GLOBAL ATTRACTORS OF QUASI-LINEAR NON-AUTONOMOUS DIFFERENCE EQUATIONS: A GROWTH MODEL WITH ENDOGENOUS POPULATION GROWTH.

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1. Introduction

The present paper is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n \omega)u_n + F(u_n, \sigma^n \omega), \quad (A \in C(\Omega, [E]),\ F \in C(E \times \Omega, E)) \tag{1}$$

where Ω is a compact metric space, E is a finite-dimensional Banach space, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time $\mathbb{Z}_+$, $[E]$ is the space of all linear operators acting on E equipped with operator norm, $C(\Omega, [E])$ (respectively, $C(E \times \Omega, E)$) is the space of all continuous functions defined on Ω (respectively, on $E \times \Omega$) with values in $[E]$ (respectively, E) equipped with compact-open topology and F is a "small" perturbation. Analogous problem it was studied by Cheban D. et al. [11], when Ω is an invariant set. In this work we consider more general case, when Ω is not invariant, but there exists a compact invariant subset $J \subseteq \Omega$ (Levinson center) which attracts every compact subset from Ω.

The obtained results are applied while studying a special class of triangular maps describing a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates and the population growth rate dynamic is described by the logistic equation (see Brianzoni S., Mammana C. and Michetti E. [3]).

We consider the Solow-Swan growth model in discrete time with differential saving and VES production function as proposed by Brianzoni et al. in [4] while assuming that the population growth rate evolves according to the logistic law as in Brianzoni et al. [3] and Cheban et al. [11]. Our main goal is to study the qualitative and quantitative long run dynamics of the economic model to show that complex futures results, as the one reached while considering the CES (Constant Elasticity of Substitution) technology.

This paper is organized as follows.

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In Section 2 we collect some notions and facts from the theory of dynamical systems (semi-group dynamical system, cocycle, full trajectory, non-autonomous dynamical system, compact global attractor) which we use in our paper.

Section 3 is devoted to the study of the existence of compact global attractors of non-autonomous dynamical systems (NDS). The sufficient conditions of existence of compact global attractors for dynamical systems with non-invariant base is given (Theorems 3.2 and 3.3).

In Section 4 we study the linear non-autonomous dynamical systems with discrete time and prove that they admit a compact global attractor and its description is given (Theorem 4.4).

In Section 5 we prove the existence of compact global attractors of quasi-linear dynamical systems and give the description of the structure of these attractors (Theorem 5.5) and dependence on small parameters (Theorem 5.6).

Section 6 is dedicated to the study a special class of the triangular maps $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ describing a triangular growth model with logistic population growth rate as studied in Brianzoni S., Mammana C. and Michetti E. [3]. We apply here our general results obtained in Sections 3-5 for studing this concrete dynamical system.

2. Some Notions and Facts from Dynamical Systems

In this Section we collect some notions and facts from the theory of dynamical systems (both with continuous and discrete time) which we use in our paper.

2.1. Triangular maps and non-autonomous dynamical systems.

Let W and Ω be two complete metric spaces and denote by $X := W\times\Omega$ their Cartesian product. Recall [8] that a continuous map $F : X \to X$ is called triangular if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e., $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \tag{2}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}$ is the set of all integer numbers and $\mathbb{Z}_+ := \{n \in \mathbb{Z} :\ n \geq 0\}$.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega)\ (\omega \in \Omega), \tag{3}$$

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

(i) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
(ii) $\varphi(n + m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n\omega$;
(iii) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ is continuous.

Denote by (Ω, Z_+, σ) the semi-group dynamical system generated by positive powers of the map $g : \Omega \to \Omega$, i.e., $\sigma(n, \omega) := g^n\omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Recall [6, 16] that a triple $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ (or briefly φ) is called a cocycle over the semi-group dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W.

Let $X := W \times \Omega$ and $(X, \mathbb{Z}_+, \pi)$ be a semi-group dynamical system on X, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then $(X, \mathbb{Z}_+, \pi)$ is called [16] a skew-product dynamical system, generated by cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$.

Remark 2.1. *Thus, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa, i.e., having a cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ we can define a triangular map $F : W \times \Omega \to W \times \Omega$ by the equality*

$$F(u, \omega) := (f(u, \omega), g(\omega)),$$

where $f(u, \omega) := \varphi(1, u, \omega)$ and $g(\omega) := \sigma(1, \omega)$ for all $u \in W$ and $\omega \in \Omega$. The semi-group dynamical system defined by the positive powers of the map $F : X \to X$ $(X := W \times \Omega)$ coincides with the skew-product dynamical system, generated by cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$

Taking into consideration this remark we can study triangular maps in the framework of cocycles with discrete time.

Let $(X, \mathbb{Z}_+, \pi)$ (respectively, $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$) be a semi-group dynamical system (respectively, a cocycle). A map $\gamma : \mathbb{Z} \to X$ is called an entire trajectory of the semi-group dynamical system $(X, \mathbb{Z}_+, \sigma)$ passing through the point $x \in X$ if $\gamma(0) = x$ and $\gamma(n + m) = \pi(m, \gamma(n))$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$.

Denote by $\Phi_\omega(\sigma)$ the set of all the entire trajectories of the semi-group dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ passing through the point $\omega \in \Omega$ at the initial moment $n = 0$ and $\Phi(\sigma) := \bigcup\{\Phi_\omega(\sigma) \mid \omega \in \Omega\}$.

A map $\nu : \mathbb{Z} \to W$ is called an entire trajectory of the cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ passing through the point $(u, \omega) \in W \times \Omega$ if there exits $\mu \in \Phi_\omega(\sigma)$ such that: $\nu(0) = u$ and $\nu(n + m) = \varphi(m, \nu(n), \mu(n))$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$.

Let Ω be a complete metric space, $(X, \mathbb{Z}_+, \pi)$ (respectively, $(\Omega, \mathbb{Z}_+, \sigma)$) be a semi-group dynamical system on X (respectively, Ω), and $h : X \to \Omega$ be a homomorphism of $(X, \mathbb{Z}_+, \pi)$ onto $(\Omega, \mathbb{Z}_+, \sigma)$. Then the triple $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is called a non-autonomous dynamical system (NDS).

Let W and Ω be complete metric spaces, $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system on Y and $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber W (or, for short, φ). We denote by $X := W \times \Omega$ and define on X a skew product dynamical system $(X, \mathbb{Z}_+, \pi)$ $(\pi = (\varphi, \sigma)$, i.e., $\pi(t, (w, \omega)) = (\varphi(t, w, \omega), \sigma(t, \omega))$ for all $t \in \mathbb{Z}_+$ and $(w, \omega) \in W \times \Omega)$. Then the triple $\langle (X, \mathbb{Z}_+, \pi), ((\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is a non-autonomous dynamical system generated by cocycle φ), where $h = pr_2 : X \mapsto \Omega$ is the projection on the second component.

2.2. **Global attractors of dynamical systems.** Let $\mathfrak{M}$ be a family of subsets from X.

A semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$ will be called $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(n, M) \subseteq B(K, \varepsilon)$ for any $n \geq L(\varepsilon, M)$, where K is a certain fixed subset from X depending only on $\mathfrak{M}$. In this case we will call K an attracting set for $\mathfrak{M}$.

For the applications the most important ones are the cases when K is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, or $\mathfrak{M} := B(X)$ where $C(X)$ (respectively, $B(X)$) is the family of all compact (respectively, bounded) subsets from X.

The system $(X, \mathbb{Z}_+, \pi)$ is called:

– point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{n \to +\infty} \rho(\pi(n, x), K) = 0; \tag{4}$$

– compactly dissipative if the equality (4) takes place uniformly w.r.t. x on the compact subsets from X.

Let $(X, \mathbb{Z}_+, \pi)$ be a compactly dissipative semi-group dynamical system and K be an attracting set for $C(X)$. We denote by

$$J := \Omega(K) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \pi(m, K)},$$

then the set J does not depend of the choice of K and is characterized by the properties of the semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$. The set J is called a Levinson center of the semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$.

Let $(X, \mathbb{Z}_+, \pi)$ be a dynamical system and $x \in X$. Denote by $\omega_x := \Omega(\{x\})$ the ω-limit set of point x.

