MARKUS-SELL'S THEOREM FOR ASYMPTOTICALLY ALMOST PERIODIC SYSTEMS

DAVID CHEBAN AND C. MAMMANA

Abstract. This paper is dedicated to the study of asymptotic stability of asymptotically almost periodic systems. We formulate and prove the analog of Markus-Sell's theorem for asymptotically almost periodic systems (both finite and infinite dimensional cases). We study this problem in the framework of general non-autonomous dynamical systems.The obtained general results we apply to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Difference Equations, Functional Differential Equations and Semi-Linear Parabolic Equations.

1. Introduction

Denote by $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, $F(t,x) := f(x) + p(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ the right hand side of system (1), where $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the space of all continuous functions $F : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ equipped with the compact open topology.

A system of differential equation

$$x' = f(x) + p(t,x) \tag{1}$$

is said to be asymptotically autonomous, if the function $p \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following condition

$$\lim_{t\to\infty} |p(t,x)| = 0 \tag{2}$$

uniformly in x on every compact subset from $\mathbb{R}^n$, where $|\cdot|$ is a norm on $\mathbb{R}^n$. Autonomous system

$$x' = f(x) \tag{3}$$

is called a limiting system for (1).

Example 1.1. *(***Bessel's equation***) Consider the equation*

$$t^2 x'' + t x' + (t^2 - \alpha^2)x = 0,$$

or equivalently

$$\begin{cases} x' = y \\ y' = -\frac{1}{t}y + (\frac{\alpha^2}{t^2} - 1)x, \end{cases}$$

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with limiting system

$$\begin{cases} x' = y \\ y' = -x \ . \end{cases}$$

Denote by $C^1(\mathbb{R}^n, \mathbb{R}^n)$ the space of all continuously differentiable functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$.

Theorem 1.2. *(L. Markus [10]) Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $O = (0,0)$ be a critical point of limiting system (3), i.e, $f(0) = 0$. Assume that the variational system of (3) based on origin O have characteristic values with negative real parts. Then there exists a neighborhood U of O and a time T such that $\lim\limits_{t\to\infty} |x(t)| = 0$ for any solution of equation (1) intersecting U no later than T, i.e., the origin is an attracting point for (1).*

Let $(C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}_+, \sigma)$ be the shift dynamical system [11, 12] (or Bebutov's dynamical system) on $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. For every function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ we denote by $H^+(F) := \overline{\{F_\tau : \tau \in \mathbb{R}_+\}}$ the closure of all positive translations of the function F and by Ω_F its ω-limit set, i.e., $\Omega_F := \{G :$ there exists a sequence $\tau_n \to +\infty$ such that $F_{\tau_n} \to G\}$, where F_τ is τ-shift of the function F, i.e., $F_\tau(t,x) := F(t+\tau, x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ be an arbitrary function. Consider the equation

$$x' = F(t,x). \tag{4}$$

Along with equation (4) we consider its H^+-class, i.e., the following family of equations

$$y' = G(t,y) \quad (G \in H^+(F)).$$

Example 1.3. *1. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically autonomous, i.e., $F(t,x) = f(x) + p(t,x)$ and p satisfies condition (2). In this case $\Omega_F = \{f\}$, i.e., its ω-limit set contains a single function.*

2. Let $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ be asymptotically T periodic, i.e., $F(t,x) = f(t,x) + p(t,x)$, $f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and p satisfies condition (2). In this case $\Omega_F = \{f_\tau : \tau \in [0,T)\}$, i.e., its ω-limit set contains a continuum functions and it is homeomorphic to unitary circle.

3. If $F \in C(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ is asymptotically quasi periodic, i.e., $F(t,x) = f(t,x) + p(t,x)$, where $f(t,x)$ is a quasi periodic function with the spectrum of frequency $\nu_1, \nu_2, \ldots, \nu_m$ and p satisfies condition (2). In this case its ω-limit set is homeomorphic to an m-torus.

Theorem 1.4. *(G. Sell [11, Ch.VIII]) Let $F \in C(\mathbb{R}^n, \mathbb{R}^n)$ be regular, asymptotically autonomous and $O \in \mathbb{R}^n$ be the null solution equation (4), i.e, $F(t,0) = 0$ for all $t \in \mathbb{R}_+$. Assume that the null solution of limiting equation (3) is uniformly asymptotically stable. Then the null solution of equation (4) is uniformly asymptotically stable.*

Remark 1.5. ***1.*** *Note that Theorem 1.4 generalizes Theorem of L. Markus in the following directions:*

a. *right hand side f of the limiting equation is only continuous (L. Markus, $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$);*
b. *the null solution of limiting equation (3) is only uniformly asymptotically stable (L. Markus, O is uniformly exponentially stable (In fact, $Re\lambda_i < 0$ $(i = 1,\dots,n)$, $\lambda_1,\dots,\lambda_n$ are characteristic values of the origin for the variational equation for (3)).*

2. *It is easy to see that there are examples with uniformly asymptotically stable origin which is not uniformly exponentially stable. For example $x' = -x^3$ $(n = 1)$.*

Consider a differential equation

$$x' = f(t,x) \quad (f \in C(\mathbb{R}\times W,\mathbb{R}^n)), \tag{5}$$

where W is an open subset from $\mathbb{R}^n$ containing the origin (i.e., $0 \in W$), $C(\mathbb{R}\times W,\mathbb{R}^n)$ is the space of all continuous functions $f : \mathbb{R}\times W \mapsto \mathbb{R}^n$ equipped with compact open topology. Denote by $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$ the *shift dynamical system* [5, 11] on the space $C(\mathbb{R}\times W,\mathbb{R}^n)$ (*dynamical system of translations* or *Bebutov's dynamical system*), i.e., $\sigma(\tau,f) := f_\tau$ for all $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R}\times W,\mathbb{R}^n)$, where $f_\tau(t,x) := f(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times W$.

