsilon}$ in $\mathbb{T}$ such that

$$\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon \tag{12}$$

for all $t \in \mathbb{T}$ and $\tau \in \mathcal{P}_{\varepsilon}$.

Seifert's definition. In the work [31] it was introduced another definition of almost periodicity for semi-group dynamical systems. Namely, the point $x \in X$ is called *almost periodic* (in the semi-group dynamical system $(X, \mathbb{T}, \pi)$) if for any positive number ε there exits a relatively dense subset $\mathcal{P}_\varepsilon$ in $\mathbb{S}$ such that (12) holds for all $t \in \mathbb{T}$ and $\tau \in \mathcal{P}_\varepsilon$ with the condition $t + \tau \in \mathbb{T}$.

Bhatia & Chow's definition seems to be more appropriate (in our opinion) to study the problem of almost periodicity of solutions of difference equations defined only on the semi-axis $\mathbb{Z}_+$. We study below the relationship between these two definitions introduced above.

Remark 2.4. *It is easy to see that almost periodicity in the sense of Seifert* [31] *implies almost periodicity in the sense of Bhatia & Chow* [7]*. Now we will show that the converse also holds and, consequently both concepts are equivalent, but let us first recall two results which will be necessary.*

Lemma 2.5. ([16, ChI]) *Let $x \in X$ be an almost periodic point (in the sense of Bhatia & Chow). Then, the following statements hold:*

(i) *for every $\varepsilon > 0$ there exists a subset $\mathcal{P}_\varepsilon$ which is relatively dense in $\mathbb{T}$ and such that*
$$\rho(\pi(t+\tau, p), \pi(t, p)) < \varepsilon$$
for all $t \in \mathbb{T}$, $\tau \in \mathcal{P}_\varepsilon$ and $p \in H(x) := \overline{\{\pi(t, x) :\ t \in \mathbb{T}\}}$;

(ii) *the set $H(x)$ is uniformly Lyapunov stable (in the positive direction), i.e., for all $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\rho(p, q) < \delta$ $(p, q \in H(x))$ implies $\rho(\pi(t, p), \pi(t, q)) < \varepsilon$ for all $t \geq 0$;*

(iii) *the dynamical system $(H(x), \mathbb{T}, \pi)$ is distal, i.e.,*
$$\inf_{t\in\mathbb{T}} (\pi(t, p), \pi(t, q)) > 0$$
for all $p, q \in H(x)$ $(p \neq q)$.

Lemma 2.6. ([30, Ch.I])*. Let $(X, \mathbb{S}_+, \pi)$ be a semigroup dynamical system and assume that for any $t \in \mathbb{S}_+$ the map $\pi^t = \pi(\cdot, t) : X \to X$ is a homeomorphism and $\tilde{\pi} : X \times \mathbb{S} \to X$ is the map defined by the equality*
$$\tilde{\pi}(x, t) := \begin{cases} \pi(x, t), & (x, t) \in X \times \mathbb{S}_+, \\ (\pi^{-t})^{-1}(x), & (x, t) \in X \times \mathbb{S}_-. \end{cases}$$
Then, the triple $(X, \mathbb{S}, \tilde{\pi})$ is a group dynamical system.

To formulate the next statement we need the notion of V-monotony (see, for example, [15]) for **a group** dynamical system.

Let $V : X \times X \to \mathbb{R}_+$ be a continuous function. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be *V-monotone*, if $V(\pi(t, x_1), \pi(t, x_2)) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \times X$ and $t \geq 0$.

Lemma 2.7. *Let $x \in X$ be an almost periodic point (in the sense of Bhatia & Chow), then the following statements hold:*

(i) *the set $H(x)$ is compact;*

(ii) *there exists a sequence $\{t_n\} \subseteq \mathbb{T}$ such that $t_n \to +\infty$ and $\pi(t_n, p) \to p$ as $n \to \infty$ uniformly with respect to $p \in H(x)$;*

(iii) *the set $H(x)$ is invariant, i.e., $\pi(t, H(x)) = H(x)$ for all $t \in \mathbb{T}$;*

(iv) *there exists a group dynamical system* $(H(x), \mathbb{S}, \tilde{\pi})$ *such that* $\tilde{\pi}(t,p) = \pi(t,p)$ *for all* $t \in \mathbb{T}$ *and* $p \in H(x)$*, i.e., the semi-group dynamical system* $(H(x), \mathbb{T}, \pi)$ *admits a group extension on* $H(x)$*;*

(v) *the dynamical system* $(H(x), \mathbb{T}, \pi)$ *is* V*-monotone, where*

$$V(p,q) := \sup\{\rho(\pi(t,p), \pi(t,q)) : \ t \in \mathbb{T}\}; \tag{13}$$

(vi) *the group dynamical system* $(H(x), \mathbb{S}, \tilde{\pi})$ *is bilaterally Lyapunov stable, i.e., for all* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\rho(p,q) < \delta$ $(p, q \in H(x))$ *implies* $\rho(\tilde{\pi}(t,p), \tilde{\pi}(t,q)) < \varepsilon$ *for all* $t \in \mathbb{S}$*;*

(vii) *the point* $x \in X$ *is almost periodic with respect to the group dynamical system* $(H(x), \mathbb{S}, \tilde{\pi})$*.*

Proof. Let ε be an arbitrary positive number. Then, by the almost periodicity of x there exists a relatively dense subset $\mathcal{P}_{\varepsilon/4}$ such that

$$\rho(\pi(t+\tau, x), \pi(t,x)) < \varepsilon/4 \tag{14}$$

for all $t \in \mathbb{T}$ and $\tau \in \mathcal{P}_{\varepsilon/4}$ and, consequently, we have

$$\begin{aligned} \rho(\pi(t+\tau_1, x), \pi(t+\tau_2, x)) &\leq \rho(\pi(t+\tau_1, x), \pi(t,x)) + \rho(\pi(t,x), \pi(t+\tau_2, x)) \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2 \end{aligned} \tag{15}$$

for all $t \in \mathbb{T}$ and $\tau_1, \tau_2 \in \mathcal{P}_{\varepsilon/3}$. Denote by $\alpha_\varepsilon := \inf\{\tau : \ \tau \in \mathcal{P}_{\varepsilon/3}\}$, then from (15) we obtain

$$\rho(\pi(t+\alpha, x), \pi(t+\tau, x)) \leq \varepsilon/2$$

for all $t \in \mathbb{T}$ and $\tau \in \mathcal{P}_{\varepsilon/4}$. Let now $s > \alpha$ and $l(\varepsilon/4)$ the positive number from the definition of relatively density of $\mathcal{P}_{\varepsilon/4}$, then we can find a number $\tau \in \mathcal{P}_{\varepsilon/4}$ such that $0 \leq s - \tau \leq l(\varepsilon/4)$ and hence, taking into account (14) and (15), we have

$$\begin{aligned} \rho(\pi(s,x), \pi(s-\tau+\alpha)) &\leq \rho(\pi(s,x), \pi(s+\tau, x)) + \rho(\pi(s+\tau, x), \pi(s-\tau+\alpha)) \\ &< \varepsilon/4 + \varepsilon/2 < \varepsilon, \end{aligned}$$

i.e., $\pi(s,x) \in B(\pi([0, l(\varepsilon/4)+\alpha], x), \varepsilon)$. Since the set $Q_\varepsilon := \pi([0, l(\varepsilon/4)+\alpha], x)$ is compact and the space X is complete, then by the Hausdorff theorem the set $\{\pi(s,x) : \ s \in \mathbb{T}\}$ is relatively compact. This means that the set $H(x) = \overline{\{\pi(t,x) : \ t \in \mathbb{T}\}}$ is compact.

By Lemma 2.5 for $\varepsilon_n := 1/n$ there exists a number $t_n \geq n$ $(t_n \in \mathbb{T})$ such that $\rho(\pi(t_n, p), p)) < 1/n$ $(\forall \ p \in H(x))$ and, consequently, $\{\pi(t_n, p)\} \to p$ as $n \to +\infty$ uniformly with respect to $p \in H(x)$.