3. Global attractors of non-autonomous systems

Section 3 is devoted to the study of the existence of compact global attractors of non-autonomous dynamical systems (NDS). The sufficient conditions of existence of compact global attractors for dynamical systems with non-invariant base is given.

Let $(Y, \mathbb{T}_2, \sigma)$ be a compactly dissipative dynamical system, J_Y its Levinson center and $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$ be a NDS. Denote by $\tilde{X} := h^{-1}(J_Y) = \{x \in X : h(x) = y \in J_Y\}$, then evidently the following statements are fulfilled:

(i) $\tilde{X}$ is closed;
(ii) $\pi(t, \tilde{X}) \subseteq \tilde{X}$ for all $t \in \mathbb{T}_1$ and, consequently, on the set $\tilde{X}$ is induced by $(X, \mathbb{T}_1, \pi)$ a dynamical system $(\tilde{X}, \mathbb{T}_1, \pi)$;
(iii) the triplet $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h \rangle$ is a NDS.

A non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$ is said to be compact dissipative if the dynamical systems $\langle (X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative.

Lemma 3.1. *Let $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$ be a compact dissipative NDS. Then the following statements hold:*

(i) *for all compact subset* $K \subseteq X$ *the set* $\Sigma_K^+ := \{\pi(t,x) : \ t \geq 0, \ x \in K\}$ *is relatively compact;*
(ii) *the dynamical system* $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma)$ *is compactly dissipative and the Levinson center* $J_{\tilde{X}}$ *coincides with* J_X*;*
(iii) $h(J_{\tilde{X}}) = J_Y$.

Proof. This statement is evident with except of the equality $h(J_{\tilde{X}}) = J_Y$. Note that $J_{\tilde{X}}$ is a maximal compact invariant set of $(\tilde{X},\mathbb{T}_1,\pi)$ and, consequently, it is invariant with respect to $(X,\mathbb{T}_1,\pi)$ too. Since J_X is a maximal compact invariant set of $(X,\mathbb{T}_1,\pi)$, then $J_{\tilde{X}} \subseteq J_X$. Since $h(J_X) = J_Y$, then $J_X \subseteq \tilde{X}$ and by construction of $(\tilde{X},\mathbb{T}_1,\pi)$ the set J_X is invariant with respect to $(\tilde{X},\mathbb{T}_1,\pi)$. Taking into account that $J_{\tilde{X}}$ is the maximal compact invariant set of $(\tilde{X},\mathbb{T}_1,\pi)$ we obtain $J_X \subseteq J_{\tilde{X}}$. Lemma is proved. □

Below we will prove that Lemma 3.1 is reversible.

Theorem 3.2. *Let* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ *be a NDS and* $(Y,\mathbb{T}_2,\sigma)$ *be compact dissipative. Then the dynamical system* $(X,\mathbb{T}_1,\pi)$ *will be compactly dissipative if and only if the following conditions hold:*

(i) *for all compact subset* $K \subseteq X$ *the set* Σ_K^+ *is relatively compact;*
(ii) *the dynamical system* $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma)\rangle$ *is compactly dissipative and the Levinson center* $J_{\tilde{X}}$ *coincides with* J_X*;*
(iii) $h(J_{\tilde{X}}) = J_Y$.

Proof. The necessity of this statement it follows from Lemma 3.1.

Sufficient. Let the $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma),h\rangle$ be compact dissipative and the set Σ_K^+ be relatively compact for every compact subset $K \subseteq X$. We will show that the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative. To this end we will prove that the compact subset $M := J_{\tilde{X}} \subseteq X$ attracts every compact subset K from X. In fact, if K is an arbitrary compact subset of X, then under the conditions of theorem the set Σ_K^+ is relatively compact. By lemma 1.3 [6, Ch.I] the set

$$\Omega(K) := \bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi(t,K)}$$

is nonempty, compact, invariant and attracts K, i.e.,

$$\lim_{t\to+\infty}\sup_{x\in K}\rho(\pi(t,x),\Omega(K)) = 0. \tag{5}$$

Note that $h(\Omega(K)) \subseteq \Omega(h(K))$. In fact, let $y \in h(\Omega(K))$, then there exists a point $x \in \Omega(K)$ such that $y = h(x)$. By Lemma 1.3 [6, Ch.I] there exists sequences $\{x_n\} \subseteq K$ and $t_n \to +\infty$ ($t_n \in \mathbb{T}_1$) such that $x = \lim_{n\to\infty}\pi(t_n,x_n)$ and, consequently, $y = \lim_{n\to\infty}h(\pi(t_n,x_n)) = \lim_{n\to\infty}\sigma(t_n,h(x_n))$. Since $h(K)$ is a compact subset of Y and $(Y,\mathbb{T}_2,\sigma)$ is compactly dissipative, then $\Omega(h(K))$ is a nonempty compact and invariant set. Taking into account that the Levinson center J_Y of $(Y,\mathbb{T}_2,\sigma)$ is its maximal compact invariant set, then we obtain $\Omega)h(K)) \subseteq J_Y$ and, consequently, $\Omega(K) \subseteq h^{-1}(\Omega(h(K)) \subseteq h^{-1}(J_Y) = \tilde{X}$. Since Levinson center $J_{\tilde{X}}$ of $(\tilde{X},\mathbb{T}_1,\pi)$ is the maximal compact invariant set in $\tilde{X}$, then $\Omega(K) \subseteq J_{\tilde{X}} = M$. From the last inclusion and equality (5) we obtain

$$\lim_{t\to+\infty} \sup_{x\in K} \rho(\pi(t,x), M) = 0. \tag{6}$$

Theorem is proved. □

Theorem 3.3. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS, $(Y,\mathbb{T}_2,\sigma)$ be compactly dissipative and $(X,\mathbb{T}_1,\pi)$ be locally compact. Then the dynamical system $\langle (X,\mathbb{T}_1,\pi)$ will be compactly dissipative if and only if the following conditions hold:*

(i) *for every point $x \in X$ the set Σ_x^+ is relatively compact;*
(ii) *the dynamical system $\langle (\tilde{X},\mathbb{T}_1,\pi)$, $(J_Y,\mathbb{T}_2,\sigma)$ is compactly dissipative.*

Proof. The necessity of Theorem it follows from Theorem 3.2. To prove the sufficiency, according to Theorem 3.2, it is sufficient to show that the set Σ_K^+ is relatively compact for all compact subset $K \subseteq X$. To this end we note that the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative. In fact, denote by $M := J_{\tilde{X}}$, then reasoning as in the proof of Theorem 3.2 it is easy to show that the ω-limit set ω_x of the point x is a nonempty, compact invariant set and $\omega_x \subseteq M$. This means that the dynamical system $(X,\mathbb{T}_1,\pi)$ is pointwise dissipative. Since dynamical system $(X,\mathbb{T}_1,\pi)$ is locally compact, then by Theorem 1.10 [6, Ch.I] this system is also compactly dissipative. Conform to Theorem 1.15 [6, Ch.I] for all compact subset $K \subseteq X$ the set Σ_K^+ is relatively compact. □

Corollary 3.4. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS, $(Y,\mathbb{T}_2,\sigma)$ be compactly dissipative and $(X,\mathbb{T}_1,\pi)$ be completely continuous. Then the dynamical system $(X,\mathbb{T}_1,\pi)$ will be compactly dissipative if and only if the following conditions hold:*

(i) *for every point $x \in X$ the set Σ_x^+ is bounded;*
(ii) *the dynamical system $\langle (\tilde{X},\mathbb{T}_1,\pi)$, $(J_Y,\mathbb{T}_2,\sigma)$ is compactly dissipative.*

Proof. This statement follows directly from Theorem 3.3. To this end it is sufficient to note that every completely continuous dynamical system is locally compact and every bounded semi-trajectory Σ_x^+ is relatively compact, if $(X,\mathbb{T}_1,\pi)$ is completely continuous. □

Let $\mathbb{T}' \subset \mathbb{S}$ $(\mathbb{T} \subset \mathbb{T}')$. A continuous mapping $\gamma_x : \mathbb{T} \to X$ is called a motion of the set-valued dynamical system $(X,\mathbb{T},\pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}'$, if

a. $\gamma_x(0) = x$;
b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ for all $t_1, t_2 \in \mathbb{T}'$ $(t_2 > t_1)$.