Below we will use the following conditions:

(A): for all $(t_0,x_0) \in \mathbb{R}_+\times W$ the equation (5) admits a unique solution $x(t;t_0,x_0)$ with initial data (t_0,x_0) and defined on $\mathbb{R}_+ := [0,+\infty)$, i.e., $x(t_0;t_0,x_0) = x_0$;

(B): the hand right side f *is positively compact*, if the set $\Sigma_f^+ := \{f_\tau :\ \tau \in \mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R}\times W,\mathbb{R}^n)$;

(C): the equation

$$y' = g(t,y), \quad (g \in \Omega_f) \tag{6}$$

is called a *limiting equation* for (5), where Ω_f is the ω-limit set of f with respect to the shift dynamical system $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$, i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\} \to +\infty$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): *equation* (5) (or its hand right side f) *is regular* , if for all $p \in H^+(f)$ the equation

$$y' = p(t,y)$$

admits a unique solution $\varphi(t,x_0,p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0,x_0,p) = x_0$ for all $x_0 \in W$, where $H^+(f) = \overline{\{f_\tau :\ \tau \in \mathbb{R}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{R}\times W,\mathbb{R}^n)$;

(E): equation (5) admits a null (trivial) solution, i.e., $f(t,0) = 0$ for all $t \in \mathbb{R}_+$.

The trivial solution of equation (5) is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ $(\delta \in (0,\varepsilon))$ such that $|x| < \delta$ implies $|\varphi(t,x,f_\tau)| < \varepsilon$ for all $t,\tau \in \mathbb{R}_+$;

(ii) *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t\to+\infty} |\varphi(t,x,f_\tau)| = 0$$

uniformly in $|x| \le a$ and $\tau \in \mathbb{R}_+$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

In connection with Theorems 1.2 and 1.4 by G. Sell was formulated the following problem.

G. Sell's conjecture ([11, Ch.VIII,p.134]). Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and f be positively pre-compact. Assume that W contains the origin 0 and $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. Assume further that there exists a positive number a such that the equality

$$\lim_{t \to +\infty} |\varphi(t, x, g)| = 0$$

takes place uniformly in $|x| \leq a$ and $g \in \Omega_f$. Then the trivial solution of (5) is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Z. Artstein [1] and Bondi P. et al. [2].

Remark 1.6. *1. Bondi P. et al.* [2] *proved this conjecture under the additional assumption that the function f is locally Lipschitzian.*

2. Artstein Z. [1] *proved this statement without Lipschitzian condition. In reality he proved a more general statement. Namely, he supposed that only limiting equations for (5) are regular, but the function f is not obligatory regular.*

3. By D. Cheban [7] *was formulated G. Sell's conjecture for abstract NDSs (the both with continuous and discrete time). In* [7] *it is given a positive answer to this conjecture and also are presented some applications of this result to different classes of evolution equations: infinite-dimensional differential equations, functional-differential equations and semi-linear parabolic equations .*

In the paper [13] it was published the following false [3] statement.

Theorem 1.7. *Let f be a regular function with $f(t, 0) = 0$ for all $t \geq 0$. If there exits a function $g \in \Omega_f$ such that the null solution of equation (6) is uniformly asymptotically stable, then the null solution of equation (5) is uniformly asymptotically stable.*

Bondi P. at al. [3] give the following counterexample to Theorem 1.7.

$$ax'' + bx' + cx = x \sin \sqrt{t} \quad (x \in \mathbb{R},\ t \in \mathbb{R}_+,\ a, b > 0,\ c \in (0, 1)). \tag{7}$$

For every $\mu \in [-1, 1]$

$$ax'' + bx' + cx = \mu x \tag{8}$$

is a limiting equation for (7). For $\mu \in [-1, c)$ the null solution of equation (8) is uniformly asymptotically stable, but the null solution of equation (7) is not uniformly stable.

Recall that a function $f \in C(\mathbb{R} \times W, E)$ is called:

- almost periodic (respectively, almost recurrent) in $t \in \mathbb{R}$ uniformly in u on every compact subset K from W, if for an arbitrary number $\varepsilon > 0$ and compact subset $K \subseteq W$ there exists a positive number $L = L(K, \varepsilon)$ such that on every segment $[a, a+L]$ ($a \in \mathbb{R}$) of the length L there exists at least one number τ such that
$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+s+\tau, u) - f(t+s)| < \varepsilon$$
(respectively,
$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+\tau, u) - f(t,u)| < \varepsilon)$$
for all $s \in \mathbb{R}$;
- recurrent (in $t \in \mathbb{R}$ uniformly in u on every compact subset K from W, if the function $f \in C(\mathbb{R} \times W, E)$ is almost recurrent and $H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ is compact;
- asymptotically recurrent (respectively, almost periodic) in $t \in \mathbb{R}$ uniformly in u on every compact subset K from W, if there exits two functions $P, R \in C(\mathbb{R} \times W, E)$ such that:
  (a)
$$f(t,u) = P(t,u) + R(t,x) \tag{9}$$
  for all $(t,u) \in \mathbb{R} \times E$;
  (b) the function P is recurrent (respectively, almost periodic) in $t \in \mathbb{R}$ uniformly in u on every compact subset K from W and
$$\lim_{t \to +\infty} \max_{u \in K} |R(t,u)| = 0 \tag{10}$$
  for every compact subset $K \subseteq W$.

From the main result of this paper (Theorems 3.7 and 3.8) it follows that Theorem 1.7 is true if the function f is asymptotically recurrent (in particularly, asymptotically almost periodic) in $t \in \mathbb{R}$ uniformly in u on every compact subset Q from $\mathbb{R}^n$.

The aim of this paper is investigation the problem of asymptotic stability of trivial solution for asymptotically almost periodic (respectively, asymptotically recurrent) systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS). We formulate and prove the analog of Theorem 1.7 for abstract non-autonomous dynamical systems. The obtained result we apply to different classes of evolution equations: Ordinary Differential Equations (both finite and infinite-dimensional cases), Difference Equations, Functional Differential Equations, Semi-Linear Parabolic Equations .