By the second statement we have $H(x) = \omega_x$. For the ω-limit set ω_x we have $\pi(t, \omega_x) \subseteq \omega_x$ for all $t \in \mathbb{T}$. Now it is sufficient to establish that under the conditions of Lemma we have the inverse inclusion too. In fact, let $t \in \mathbb{T}$ and $p \in \omega_x$, then there exists a sequence $\tau_n \to +\infty$ such that $\pi(\tau_n, x) \to p$. Let n be sufficiently large (such that $\tau_n > t$) and consider the sequence $\{\pi(\tau_n - t, x)\}$. Since $H(x)$ is a compact set, then without loss of generality, we may suppose that the sequence $\{\pi(\tau_n - t, x)\}$ is convergent. Denote by $\bar{p}$ its limit, then we have $\pi(t, \bar{p}) = p$. It is evident that $\bar{p} \in \omega_x$ and, consequently, $\omega_x \subseteq \pi(t, \omega_x)$ for all $t \in \mathbb{T}$.

Consider a semi-group dynamical system $(H(x), \mathbb{T}, \pi)$. Without loss of generality we may suppose that $\mathbb{T} = \mathbb{S}_+$. Under the conditions of Lemma the set $H(x) = \omega_x$ is a compact and invariant set, in particular, $\pi(t, \cdot)$ is a mapping from $H(x)$ onto

$H(x)$. By Lemma 2.5 the dynamical system $(H(x),\mathbb{T},\pi)$ is distal, and, consequently, $\pi(t,p)\neq\pi(t,q)$ for all $p,q\in H(x)$ $(p\neq q)$ and $t\in\mathbb{T}$. Thus $\pi(t,\cdot)$ is an homeomorphism from $H(x)$ onto itself. Now to finish the proof of the fourth statement it is sufficient to apply Lemma 2.6.

Denote by $V: H(x)\times H(x)\mapsto\mathbb{R}_+$ the mapping defined by equality (13). Note that V is a new metric on the space $H(x)$ topologically equivalent to ρ. Observe that

$$|V(u,v)-V(p,q)|\le V(u,p)+V(v,q) \tag{16}$$

for all $u,v,p,q\in H(x)$. Since the dynamical system $(H(x),\mathbb{T},\pi)$ is Lyapunov stable, then $V(u,p)+V(v,q)\to 0$ as $u\to p$ and $v\to q$, hence from (16) it follows the continuity of V. Finally, notice that by definition of V we have $V(\pi(t,p),\pi(t,q))\le V(p,q)$ for all $t\in\mathbb{T}$ and $p,q\in H(x)$. Thus the fifth statement is proved.

Let $p,q\in H(x)$, consider the function $\psi(t):=V(\tilde{\pi}(t,p),\tilde{\pi}(t,p))$ (for all $t\in\mathbb{S}$). Note that $\psi:\mathbb{S}\mapsto\mathbb{R}_+$ is a continuous mapping and

$$\psi(t_2)=V(\tilde{\pi}(t_2,p),\tilde{\pi}(t_2,p))=V(\tilde{\pi}(t_2-t_1,\tilde{\pi}(t_1,p)),\tilde{\pi}(t_2-t_1,\tilde{\pi}(t_1,p)))\le \\ V(\tilde{\pi}(t_1,p),\tilde{\pi}(t_1,p))=\psi(t_1)$$

for all $t_1\le t_2$ $(t_1,t_2\in\mathbb{S})$. Thus ψ is a monotone decreasing function and, consequently, there exists the limit $\lim\limits_{t\to+\infty}\psi(t)=C$, where C is a nonnegative constant. By the second statement of the Lemma, there exists a sequence $t_n\to+\infty$ such that $\pi(t_n,p)\to p$ and $\pi(t_n,q)\to q$ as $n\to\infty$. Since the function $V: H(x)\times H(x)\to\mathbb{R}_+$ is continuous, we have

$$V(\tilde{\pi}(s,p),\tilde{\pi}(s,q))=\lim_{n\to\infty}\psi(s+t_n)=C \tag{17}$$

for all $s\in\mathbb{S}$. Using the identity (17) it is not difficult to finish the proof of the sixth statement. Indeed, if we suppose that it is not true, then there are a positive number $\varepsilon_0>0$, sequences $\{s_n\}\subseteq\mathbb{S}$, $\{\delta_n\}$ and $\{p_n\},\{q_n\}\subseteq H(x)$ such that $\delta_n>0$, $\delta_n\to 0$ as $n\to\infty$,

$$\rho(p_n,q_n)<\delta_n \text{ and } \rho(\tilde{\pi}(s_n,p_n),\tilde{\pi}(s_n,q_n)\ge\varepsilon_0. \tag{18}$$

Now, without loss of generality, we may suppose that $s_n\to-\infty$. Since $H(x)$ is compact, then we may suppose that the sequence $\{\tilde{\pi}(s_n,p_n)\}$ (respectively, $\{\tilde{\pi}(s_n,q_n)\}$) is convergent. Denote by $\bar{p}$ (respectively, $\bar{q}$) its limit. Note that

$$\begin{aligned} V(\tilde{\pi}(s_n,,p_n),\tilde{\pi}(+s_n,q_n)) &= V(\tilde{\pi}(-s_n+s_n,p_n),\tilde{\pi}(-s_n+s_n,q_n)) \\ &= V(\tilde{\pi}(-s_n,\tilde{\pi}(+s_n,p_n)),\tilde{\pi}(-s_n,\tilde{\pi}(+s_n,q_n)) \\ &= V(p_n,q_n)\to 0 \end{aligned}$$

as $n\to\infty$ and, consequently, $\bar{p}=\bar{q}$. On the other hand, taking limit in (18) as $n\to\infty$ we obtain $\rho(\bar{p},\bar{q})\le\varepsilon_0$. The obtained contradiction proves our statement.

The seventh statement follows from the statements (i), (vi) and Markov's Theorem (see, for example, [37, ChV]). □

Remark 2.8. *The statements (i),(iv) and (vii) were established in the work* [7] *(see also the book* [30, Ch.II]*). We include here these statements with their proofs for completeness and the convenience of the reader.*

Theorem 2.9. *Let $(X,\mathbb{T},\pi)$ be a semi-group dynamical system and $x\in X$. The following statement are equivalent:*

(i) *the point x is almost periodic in the sense of Bhatia & Chow;*
(ii) *the point x is almost periodic in the sense of Seifert.*

Proof. This statement follows from Remark 2.4 and Lemma 2.7 (item (vii)). □

3. Comparability by recurrence of motions of dynamical systems

One of the fundamental question of the qualitative theory of non-autonomous differential/difference equations is the problem of almost periodicity, or more generally Poisson stability (in particular, Levitan almost periodcity, Bochner almost automorphy, almost recurrence in the sense of Bebutov, recurrence in the sense of Birkhoff, etc) of solutions. B. A. Shcherbakov [32, 33, 34] introduced the notion of comparability and uniform comparability for the motions of dynamical systems by the character of their recurrence which plays a very important role in the study of Poisson stability of the solutions of differential/difference equations. B. A. Shcherbakov also formulated and proved the principle of compatibility (a series of abstract results which permit in many cases to solve the problem of Poisson stability of solutions for some classes of differential/difference equations) of solutions by the character of recurrence.

Note that the theory developed by B. A. Shcherbakov is appropriate for differential/difference equations defined on the whole axis $\mathbb{S}$ ($\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$). However, as our aim is to apply this theory to equations defined only on the semi-axis, we need to modify this theory so that it can be applied to more general cases.

Therefore we will prove some generalizations of these results which will allow us to apply Shcherbakov's principle to a wider class of differential/difference equations.

3.1. **B. A. Shcherbakov's principle of comparability of motions by their character of recurrence.**

In this subsection we will recall a short survey of some notions and results stated and proved by to B. A. Shcherbakov [32, 33, 34]. The main reason is that these results were published in Russian and may be a little unknown for many readers.

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems. A point $x \in X$ is said to be comparable with $y \in Y$ by the character of recurrence, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of y is an ε–shift for x, i.e., $d(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$, where d (respectively, ρ) is the distance on Y (respectively, on X).

Theorem 3.1. *The following conditions are equivalent:*

(i) *the point $x \in X$ is comparable with y by the character of recurrence;*
(ii) *there exists a continuous mapping $h : \Sigma_y = \{\sigma(t, y) : t \in \mathbb{T}\} \to \Sigma_x = \{\pi(t, x) : t \in \mathbb{T}\}$ such that $h(\sigma(t, y)) = \pi(t, x)$ for all $t \in \mathbb{T}$;*
(iii) $\mathfrak{N}_y \subseteq \mathfrak{N}_x$;
(iv) $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$.

Theorem 3.2. *Let $x \in X$ be comparable with $y \in Y$. If the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point $x \in X$.*

A point $x \in X$ is called *uniformly comparable with $y \in Y$ by the character of recurrence*, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of $\sigma(t, y)$ is an ε–shift for $\pi(t, x)$ for all $t \in \mathbb{T}$, i.e., $d(\sigma(t + \tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t + \tau, x), x) < \varepsilon$ for all $t \in \mathbb{T}$ (or equivalently, $d(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{T}$).