The set of all motions of $(X,\mathbb{T},\pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\mathcal{F}_x(\pi)$ and we define $\mathcal{F}(\pi) := \bigcup\{\mathcal{F}_x(\pi) \mid x \in X\}$ (or simply $\mathcal{F}$).

Let X be a metric space and Y be a topological space. The set-valued mapping $\gamma : Y \to C(X)$ is said to be upper semi-continuous (or β-continuous), if $\lim\limits_{y\to y_0} \beta(\gamma(y),\gamma(y_0)) = 0$ for all $y_0 \in Y$.

Let (X, h, Y) be a fiber space, i.e., $h : X \mapsto Y$ is a continuous mapping from X onto Y. The mapping $\gamma : Y \to C(X)$ is called a section (selector) of the fiber space (X, h, Y), if $h(\gamma(y)) = y$ for all $y \in Y$.

Remark 3.5. *Let $X := W \times Y$. Then $\gamma : Y \to C(X)$ is a section of the fiber space (X, h, Y) ($h := pr_2 : X \to Y$), if and only if $\gamma = (\psi, Id_Y)$ where $\psi : W \to C(W)$.*

Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two dynamical systems. The mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (respectively homeomorphic) and $h(\pi(x, t)) = \sigma(h(x), t)$ $(t \in \mathbb{T}_1, \ x \in X)$.

A mapping $\gamma : Y \to C(X)$ is called an invariant section of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, if it is a section of the fiber space (X, h, Y) and $\gamma(Y)$ is an invariant subset of the dynamical system $(X, \mathbb{T}, \pi)$ (or, equivalently, $\pi^t\gamma(y) = \gamma(\sigma^t y)$ for all $t \in \mathbb{T}$ and $y \in Y$).

Denote by $\alpha : C(X) \times C(X) \to \mathbb{R}_+$ the Hausdorff distance on $C(X)$, i.e.,

$$\alpha(A, B) := \max(\beta(A, B), \beta(B, A)),$$

where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-deviation of the set $A \subseteq X$ from the set $B \subseteq X$ and $\rho(a, B)$ the distance from the point a to the set A.

Theorem 3.6. [7, Ch.V],[9, 10] *Let Λ be a metric space, $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ ($\lambda \in \Lambda$) be a family of non-autonomous dynamical system and suppose the following conditions are fulfilled:*

(i) *the space Y is compact;*
(ii) *Y is invariant, i.e., $\sigma^t Y = Y$ for all $t \in \mathbb{T}_2$;*
(iii) *the non-autonomous dynamical systems $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ are equicontracting in the extended sense, i.e., there exist positive numbers N and ν such that*

$$\rho(\pi_\lambda(t, x_1), \pi_\lambda(t, x_2)) \le Ne^{-\nu t}\rho(x_1, x_2)$$

for all $\lambda \in \Lambda$, $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$ and $t \in \mathbb{T}_1$;
(iv) *for each $t \in \mathbb{T}_1$ the mapping $(\lambda, x) \to \pi_\lambda(t, x)$ from $\Lambda \times X$ into X is continuous;*
(v) *$\Gamma(Y, X) = \{\gamma \mid \gamma : Y \to K(X)$ is a set-valued β–continuous mapping and $h(\gamma(y)) = y$ for all $y \in Y\} \neq \emptyset$.*

Then

(i) *for each $\lambda \in \Lambda$ there exists a unique invariant section $\gamma_\lambda \in \Gamma(Y, X)$ of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$;*
(ii) *the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ is compactly dissipative (i.e., $(X, \mathbb{T}_1, \pi_\lambda)$ is compactly dissipative) and its Levinson center $J^\lambda = \gamma_\lambda(Y)$;*
(iii) *$\pi^t_\lambda J^\lambda_y = J^\lambda_{\sigma(t,y)}$ for all $t \in \mathbb{T}_1$ and $y \in Y$;*
(iv) *the mapping $\lambda \to \gamma_\lambda$ is continuous, i.e.,*

$$\lim_{\lambda\to\lambda_0} \sup_{y\in Y} \alpha(\gamma_\lambda(y), \gamma_{\lambda_0}(y)) = 0;$$

(v) *if* $(Y, \mathbb{T}_2, \sigma)$ *is a group-dynamical system (i.e.,* $\mathbb{T}_2 = \mathbb{S}$*), then the unique invariant section* γ_λ *of the non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is one-valued (i.e.,* $\gamma_\lambda(y)$ *consists a single point for any* $y \in Y$*) and*

$$\rho(\pi_\lambda(t, x), \pi_\lambda(t, \gamma_\lambda(h(x)))) \le N e^{-\nu t} \rho(x, \gamma_\lambda(h(x)))$$

for all $x \in X$ *and* $t \in \mathbb{T}$.

4. Linear non-autonomous dynamical systems

In Section 4 we study the linear non-autonomous dynamical systems with discrete time and prove that they admit a compact global attractor and its description is given.

Let Ω be a complete metric space and $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system on Ω with discrete time.

Below in this section we will suppose that the set Ω is invariant, i.e., $\sigma(n, \Omega) = \Omega$ for all $n \in \mathbb{Z}_+$. Let E be a finite-dimensional Banach space with the norm $|\cdot|$ and W be a complete metric space. Denote by $[E]$ the space of all linear continuous operators on E and by $C(\Omega, W)$ the space of all the continuous functions $f : \Omega \to W$ endowed with the compact-open topology, i.e., the uniform convergence on compact subsets in Ω. The results of this section will be used in the next sections.

Consider a linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n \quad (\omega \in \Omega) \tag{7}$$

and an inhomogeneous equation

$$u_{n+1} = A(\sigma(n, \omega))u_n + f(\sigma(n, \omega)), \tag{8}$$

where $A \in C(\Omega, [E])$ and $f \in C(\Omega, E)$.

Let $U(n, \omega)$ be the Cauchy operator of linear equation (7).

We will say that equation (7) is uniformly exponential stable if there exist constants $0 < q < 1$ and $N > 0$ such that

$$\|U(n, \omega)\| \le N q^n \tag{9}$$

for all $\omega \in \Omega$ and $n \in \mathbb{Z}_+$.

Let (X, ρ) be a metric space with distance ρ. Denote by $C(\mathbb{Z}, X)$ the space of all the functions $f : \mathbb{Z} \to X$ equipped with a product topology. This topology can be metrizable. For example, by the equality

$$d(f_1, f_2) := \sum_{1}^{+\infty} \frac{1}{2^n} \frac{d_n(f_1, d_2)}{1 + d_n(f_1, d_2)},$$

where $d_n(f_1, d_2) := \max\{\rho(f_1(k), f_2(k)) \mid k \in [-n, n]\}$, a distance is defined on $C(\mathbb{Z}, X)$ which generates the pointwise topology.

If Ω is compact, then $C(\Omega, E) := \{f \in C(\Omega, E) : \ \|f\| := \max_{\omega \in \Omega} |f(\omega)|\}$. Note that the space $C(\Omega, E)$ equipped with the norm $\|\cdot\|$ is a Banach space.

Remark 4.1. *1. Let $(\Omega, \mathbb{Z}_+, \sigma)$ be a compactly dissipative system. The skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$), generated by cocycle φ admits is compact dissipative if and only if the cocycle φ is compactly dissipative.*

A family of subsets $\mathbb{I} := \{I_\omega |\ \omega \in \Omega\}$ ($I_\omega \subseteq E$) is called positively invariant with respect to cocycle φ, if $\varphi(t, I\omega, \omega) \subseteq I_{\sigma(t,\omega)}$ for all $t \in \mathbb{Z}_+$ and $\omega \in \Omega$.

A bounded subset $\mathcal{M} \subseteq X$ is said to be absorbing for dynamical system $(X, \mathbb{Z}_+, \pi)$ if for any bounded subset $M \subseteq X$ there exists a positive number L such that $\pi(t, M) \subseteq \mathcal{M}$ for all $t \geq L$.