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, point/compact dissipativity, recurrence, shift dynamical systems, etc) and facts from the theory of dynamical systems which will be necessary in this paper.

Section 3 is devoted to the study of asymptotic stability of NDS with asymptotically recurrent base. The main result of paper (Theorem 3.7 and Theorem 3.8) contain some tests of asymptotic stability for asymptotically recurrent (in particularly, asymptotically almost periodic) NDS.

Finally, Section 4 contains a series of applications of our general results from Sections 2-3 for Ordinary Differential Equations (Theorem 4.1 and Theorem 4.2), Functional-Differential Equations with finite delay (Theorem 4.7) and Semi-Linear Parabolic Equations (Theorem 4.9).

2. Asymptotic Stability of Dynamical Systems

In this section we collect some facts about stability and asymptotic stability of dynamical systems (both autonomous and non-autonomous) which use in this paper.

2.1. **Compact Global Attractors of Dynamical Systems.** Let X be a topological space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of the nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ be one of the sub-semigroups $\mathbb{R}_+$ (respectively, $\mathbb{Z}_+$) or $\mathbb{R}$ (respectively, $\mathbb{Z}$).

Triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions: $\pi(0, x) = x$ and $\pi(s, \pi(t, x)) = \pi(s + t, x)$ ($\forall t, s \in \mathbb{T}$) is called a *dynamical system*. If $\mathbb{T} = \mathbb{R}$ ($\mathbb{R}_+$) or $\mathbb{Z}$ ($\mathbb{Z}_+$), then $(X, \mathbb{T}, \pi)$ is called a *group (semi-group) dynamical system*. In the case, when $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a *flow*, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a *cascade (discrete flow)*.

Below X will be a complete metric space with the distance ρ.

The function $\pi(\cdot, x) : \mathbb{T} \to X$ is called a *motion* passing through the point x at the moment $t = 0$ and the set $\Sigma_x := \pi(\mathbb{T}, x)$ is called a *trajectory* of this motion.

A nonempty set $M \subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to dynamical system $(X, \mathbb{T}, \pi)$ or, simple, positively invariant (negatively invariant, invariant), if $\pi(t, M) \subseteq M$ ($M \subseteq \pi(t, M), \pi(t, M) = M$) for every $t \in \mathbb{T}_+ := \{t \in \mathbb{T} :\ t \geq 0\}$.

A closed positively invariant set (respectively, invariant set), which does not contain own closed positively invariant (respectively, invariant) subset, is called *minimal*.

Let $M \subseteq X$. The set

$$\omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}$$

is called ω-limit for M.

The set $W^s(\Lambda)$, defined by equality

$$W^s(\Lambda) := \{x \in X |\ \lim_{t \to +\infty} \rho(\pi(t, x), \Lambda) = 0\}$$

is called a *stable manifold* of the set $\Lambda \subseteq X$.

The set M is called:

- *orbital stable*, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \geq 0$;

- *attracting*, if there exists $\gamma > 0$ such that $B(M,\gamma) \subset W^s(M)$, where $B(M,\gamma) := \{x \in X : \rho(x,M) < \gamma\}$;
- *asymptotic stable*, if it is orbital stable and attracting;
- *uniform attracting*, if there exists $\gamma > 0$ such that

$$\lim_{t\to+\infty} \sup_{x\in B(M,\gamma)} \rho(\pi(t,x),M) = 0.$$

The system $(X,\mathbb{T},\pi)$ is called:

– *compactly dissipative* if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t\to+\infty} \beta(\pi(t,M),K) = 0 \tag{11}$$

for all compact subset $M \subseteq X$, where $\beta(A,B) := \sup_{a\in A} \rho(a,B)$ and $\rho(a,B) := \inf_{b\in B} \rho(a,b)$;

– *local completely continuous* (*compact*) if for all point $p \in X$ there are two positive numbers δ_p and l_p such that the set $\pi(l_p, B(p,\delta_p))$ is relatively compact.

Let $(X,\mathbb{T},\pi)$ be compactly dissipative and K be a compact set attracting every compact subset from X. Let us set $J = \omega(K)$. It can be shown [5, Ch.I] that the set J doesn't depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X,\mathbb{T},\pi)$ itself. The set J is called a *Levinson center* of the compactly dissipative dynamical system $(X,\mathbb{T},\pi)$.

Lemma 2.1. [7] *Let $(X,\mathbb{T},\pi)$ be a dynamical system and $x \in X$ be a point with relatively compact semi-trajectory $\Sigma_x^+ := \{\pi(t,x) : t \geq 0\}$. Then the following statements hold:*

(i) *the dynamical system $(X,\mathbb{T},\pi)$ induces on the $H^+(x) := \overline{\Sigma}_x^+$ a dynamical system $(H^+(x),\mathbb{T}_+,\pi)$, where by bar is denoted the closure of Σ_x^+ in the space X;*
(ii) *the dynamical system $(H^+(x),\mathbb{T}_+,\pi)$ is compactly dissipative;*
(iii) *Levinson center $J_{H^+(x)}$ of $(H^+(x),\mathbb{T}_+,\pi)$ coincides with ω-limit set ω_x of the point x.*

2.2. **G. Sell's conjecture for non-autonomous dynamical systems.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $(Y,\mathbb{T}_2,\sigma)$ be a dynamical system on metric space Y. Recall that a triplet $\langle W,\varphi,(Y,T_2,\sigma)\rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y,\mathbb{T}_2,\sigma)$ with the fiber W, if the following conditions are fulfilled:

(i) $\varphi(0,u,y) = u$ for all $u \in W$ and $y \in Y$;
(ii) $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau \in \mathbb{T}_1$, $u \in W$ and $y \in Y$;
(iii) the mapping $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ is continuous.