Theorem 3.3. *The following condition are equivalent:*

(i) *the point $x \in X$ is uniformly comparable with $y \in Y$ by the character of recurrence;*
(ii) *there exists a uniformly continuous mapping $h : \Sigma_y \to \Sigma_x$ with the following property $h(\sigma(t, y)) = \pi(t, x)$ for all $t \in \mathbb{T}$.*

Denote by $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T}\ :\ \text{such that } \{\pi(t_n, x)\} \text{ converges }\}$, $\mathfrak{M}_x^\infty := \{\{t_n\} \in \mathfrak{M}_x\ :\ \text{such that } t_n \to \infty \text{ as } n \to \infty\}$.

Theorem 3.4. ([16, Ch.II],[18]) *The following conditions are equivalent:*

(i) *$\mathfrak{M}_y \subseteq \mathfrak{M}_x$;*
(ii) *there exists a continuous mapping $h : \overline{\Sigma}_y \to \overline{\Sigma}_x$ with the following properties:*
  (a) *$h(y) = x$;*
  (b) *$h(\sigma(t, q)) = \pi(t, h(q))$ for all $t \in \mathbb{T}$ and $q \in \overline{\Sigma}_y$.*

Corollary 3.5. *If $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, then $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.*

Proof. This statement follows from Theorems 3.1 and 3.4. □

Theorem 3.6. *Let y be stable in the sense of Lagrange. Then, the following conditions are equivalent:*

(i) *the point $x \in X$ is uniformly comparable with $y \in Y$ by the character of recurrence;*
(ii) *$\mathfrak{M}_y \subseteq \mathfrak{M}_x$.*

Theorem 3.7. *Let y be τ–periodic ($\tau > 0$). Then, the following conditions are equivalent:*

(i) *the point $x \in X$ is comparable with $y \in Y$ by character of recurrence;*
(ii) *the point $x \in X$ is uniformly comparable with $y \in Y$ by character of recurrence.*

Denote by $\mathfrak{P}_y := \{\{t_n\} : \text{such that (11) holds}\}$.

Remark 3.8. *The point $y \in Y$ is uniformly Poisson stable (with respect to the dynamical system $(Y, \mathbb{T}, \sigma)$) if and only if $\mathfrak{P}_y \neq \emptyset$.*

Theorem 3.9. *Let $x \in X$ be uniformly comparable with $y \in Y$ by the character of recurrence. If the point $y \in Y$ is recurrent (respectively, almost periodic, uniformly Poisson stable), then so is the point $x \in X$.*

Proof. When the point y is recurrent (respectively, almost periodic), the statement was proved by B. Shcherbakov [33]. Let now y be uniformly Poisson stable. Since the point x is uniformly comparable with y by the character of recurrence, then there exists a uniformly continuous mapping $h : \Sigma_y \to \Sigma_x$ such that $h(\sigma(t,y)) = \pi(x,t)$ for all $t \in \mathbb{T}$. Let ε be an arbitrary positive number and $\delta = \delta(\varepsilon) > 0$ taken from the uniform continuity of the mapping h. If $\{t_n\} \in \mathfrak{P}_y$, then for given $\delta = \delta(\varepsilon) > 0$ there exists a natural number $N(\varepsilon)$ such that

$$d(\sigma(t+t_n,y),\sigma(t,y)) < \delta \tag{19}$$

for all $n \geq N(\varepsilon)$ and $t \in \mathbb{T}$. According to the choice of δ and the uniform comparability of x with y, we have from (19) that

$$\rho(\pi(t+t_n,x),\pi(t,x)) < \varepsilon$$

for all $n \geq N(\varepsilon)$ and $t \in \mathbb{T}$. This means that $\{t_n\} \in \mathfrak{P}_x$ and, consequently, $\mathfrak{P}_x \neq \emptyset$. Thus the point x is uniformly Poisson stable. $\square$

Let $x \in X$ be an almost periodic (respectively, almost automorphic) point of the dynamical system $(X,\mathbb{T},\pi)$. If the space X is linear (in particular, it is a Banach space), then it can be defined the Fourier modulus $\mathcal{M}_x$ of the point x (see, for example, [25] and [36]). **(SHOULD WE INSERT THE DEFINITION HERE?)**

Theorem 3.10. *Let X and Y be two linear metric space, $(X,\mathbb{T},\pi)$ (respectively, $(Y,\mathbb{T},\sigma)$) be a dynamical system on X (respectively, on Y) and $y \in Y$ be an almost periodic (respectively, almost automorphic) point. Then the following conditions are equivalent:*

1. *the point $x \in X$ is uniformly comparable with y by the character of recurrence;*
2. $\mathfrak{M}_y \subseteq \mathfrak{M}_x$*;*
3. *the point x is almost periodic (respectively, almost automorphic) and $\mathcal{M}_x \subseteq \mathcal{M}_y$.*

Proof. Let $\mathbb{T} = \mathbb{S}$. The equivalence of conditions 1. and 2. is established in Theorem 3.6. The equivalence of conditions 2. and 3. is a classical result (see, for example, [25]) if the point y is almost periodic. The case in which y is almost automorphic, the equivalence of conditions 2. and 3. was established in the work [36].

If $\mathbb{T} = \mathbb{S}_+$ the equivalence of conditions 1.-3. can be established using the same arguments (with slight modifications) as in the case $\mathbb{T} = \mathbb{S}$. $\square$

Theorem 3.11. *Let $y \in Y$ be an almost automorphic point. If the point $x \in X$ is uniformly comparable with y by the character of recurrence, then x is also almost automorphic and $\mathcal{M}_x \subseteq \mathcal{M}_y$.*

Proof. Let $\mathbb{T} = \mathbb{S}$. Let $y \in Y$ be an almost automorphic point and $x \in X$ is uniformly comparable with y by the character of recurrence. Since the point $y \in Y$ is stable in the sense of Lagrange, then the point x also is stable in the sense of Lagrange and, by Theorem 3.6, we have $\mathfrak{M}_y \subseteq \mathfrak{M}_x$. According to Corollary 3.5 we obtain $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ and, by Remark 2.2, the point x is also almost automorphic. Now the inclusion $\mathcal{M}_x \subseteq \mathcal{M}_y$ follows from Theorem 3.8 [36, Part I].

If $\mathbb{T} = \mathbb{S}_+$, the statement can be established using the same arguments (with slight modifications) from the case $\mathbb{T} = \mathbb{S}$. □

3.2. **Some generalization of B. A. Shcherbakov's results.** In this Subsection we will give some generalization of B. A. Shcherbakov's results concerning the comparability of points by the character of their recurrence. Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of group $\mathbb{S}$ ($\mathbb{T}_i = \mathbb{S}$ or $\mathbb{S}_+$ and $i = 1, 2$). Consider two dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$.

Let $\mathfrak{M}_x^{+\infty} := \{\{t_n\} \in \mathfrak{M}_x \ : \text{ such that } t_n \to +\infty \text{ as } n \to \infty\}$ and $\mathfrak{N}_x^{+\infty} := \{\{t_n\} \in \mathfrak{N}_x \ : \text{ such that } t_n \to +\infty \text{ as } n \to \infty\}$.

Denote by $\mathfrak{M}_{y,q}^{+\infty} := \{\{t_n\} \in \mathfrak{M}_y^{+\infty} \ : \text{ such that } \sigma(t_n, y) \to q \text{ as } n \to \infty\}$ and $\mathfrak{M}_y^{+\infty}(M) := \bigcup\{\mathfrak{M}_{y,q}^{+\infty} : \ q \in M\}$.