Recall [6, Ch.II] that a cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or shortly φ) is said to be:

(i) *asymptotically compact* if for every bounded, positively invariant with respect to cocycle φ subset $M \subseteq E$ there exists a compact subset $K \subseteq E$ such that

$$\lim_{t\to\infty} \beta(\varphi(t, M, \Omega), K) = 0 \tag{10}$$

(ii) *compactly dissipative*, if there exists a compact subset $K \subseteq E$ such that

$$\lim_{t\to\infty} \beta(\varphi(t, M, \Omega), K) = 0; \tag{11}$$

for every compact subsets $M \in C(E)$;

(iii) ultimately bounded, if there exists a number $R > 0$ such that for every bounded subset $M \subseteq E$ there exists a number $L = L(M)$ such that $|\varphi(t, u, \omega) \leq R$ for all $t \geq L$ and $(u, \omega) \in M \times \Omega$.

Remark 4.2. *a. A cocycle φ is ultimately bounded if and only if for skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by φ there exists a bounded absorbing set.*

b. Let $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber E and $(X, \mathbb{Z}_+, \pi)$ be a skew-product dynamical system generated by φ ($X := E$, $\pi := (\varphi, \sigma)$).

(i) *$\mathbb{I} = \{I_\omega |\ \omega \in \Omega\}$ is positively invariant with respect to cocycle φ if the set $M := \{(u, \omega)|\ u \in I_\omega,\ \omega \in \Omega\}$ is positively invariant with respect to skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by φ.*

(ii) *If a subset $M \subseteq X$ is positively invariant with respect to skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, then the family of subsets $\mathbb{I} = \{I_\omega |\ \omega \in \Omega\}$ is positively invariant with respect to cocycle φ, where $I_\omega := \{u \in E |\ (u, \omega) \in M\}$.*

(iii) *A cocycle φ is asymptotically compact (respectively, compactly dissipative) if and only if the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by φ is so.*

Lemma 4.3. *Suppose that the following conditions are fulfilled:*

(i) *Ω is a compact metric space;*

(ii) *φ is asymptotically compact and ultimately bounded.*

Then φ is compactly dissipative.

Proof. Denote by $\tilde{K} := B[0,R] \times \Omega$, where $B[0,R] := \{u \in E| \ |u| \le R\}$. Since the cocycle φ is ultimately bounded, then according to choice of R we conclude that there exists a positive number $\tilde{L}$ such that

$$\pi(t, \tilde{K}) \subseteq \tilde{K}$$

for all $t \ge \tilde{L}$. Taking into account that φ is asymptotically compact, then (see Remark 4.2 (item (iii))) there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t\to\infty} \beta(\pi(t, \tilde{K}), K) = 0.$$

Let M be an arbitrary nonempty compact subset from X and $I := \bigcup\{I_\omega| \ \omega \in \Omega\}$, where $I_\omega := \{u \in E| \ (u,\omega) \in M\}$. Since I is a bounded subset of E, then there exists a positive number $\tilde{L}_1$ such that

$$\varphi(t, I, \Omega) \subseteq B[0,R] \tag{12}$$

for all $t \ge \tilde{L}_1$. Let $L := \max\{\tilde{L}, \tilde{L}_1\}$, then from (12) we have

$$\pi(t, M) \subseteq \tilde{K} \tag{13}$$

for all $t \ge L$. According to (12) and (13) we obtain

$$\lim_{t\to\infty} \beta(\pi(t, M), K) = 0.$$

Thus the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compactly dissipative. To finish the proof it is sufficient to refer to Remark 4.2 (item (iii)). □

Theorem 4.4. *Suppose that the following conditions hold:*

(i) Ω *is compact;*
(ii) *the linear equation* (7) *is exponentially stable;*
(iii) $f \in C(\Omega, E)$.

Then the following statements hold:

(i) *the non-autonomous dynamical system* $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ *generated by equation (8) is compact dissipative;*
(ii) *the non-autonomous dynamical system* $\langle (\tilde{X}, \mathbb{Z}_+, \pi), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$ *admits a unique upper-semi-continuous invariant section* $\gamma : J_\Omega \mapsto \tilde{X}$*;*
(iii) *the set* $\mathbb{I} := \{I_\omega := \gamma(\omega)| \ \omega \in J_\Omega\}$ *is invariant with respect to (8), i.e.,* $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{Z}_+$ *and* $\omega \in J_\Omega$*, where* φ *is the cocycle generated by equation (8).*

Proof. To prove the first statement we note that according to formula of the variation of constants (see, for example, [12, Ch.I],[13, Ch.VII]) we have

$$\varphi(t, u, \omega) = U(t,\omega)u + \sum_{0}^{t-1} U(t-s, \sigma(s,\omega)) f(\sigma(s,\omega)). \tag{14}$$

From (14) we obtain

$$\begin{aligned} |\varphi(t,u,\omega)| &\le |U(t,\omega)u| + \textstyle\sum_0^{t-1} |U(t-s,\sigma(s,\omega))f(\sigma(s,\omega))| \le \\ &\mathcal{N}q^t|u| + \textstyle\sum_0^{t-1} \mathcal{N}q^{t-s}||f|| \le \mathcal{N}q^t|u| + \mathcal{N}||f||\tfrac{1-q^t}{1-q} \end{aligned} \tag{15}$$

for all $t \in \mathbb{Z}_+$, $u \in E$ and $\omega \in \Omega$, where $||f|| := \max_{\omega\in\Omega} |f(\omega)|$. Passing into limit in (15) as $t \to \infty$ we will have

$$\lim_{t\to\infty} \sup_{|u|\le r,\ \omega\in\Omega} |\varphi(t,u,\omega)| \le \frac{\mathcal{N}||f||}{1-q}. \tag{16}$$

Since the space E is finite-dimensional and taking into account inequality (16) we conclude that the cocycle φ is compactly dissipative and according to Lemma 4.3 the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by cocycle φ is compactly dissipative.

To prove the second statement consider a non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ generated by equation (8) (or equivalently, by cocycle φ). Since the dynamical systems $(X, \mathbb{Z}_+, \pi)$ and $(\Omega, \mathbb{Z}_+, \sigma)$ are compactly dissipative and $h : X \mapsto \Omega$ is an homomorphism from $(X, \mathbb{Z}_+, \pi)$ onto $(\Omega, \mathbb{Z}_+, \sigma)$, then by Theorem 2.6 [6, Ch.II] we have $h(J_X) = J_\Omega$ where J_X (respectively, J_Ω) is the Levinson center of dynamical system $(X, \mathbb{Z}_+, \pi)$ (respectively, $(\Omega, \mathbb{Z}_+, \sigma)$). Consider the non-autonomous dynamical system $\langle (\tilde{X}, \mathbb{Z}_+, \pi), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$ associated by $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$. Let $\omega \in J_\Omega$ be an arbitrary point and $x_i := (u_i, \omega) \in \tilde{X}$ ($u_i \in E$ and $i = 1, 2$) and $\pi(t, x_i) = (\pi(t, u_i, \omega), \sigma(t, \omega))$, where $\varphi(t, u, \omega)$ is a unique solution of equation (8) with initial data $\varphi(0, u, \omega) = u$. Since

$$\rho(\pi(t,x_1),\pi(t,x_2)) = |\varphi(t,u_1,\omega) - \varphi(t,u_2,\omega)| =$$
$$|U(t,\omega)(u_1-u_2)| \le Nq^t|u_1-u_2| = Nq^t\rho(x_1,x_2)$$

for all $t \in \mathbb{Z}_+$, then by Theorem 3.6 there exists a unique upper semi-continuous invariant section for non-autonomous dynamical system $\langle (\tilde{X}, \mathbb{Z}_+, \pi), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$. Theorem is proved. □

A dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ admits a two-sided extension, if there exists a two-sided dynamical system $(\Omega, \mathbb{Z}, \tilde{\sigma})$ on Ω such that $\tilde{\sigma}(t, \omega) = \sigma(t, \omega)$ for all $t \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Remark 4.5. *1. Note that if the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ admits a two-sided extension, then under the conditions of Theorem 4.4 the set $\gamma(\omega)$ consists a single point.*

2. If the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ does not admit a two-sided extension (i.e., the mapping $\sigma(1, \cdot) : \Omega \mapsto \Omega$ is not invertible), then under the conditions of Theorem 4.4 the set $\gamma(\omega)$ contains more than one point (see, for example, [7, Ch.IV,p.130]*).*

5. Global attractors of quasi-linear triangular systems

In Section 5 we prove the existence of compact global attractors of quasi-linear dynamical systems and give the description of the structure of these attractors and dependence on small parameters.