Example 2.2. (**Bebutov's dynamical system**) Let X, W be two metric space. Denote by $C(\mathbb{T}\times W,X)$ the space of all continuous mappings $f : \mathbb{T}\times W \mapsto X$ equipped with the compact-open topology and σ be the mapping from $\mathbb{T}\times C(\mathbb{T}\times W,X)$ into $C(\mathbb{T}\times W,X)$ defined by the equality $\sigma(\tau,f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in$

$C(\mathbb{T}\times W,X)$, where f_τ is the τ-translation (shift) of f in t, i.e., $f_\tau(t,x)=f(t+\tau,x)$ for all $(t,x)\in\mathbb{T}\times W$. Then [5, Ch.I] the triplet $(C(\mathbb{T}\times W,X),\mathbb{T},\sigma)$ is a dynamical system on $C(\mathbb{T}\times W,X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

A function $f\in C(\mathbb{T}\times W,X)$ is said to be *recurrent* (respectively, *almost periodic*) in $t\in\mathbb{T}$ uniformly in $x\in W$ on every compact subset from W, if $f\in C(\mathbb{T}\times W,X)$ is a recurrent (respectively, almost periodic) point of the Bebutov's dynamical system $(C(\mathbb{T}\times W,X),\mathbb{T},\sigma)$.

Example 2.3. Consider differential equation (5) with regular second right hand side $f\in C(\mathbb{R}\times W,\mathbb{R}^n)$, where $W\subseteq\mathbb{R}^n$. Denote by $(H^+(f),\mathbb{R}_+,\sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$, where $H^+(f):=\overline{\{f_\tau:\ \tau\in\mathbb{R}_+\}}$. Let $\varphi(t,u,g)$ a unique solution of equation

$$y'=g(t,y),\quad (g\in H^+(f)),$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

(i) $\varphi(0,u,g)=u$ for all $u\in W$ and $g\in H^+(f)$;
(ii) $\varphi(t+\tau,u,g)=\varphi(t,\varphi(\tau,u,g),g_\tau)$ for all $t,\tau\in\mathbb{R}_+$, $u\in W$ and $g\in H^+(f)$;
(iii) the mapping $\varphi:\mathbb{R}_+\times W\times H^+(f)\mapsto W$ is continuous.

From above it follows that the triplet $\langle W,\varphi,(H^+(f),\mathbb{R}_+,\sigma)\rangle$ is a cocycle over $(H^+(f),\,\mathbb{R}_+,\,\sigma)$ with the fiber $W\subseteq\mathbb{R}^n$. Thus, every non-autonomous equation (5) with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation (5).

Suppose that $W\subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0\in W$ (0 is the null element of E) and the cocycle $\langle W,\varphi,(Y,T_2,\sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t,0,y)=0$ for all $t\in\mathbb{T}_1$ and $y\in Y$.

The trivial motion/solution of cocycle φ is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta=\delta(\varepsilon)$ ($\delta\in(0,\varepsilon)$) such that $|u|<\delta$ implies $|\varphi(t,u,y)|<\varepsilon$ for all $t\geq 0$ and $y\in Y$;
(ii) *uniformly attracting*, if there exists a positive number a such that

$$\lim_{t\to+\infty}|\varphi(t,u,y)|=0 \tag{12}$$

uniformly with respect to $|u|\leq a$ and $y\in Y$;
(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

Theorem 2.4. [7] *Let $\langle(X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),\ h\rangle$ be a NDS. Suppose that the following conditions are fulfilled:*

(i) *Y is compact;*
(ii) *the dynamical system $(X,\mathbb{T}_1,\pi)$ is locally compact;*
(iii) *the trivial section Θ of (X,h,Y) is positively invariant;*
(iv) *the trivial section $\tilde{\Theta}$ of NDS $\langle(\tilde{X},\mathbb{T}_1,\pi),\ (J_Y,\mathbb{T}_2,\sigma),\ h\rangle$ is uniformly attracting.*

Then the trivial section Θ *of non-autonomous dynamical system* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma), h\rangle$ *is uniformly stable.*

Corollary 2.5. [7] *Under the conditions of Theorem 2.4 the trivial section* Θ *of NDS* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ *is uniformly asymptotically stable.*

Remark 2.6. *Note that Corollary 2.5 gives a positive answer to G.Sell's conjecture for local-compact NDS* [7].

3. Asymptotic Stability of NDS with Asymptotically Recurrent Base

Section 3 is devoted to the study of asymptotic stability of NDS with asymptotically recurrent base. The main result of paper (Theorem 3.7 and Theorem 3.8) contain some tests of asymptotic stability for asymptotically recurrent (in particularly, asymptotically almost periodic) NDS.

Remark 3.1. *If the bundle* (X,h,Y) *is locally trivial, then for all* $y\in Y$, $x\in X_y$ *and* $y_n\to y$ *there exists a sequences* $x_n\in X_{y_n}$ *such that* $x_n\to x$.

Let $M\subseteq Y$. Denote by $\Theta_M := h^{-1}(M)\bigcap\Theta$, where Θ is the null section of the vectorial bundle (X,h,Y).

Definition 3.2. *The set* Θ_M *is said to be:*

(i) uniform stable, *if for all* $\varepsilon>0$ *there exists a number* $\delta=\delta(\varepsilon,M)>0$ *such that* $|x|<\delta$ $(x\in h^{-1}(M))$ *implies* $|\pi(t,x)|<\varepsilon$ *for all* $t\geq 0$*;*

(ii) uniform attracting, *if there exists a positive number a such that*

$$\lim_{t\to\infty}\sup_{|x|\leq a,\ x\in h^{-1}(M)}|\pi(t,x)|=0; \tag{13}$$

(iii) uniform asymptotic stable, *if it is uniform stable and uniform attracting.*

Lemma 3.3. *Suppose that the set* Θ_M $(M\subseteq Y)$ *is uniformly stable (respectively, uniformly attracting or uniformly asymptotically stable), then the set* $\Theta_{\overline{M}}$ *is also uniformly stable (respectively, uniformly attracting or uniformly asymptotically stable), where by bar is denoted the closure of* M *in* Y.

