

COMPACT GLOBAL CHAOTIC ATTRACTORS OF DISCRETE CONTROL SYSTEMS

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ABSTRACT. The paper is dedicated to the study of the problem of existence of compact global chaotic attractors of discrete control systems and to description of its structure. We consider so called switched systems with discrete time $x_{n+1} = f_{\nu(n)}(x_n)$, where $\nu : \mathbb{Z}_+ \rightarrow \{1, 2, \dots, m\}$. If $m \geq 2$ we give sufficient conditions (the family $\mathcal{M} := \{f_1, f_2, \dots, f_m\}$ of functions is contracting in the extended sense) for the existence of a compact global chaotic attractor. We study this problem in the framework of non-autonomous dynamical systems (cocycles).

1. INTRODUCTION

The aim of this paper is the study of the problem of existence of compact global chaotic attractors of discrete control systems (see, for example, Bobylev, Emel'yanov and Korovin [3], Cheban [7, 8] and the references therein). Let W be a metric space, $\mathcal{M} := \{f_i : i \in I\}$ be a family of continuous mappings of W into itself and $(W, f_i)_{i \in I}$ be the family of discrete dynamical systems, where (W, f) is a discrete dynamical system generated by positive powers of continuous map $f : W \rightarrow W$. On the space W we consider a discrete inclusion

$$u_{t+1} \in F(u_t)$$

associated by $\mathcal{M} := \{f_i : i \in I\}$ (denotation $DI(\mathcal{M})$), where $F(u) = \{f(u) : f \in \mathcal{M}\}$ for all $u \in W$.

A solution of the discrete inclusion $DI(\mathcal{M})$ is called (see, for example, [3, 11]) a sequence $\{x_j \mid j \geq 0\} \subset W$ such that

$$(1) \quad x_j = f_{i_j} x_{j-1}$$

for some $f_{i_j} \in \mathcal{M}$ (trajectory of $DI(\mathcal{M})$), i.e.

$$x_j = f_{i_j} f_{i_{j-1}} \dots f_{i_1} x_0 \text{ all } f_{i_k} \in \mathcal{M}.$$

We can consider that it is a discrete control problem, where at each moment of the time j we can apply a control from the set \mathcal{M} , and $DI(\mathcal{M})$ is the set of possible trajectories of the system.

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The problem of existence of compact global attractors for a discrete inclusion arise in a number of different areas of mathematics (see, for example, [8, 9] and the references therein).

A sequence $\omega : \mathbb{Z}_+ \mapsto \{1, 2, \dots, m\}$ is called $m \in \mathbb{N} := \{1, 2, \dots\}$ periodic, if $\omega(n + m) = \omega(n)$ for all $n \in \mathbb{Z}_+$.

A point $x_0 \in W$ is said to m -periodic for $DI(\mathcal{M})$ if there exists an m -periodic sequence $\omega : \mathbb{Z}_+ \mapsto \{1, 2, \dots, m\}$ such that solution $\{x(k)\}_{k \in \mathbb{Z}_+}$ of equation (1) ($\omega(i) = i_j$ for all $i \in \mathbb{Z}$ and $i_j \in \{1, 2, \dots, m\}$) with initial data $x(0) = x_0$ is m -periodic, i.e., $x(k + m) = x(k)$ for all $k \in \mathbb{Z}_+$.

It is well known the following result.

Theorem 1.1. [2, Ch.II,IV] *Let $\mathcal{M} = \{f_1, f_2, \dots, f_m\}$ be a finite family of continuous mappings from W into itself. If there exists a number $q \in (0, 1)$ such that $\rho(f_i(x_1), f_i(x_2)) \leq q\rho(x_1, x_2)$ for all $x_1, x_2 \in W$ and $i \in \{1, 2, \dots, m\}$, then the following statement hold:*

- (i) *$DI(\mathcal{M})$ admits a compact global attractor I , i.e.,*
 - (a) *I is a nonempty, compact and invariant set ($F(I) = I$, where $F(x) := \{f_1(x), f_2(x), \dots, f_m(x)\}$ for all $x \in W$ and $F(I) := \bigcup\{F(x) : x \in I\}$);*
 - (b) *$\lim_{n \rightarrow \infty} \beta(F^n(x), I) = 0$ for all $x \in W$ uniformly with respect to x on every compact subset M from W , where $\beta(A, B) := \sup_{a \in B} \rho(a, B)$ ($A, B \subseteq W$).*
- (ii) *I coincides with the closure of the all periodic points of $DI(\mathcal{M})$.*