Consider a difference equation

$$u_{n+1} = \mathcal{F}(u_n, \sigma(n, \omega)) \ (\omega \in \Omega). \tag{17}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (17) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (17) is said to be dissipative (respectively, uniform dissipative) if there exists a positive number r such that

$$\limsup_{n\to+\infty} |\varphi(n,u,\omega)| \le r \quad (\text{respectively, } \limsup_{n\to+\infty} \sup_{\omega\in\Omega, |u|\le R} |\varphi(n,u,\omega)| \le r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)), \tag{18}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (condition (ii) in Theorem 5.1).

Denote by $U(k,\omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n.$$

Theorem 5.1. [11] *Suppose that the following conditions hold:*

(i) *there are positive numbers N and $q < 1$ such that*

$$\|U(n,\omega)\| \le Nq^n \qquad (n \in Z_+); \tag{19}$$

(ii) $|F(u,\omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1-q)N^{-1})$ *for all $u \in E$ and $\omega \in \Omega$.*

Then equation (18) is uniform dissipative.

Theorem 5.2. [8] *Let (Ω, Z_+, σ) be a compactly dissipative system and φ be a cocycle generated by equation (18). Under the conditions of Theorem 5.1 the skew-product system (X, Z_+, π) ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$), generated by cocycle φ admits a compact global attractor.*

Theorem 5.3. [11] *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

(i) *the space Ω is compact;*
(ii) *there exist positive numbers N and $q < 1$ such that inequality (19) holds;*
(iii) $|F(u_1,\omega) - F(u_2,\omega)| \le L|u_1 - u_2|$ $(0 \le L < N^{-1}(1-q))$ *for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

Then there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that

$$|\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)| \le \mathcal{N}\nu^n |u_1 - u_2| \tag{20}$$

for all $u_1, u_2 \in E$, $\omega \in \Omega$ and $n \in Z_+$.

Corollary 5.4. *Under the conditions of Theorem 5.3 the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h \rangle$ ($X := E \times \Omega$, $\pi(t,(u,\omega)) = (\varphi(t,u,\omega), \sigma(t,\omega)$ and $h := pr_2 : X \mapsto \Omega$) generated by equation (18) is compactly dissipative.*

Proof. Let Ω be a compact metric space and $F \in C(\Omega \times E, E)$, then we have

$$|F(\omega,u)| \le |F(\omega,u) - F(\omega,0)| + |F(\omega,0)| \le L|u| + C, \tag{21}$$

for all $u \in E$, where $C := \max_{\omega\in\Omega} |F(\omega,0)|$. Now to finish the proof of this statement it is sufficient to apply Theorem 5.2. □

Theorem 5.5. *Let Ω be a compact metric space and φ be a cocycle generated by equation (18). Under the conditions of Theorem 5.3 the following statements hold:*

(i) *the skew-product system (X, Z_+, π) ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$), generated by cocycle φ admits a compact global attractor J_X;*
(ii) *$pr_2(J_X) = J_\Omega$, where J_Ω is Levinson center (compact global attractor) of $(\Omega, \mathbb{Z}_+, \sigma)$;*
(iii) *the set $\mathbb{I} := \{I_\omega |\ \omega \in J_\Omega\}$ ($I_\omega := pr_1(J_\omega)$ and $J_\omega := J \bigcap h^{-1}(\omega)$) is invariant with respect to φ, i.e., $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{Z}_+$ and $\omega \in J_\Omega$.*

Proof. Let $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$be a cocycle generated by equation (18) and $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ be the non-autonomous dynamical system associated by cocycle φ ($X := E \times \Omega$, $\pi(t, (u, \omega)) := (\varphi(t, u, \omega), \sigma(t, \omega))$ and $h := pr_2$). Let $x_i = (u_i, \omega) \in X$ ($u_i \in E$ and $i = 1, 2$) and $\pi(t, x_i) = (\pi(t, u_i, \omega), \sigma(t, \omega)$, where $\varphi(t, u, \omega)$ is a unique solution of equation (8) with initial data $\varphi(0, u, \omega) = u$. By Theorem 5.3 we have

$$\rho(\pi(t, x_1), \pi(t, x_2)) = |\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)| = \leq \mathcal{N} q^t |u_1 - u_2| = \mathcal{N} q^t \rho(x_1, x_2) \tag{22}$$

for all $t \in \mathbb{Z}_+$. Now to finish the proof of Theorem it is sufficient to apply Theorem 3.6 for non-autonomous dynamical system $\langle (\tilde{X}, \mathbb{Z}_+, \pi), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$. Theorem is proved. □

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n + f(\sigma(n, \omega)) + \mu F(u_n, \sigma(n, \omega)), \tag{23}$$

where $A \in C(\Omega, [E])$, $f \in C(\Omega, E)$, $F \in C(E \times \Omega, E)$ and μ is a "small" parameter.

Theorem 5.6. *Let $A \in C(\Omega, [E])$, $f \in C(\Omega, E)$, $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

(i) *the set Ω is compact;*
(ii) *there exist positive numbers $\mathcal{N}$ and $\nu < 1$ such that inequality (19) holds;*
(iii) *there exists a positive number L such that $|F(u_1, \omega) - F(u_2, \omega)| \leq L|u_1 - u_2|$ for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

Then there exists a positive number μ_0 such that:

(i) *for each $\mu \in [-\mu_0, \mu_0]$ there exists a unique upper semi-continuous invariant section γ_μ of the non-autonomous dynamical system $\langle (\tilde{X}, \mathbb{Z}_+, \pi_\mu), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$ associated by equation (23);*
(ii) *the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi_\mu), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is compactly dissipative and its Levinson's center $J^\mu = \gamma_\mu(J_\Omega)$;*
(iii) *$\pi^t_\mu J^\mu_\omega = J^\mu_{\sigma(t,\omega)}$ for all $t \in \mathbb{Z}_+$ and $\omega \in J_\Omega$;*
(iv) *the mapping $\mu \to \gamma_\mu$ is continuous, i.e.,*

$$\lim_{\mu \to 0} \sup_{\omega \in J_\Omega} \alpha(\gamma_\mu(\omega), \gamma_0(\omega)) = 0,$$

where γ_0 is a upper semi-continuous invariant section of non-autonomous dynamical system generated by linear non-homogeneous equation (8).

Proof. Let $\mu \in [-\mu_0, \mu_0]$, where $0 < \mu_0 < (L\mathcal{N})^{-1}(1-q)$, and $\langle (X, \mathbb{Z}_+, \pi_\mu), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ be the non-autonomous dynamical system generated by equation (23). Then for $x_i = (u_i, \omega) \in X$ ($u_i \in E$ and $i = 1, 2$) and $\pi_\mu(t, x_i) = (\varphi_\mu(t, u_i, \omega), \sigma(t, \omega)$, where $\varphi_\mu(t, u, \omega)$ is a unique solution of equation (23) with initial data $\varphi_\mu(0, u, \omega) = u$, by Theorem 5.3 we will have

$$\rho(\pi_\mu(t, x_1), \pi_\mu(t, x_2)) = |\varphi_\mu(t, u_1, \omega) - \varphi_\mu(t, u_2, \omega)| = \\ \leq \mathcal{N}\nu^t|u_1 - u_2| = \mathcal{N}\nu^t\rho(x_1, x_2) \tag{24}$$

for all $t \in \mathbb{Z}_+$. Now to finish the proof of Theorem it is sufficient to apply Theorem 3.6 for non-autonomous dynamical system $\langle (\tilde{X}, \mathbb{Z}_+, \pi_\mu), (J_\Omega, \mathbb{Z}_+, \sigma), h\rangle$ generated by equation (23). Theorem is proved. □

Remark 5.7. *All the results of Section 5 remain true, if we replace the phase space E by positively invariant (with respect to cocycle φ generated by (17)) subset $V \subset E$.*

6. Application: a growth model with VES technology

6.1. **The model.** The Solow-Swan growth model (see [21] and [22]) with VES (Variable Elasticity of Substitution) technology has been studied by Karagiannis et al. [14] while assuming continuous time: the authors show that the model can exhibit unbounded endogenous growth despite the absence of exogenous technical change and the presence of non-reproducible factors. Anyway their model is unable to produce economic fluctuations.