Proof. Let Θ_M be uniformly stable, $\varepsilon>0$ and $\delta=\delta(\varepsilon,M)>0$ a positive number figuring in the definition of uniform stability of Θ_M. Denote by $\nu(\varepsilon):=\frac{\delta(\varepsilon/2,M)}{2}$. Let now $x\in h^{-1}(\overline{M})$ with $|x|<\nu(\varepsilon)$, $y:=h(x)\in\overline{M}$ and $y_n\in M$ such that $y_n\to y$. By Remark 3.1 there exist sequences $y_n\to y$ $(y_n\in M)$ and $x_n\in X_{y_n}$ such that $x_n\to x$. According to choose of the number $\nu(\varepsilon)$ there exists $n_0\in\mathbb{N}$ such that $|x_n|\leq\nu(\varepsilon)<\frac{\delta(\varepsilon/2,M)}{2}$ and, consequently,

$$|\pi(t,x_n)|<\varepsilon/2 \tag{14}$$

for all $t\geq 0$. Passing into limit in (14) as n goes to ∞ we obtain $|\pi(t,x)|\leq\varepsilon/2<\varepsilon$ for all $t\geq 0$.

Let now Θ_M be uniformly attracting, $a>0$ be a positive number figuring in (13), and ε be an arbitrary positive number. According to equality (13) there exists a positive number $L=L(\varepsilon/2,M)$ such that

$$|\pi(t,x)|<\varepsilon/2 \tag{15}$$

for all $t \ge L$, $|x| \le a$ with $x \in h^{-1}(M)$. Let now $x \in h^{-1}(\overline{M})$ with $|x| \le a/2$, $y_n \to y$ ($y_n \in M$) and $x_n \to x$ ($x_n \in X_{y_n}$), then there exists a number $n_0 \in \mathbb{N}$ such that $|x_n| \le a$ for all $n \ge n_0$. From (15) we obtain

$$|\pi(t, x_n)| \le \varepsilon/2 \tag{16}$$

for all $t \ge L$ and $n \ge n_0$. Passing into limit in (16) as n tends to ∞ we obtain $|\pi(t, x)| < \varepsilon$ for all $t \ge L$ and $x \in h^{-1}(\overline{M})$ with $|x| \le b := a/2$. This means that

$$\lim_{t\to\infty} \sup_{|x|\le b,\ x\in h^{-1}(\overline{M})} |\pi(t, x)| = 0,$$

i.e., the set $\Theta_{\overline{M}}$ is uniformly attracting.

Since the set $\Theta_{\overline{M}}$ is uniformly stable and uniformly attracting, then it is uniformly asymptotically stable. The lemma is completely proved. □

Definition 3.4. *Let $y_0 \in Y$, the null element $\theta_{y_0} \in X_{y_0}$ is said to be* uniformly stable *(respectively,* uniformly attracting *or* uniformly asymptotically stable*), if the set $\Theta_{\Sigma^+_{y_0}}$ is so, where $\Sigma^+_{y_0} := \{\sigma(t, y_0) :\ t \ge 0\}$ is the positive semi-trajectory of the point y_0.*

Corollary 3.5. *Let $y_0 \in Y$ and θ_{y_0} be uniformly stable (respectively, uniformly attracting or uniformly asymptotically stable), then the set $\Theta_{H^+(y_0)}$ is also uniformly stable (respectively, uniformly attracting or uniformly asymptotically stable).*

Proof. This statement follows directly from Lemma 3.3 if we apply it to the set $M = \Sigma^+_{y_0}$. □

Let $y \in Y$ be an asymptotically recurrent point, i.e., there exists a recurrent point $q \in Y$ such that

$$\lim_{t\to+\infty} \rho(\sigma(t, y), \sigma(t, q)) = 0. \tag{17}$$

Denote by $P_y := \{q \in Y :\ q$ is a recurrent point such that (17) holds$\}$.

Remark 3.6. *1. If the point $y \in Y$ is asymptotically recurrent, then its ω-limit set ω_y is a compact and minimal set of dynamical system $(Y, \mathbb{T}, \sigma)$.*

2. There exist points $y \in Y$ (see, for example, [6, Ch.I,p.13] *Example 1.42) with the properties that Σ^+_y is relatively compact and ω_y is compact and minimal, but y is not asymptotically recurrent.*

Theorem 3.7. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a NDS. Suppose that the following conditions are fulfilled:*

(i) *the dynamical system $(X, \mathbb{T}_1, \pi)$ is locally compact;*
(ii) *the space Y is compact;*
(iii) *Levinson center J_Y of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal;*
(iv) *the null section Θ of the bundle (X, h, Y) is positively invariant;*
(v) *there exists a point $q \in J_Y$ such that $\theta_q \in \Theta$ is uniformly attracting.*

Then the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is uniformly asymptotically stable.

Proof. According to Corollary 2.5 the set $\Theta_{H^+(q)}$is uniformly attracting. Since q is recurrent and the set J_Y is minimal we obtain $J_Y = H^+(q)$. Thus the trivial section $\tilde{\Theta}$ of the NDS $\langle(\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$ is uniformly attracting. Now to finish the proof of Theorem 3.7 it is sufficient to apply Theorem 2.4 and Corollary 2.5. □

Theorem 3.8. *Let $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS. Suppose that the following conditions are fulfilled:*

(i) *the dynamical system $(X,\mathbb{T}_1,\pi)$ is locally compact;*
(ii) *there exists an asymptotically recurrent point $y \in Y$ such that $Y = H^+(y)$;*
(iii) *the null section Θ of the bundle $(X,h,Y))$ is positively invariant;*
(iv) *there exists a recurrent point $q \in P_y$ such that $\theta_q \in \Theta$ is uniformly attracting.*

Then the trivial section Θ of $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is uniformly asymptotically stable.