In the book [8] (Chapter VI) it was generalized this theorem for the finite family $\mathcal{M} = \{f_1, f_2, \dots, f_m\}$ when it is contracting in the generalized sense, i.e., there are two positive numbers \mathcal{N} and $q \in (0, 1)$ such that

$$\rho(f_{i_n} f_{i_{n-1}} \dots f_{i_1}(x_1), f_{i_n} f_{i_{n-1}} \dots f_{i_1}(x_2)) \leq \mathcal{N} q^n \rho(x_1, x_2)$$

for all $x_1, x_2 \in W$ and $n \in \mathbb{N}$, where $i_1, i_2, \dots, i_n \in \{1, 2, \dots, m\}$.

In this paper we consider an arbitrary family of discrete dynamical systems (W, f) ($f \in \mathcal{M}$, \mathcal{M} contains, generally speaking, an infinite number of mappings f) on the complete metric space W and we give the conditions which guarantee the existence of compact global attractor for \mathcal{M} . If \mathcal{M} consists of a finite number of maps, then we prove that \mathcal{M} admits a compact global chaotic attractor. We study this problem in the framework of non-autonomous dynamical systems (cocycles).

This paper is organized as follows.

In Section 2 we give some notions and facts (set-valued dynamical systems, compact global attractors, an ensemble (coolage) of dynamical systems, cocycles) from the theory of set-valued dynamical systems which we use in our paper.

Section 3 is dedicated to the study of compact global chaotic attractors of discrete control systems. We give also the description of the dynamics of global attractors for this type of control systems. The main result of Section 3 (Theorem 3.2) contains the conditions of existence of chaotic attractor for discrete control systems.

In Section 4 we study the problem of existence of compact global attractor for discrete control system in the case when \mathcal{M} contains an infinite number of mappings f and they are not (in general) invertible. The main result (Theorem 4.9 and Theorem 4.12) of Section 4 give the conditions of existence of compact global attractors and describes its dynamics.

Section 5 contains some applications of general results obtained in Sections 3 and 4 for certain classes of control systems with continuous time.

2. SOME NOTIONS AND FACTS FROM DYNAMICAL AND CONTROL SYSTEMS

In this Section we collect some notions and facts from the theory of set-valued dynamical systems which we use in our paper.

2.1. Set-valued dynamical systems and their compact global attractors.

Let (X, ρ) be a complete metric space, \mathbb{S} be a group of real (\mathbb{R}) or integer (\mathbb{Z}) numbers, \mathbb{T} ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a semi-group of additive group \mathbb{S} . If $A \subseteq X$ and $x \in X$, then we denote by $\rho(x, A)$ the distance from the point x to the set A , i.e. $\rho(x, A) = \inf\{\rho(x, a) : a \in A\}$. We denote by $B(A, \varepsilon)$ an ε -neighborhood of the set A , i.e. $B(A, \varepsilon) = \{x \in X : \rho(x, A) < \varepsilon\}$, by $C(X)$ we denote the family of all non-empty compact subsets of X . For every point $x \in X$ and number $t \in \mathbb{T}$ we put in correspondence a closed compact subset $\pi(t, x) \in C(X)$. So, if $\pi(P, A) = \bigcup\{\pi(t, x) : t \in P, x \in A\}$ ($P \subseteq \mathbb{T}$), then

- (i) $\pi(0, x) = x$ for all $x \in X$;
- (ii) $\pi(t_2, \pi(t_1, x)) = \pi(t_1 + t_2, x)$ for all $x \in X$;
- (iii) $\lim_{x \rightarrow x_0, t \rightarrow t_0} \beta(\pi(t, x), \pi(t_0, x_0)) = 0$ for all $x_0 \in X$ and $t_0 \in \mathbb{T}$, where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-deviation of the set $A \subseteq X$ from the set $B \subseteq X$.

In this case it is said [17] that there is defined a set-valued semi-group dynamical system.