Theorem 3.12. ([12],[16, Ch.II]) *Let $\mathfrak{M}_y^{+\infty}(M) \subseteq \mathfrak{M}_x^{+\infty}$, then the following statements take place:*

(i) $\mathfrak{M}_y^{+\infty}(\Sigma_M) \subseteq \mathfrak{M}_x^{+\infty}$, *where* $\Sigma_M := \{\sigma(t, q) : \ t \in \mathbb{T}_2 \text{ and } q \in M\}$;
(ii) *for every $q \in \Sigma_M$ there exists a unique $p \in \omega_x$ such that*

$$\mathfrak{M}_{y,q}^{+\infty} \subseteq \mathfrak{M}_{x,p}^{+\infty}; \tag{20}$$

(iii) *the mapping $h : \Sigma_M \to \omega_x$ defined by the equality $h(q) = p$ for all $q \in \Sigma_M$, where the point $p \in \omega_x$ is defined by (20), is continuous and*

$$h(\sigma(t, q)) = \pi(t, h(q))$$

for all $q \in \Sigma_M$ and $t \in \mathbb{T}_1$;
(iv) *if the point $y \in Y$ is Poisson stable (in the positive direction), then the point x is also Poisson stable (in the positive direction) and $h(y) = x$.*

Theorem 3.13. *Let $y \in Y$ be a Poisson stable (in the positive direction) point. Then, the following conditions are equivalent:*

a. $\mathfrak{M}_y \subseteq \mathfrak{M}_x$;
b. $\mathfrak{M}_y^{\infty} \subseteq \mathfrak{M}_x^{\infty}$ *and* $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$;
c. *there exists a continuous mapping $h : \omega_y \to \omega_x$ with the properties:*
(i)

$$h(y) = x; \tag{21}$$

(ii)

$$h(\sigma(t, q)) = \pi(t, h(q)) \tag{22}$$

for all $t \in \mathbb{T}$ and $q \in \omega_y$.

Proof. To prove this theorem it is sufficient to establish the implication b.⇒ c. According to Theorem 3.12 there exists a continuous mapping $h : \omega_y \to \omega_x$ with properties (21) and (22). Implication c. ⇒ a. is evident and the proof is complete. □

Theorem 3.14. *Let $y \in \omega_y$. Then, the following conditions are equivalent:*

a. $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$;
b. $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$.

Proof. The implication $a. \Rightarrow b.$ is evident. Now we will establish the converse. If $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$ then, by Theorem 3.12, there exists a continuous function $h : \Sigma_y \to \omega_x$ satisfying the condition $h(\sigma(t, y)) = \pi(t, h(y))$ for all $t \in \mathbb{T}_1$. Note that under condition b. we have $h(y) = x$. In fact $\mathfrak{M}_{y,y}^{+\infty} = \mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty} = \mathfrak{M}_{x,x}^{+\infty}$ and, consequently, by Theorem 3.12, $h(y) = x$. Now to finish the proof it is sufficient to apply Theorem 3.1. □

Theorem 3.15. *Let $y \in \omega_y$. Then, the following conditions are equivalent:*

a. $\mathfrak{M}_y^{\infty} \subseteq \mathfrak{M}_x^{\infty}$ *and* $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$;
b. $\mathfrak{M}_y^{+\infty} \subseteq \mathfrak{M}_x^{+\infty}$ *and* $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$.

Proof. The implication $a. \Rightarrow b.$ is evident. If $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$, then by Theorem 3.12 there exists a continuous function $h : \omega_y \to \omega_x$ satisfying the condition $h(\sigma(t, q)) = \pi(t, h(q))$ for all $q \in \omega_y,\ t \in \mathbb{T}_1$ and $h(y) = x$. Since $y \in \omega_y$, then $\overline{\Sigma}_y = \omega_y$. Now, to finish the proof it is sufficient to apply Theorems 3.4 and 3.13. □

3.3. **Compatibility and Uniform Compatibility of Motions by the Character of Recurrence in the Sense of Shcherbakov.** We will prove now the main abstract results in this paper. First, we start with the following definitions.

Let (X, h, Ω) be a fiber space, i.e., X and Ω are two metric spaces and $h : X \to \Omega$ is a homomorphism from X onto Ω. The subset $M \subseteq X$ is said to be *conditionally relatively compact* [13, 14], if the pre-image $h^{-1}(\Omega') \bigcap M$ of every relatively compact subset $\Omega' \subseteq \Omega$ is a relatively compact subset of X, in particular $M_\omega := h^{-1}(\omega) \bigcap M$ is relatively compact for every ω. The set M is called *conditionally compact* if it is closed and conditionally relatively compact.

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on the metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, if the following conditions are fulfilled:

(i) $\varphi(0, u, y) = u$ **for all** $u \in W$ **and** $y \in Y$**;**
(ii) $\varphi(t+\tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ **for all** $t, \tau \in \mathbb{T}_1$**,** $u \in W$ **and** $y \in Y$**;**
(iii) **the mapping** $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ **is continuous.**

Example 3.16. Consider the next difference equation

$$x(t+1) = f(t, x(t)) \tag{23}$$

with right hand side $f \in C(\mathbb{Z}_+ \times W, \mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f), \mathbb{Z}_+, \sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{Z}_+ \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{Z}_+\}}$, and $\sigma(t, g) = g_t$, for all $g \in H^+(f)$, where g_t is defined by $g_t(x, s) = g(x, t+s)$. Let $\varphi(t, u, g)$ denote the unique solution of equation

$$y(t+1) = g(t, y(t)), \quad (g \in H^+(f)).$$

Then, from the general properties of the solutions of non-autonomous difference equations it follows that the following statements hold:

(i) $\varphi(0,u,g) = u$ **for all** $u \in W$ **and** $g \in H^+(f)$**;**
(ii) $\varphi(t+\tau,u,g) = \varphi(t,\varphi(\tau,u,g),g_\tau)$ **for all** $t,\tau \in \mathbb{Z}_+$**,** $u \in W$ **and** $g \in H^+(f)$**;**
(iii) **the mapping** $\varphi : \mathbb{Z}_+ \times W \times H^+(f) \mapsto W$ **is continuous.**

Consequently, the triplet $\langle W,\varphi,(H^+(f),\mathbb{Z}_+,\sigma)\rangle$ **is a cocycle over** $(H^+(f),$ $\mathbb{Z}_+,\ \sigma)$ **with the fiber** $W \subseteq \mathbb{R}^n$**. Thus, every non-autonomous difference equation (23) naturally generates a cocycle which plays a very important role in the qualitative study of equation (23).**

Recall [14] **that a triplet** $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ **is said to be a** ***non-autonomous dynamical system*** **(NDS), when** $(X,\mathbb{T}_1,\pi)$ **(respectively,** $(Y,\mathbb{T}_2,\sigma)$**) is a dynamical system on** X **(respectively,** Y**) and** h **is a homomorphism from** $(X,\mathbb{T}_1,\pi)$ **onto** $(Y,\mathbb{T}_2,\sigma)$**.**

Example 3.17. *(***NDS generated by a cocycle***.)* **Note that every cocycle** $\langle W,\ \varphi,\ (Y,\ \mathbb{T}_2,\ \sigma)\rangle$ **naturally generates a NDS. Indeed, let** $X := W \times Y$ **and assume that** $(X,\mathbb{T}_1,\pi)$ **is a skew-product dynamical system on** X **(i.e.,** $\pi(t,x) := (\varphi(t,u,y),\sigma(t,y))$ **for all** $t \in \mathbb{T}_1$ **and** $x := (u,y) \in X$**). Then the triplet** $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$**, where** $h := pr_2 : X \mapsto Y$ **is the second projection (i.e.,** $h(u,y) = y$ **for all** $u \in W$ **and** $y \in Y$**), is a NDS.**

Lemma 3.18. ([8]) *Let* $\langle W,\varphi,(\Omega,\mathbb{T}_2,\lambda)\rangle$ *be a cocycle and* $\langle (X,\mathbb{T}_1,\pi),(\Omega,\mathbb{T}_2,\lambda),h\rangle$ *be a non-autonomous dynamical system associated to the cocycle* φ*. Suppose that* $x_0 := (u_0,\omega_0) \in X := W \times \Omega$*, and that the set* $Q_{(u_0,\omega_0)} := \overline{\{\varphi(t,u_0,\omega_0) \mid t \in \mathbb{T}_1\}}$ *(respectively,* $Q^+_{(u_0,\omega_0)} := \overline{\{\varphi(t,u_0,\omega_0) \mid t \in \mathbb{T}_1,\ t \geq 0\}}$*) is compact.*

Then, the set $H(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{T}\}}$ *(respectively,* $\overline{\{\pi(t,x_0) \mid t \in \mathbb{T}_1,\ t \geq 0\}}$ $:= H^+(x_0)$*) is conditionally compact.*

Let $\langle (X,\mathbb{T}_1,\pi),(\Omega,\mathbb{T}_2,\lambda),h\rangle$ be a non-autonomous dynamical system and $\omega \in \Omega$ be a positively Poisson stable point. Denote by

$$\mathcal{E}^+_\omega := \{\xi \mid \quad \exists\{t_n\} \in \mathfrak{N}_\omega \quad \text{such that} \quad \pi(t_n,\cdot)|_{X_\omega} \to \xi \ \text{ and } \ t_n \to +\infty\},$$

where $X_\omega := \{x \in X| \quad h(x) = \omega\}$ and $\to$ means the pointwise convergence.