More recently in [4], Brianzoni et al. studied the discrete time Solow-Swan growth model, where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm and Kaas [5] and where the production function $f : R_+ \to R_+$, mapping capital per worker u into output per worker $f(u)$, is of the VES type. Following Kargiannis et al. [14], they considered the specification of the VES production function in intensive form given by Revamkar [15] as follows:

$$f(u) = Ak^{a\gamma}[1 + bau]^{(1-a)\gamma} \ (u \geq 0), \tag{25}$$

being $A > 0$, $a \in (0, 1]$, $b \geq -1$ and $1/u \geq -b$, while assuming that the production function exhibits constant return to scale, i.e., $\gamma = 1$.

Anyway, in their work the authors assume that the labor force grows at a constant rate. This last hypothesis is usually assumed in standard economic growth theory, however, this assumption is unable to explain possible fluctuations in the growth rate. For this reason a number of economic growth model with endogenous population growth has been proposed (see, for instance, Brianzoni et al. [1, 2, 3]). In particular Brianzoni et al. [3] and Cheban et al. [11] recently investigated the neoclassical growth model with differential saving and CES production function under the assumption that the labor force dynamics is described by the logistic equation. Such a law satisfies the following economic properties: (1) when population is small in proportion to the environmental carrying capacity, then it grows at a positive constant rate and (2) when population is larger in proportion to the environmental carrying capacity, the resources become relatively more scarce and, as result, this must affect the population growth rate negatively.

In the present work we consider the Solow-Swan growth model in discrete time with differential saving and VES production function as proposed by Brianzoni et al. in [4] while assuming that the population growth rate evolves according to the logistic law as in Brianzoni et al. [3] and Cheban et al. [11]. Our main goal is to study the qualitative and quantitative long run dynamics of the economic model to show that complex futures results, as the one reached while considering the CES (Constant Elasticity of Substitution) technology.

Let us consider the following equation describing the evolution of the capital per capita u in the standard neoclassical Solow-Swan growth model with differential saving (see [4]):

$$F(u,\omega) = \frac{1}{1+\omega}[(1-\delta)u + s_w w(u) + s_r u f'(u)], \tag{26}$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively. The wage rate equals the marginal product of labor which is $w(u) := f(u) - uf'(u)$, furthermore shareholders receive the marginal product of capital $f'(u)$ which implies that the total capital income per worker is $uf'(u)$.

Observe that $\omega \geq 0$ represents the labor force growth rate: in our formulation we let it vary with time. More precisely we add a further assumption, that is the population growth rate evolves according to the logistic law that is $\omega' = \mu\omega(1-\omega)$.

Consider the case $b \geq 0$. By substituting the VES production function given by (25) (with $\gamma = 1$) in (26) we obtain the following map describing the evolution of the capital accumulation:

$$H(u,\omega) = \frac{1}{1+\omega}\{(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\} \tag{27}$$

The resulting system, $T = (\omega', u')$, describing capital per worker (u) and population growth rate (ω) dynamics, is given by:

$$T := \begin{cases} u' = \frac{1}{1+\omega}\left[(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\right] \\ \\ \omega' = \mu\omega(1-\omega) \end{cases} \tag{28}$$

where $\mu \in (0,4]$ for the dynamics generated by the logistic map not being explosive.

We get a discrete-time dynamical system described by the iteration of a map of the plane of triangular type. In fact the second component of the previous system does not depend on u, therefore the map is characterized by the triangular structure:

$$T := \begin{cases} u' = g(u,\omega) \\ \\ \omega' = f(\omega) \end{cases} . \tag{29}$$

As a consequence, the dynamics of the map T are influenced by the dynamics of the one–dimensional map f, that is the well-known logistic map.

6.2. **Dynamics of the logistic map** $f_\lambda(\omega) = \lambda\omega(1-\omega)$**.** We recall some general results for map f_λ (see, for example, [17]). For $\lambda \in (0,4]$ the map f_λ acts from interval $[0,1]$ into itself and, consequently, it admits a compact global attractor $I_\lambda \subseteq [0,1]$. Since I_λ is connected (see, for example, Theorem 1.33 [6]) and $0 \in I_\lambda$, then $I_\lambda = [0, a_\lambda]$ $(a_\lambda \le 1)$.

(i) If $0 < \lambda \le \lambda_0 := 1$, then $I_\lambda = \{0\}$.
(ii) If $\lambda_0 < \lambda < \lambda_1 := 3$, then the map f_λ has two fixed points: $\omega = 0$ is a repelling fixed point and $p_0 = 1 - 1/\lambda$ is an attracting fixed point. If $\omega \in I_\lambda \setminus \{0, p_0\}$, then $\alpha_\omega = 0$ and $\omega_x = p_0$.
(iii) If $\lambda_1 < \lambda \le \lambda_2 := 1 + \sqrt{6}$, then the map f_λ has one repelling fixed point $x = 0$ and there is an attracting 2-periodic point p_1.
(iv) There exists a increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that
  (a) $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\ldots$.
  (b) If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2,3,\ldots)$, then the map f_λ has one repelling fixed point $\omega = 0$ and there is an attracting 2^k-periodic point p_k.
(v) For all $0 < \lambda < \lambda_\infty$ the structure of the attractor I_λ is sufficiently simple. Every trajectory is asymptotically periodic. There exists a unique attracting 2^m-periodic point p (the number m depends on λ) which attracts all trajectory from $[0,1]$, except for a countable set of points. For $\lambda \ge \lambda_\infty$ the attractor I_λ is more complicated, in particularly, it may be a strange attractor (see [17]).

Let $(X, \mathbb{Z}_+, \pi)$ be a semi-group dynamical system with discrete time. A number m is called an ε-shift (respectively, ε-almost period) of the point x if $\rho(\pi(m,x),x) < \varepsilon$ (respectively, $\rho(\pi(m+n,x),\pi(n,x)) < \varepsilon$ for all $n \in \mathbb{Z}_+$).

The point x is called almost recurrent (respectively, almost periodic) if for any $\varepsilon > 0$ there exists a positive number $l \in \mathbb{Z}_+$ such that on every segment (in $\mathbb{Z}_+$) of length l there may be found an ε-shift (respectively, ε-almost period) of the point x.

The point x is said to be recurrent, if it is almost recurrent and its trajectory is relatively compact.

(vi) Denote by $Per(f_\lambda)$ the set of all periodic points of f_λ. If $\lambda = \lambda_\infty$, then the map f_λ has the 2^i-periodic point p_i for all $i \in \mathbb{Z}_+$ (all the points p_i are repelling). The boundary $K = \partial Per(f_\lambda)$ of set $P(f_\lambda)$ is a Cantor set. The set K is an almost periodic minimal and it does not contain periodic points. The set K attracts all trajectory from $[0,1]$, except for a countable set of points $P = \cup_{i=0}^{\infty} f_\lambda^{-i}(Per(f_\lambda))$. If $x \in [0,1] \setminus P$, then $\omega_x = K$ (see [17]).

6.3. **Existence of an attractor for** $b \in (0, +\infty)$**.**

Lemma 6.1. *The function $H(u,\omega)$ can be presented in the following form*

$$H(u,\omega) = \frac{1}{1+\omega}\{(1 - \delta + s_r ab(\frac{A}{ab})^a)u\} + R(u,\omega), \tag{30}$$

where $R(u,\omega)$ is bounded, i.e., there exists a positive constant C such that $|R(u,\omega)| \le C$ for all $\omega \in [0,1]$ and $u \in [0,+\infty)$.

Proof. It easy to see that the function $R(u,\omega)$ is continuous on $[0,1]\times\mathbb{R}_+$ and, consequently, it is bounded on $[0,1]\times[0,\alpha]$ for all $\alpha>0$. In particularly, for all $\alpha>0$ there exists a constant $C(\alpha)>0$ such that

$$|R(u,\omega)|\le C(\alpha) \tag{31}$$

for all $(u,\omega)\in[0,1]\times[0,\alpha]$.

Let now $\beta\in(0,1)$ be a positive number. Note that

$$(1+x)^\alpha = 1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n+\ldots \tag{32}$$

and the series (32) converges uniformly on $[-\beta,\beta]$.

From (32) we have

$$(1-\tfrac{1}{1+abu})^a = 1-\tfrac{a}{1!}(-\tfrac{1}{1+abu}+\tfrac{a(a-1)}{2!}\tfrac{1}{(1+abu)^2}+\ldots+ \\ \tfrac{a(a-1)\ldots(a-n+1)}{n!}\tfrac{1}{(1+abu)^n}+\ldots \tag{33}$$

for all $a,b>0$, $u\ge 0$ and the convergence of (33) is uniform with respect to $u\in[\frac{1-\beta}{ab},+\infty)$.