Proof. Since the point $y \in Y$ is asymptotically recurrent, $q \in P_y$ and $Y = H^+(y)$, then Y is compact, the dynamical system $(Y,\mathbb{T}_2,\sigma)$ is compactly dissipative and its Levinson center J_Y coincides with $\omega_y = H^+(q)$ and, consequently, it is minimal. Now to finish the proof of Theorem 3.8 it is sufficient to apply Theorem 3.7. □

Recall that a subset $M \subseteq X$ is called transitive, if there exists a point $p \in X$ such that $M = H(q) := \overline{\{\pi(t,p) :\ t \in \mathbb{R}\}}$, where by bar is denoted a closure in the space X.

A point $x \in X$ is said to be:

- Poisson stable (in the positive direction), if $x \in \omega_x$;
- asymptotically Poisson stable, if there exists a Poisson stable point $p \in X$ such that

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,p)) = 0. \tag{18}$$

Lemma 3.9. *If the point $x \in X$ is asymptotically Poisson stable, then its ω-limit set ω_x is transitive.*

Proof. Let x be an asymptotically Poisson stable point and ω_x be its ω-limit set. Then there exists a Poisson stable point $p \in X$ such that (18) holds. From (18) it follows the equality $\omega_x = \omega_p$. On the other hand from the Poisson stability of x we obtain $\omega_p = H(p)$ and, consequently, $\omega_x = H(p)$. Lemma is proved. □

Remark 3.10. *All results of Section 3 remain true, if we replace the minimality of J_Y by its transitivity.*

Remark 3.11. *All results of Sections 2–3 remain true if*

1. *we replace the positive invariance of the trivial section Θ by the following condition: there exists a compact positively invariant set $M \subseteq X$ such that $M_y := \{x \in M :\ h(x) = y\}$ consists a single point for all $y \in Y$;*

2. *the compact metric space Y we replace by an arbitrary compact regular topological space.*

4. Some applications

Section 4 contains a series of applications of our general results from Sections 2-3 for Ordinary Differential Equations, Functional-Differential Equations with finite delay and Semi-Linear Parabolic Equations.

4.1. **Ordinary differential equations.** Denote by $C(\mathbb{S}\times W, E)$ the space of all continuous mappings $f:\mathbb{S}\times W\mapsto E$ equipped with the compact open topology. On the space $C(\mathbb{S}\times W, E)$ it is defined a shift dynamical system [5, ChI] (dynamical system of translations or Bebutov's dynamical system) $(C(\mathbb{S}\times W, E), \mathbb{S}, \sigma)$, where σ is a mapping from $\mathbb{S}\times C(\mathbb{S}\times W, E)$ onto $C(\mathbb{S}\times W, E)$ defined as follow $\sigma(\tau, f) := f_\tau$ for all $(\tau, f)\in\mathbb{S}\times C(\mathbb{S}\times W, E)$, where f_τ is the τ-translation of f in t, i.e., $f_\tau(t,x) := f(t+\tau, x)$ for all $(t,x)\in\mathbb{S}\times W$. Consider a differential equation

$$u' = f(t,u), \tag{19}$$

where $f\in C(\mathbb{R}\times W, E)$.

If the function f is regular, then equation (19) naturally defines a cocycle $\langle W,\ \varphi,\ (H^+(f),\ \mathbb{R}_+,\ \sigma)\rangle$, where $(H^+(f), \mathbb{R}_+, \sigma)$ is a (semi-group) dynamical system on $H^+(f)$ induced by Bebutov's dynamical system.

Applying the general results from Sections 2-3 we will obtain a series of results for equation (19). Below we formulate some of them.

Denote by $\Omega_f := \{g\in H^+(f):$ there exists a sequence$\tau_n\to+\infty$ such that $g = \lim\limits_{n\to\infty} f_{\tau_n}\}$ the ω-limit set of f.

Theorem 4.1. *Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *the set $H^+(f)$ is compact;*
(iii) *the ω-limit set Ω_f of function f is a compact minimal set of Bebutov's dynamical system $(C(\mathbb{R}\times W, E), \mathbb{R}, \sigma)$;*
(iv) *$f(t,0)=0$ for all $t\in\mathbb{R}_+$;*
(v) *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*
(vi) *there exists a function $P\in\Omega_f$ such that the trivial solution of equation*

$$x' = P(t,x) \tag{20}$$

is uniformly attraction, i.e., there exists a positive number a such that

$$\lim_{t\to+\infty}\ \sup_{|v|\le a,\tau\ge 0} |\varphi(t, v, P_\tau)| = 0. \tag{21}$$

Then the null solution of equation (19) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and by Lemma 2.1 its

Levinson center (maximal compact invariant set) $J_{H^+(f)}$ coincides with ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote by $X := W \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ, i.e., $\pi(t, (v, g)) := (\varphi(t, v, g), \sigma(t, g))$ for all $t \in \mathbb{R}_+$ and $(v, g) \in X$. Now consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ ($h := pr_2$) associated by equation (19). It easy to verify that this NDS posses the following properties:

(i) by Lemma 2.1 the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
(iv) according to (21) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h\rangle$ is uniformly attracting (see Lemma 3.3) because $|\pi(t, x)| = |\varphi(t, v, g)|$ for all $t \in \mathbb{R}_+$ and $x := (v, g) \in X$.

Now to finish the proof it is sufficient to apply Corollary 2.5. □

Theorem 4.2. *Assume that the following conditions are fulfilled:*

(i) *the function f is regular;*
(ii) *f is asymptotically recurrent in t uniformly in x on every compact subset from W;*
(iii) *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
(iv) *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*
(v) *the trivial solution of equation (20) is uniformly attracting, i.e., there exists a positive number a such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, \tau\ge 0} |\varphi(t, v, P_\tau)| = 0.$$

Then the null solution of equation (19) is uniformly asymptotically stable.

Proof. This statement it follows directly from Lemma 3.3 and Corollary 3.5 using the same arguments as in the proof of Theorem 4.1. □

Remark 4.3. *1. Note that the compactness and minimality of the set Ω_f in Theorem 4.1 are essential. In fact, in the work* [3] *there is an example of equation of type (19), where all conditions of Theorem 4.1 are fulfilled with the exception of minimality of Ω_f and for which equation the trivial solution is not uniformly stable.*

2. In the case, when the function f is asymptotically stationary (i.e., the function P figuring in (22) does not depend on $t \in \mathbb{R}$) and E is finite-dimensional Theorem 4.2 coincides with one result of G. Sell (see [11, Ch.VIII, p.135] *Theorem 10) and L. Markus* [10].