Recall that if X is a compact metric space, then X^X denotes the collection of all maps from X to itself, provided with the product topology, or, in other words, the topology of pointwise convergence. By Tychonoff's theorem, X^X is compact.

X^X possesses a semi-group structure defined by the composition of maps.

Remark 3.19. *Let* $\omega \in \Omega$ *be a Poisson stable point,* $\langle (X,\mathbb{T}_1,\pi),\ (\Omega,\mathbb{T}_2,\lambda),h\rangle$ *be a non-autonomous dynamical system, and* X *be a conditionally compact space, then (see* [13],[14, Ch.IX]*)* $\mathcal{E}^+_\omega$ *is a nonempty compact sub-semigroup of the semigroup* $X_\omega^{X_\omega}$ *(w.r.t. composition of mappings).*

Theorem 3.20. [17, Ch.VI] *Let* X *be a conditionally compact metric space and* $\langle (X,\mathbb{T}_1,\pi),\ (\Omega,\mathbb{T}_2,\lambda),h\rangle$ *be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *There exists a Poisson stable point* $\omega \in \Omega$;
(ii) $\lim\limits_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all* $x_1,x_2 \in X_\omega := h^{-1}(\omega) = \{x \in X : h(x) = \omega\}$.

Then there exists a unique point $x_\omega \in X_\omega$ *such that* $\xi(x_\omega) = x_\omega$ *for all* $\xi \in \mathcal{E}^+_\omega$.

Now we will prove several results which are crucial for our objectives in the next section. These results extend analogous ones in [8] to the case of semigroup dynamical systems.

Lemma 3.21. *Let* X *be a conditionally compact metric space and* $\langle (X,\mathbb{T}_1,\pi), (\Omega,\mathbb{T}_2,\lambda), h\rangle$ *be a non-autonomous dynamical system, and* $x_0 \in X$. *Suppose that the following conditions are fulfilled:*

(i) *the point* $\omega := h(x_0) \in \Omega$ *is Poisson stable;*
(ii) *there exists a point* $x_\omega \in X_\omega$ *such that* $\xi(x_\omega) = x_\omega$ *for all* $\xi \in \mathcal{E}^+_\omega$.

Then the point x_ω *is comparable by character of recurrence with* ω.

Proof. Let $\{t_n\} \in \mathfrak{N}^{+\infty}_\omega$. We will show that $\{\pi(t_n,x_\omega\}$ converges to x_ω as $n \to \infty$. If we suppose that it is not true, then there exist two subsequences $\{t^i_n\} \subseteq \{t_n\}$ $(i = 1,2)$ such that $\pi(t^i_n) \to x^i$ $(i = 1,2)$ as $n \to \infty$ and $x^1 \neq x^2$. Since the space X is conditionally compact and $\{t_n\} \in \mathfrak{N}^{+\infty}_\omega$, then without loss of generality we may suppose that $\{\pi^{t^i_n}\}$ is convergent. Denote by $\xi^i := \lim\limits_{n\to\infty} \pi^{t^i_n}$, then $\xi^i \in E^+_\omega$ $(i = 1,2)$ and, consequently, $x^i = \xi^i(x_\omega)$. On the other hand by conditions of Lemma we have $\xi^i(x_\omega) = x_\omega$ and, consequently, $x^1 = x^2 = x_\omega$. The obtained contradiction prove our statement, i.e., $\mathfrak{N}^{+\infty}_\omega \subseteq \mathfrak{N}^{+\infty}_x$. Now, to finish the proof, it is sufficient to apply Theorems 3.1 and 3.14. □

Corollary 3.22. *Let* X *be a conditionally compact metric space and* $\langle (X,\mathbb{T}_1,\pi), (\Omega,\mathbb{T}_2,\lambda), h\rangle$ *be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *There exists a Poisson stable point* $\omega \in \Omega$;
(ii) $\lim\limits_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all* $x_1,x_2 \in X_\omega := h^{-1}(\omega) = \{x \in X : h(x) = \omega\}$.

Then, there exists a unique point $x_\omega \in X_\omega$ *which is comparable with* $\omega \in \Omega$ *by the character of recurrence, such that*

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,x_\omega)) = 0 \tag{24}$$

for all $x \in X_\omega$.

Proof. According to Theorem 3.20, there exists a unique point $x_\omega \in X_\omega$ such that $\xi(x_\omega) = x_\omega$ for all $\xi \in \mathcal{E}^+_\omega$. To finish the proof it is sufficient to apply Lemma 3.21. □

Corollary 3.23. *Let* $\omega \in \Omega$ *be a stationary (respectively,* τ*-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point. Then under* *the conditions of Corollary 3.22 there exists a unique stationary (respectively, τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point $x_\omega \in X_\omega$ such that equality (24) holds for all $x \in X_\omega$.*

Theorem 3.24. *Let X be a compact metric space and $\langle (X,\mathbb{T},\pi),\ (\Omega,\mathbb{T},\lambda),h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $\omega \in \Omega$ is recurrent;*
(ii) $\lim\limits_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.*

Then there exists a unique point $x_\omega \in X_\omega$ which is uniformly comparable with $\omega \in \Omega$ by the character of recurrence, and such that (24) holds for all $x \in X_\omega$.

Proof. By Theorem 3.20 there exists a unique fixed point $x_\omega \in X_\omega$ of the semigroup $\mathcal{E}^+_\omega$. Thanks to Corollary 3.23, the point x_ω is recurrent. To prove this statement it is sufficient to show that the point x_ω is as required. Let $M := \overline{\{\pi(t,x_\omega)\ : t \in \mathbb{T}\}}$. Then, it is a compact minimal set because the point x_ω is recurrent. We will show that $M_q := M \cap X_q$ (for all $q \in H(\omega) := \overline{\{\sigma(t,\omega)\ : t \in \mathbb{T}\}}$) consists of a single point. If we suppose that it is not true, then there exist $q_0 \in H(\omega)$ and $x_1, x_2 \in M_{q_0}$ such that $x_1 \neq x_2$. By Corollary 3.22 there exists a unique point $x_{q_0} \in M_{q_0}$ which is comparable with point q_0 by the character of recurrence. Without loss of generality, we can suppose that $x_{q_0} = x_1$. Since the set M is minimal, there exists a sequence $\{t_n\} \in \mathfrak{N}^{+\infty}_{q_0}$ such that $\{\pi(t_n,x_1)\} \to x_2$. On the other hand, in view of the inclusion $\mathfrak{N}^{+\infty}_{q_0} \subseteq \mathfrak{N}^{+\infty}_{x_1}$, we have $\{\pi(t_n,x_1)\} \to x_1$ and, consequently, $x_1 = x_2$. This contradiction proves our statement.

Now we will prove that $\mathfrak{M}^{+\infty}_\omega \subseteq \mathfrak{M}^{+\infty}_{x_\omega}$. Let $\{t_n\} \in \mathfrak{M}^{+\infty}_\omega$, then $\{t_n\} \in \mathfrak{M}^{+\infty}_{x_\omega}$. Arguing once more by contraction, if we suppose that it is not true, then there are two subsequences $\{t_{n^i_k}\}$ $(i = 1,2)$ such that $\lim\limits_{k\to+\infty} \pi(t_{n^i_k}, x_\omega) = x_i$ (i=1,2) and $x_1 \neq x_2$. Denote by $q_0 := \lim\limits_{n\to+\infty} \sigma(t_n,\omega)$, then $q_0 \in H(\omega)$ and $x_1, x_2 \in M_{q_0}$. But this is a contradiction, since we proved above that M_q consisted of a single point for all $q \in H(\omega)$. Taking into account Corollary 3.22 to finish the proof of Theorem it is sufficient to apply Theorem 3.15. $\square$

Corollary 3.25. *Let $\omega \in \Omega$ be a stationary (respectively, τ-periodic, Bohr almost periodic, recurrent) point. Then, under the conditions of Theorem 3.24, there exists a unique stationary (respectively, τ-periodic, Bohr almost periodic, recurrent) point $x_\omega \in X_\omega$ such that (24) is fulfilled for all $x \in X_\omega$.*

4. Some applications

Our results from Sections 2-3 can be applied to study the problem of Poisson stability for different classes of differential/difference equations (both on the whole and/or half axis). Below we apply these results to study the problem of almost periodicity (respectively, Levitan almost periodicity, Bochner almost automorphy, or Poisson stability) of solutions for linear difference equations and we obtain some new and interesting results in this direction.