Since

$$H(u,\omega)-\tfrac{1}{1+\omega}\{(\tfrac{A}{ab})^a[s_w(1-a)+s_r(a+abu)]\}= \\ \tfrac{1}{1+\omega}\{(1-\delta)u+(\tfrac{A}{ab})^a((1-\tfrac{1}{1+abu})^a-1)[s_w(1-a)+s_r(a+abu)]\} \tag{34}$$

and $\frac{1}{1+abu}\in[0,1]$, then from (32) and (34) we obtain

$$H(u,\omega)-\frac{1}{1+\omega}\{(\frac{A}{ab})^a[s_w(1-a)+s_r(a+abu)]\}=$$

$$\frac{1}{1+\omega}\{(1-\delta)u+(\frac{A}{ab})^a[(1-\frac{1}{1+abu})^a-1][s_w(1-a)+s_r(a+abu)]=$$

$$\frac{1}{1+\omega}\{(1-\delta)u+(\frac{A}{ab})^a[-a\frac{1}{1+abu}+\frac{a(a-1)}{2!}\frac{1}{(1+abu)^2}+\ldots][s_w(1-a)+s_r(a+abu)]=$$

$$\frac{1}{1+\omega}\{(1-\delta)u+(\frac{A}{ab})^a\frac{1}{1+abu}[-a+\frac{a(a-1)}{2!}\frac{1}{1+abu}+\ldots][s_w(1-a)+s_r(a+abu)].$$

Passing into limit as $u\to+\infty$ we obtain

$$\lim_{u\to+\infty}\big(H(u,\omega)-\frac{1}{1+\omega}\{(\frac{A}{ab})^a[s_w(1-a)+s_r(a+abu)]\}\big)=(\frac{A}{ab})^a s_r(-a)$$

and, consequently, $H(u,\omega)-\frac{1}{1+\omega}\{(\frac{A}{ab})^a[s_w(1-a)+s_r(a+abu)]\}$ is bounded on $[\frac{1-\beta}{ab},+\infty)$, i.e., there exists a constant $C_1(\beta)>0$ such that

$$|R(u,\omega)|\le C_1(\beta) \tag{35}$$

for all $(u,\omega)\in[\frac{1-\beta}{ab},+\infty)\times[0,1]$. From (31) and (35) we obtain

$$|R(u,\omega)|\le C$$

for all $(u,\omega)\in\mathbb{R}_+\times[0,1]$, where $C:=\max\{C(\frac{1-\beta}{ab}),C_1(\beta)\}$. The lemma is proved. □

Lemma 6.2. [11] *Let $(R_+\times[0,1],T)$ be a triangular map admitting a compact global attractor $J\subset R_+\times[0,1]$. If $p\in[0,1]$ is a m-periodic point of the map $T_1:[0,1]\mapsto[0,1]$ ($T=(T_2,T_1)$), then*

(i) *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \leq b_p$);*
(ii) *there exists $q \in I_p = [a_p, b_p]$ such that (q, p) is an m-periodic point of the map T.*

Theorem 6.3. *Let $b > 0$ and $\delta > s_r ab(\frac{A}{ab})^a$, then the following statements hold:*

(i) *the dynamical system $(\mathbb{R}_+ \times [0,1], T)$, generated by map (28), admits a compact global attractor $J \subset \mathbb{R}_+ \times [0,1]$;*
(ii) *if $p \in [0,1]$ is an m-periodic point of the map $T_1 : [0,1] \mapsto [0,1]$ ($T = (T_2, T_1)$), then*
   (a) *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in \mathbb{R}_+$ and $a_p \leq b_p$);*
   (b) *there exists $q \in I_p = [a_p, b_p]$ such that (q, p) is an m-periodic point of the map T.*
(iii) *if $p \in [0,1]$ is recurrent, then there exits at least one recurrent point $q \in J_p$.*

Proof. Let $b > 0$, then by Lemma 6.1 the function T_1 can be written in the form

$$T_1(u,\omega) = \frac{1}{1+\omega}\{(1-\delta+s_r ab(\frac{A}{ab})^a)u\} + R(u,\omega), \tag{36}$$

where $R(u,\omega)$ is bounded, i.e., there exists a positive constant M such that $|R(u,\omega)| \leq M$ for all $\omega \in [0,1]$ and $u \in [0,+\infty)$.

Since $0 \leq \frac{1}{1+\omega} \leq 1$ for all $\omega \in [0,1]$, then from (36) we obtain

$$0 \leq T_1(u,\omega) \leq \alpha u + M \tag{37}$$

for all $(u,\omega) \in R_+ \times [0,1]$, where $\alpha := 1 - \delta + s_r ab(\frac{A}{ab})^a < 1$.

Since the map T is triangular, to prove the first statement of Theorem it is sufficient to apply Theorem 5.2 (see also Remark 5.7). The second statement follows from Lemma 6.2.

Let $p \in [0,1]$ be a recurrent point. Since J (Levinson center) is a compact invariant subset of the dynamical system $(\mathbb{R}_+ \times [0,1], T)$, then according to Theorem 3.4.5 [20, Ch.III] (see also [18] and [19]) there exists at least one recurrent point $q \in J_p$. □

6.4. **Structure of the attractor.**

Lemma 6.4. *The following statements hold:*

(i) *let $f(u) := Au^a(1+abu)^{-a}$, then*

$$f'(u) = \frac{af(u)}{u(1+abu)}; \tag{38}$$

(ii) *if $H(u) = (1-\delta)u + f(u)[s_w(1-a) + s_r(a+abu)]$, then*

$$H'(u) = 1 - \delta + \big(\frac{a}{u(1+abu)}[s_w(1-a) + s_r(a+abu)] + s_r ab\big) f(u). \tag{39}$$

Proof. This statement is evident. □

Lemma 6.5. *The following statements hold:*

(i) *$H'(u) \geq 1 - \delta > 0$ for all $u \in (0,+\infty)$;*

(ii)

$$\lim_{u\to\infty} H'(u) = 1-\delta \quad \textit{and} \quad \lim_{u\to+0} H'(u) = +\infty; \tag{40}$$

(iii) *there exists a positive number* u_0 *such that* $H(u) > 2u$ *for all* $u \in [0, u_0]$*;*

Proof. The first and second statements are evident. To prove the third statement we note that from (40) it follows the existence a positive number u_0 such that $H'(u) \geq 3$ for all $u \in (0, u_0]$. Let now $\alpha \in (0, u_0)$, then we have

$$H(u) - H(\alpha) = H'(\theta)(u-\alpha) \geq 3(u-\alpha) \tag{41}$$

for all $u \in (0, u_0)$, where $\theta \in (\alpha, u)$. Passing into limit in (41) as $\alpha \to 0$ and taking into account the continuity of $H(u)$ at the point $u = 0$ and the equality $H(0) = 0$ we obtain $H(u) \geq 3u > 2u$ for all $u \in (0, u_0)$. Lemma is proved. □

Corollary 6.6. *The function* $H(u)$ *is strict monotone increasing, i.e.,* $u_1 < u_2$ *implies* $H(u_1) < H(u_2)$ *for all* $u_1, u_2 \in \mathbb{R}_+$*.*

Lemma 6.7. *Let* $(\mathbb{R}_+ \times [0,1], T)$*be a dynamical system generated by map (28), i.e.,* $T^t(u,\omega) = (\varphi(t,u,\omega), f^t(\omega))$*, then* $\varphi(t,u,\omega) \geq \lambda u$ *for all* $t \in \mathbb{Z}_+$*,* $u \in (0, u_0]$ *and* $\omega \in [0,1]$*, where* $\lambda \in (1, 3/2]$*.*