3. Note that assumption of the minimality of Ω_f is not necessary in this Subsection (see Remark 3.10 and Lemma 3.9). All the results from Subsection 4.1 remain true, if we replace the minimality of Ω_f by its transitivity (this means that there exists a function $P \in \Omega_f$ such that $\Omega_f = H(P) := \overline{\{P_\tau :\ \tau \in \mathbb{R}\}}$).

4.2. **Difference equations.** Consider a difference equation

$$u(t+1) = f(t, u(t)), \tag{22}$$

where $f \in C(\mathbb{Z} \times W, E)$.

Along with equation (22) we consider the family of equations

$$v(t+1) = g(t, v(t)), \tag{23}$$

where $g \in H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{Z}_+\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z} \times W, E)$. Let $\varphi(t, v, g)$ be a unique solution of equation (23) with initial data $\varphi(0, v, g) = v$. Denote by $(H^+(f), \mathbb{Z}_+, \sigma)$ the shift dynamical system on $H^+(f)$, then the triplet $\langle W, \varphi, (H^+(f), \mathbb{Z}_+, \sigma)\rangle$ is a cocycle (with discrete time) over $(H^+(f), \mathbb{Z}_+, \sigma)$ with the fibre W.

Applying the results from Sections 2-3 we will obtain the following result for difference equation (22).

Theorem 4.4. *Assume that the following conditions are fulfilled:*

(i) *the function $f \in C(\mathbb{Z} \times W, E)$ is asymptotically recurrent in t uniformly in x on every compact subset from W;*
(ii) *$f(t, 0) = 0$ for all $t \in \mathbb{Z}_+$;*
(iii) *there exists a neighborhood U of the origin 0 and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact;*
(iv) *the trivial solution of equation*

$$x(t+1) = P(t, x)$$

is uniformly attracting, i.e., there exists a positive number a such that

$$\lim_{t \to +\infty} \sup_{|v| \le a, \tau \ge 0} |\varphi(t, v, P_\tau)| = 0.$$

Then the null solution of equation (22) is uniformly asymptotically stable.

4.3. **Functional differential-equations (FDEs) with finite delay.** We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

Let us first recall some notions and notations from [8]. Let $r > 0$, $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a, b] = [-r, 0]$, then we set $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \ge 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for all $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t + \theta)$, $-r \le \theta \le 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{24}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is defined (see, for example, [5, ChI]) a shift dynamical system $(C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $\sigma(\tau, f) := f_\tau$ for all $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f_τ is τ-translation of f, i.e., $f_\tau(t, \phi) := f(t + \tau, \phi)$ for all $(t, \phi) \in \mathbb{R} \times \mathcal{C}$.

Let us set $H^+(f) := \overline{\{f_s : s \in \mathbb{R}_+\}}$, where by bar we denote the closure in $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$.

Along with the equation (24) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{25}$$

where $g \in H^+(f)$.

Below, in this subsection, we suppose that equation (24) is regular.

Remark 4.5. 1. Denote by $\tilde{\varphi}(t, u, f)$ the solution of equation (24) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0, u, f) = u \in \mathcal{C}$, i.e., $\varphi(s, u, f) = u(s)$ for all $s \in [-r, 0]$. By $\varphi(t, u, f)$ we will denote below the trajectory of equation (24), corresponding to the solution $\tilde{\varphi}(t, u, f)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t, u, f)(s) := \tilde{\varphi}(t + s, u, f)$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r, 0]$.

2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (24) as synonym concepts.

It is well known [4, 11] that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H^+(f) \mapsto \mathbb{R}^n$ possesses the following properties:

(i) $\varphi(0, v, g) = u$ for all $v \in \mathcal{C}$ and $g \in H^+(f)$;
(ii) $\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), \sigma(\tau, g))$ for all $t, \tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H^+(f)$;
(iii) the mapping φ is continuous.

Thus, a triplet $\langle \mathcal{C}, \varphi, (H^+(f), \mathbb{R}_+, \sigma) \rangle$ is a cocycle which is associated to equation (24). Applying the results from Sections 2-3 we will obtain certain results for functional differential equation (24).

A function $f \in C(\mathbb{R} \times W, \mathcal{C})$ is said to be completely continuous, if the set $f(\mathbb{R}_+ \times A)$ is bounded for all bounded subset $A \subseteq \mathcal{C}$.

Lemma 4.6. [7] *Suppose that the following conditions hold:*

(i) *the function $f \in C(\mathbb{R} \times W, \mathcal{C})$ is regular and completely continuous;*
(ii) *the set $H^+(f)$ is compact.*

Then the cocycle φ associated by (24) is completely continuous, i.e., for all bounded subset $A \subseteq W$ there exists a positive number $l = l(A)$ such that the set $\varphi(l, A, H^+(f))$ is relatively compact in $\mathcal{C}$.

Theorem 4.7. *Assume that the following conditions are fulfilled:*

a. *the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is regular and completely continuous;*
b. *f is asymptotically recurrent in t uniformly in x on every compact subset from $\mathcal{C}$;*
c. *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
d. *the trivial solution of equation*

$$x' = P(t, x_t)$$

is uniformly attraction.

Then the null solution of equation (24) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and by Lemma 2.1 its Levinson center $J_{H^+(f)}$ coincides with ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote by $X := \mathcal{C} \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ. Now consider a NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ $(h := pr_2)$ associated by equation (24). It easy to verify this NDS possesses the following properties:

(i) the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
(iv) according to item e. the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h\rangle$ is uniformly attracting because $|\pi(t, x)| = |\varphi(t, v, g)|$ for all $t \in \mathbb{R}_+$ and $x := (v, g) \in X$;
(v) according to Lemma 4.6 the dynamical system $(X, \mathbb{R}_+, \pi)$ is completely continuous.