4.1. **Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** Let $(X,\mathbb{T},\pi)$ be a dynamical system on X, Y be a complete pseudo metric space, and P be a family of pseudo metrics on Y. We denote by $C(X,Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d^p_K\}$ $(p \in P,\ K \in \mathcal{K}(X))$, where

$$d^p_K(f,g) := \sup_{x\in K} p(f(x), g(x))$$

and $\mathcal{K}(X)$ denotes the family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X,Y) \to C(X,Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau,x))$, $x \in X$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1+\tau_2},\ \forall \tau_1,\tau_2 \in \mathbb{T}$;
c. σ_τ is continuous $\forall \tau \in \mathbb{T}$.

Furthermore, $(C(X,Y),\mathbb{T},\sigma)$ is a dynamical system (see [14] for the details).

Let us now recall an example of dynamical system of the form $(C(X,Y),\mathbb{T},\sigma)$ which is useful in applications.

Example 4.1. *Let $X = \mathbb{T}$, and denote by $(X,\mathbb{T},\pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t,x) := x + t$. The dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$ is called* Bebutov's dynamical system [32] *(a dynamical system of translations, or shifts dynamical system). For example, the equality*

$$d(f,g) := \sup_{L>0} \max\{d_L(f,g), L^{-1}\},$$

*where $d_L(f,g) := \max_{|t|\le L} \rho(f(t),g(t))$, defines a complete metric (*Bebutov's metric*) on the space $C(\mathbb{T},Y)$ which is compatible with the compact-open topology on $C(\mathbb{T},Y)$.*

We say that the function $\varphi \in C(\mathbb{T},Y)$ possesses a *property* (A), if the motion $\sigma(\cdot,\varphi) : \mathbb{T} \to C(\mathbb{T},Y)$ possesses this property in the Bebutov dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$, generated by the function φ. As property (A) we can take periodicity, almost periodicity, almost automorphy, recurrence, etc.

4.2. **Compatible and Uniformly Compatible Solutions of Linear Difference Equations.** In this subsection we will apply our abstract theory, previously developed in Section 3, to analyze two important applications: non-homogeneous linear difference equations (on $\mathbb{Z}_+$ or/and on $\mathbb{Z}$), and non-homogeneous linear functional-difference equations.

4.2.1. *Linear Difference Equations (on $\mathbb{Z}_+$ or/and on $\mathbb{Z}$).* Consider the following difference equation

$$u(t+1) = A(t)u(t) + f(t) \tag{25}$$

with positively Poisson stable coefficients $A(t)$ and $f(t)$ (i.e., there exists a sequence $t_n \to +\infty$ ($t_n \in \mathbb{Z}_+$) such that $(A_{t_n}, f_{t_n}) \to (A, f)$ as $n \to +\infty$) and its corresponding homogeneous equation

$$u(t+1) = A(t)u(t), \tag{26}$$

where $(A, f) \in C(\mathbb{T}, [E]) \times C(\mathbb{T}, E)$, and A_τ, f_τ are defined as $A_\tau(t) = A(t+\tau)$, $f_\tau(t) := f(t+\tau)$ for $t \in \mathbb{T}$. Along with equations (25) and (26), we consider also the H-class of equation (25) (respectively, (26)), which is the family of equations

$$v(t+1) = B(t)v(t) + g(t), \tag{27}$$

(respectively,

$$v(t+1) = B(t)v(t) \tag{28}$$

with $(B, g) \in H(A, f) := \overline{\{(A_\tau, f_\tau) \mid \tau \in \mathbb{T}\}}$ (respectively, $B \in H(A)$), where the bar denotes the closure in $C(\mathbb{T}, [E]) \times C(\mathbb{T}, E)$ (respectively, $C(\mathbb{T}, [E])$). Let $\varphi(t, v, (B, g))$ (respectively, $\varphi(t, v, B)$) be the solution of equation (27) (respectively, (28)) which satisfies the condition $\varphi(0, v, (B, g)) = v$ (respectively, $\varphi(0, v, B) = v$) and defined on $\mathbb{Z}_+$.

We set now $Y := H(A, f)$, and denote the dynamical system of shifts on $H(A, f)$ by $(Y, \mathbb{T}, \sigma)$. Consider $X := E \times Y$, and define a dynamical system on X by setting $\pi(\tau, (v, B, g)) := (\varphi(\tau, v, (B, g)), B_\tau, g_\tau)$ for all $(v, (B, g)) \in E \times Y$ and $\tau \in \mathbb{Z}_+$. Then $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is a non-autonomous dynamical system, where $h := pr_2 : X \to Y$ denotes the projection over the second variable, i.e., $h(e, y) = y$ for $(e, y) \in X$.

Now we apply the results of Section 3 to this system, and obtain some results concerning the difference equation (27).

A solution $\varphi \in C(\mathbb{T}, E)$ of equation (25) is called [34] *compatible by the character of recurrence* if $\mathfrak{N}^{+\infty}_{(A,f)} \subseteq \mathfrak{N}^{+\infty}_{\varphi}$, where $\mathfrak{N}^{+\infty}_{(A,f)} := \{\{t_n\} \subset \mathbb{Z}_+ \mid (A_{t_n}, f_{t_n}) \to (A, f)$ and $t_n \to +\infty\}$ (respectively, $\mathfrak{N}^{+\infty}_{\varphi} := \{\{t_n\} \subset \mathbb{Z}_+ \mid \varphi_{t_n} \to \varphi$ and $t_n \to +\infty\}$).

The next result generalizes Theorem 4.12 in [8]**.**

Theorem 4.2. *Let $(A, f) \in C(\mathbb{T}, [E]) \times C(\mathbb{T}, E)$ be positively Poisson stable. Suppose that the following conditions hold:*

(i) *equation (25) admits a relatively compact on $\mathbb{Z}_+$ solution $\varphi(t, u_0, (A, f))$, i.e., the set $Q_{(u_0,(A,f))} := \overline{\varphi(\mathbb{Z}_+, u_0, (A, f))}$ is compact in E;*
(ii) *all the relatively compact on $\mathbb{Z}$ solutions of equation (26) tend to zero as the time t tends to $+\infty$, i.e., $\lim\limits_{t \to +\infty} |\varphi(t, u, A)| = 0$ if $\varphi(t, u, A)$ is relatively compact (this means that the set $\varphi(\mathbb{Z}, u, A))$ is relatively compact in E).*

Then, equation (25) has a unique compatible solution $\varphi(n, \bar{u}, f)$ with values from the compact $Q_{(u_0,(A,f))}$.

Proof. Denote by $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ the non-autonomous dynamical system, generated by equation (25) (see construction above). By Lemma 3.18, the set $H^+(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{Z}_+\}}$)

is conditionally compact. Let now $x_1, x_2 \in H(x_0) \cap X_{(A,f)}$, where $X_{(A,f)} := E \times \{(A,f)\}$ (i.e., $x_i = (u_i,(A,f))$ and $u_i \in E$ (i=1,2)), then

$$\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{t\to+\infty} |\varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))| = 0.$$

Now, to finish the proof it is sufficient to take into account Corollary 3.22. □

Corollary 4.3. *Under the conditions of Theorem 4.2, if $(A,f) \in C(\mathbb{T},[E]) \times C(\mathbb{T},E)$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (25) admits a unique τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.2 and Corollary 3.23. □

Corollary 4.4. *Under the conditions of Theorem 4.2, if $(A,f) \in C(\mathbb{T},[E]) \times C(\mathbb{T},E)$ is almost automorphic, then equation (25) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t,\bar{u},(A,f))$ is relatively compact, it easily follows that $\bar{\varphi} := \varphi(\cdot,\bar{u},(A,f)) \in C(\mathbb{T},E)$ is a Lagrange stable point of the dynamical system $(C(\mathbb{T},E),\mathbb{T},\sigma)$. On the other hand, by Corollary 4.3 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, it is almost automorphic. □

Corollary 4.5. *Under the conditions of Theorem 4.2, if $(A,f) \in C(\mathbb{T},[E]) \times C(\mathbb{T},E)$ is Bohr almost periodic, then equation (25) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.4 because every Bohr almost periodic function is almost automorphic. □

A solution $\varphi \in C(\mathbb{T},E)$ of equation (25) is called [32, 34] *uniformly compatible by the character of recurrence*, if $\mathfrak{M}^{+\infty}_{(A,f)} \subseteq \mathfrak{M}^{+\infty}_{\varphi}$, where $\mathfrak{M}^{+\infty}_{(A,f)} := \{\{t_n\} \subset \mathbb{T} \mid$ such that $t_n \to +\infty$ and the sequence $\{(A_{t_n}, f_{t_n})\}$ is convergent$\}$ (respectively, $\mathfrak{M}^{+\infty}_{\varphi} := \{\{t_n\} \subset \mathbb{T} \mid$ such that $t_n \to +\infty$ and the sequence $\{\varphi_{t_n}\}$ is convergent$\}$).