Proof. Note that $H(u,\omega) = \frac{1}{1+\omega}H(u)$ and $\varphi(t,u,\omega)$ is a unique solution of equation

$$u_{t+1} = H(u_t, f^t(\omega)) \tag{42}$$

with initial data $u_0 = u$. Let $u \in [0, u_0]$, then by Lemma 6.5 we have $\varphi(1,u,\omega) = H(u,\omega) \geq 3u$ for all $u \in [0, u_0]$ because $\frac{1}{1+\omega} \in [1/2, 1]$ for all $\omega \in [0,1]$. If we suppose that $\varphi(t,u,\omega) \geq \lambda u$ for all $t = 1, 2, \ldots, n$, then we obtain $\varphi(t+1,u,\omega) = \varphi(1, \varphi(t,u,\omega), f(\omega)) \geq \lambda\varphi(t,u,\omega) \geq \lambda^2 u > \lambda u$. Lemma is proved. □

Corollary 6.8. *Let* $(\mathbb{R}_+ \times [0,1], T)$ *be a dynamical system generated by map (28), then for all* $u \in [0, u_0]$ *the set* $\mathbb{R}_u \times [0,1]$*, where* $\mathbb{R}_u := [u, +\infty)$*, is a positively invariant set of* $(\mathbb{R}_+ \times [0,1], T)$*.*

Remark 6.9. *1. Since the dynamical system* $(\mathbb{R}_+ \times [0,1], T)$ *is compact dissipative, then there exists a positive number* $r_0 > u_0$ *such that the set* $U_{r_0} = [0, r_0) \times [0,1]$ *contains an open, positively invariant neighborhood* U *of the Levinson center* J *such that* $[0, k_0) \times [0,1] \subset U$*.*

The set Θ is said to be

- stable in the negative direction, if for every positive number ε there exists $\delta = \delta(\varepsilon) > 0$ such that for all entire trajectory $\gamma = (\nu, \mu) \in \Phi_{(u,\omega)}(T)$ of dynamical system $(\mathbb{R}_+ \times [0,1], T)$ with $|u| < \delta$ we have $|\nu(t)| < \varepsilon$ for all $t \leq 0$;
- an attracting set in the negative direction, if there exists a positive number α such that $\lim_{t\to-\infty} |\nu(t)| = 0$ for all entire trajectory ν (for the cocycle φ) with condition $|\nu(0)| = |u| \leq \alpha$;
- asymptotically stable in the negative direction, if Θ is stable and attractive in the negative direction.

Theorem 6.10. *Let $(\mathbb{R}_+ \times [0,1], T)$ be a dynamical system generated by map (28), then the following statement hold:*

(i) *the set $\Theta = \{0\} \times [0, a_\lambda]$ is a unique compact invariant set containing in the neighborhood $U := [0, r_0) \times [0, 1)$ of the set Θ, i.e., Θ is a locally maximal compact invariant set of the dynamical system $(\mathbb{R}_+ \times [0,1], T)$;*
(ii) *for all $x \in \overline{U} = [0, k_0] \times [0, 1]$ we have $\omega_x \bigcap \Theta = \emptyset$;*
(iii) *the set Θ is asymptotically stable.*

Proof. Suppose that the first statement is not true, then there exists a compact invariant set $\Theta' \subset U$ such that:

(i) $\Theta \subseteq \Theta'$;
(ii) $\Theta' \neq \Theta$.

Let $x_0 = (\bar{u}, \omega_0) \in \Theta' \setminus \Theta$, then $0 < \bar{u} < u_0$ and there exists an entire trajectory $\gamma = (\nu, \mu)$ such that $\gamma(0) = (\bar{u}, \omega_0)$ and $\gamma(t) \in \Theta'$ for all $t \in \mathbb{Z}$. According to Lemma 6.7 we have

$$\nu(t_2) = \varphi(t_2 - t_1, \nu(t_1), \mu(t_1)) \geq \lambda \nu(t_1) \tag{43}$$

for all $t_2 > t_1$. From (43) we obtain $\nu(t) \geq \lambda^t |\bar{u}|$ for all $t \in \mathbb{Z}_+$. This means that the sequence $\{\nu(t)\}_{t \geq 0}$ is unbounded. The obtained contradiction show that our assumption that $\Theta' \neq \Theta$ is not true, i.e., $\Theta' = \Theta$.

To prove the second statement we suppose that it is not true, then there exists a point $x_0 \in \overline{U}$ such that $\omega_{x_0} \bigcap \Theta \neq \emptyset$, then there exists a sequence $t_n \to +\infty$ such that $\nu(t_n) \to 0$ as $n \to \infty$. From this fact follows existence sequences $\{k_n\}$, $\{u_n\}$ and $\{s_n\}$ such that:

(i) $u_n \geq u_0$ and converges to point $\bar{u} \geq u_0$;
(ii) $s_n \to +\infty$ as $n \to \infty$;
(iii) $\omega_n \in [0, a_\lambda]$ and $\varphi(s, u_n, \omega_n) \in U$ for all $0 < s \leq s_n$;
(iv) $\varphi(s_n, u_n, \omega_n) = \bar{u}_n := \varphi(t_n, \bar{u}, \omega_0) \to 0$ as $n \to +\infty$.

Without loss of generality we can suppose that the sequence $\{\omega_n\}$ is convergent. Denote by $\omega_0 = \lim_{n \to \infty} \omega_n$, then $\varphi(t, \bar{u}, \omega_0) \leq u_0$ for all $t \geq 0$. Reasoning as above it is not difficult to establish that $\varphi(t, \bar{u}, \omega_0) \geq \lambda^t |\bar{u}| \to +\infty$ as $t \to +\infty$. The obtained contradiction completes the second statement.

Let now prove the stability of the set Θ in the negative direction. Suppose that this statement is not true, then there are $0 < \varepsilon_0 < u_0$, $\delta_n \to 0$ $(\delta_n > 0)$, $\varepsilon_0 \leq |u_n| < r_0$, $\omega_n \in [0, a_\lambda]$, and $t_n \to +\infty$ such that

(i) $u_n \to u_0$ and $\omega_n \to \omega_0$ as $n \to +\infty$;
(ii)
$$(\varphi(t_n, u_n, f^{t_n}(\omega_n)) \to (0, \tilde{\omega})$$
as $n \to +\infty$;
(iii)
$$|\varphi(t, u_n, \omega_n)| < \varepsilon_0 \tag{44}$$
for all $t \in [0, t_n)$.

Passing into limit in (44) as $n \to \infty$ we obtain $|\varphi(t, u_0, \omega_0)| < \varepsilon_0$ for all $t \in \mathbb{Z}_+$. Thus the ω-limit set ω_{x_0} of point $x_0 := (u_0, \omega_0)$ is a nonempty, compact, invariant set and $\omega_{x_0} \subseteq B[\Theta, \varepsilon_)]$. Since the set Θ is local maximal, then $\omega_0 \subseteq \Theta$. The last inclusion contradicts to seconde statement of Theorem. Obtained contradiction completes the proof of our statement.

Finally, we establish that Θ is an attracting set. To this end we consider an arbitrary entire trajectory (ν, γ) of the dynamical system $(\mathbb{R}_+ \times [0,1], T)$ passing through the point $(u, \omega) \in U = [0, r_0] \times \mathbb{R}_+$ with the condition $|\nu(s)| \le u_0$ (respectively, $(\nu(s), \gamma(s)) \in U$) for all $s \in \mathbb{Z}_-$. It is clear that the α-limit set $\alpha_{(\nu,\gamma)}$ is a nonempty and compact set and it consists from the entire trajectories of the dynamical system $(\mathbb{R}_+ \times [0,1], T)$. Since Θ is a maximal compact invariant set in U, then $\alpha_{(\nu,\gamma)} \subseteq \Theta$ and, consequently, $\lim\limits_{s \to -\infty} |\nu(s)| = 0$. Theorem is completely proved. □

6.5. **Conclusion.** Under the conditions of Theorem 6.3 the mapping $T = (T_2, T_1)$ $(T_1 = f_\lambda)$ admits a compact global attractor $J_\lambda \subset R_+ \times [0,1]$. There exists an increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that

- (i) $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\dots$.
- (ii) If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2, 3, \dots)$, then the map $T = (T_2, T_1)$ has at least one fixed point $(q_0, 0) \in J_\lambda$ and there is a 2^k-periodic point $(q_k, p_k) \in J_\lambda$.
- (iii) For $\lambda \ge \lambda_\infty$ the set J_λ may be a strange attractor. For example, under the conditions of Theorem 6.3, for $\lambda = \lambda_\infty$ the attractor J_λ contains a compact minimal set (but not periodic) consisting from recurrent motions.

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