Now to finish the proof it is sufficient to apply Lemma 3.3 and Corollary 3.5. $\square$

4.4. **Semi-linear parabolic equations.** Let E be a Banach space, and let $A : D(A) \to E$ be a linear closed operator with the dense domain $D(A) \subseteq E$.

An operator A is called [9] sectorial if for some $\varphi \in (0, \pi/2)$, some $M \geq 1$, and some real a, the sector

$$S_{a,\varphi} := \{\lambda : \varphi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$.

If A is a sectorial operator, then there exists an $a_1 \geq 0$ such that $Re\, \sigma(A + a_1 I) > 0$ $(\sigma(A) := \mathbb{C} \setminus \rho(A))$. Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [9]

$$A_1^{-\alpha} := \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha := D(A_1^\alpha)$, and let us equip the space E^α with the graph norm $|x|_\alpha := |A_1^\alpha x|$ $(x \in E)$, $E^0 := E$, and $E^1 := D(A)$. Then E^α is a Banach space with the norm $|\cdot|_\alpha$ and is densely and continuously embedded in E.

Consider differential equation

$$x' + Ax = f(t, x), \tag{26}$$

where $f \in C(\mathbb{R} \times E^\alpha, E)$ and $C(\mathbb{R} \times E^\alpha, E)$ is the space of all the continuous functions equipped with compact open topology.

Along with equation (26), consider family of equations

$$y' + Ay = g(t, y), \tag{27}$$

where $g \in H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{R}_+\}}$.

Recall that a function f is said to be regular, if for every $(v, g) \in E^\alpha \times H^+(f)$ equation (27) admits a unique solution [9, Ch.III] $\varphi(t, v, g)$ with initial data $\varphi(0, v, g) = v$ and the mapping $\varphi : \mathbb{R}_+ \times E^\alpha \times H^+(F) \mapsto E^\alpha$ is continuous.

Regularity conditions for f are given in Theorems 3.3.3, 3.3.4, 3.3.6, and 3.4.1 in [9, Ch.III].

Assuming that f is regular, a non-autonomous dynamical system can be associated in a natural way with equation (26). Namely, we set $Y := H^+(f)$ and by $(Y, \mathbb{R}_+, \sigma)$ denote the dynamical system of translations on Y. Further, let $X := E^\alpha \times Y$, and let $(X, \mathbb{R}_+, \pi)$ be the dynamical system on X defined by the relation $\pi^\tau(v, g) = \langle \varphi(\tau, v, g), g_\tau \rangle$. Finally, by setting $h = pr_2 : X \to Y$, we obtain the non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$ determined by equation (26).

Recall that a function $f \in C(\mathbb{R} \times E^\alpha, E)$ is said to be locally Hölder continuous in t and locally Lipschitz in x, if for every $(t_0, x_0) \in \mathbb{R} \times E^\alpha$ there exists a neighborhood V ($(t_0, x_0) \in V$) and positive numbers L and θ such that

$$|f(t_1, x_1) - f(t_2, x_2)| \le L(|t_1 - t_2|^\theta + |x_1 - x_2|_\alpha)$$

for all $(t_i, x_i) \in V$ $(i = 1, 2)$.

Lemma 4.8. *Suppose that the following conditions are fulfilled:*

(i) *A is a sectorial operator;*
(ii) *the resolvent of operator A is compact;*
(iii) *$0 \le \alpha < 1$ and $f \in C(\mathbb{R} \times E^\alpha, E)$;*
(iv) *the function f is locally Hölder continuous in t and locally Lipschitz in x;*
(v) *the set $f(\mathbb{R}_+ \times B)$ is bounded in E for all bounded subset B from E^α.*

Under the conditions listed above, if the function f is regular and the set $H^+(f)$ is compact, then the cocycle φ associated by equation (26) is completely continuous.

Proof. This statement can be proved with the slight modification of the proof of Theorem 3.3.6 [9, Ch.III]. □

Applying results from Sections 3-4 we obtain the following result for evolution equation (26).

Theorem 4.9. *Assume that the following conditions hold:*

a. *the function $f \in C(\mathbb{R} \times E^\alpha, E)$ is asymptotically recurrent in t uniformly in x on every compact subset from E^α;*
b. *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
c. *the set $f(\mathbb{R}_+ \times B)$ is bounded in E for all bounded subset B from E^α;*
d. *the trivial solution of equation*

$$x' = Ax + P(t, x)$$

is uniformly attracting.

Then the null solution of equation (26) is uniformly asymptotically stable.

Proof. Consider the dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$. Since the space $H^+(f)$ is compact, then according to Lemma 2.1 $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center $J_{H^+(f)}$ coincides with ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote by $X := E^\alpha \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ. Consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ $(h := pr_2)$ associated by equation (26). It easy to verify that this NDS possesses the following properties:

(i) the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compact dissipative and by Lemma 2.1 its Levinson center J_Y coincides with Ω_f;
(ii) the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h\rangle$ coincides with $\{0\} \times Y$;
(iii) Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
(iv) according to item b. the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi)$, $(J_Y, \mathbb{R}_+, \sigma)$, $h\rangle$ is uniformly attracting because $|\pi(t, x)| = |\varphi(t, v, g)|$ for all $t \in \mathbb{R}_+$ and $x := (v, g) \in X$;
(v) by Lemma 4.8 the cocycle φ and, consequently, the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ too, is completely continuous.

Now to finish the proof it is sufficient to apply Lemma 3.3 and Theorem 3.8. □

Remark 4.10. *Theorem 4.9 for asymptotically autonomous equations it was established in* [9] *(Chapter IV, Theorem 4.3.7).*

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(D. Cheban) State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova

E-mail address, D. Cheban: cheban@usm.md

(C. Mammana) Dept. of Economic and Financial Institutions, University of Macerata, str. Crescimbeni 20, I–62100 Macerata, Italy

E-mail address, C. Mammana: mammana@unimc.it