Theorem 4.6. *Let $(A,f) \in C(\mathbb{T},[E]) \times C(\mathbb{T},E)$ be recurrent. Suppose that the following conditions hold:*

(i) *equation (25) admits a solution $\varphi(t,u_0,(A,f))$ which is relatively compact on $\mathbb{Z}_+$;*
(ii) *for all $B \in H(A)$ the solutions of equation (28), which are relatively compact on $\mathbb{Z}$, tend to zero as the time tends to $+\infty$, i.e., $\lim_{t\to+\infty} |\varphi(t,u,B)| = 0$, if $\varphi(t,u,B)$ is relatively compact on $\mathbb{Z}$.*

Then equation (25) has a unique uniformly compatible solution $\varphi(t,\bar{u},f)$ with values from the compact $Q_{(u_0,(A,f))}$.

Proof. Denote by $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ the non-autonomous dynamical system, generated by equation (25). Under the conditions of the theorem the set $H^+(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{Z}_+\}}$) is compact.

Let now $x_1, x_2 \in H(x_0) \cap X_{(B,g)}$, where $(B,g) \in H(A,f)$ and $X_{(B,g)} := E \times \{(B,g)\}$ (i.e., $x_i = (u_i, (B,g))$ and $u_i \in E$ (i=1,2)). Then

$$\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{t\to+\infty} |\varphi(t,u_1,(B,g)) - \varphi(y,u_2,(B,g))| = 0.$$

Now to finish the proof it is sufficient to apply Theorem 3.24. □

Corollary 4.7. *Under the conditions of Theorem 4.6, if $(A,f) \in C(\mathbb{T},[E]) \times C(\mathbb{T},E)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (25) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.6 and Corollary 3.25. □

To conclude this subsection we consider particular examples that illustrate the above results.

Example 4.8. Let $a \in C(\mathbb{R},\mathbb{R})$ be the Bohr almost periodic function defined by the equality

$$a(t) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}} \sin \frac{t}{2k+1}. \tag{29}$$

Note that $a(t+t_n) \to -a(t)$ uniformly on $\mathbb{R}$, where $t_n := (2n+1)!!$. Therefore, $-a \in H(a) := \overline{\{a_\tau \mid \tau \in \mathbb{R}\}}$. In the work [8] it is proved that the module of all non-zero solutions of the equation

$$x' = a(t)x \tag{30}$$

tends to $+\infty$ as $|t| \to +\infty$, whereas those of the equation

$$y' = b(t)y, \tag{31}$$

with $b := -a \in H(a)$ tend to zero.

Thus, if $g \in C(\mathbb{R},\mathbb{R})$ is a Bohr almost periodic function and the equation

$$y' = b(t)y + g(t) \tag{32}$$

admits a bounded solution, then according to Theorem 4.1 and Corollary 4.4 from [8] it has a unique almost automorphic solution.

Below we will construct a discrete analog of example (32). To this end we will use the so called procedure of discretization [11].

Along with the homogeneous equation (30), we will consider the corresponding non-homogeneous equation

$$x' = a(t)x + f(t), \tag{33}$$

where $f \in C(\mathbb{R},\mathbb{R})$.

Let $(H(a),\mathbb{R},\sigma)$ (respectively, $(H(a,f),\mathbb{R},\sigma)$) be a shift dynamical system on $H(a)$ (respectively, on $H(a,f)$), where $H(a) := \overline{\{a_\tau :\ \tau \in \mathbb{R}\}}$ (respectively, $H(a,f) := \overline{\{(a_\tau, f_\tau) :\ \tau \in \mathbb{R}\}}$), where a_τ (respectively, (a_τ, f_τ)) is the τ shift of a (respectively, (a,f)) and by bar is denoted the closure in the space $C(\mathbb{R},\mathbb{R})$ (respectively, in $C(\mathbb{R},\mathbb{R}) \times C(\mathbb{R},\mathbb{R})$). Denote by $\langle \mathbb{R}, \tilde{\varphi}, (H(a),\mathbb{R},\tilde{\sigma})\rangle$ (respectively, by

$\langle \mathbb{R}, \tilde{\psi}, (H(a), \mathbb{R}, \tilde{\sigma})\rangle$ the cocycle generated by (30) (respectively, by (33)). Let now $\langle \mathbb{Z}, \varphi, (H(a), \mathbb{R}, \sigma)\rangle$ (respectively, by $\langle \mathbb{Z}, \psi, (H(a), \mathbb{Z}, \sigma)\rangle$) be the discretization (for more details see [11]) of the cocycle $\tilde{\varphi}$ (respectively, $\tilde{\psi}$). Then $\psi(n, x, a, f)$ is a solution of the scalar difference equation

$$x(t+1) = A(t)x(t) + F(t), \tag{34}$$

where $A(t) := U(1, a_t)$, $B(t) := \int_0^1 U(1, a_t)U^{-1}(s, a_t)f_t(s)$ and $U(\tau, a) := \exp \int_0^\tau a(s)ds$ for all $t \in \mathbb{Z}$ and $\tau \in \mathbb{R}$. According to this construction, we have the following properties:

(i) if the functions $a, f \in C(\mathbb{R}, \mathbb{R})$ are Bohr almost periodic, then the functions $A, F \in C(\mathbb{Z}, \mathbb{R})$ are also almost periodic;

(ii) every bounded solution $\varphi(t, A, x)$ of equation

$$x(t+1) = A(t)x(t) \tag{35}$$

possesses the following property $\lim\limits_{|t|\to+\infty} |\varphi(t, x, A)| = 0$, because $\varphi(t, x, A) = \tilde{\varphi}(t, x, a)$ for all $t \in \mathbb{Z}$.

Thus if equation (34) admits a bounded (on $\mathbb{Z}$) solution, then by Theorem 4.2 and Corollary 4.5 it has a unique almost automorphic solution.

4.3. Linear Functional-Difference Equations with Finite Delay.

Let $r \in \mathbb{Z}_+$, $C([a, b], E)$ be the Banach space of all functions $\varphi : [a, b] \to E$ with the norm sup. For $[a, b] := [-r, 0]$ we put $\mathcal{C} := C([-r, 0], E)$. Let $c, a \in \mathbb{Z}$, $a \geq 0$, and $u \in C([c - r, c + a], E)$. We define $u_t \in C$ for any $t \in [c, c + a]$ by the relation $u_t(s) := u(t + s), -r \leq s \leq 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C}, E)$ be the Banach space of all linear operators that act from $\mathcal{C}$ into E equipped with the operator norm, let $C(\mathbb{T}, \mathfrak{A})$ be the space of all operator-valued functions $A : \mathbb{T} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{T}, \mathfrak{A}), \mathbb{T}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{T}, \mathfrak{A})$. Let $H(A) := \overline{\{A_\tau \mid \tau \in \mathbb{T}\}}$, where A_τ is the shift of the operator-valued function A by τ and the bar denotes closure in $C(\mathbb{T}, \mathfrak{A})$.

Remark 4.9. *Notice that we will use the same notation (A_τ and u_τ) for two (slightly) different concepts, but no confusion should be with them and everything will be clear by the context.*

Example 4.10. *Consider a non-homogeneous linear functional-difference equation with finite delay (see, for example,* [27, 38]*)*

$$u(t+1) = A(t)u_t + f(t) \tag{36}$$

with positively Poisson stable coefficients $A(t)$ and $f(t)$ and the corresponding homogeneous linear equation

$$u(t+1) = A(t+1)u_t, \tag{37}$$

where $A \in C(\mathbb{T}, \mathfrak{A})$ and $f \in C(\mathbb{T}, E)$.

Remark 4.11. *1. Denote by $\varphi(t, u, A, f)$ the solution of equation (36) defined on $\mathbb{Z}_+$ (respectively, on $\mathbb{Z}$) with initial condition $\varphi(0, u, A, f) = u \in \mathcal{C}$. By $\tilde{\varphi}(t, u, A, f)$ we will denote below the trajectory of equation (36), corresponding to the solution $\varphi(t, u, A, f)$, i.e. the mapping from $\mathbb{Z}_+$ (respectively, $\mathbb{Z}$) into $\mathcal{C}$, defined by equality $\tilde{\varphi}(t, u, A, f)(s) := \varphi(t+s, u, A, f)$ for all $t \in \mathbb{Z}_+$ (respectively, $t \in \mathbb{Z}$) and $s \in [-r, 0]$.*

2. Let $\varphi(t,u_i,A,f)$ $(i=1,2)$ be two solutions of equation (36), then

$$\lim_{t\to\infty}|\varphi(t,u_1,A,f)-\varphi(t,u_2,A,f)| = \lim_{t\to\infty}|\tilde{\varphi}(t,u_1,A,f)-\tilde{\varphi}(t,u_2,A,f)|_{\mathcal{C}}.$$

Along with equation (36) (respectively, (37)) we consider the family of equations

$$v(t+1) = B(t)v_t + g(t) \tag{38}$$

(respectively,

$$v(t+1) = B(t)v_t, \tag{39}$$

where $(B,g) \in H(A,f) := \overline{\{(B_\tau,f_\tau) \mid \tau\in\mathbb{T}\}}$. (respectively, $B \in \overline{\{B_\tau \mid \tau\in\mathbb{T}\}}$ $:= H(A)$). Let $\tilde{\varphi}(t,v,(B,g))$ (respectively, $\tilde{\varphi}(t,v,B)$ be the solution of equation (38) (respectively, (39)) satisfying the condition $\tilde{\varphi}(0,v,(B,g)) = v$ (respectively, $\tilde{\varphi}(0,v,B) = v$) and defined for all $t\geq 0$. Let $Y := H(A,f)$ and denote the dynamical system of shifts on $H(A,f)$ by $(Y,\mathbb{T},\sigma)$. Let $X := \mathcal{C}\times Y$ and let $\pi := (\varphi,\sigma)$ be the dynamical system on X defined by the equality $\pi(\tau,(v,(B,g))) := (\tilde{\varphi}(\tau,v,(B,g)),B_\tau,g_\tau)$. The semi-group non-autonomous system $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ $(h := pr_2 : X\to Y)$ is generated by equation (36).

Lemma 4.12. *Let $\varphi(n,u,(A,f))$ be the solution of equation (36) which is relatively compact on $\mathbb{T}$, and $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system generated by equation (36). Then, the set*

$$H^+(u,(A,f)) := \overline{\{(\tilde{\varphi}(\tau,u,(A,f)),(A_\tau,f_\tau)) \mid \tau\geq 0\}}$$

is conditionally compact with respect to (X,h,Y).

Proof. This statement is obvious. □

Theorem 4.13. *Let $(A,f)\in C(\mathbb{T},\mathfrak{A})\times C(\mathbb{T},E)$ be positively Poisson stable. Suppose that the following conditions hold:*

- (i) *equation (36) admits a solution $\varphi(t,u_0,(A,f))$ which is relatively compact on $\mathbb{Z}_+$;*
- (ii) *all the solutions of equation (37), which are relatively compact on $\mathbb{Z}$, tend to zero as the time tends to $+\infty$, i.e., $\lim_{t\to+\infty}|\varphi(t,u,A)| = 0$ if $\varphi(t,u,A)$ is relatively compact on $\mathbb{Z}$.*

Then, equation (36) has a unique compatible solution $\varphi(t,\bar{u},A,f)$.

Proof. First of all we will prove that equation (36) admits at most one compatible solution. If we suppose that it is not true, then there would exist at least two compatible solutions $\varphi(t,u_i,(A,f))$ (i=1,2, and $u_1\neq u_2$) defined and bounded on $\mathbb{Z}$. Since (A,f) is positively Poisson stable, then $\psi(t) := \varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))$ $(t\in\mathbb{Z})$ is also positively Poisson stable. On the other hand, $\psi(t) = \varphi(t,u_1-u_2,A)$ is a solution of equation (37) which is relatively compact on $\mathbb{Z}$ and, consequently, $\lim_{t\to+\infty}|\psi(t)| = 0$. From the last equality and the Poisson stability of ψ we obtain $\psi(t) = 0$ for all $t\in\mathbb{Z}$. In particular, $u_1-u_2 = \psi(0) = 0$. This contradiction proves our statement.

Now we will prove that equation (36) admits at least one compatible solution. Denote by $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ the non-autonomous dynamical system, generated by equation (36) (see Example 4.10). By Lemma 4.12, the positively invariant set $\overline{H^+(x_0) \subset X}$ (where $x_0 := (u_0, (A, f)) \in X := C \times H(A, f)$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{Z}_+\}}$) is conditionally compact. Let now $x_1, x_2 \in H^+(x_0) \cap X_{(A,f)}$, where $X_{(A,f)} := \mathcal{C} \times \{(A, f)\}$ (i.e., $x_i = (u_i, (A, f))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{t\to+\infty} |\varphi(t, u_1, (A, f)) - \varphi(t, u_2, (A, f))|_{\mathcal{C}} = 0.$$

Now, to finish the proof, it is sufficient to apply Corollary 3.22. □

Corollary 4.14. *Under the conditions of Theorem 4.13, if $(A, f) \in C(\mathbb{Z}, \mathfrak{A}) \times C(\mathbb{Z}, E)$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (36) admits a unique τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.13 and Corollary 3.23. □

Corollary 4.15. *Under the conditions of Theorem 4.13 if $(A, f) \in C(\mathbb{T}, \mathfrak{A}) \times C(\mathbb{T}, E)$ is almost automorphic, then equation (36) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t, \bar{u}, (A, f))$ is relatively compact on $\mathbb{Z}$, then $\bar{\varphi} := \varphi(\cdot, \bar{u}, (A, f)) \in C(\mathbb{Z}, E)$ is a Lagrange stable point of the dynamical system $(C(\mathbb{Z}, E), \mathbb{Z}, \sigma)$. On the other hand, by Corollary 4.15 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, it is almost automorphic. □

Corollary 4.16. *Under the conditions of Theorem 4.13, if $(A, f) \in C(\mathbb{Z}, \mathfrak{A}) \times C(\mathbb{Z}, E)$ is Bohr almost periodic, then equation (36) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.15 because every Bohr almost periodic function is almost automorphic. □

Theorem 4.17. *Let $(A, f) \in C(\mathbb{T}, \mathfrak{A}) \times C(\mathbb{T}, E)$ be recurrent. Suppose that the following conditions hold:*

(i) *equation (36) admits a solution $\varphi(t, u_0, (A, f))$ which is relatively compact on $\mathbb{Z}_+$;*
(ii) *for all $B \in H(A)$ the solutions of equation (39), which are relatively compact on $\mathbb{Z}$, tend to zero as the time tends to $+\infty$, i.e., $\lim_{t\to+\infty} |\varphi(t, u, B)| = 0$ if $\varphi(t, u, B)$ is relatively compact on $\mathbb{Z}$.*

Then, equation (36) has a unique uniformly compatible solution $\varphi(t, \bar{u}, A, f)$.

Proof. Note that equation (36) admits at most one uniformly compatible solution. In fact, every uniformly compatible solution is compatible. On the other hand, by Theorem 4.13, equation (36) admits at most one compatible solution.

Now we will prove that equation (36) admits at least one uniformly compatible solution. Indeed, since the function $\varphi(t, u_0, (A, f))$ is relatively compact on $\mathbb{Z}_+$, then $\tilde{\varphi}(\mathbb{Z}_+, u_0, (A, f))$ is relatively compact in $\mathcal{C}$. Denote by $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ the semi-group non-autonomous dynamical system, generated by equation (36). Under the conditions of the theorem the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{Z}_+\}})$ is compact. Let now $x_1, x_2 \in H^+(x_0) \cap X_{(B,g)}$, where $(B, g) \in H(A, f)$ and $X_{(B,g)} := \mathcal{C} \times \{(B, g)\}$ (i.e. $x_i = (u_i, (B, g))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{t\to+\infty} |\varphi(t, u_1, (B, g)) - \varphi(t, u_2, (B, g))|_{\mathcal{C}} = 0.$$

To finish the proof it is sufficient now to apply Theorem 3.24. □

Corollary 4.18. *Under the conditions of Theorem 4.17, if $(A, f) \in C(\mathbb{T}, \mathfrak{A}) \times C(\mathbb{T}, E)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (36) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.17 and Corollary 3.25. □

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