Let $\mathbb{T} \subset \mathbb{T}' \subset \mathbb{S}$. A continuous mapping $\gamma_x : \mathbb{T}' \rightarrow X$ is called a motion of the set-valued dynamical system (X, \mathbb{T}, π) issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on \mathbb{T}' , if

- a. $\gamma_x(0) = x$;
- b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ for all $t_1, t_2 \in \mathbb{T}'$ ($t_2 > t_1$).

The set of all motions of (X, \mathbb{T}, π) , passing through the point x at the initial moment $t = 0$ is denoted by $\mathfrak{F}_x(\pi)$ and $\mathfrak{F}(\pi) := \bigcup\{\mathfrak{F}_x(\pi) \mid x \in X\}$ (or simply \mathfrak{F}).

The trajectory $\gamma \in \mathfrak{F}(\pi)$ defined on \mathbb{S} is called a full (entire) trajectory of the dynamical system (X, \mathbb{T}, π) .

Denote by $\Phi(\pi)$ the set of all full trajectories of the dynamical system (X, \mathbb{T}, π) and $\Phi_x(\pi) := \mathfrak{F}_x(\pi) \cap \Phi(\pi)$.

A system (X, \mathbb{T}, π) is called [6],[8] compactly dissipative, if there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t \rightarrow +\infty} \beta(\pi^t M, K) = 0;$$

for all $M \in C(X)$.

Let (X, \mathbb{T}, π) be compactly dissipative and K be a compact set attracting every compact subset of X . Let us set

$$(2) \quad J := \omega(K) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K}.$$

It can be shown [6],[8] that the set J defined by equality (2) does not depend on the choice of the attractor K , but it is characterized only by the properties of the dynamical system (X, \mathbb{T}, π) itself. The set J is called Levinson center of the compact dissipative dynamical system (X, \mathbb{T}, π) .

2.2. Discrete inclusions, ensemble of dynamical systems (collages) and cocycles. Let W be a topological space. Denote by $C(W, W)$ the space of all continuous operators $f : W \rightarrow W$ equipped with the compact-open topology. Consider a set of operators $\mathcal{M} \subseteq C(W, W)$ and, respectively, an ensemble (collage) of discrete dynamical systems $(W, f)_{f \in \mathcal{M}}$ ((W, f) is a discrete dynamical system generated by positive powers of map f).

A discrete inclusion $DI(\mathcal{M})$ is called (see, for example, [3, 11]) a set of all sequences $\{x_j\} \subset W$ ($j \in \mathbb{Z}_+$) such that

$$(3) \quad x_j = f_{i_j} x_{j-1}$$

for some $f_{i_j} \in \mathcal{M}$ (trajectory of $DI(\mathcal{M})$), i.e.

$$x_j = f_{i_j} f_{i_{j-1}} \dots f_{i_1} x_0 \text{ all } f_{i_k} \in \mathcal{M}.$$

A bilateral sequence $\{x_j\} \subset W$ ($j \in \mathbb{Z}$) is called a full trajectory of $DI(\mathcal{M})$ (entire trajectory or trajectory on \mathbb{Z}), if $x_{n+j} = f_{i_j} x_{n+j-1}$ for all $n \in \mathbb{Z}$.

Let us consider the set-valued function $\mathcal{F} : W \rightarrow C(W)$ defined by the equality $\mathcal{F}(x) := \{f(x) \mid f \in \mathcal{M}\}$. Then the discrete inclusion $DI(\mathcal{M})$ is equivalent to the difference inclusion

$$(4) \quad x_j \in \mathcal{F}(x_{j-1}).$$

Denote by \mathfrak{F}_{x_0} the set of all trajectories of discrete inclusion (4) (or $DI(\mathcal{M})$) issuing from the point $x_0 \in W$ and $\mathfrak{F} := \bigcup \{\mathfrak{F}_{x_0} \mid x_0 \in W\}$.

Below we will give a new approach concerning the study of discrete inclusions $DI(\mathcal{M})$ (or difference inclusion (4)). Denote by $C(\mathbb{Z}_+, W)$ the space of all continuous mappings $f : \mathbb{Z}_+ \rightarrow W$ equipped with the compact-open topology. Denote by $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [15, 16]) on $C(\mathbb{Z}_+, W)$, i.e. $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (i.e. $f_k(n) := f(n+k)$ for all $n \in \mathbb{Z}_+$).

We may now rewrite equation (3) in the following way:

$$(5) \quad x_{j+1} = \omega(j)x_j, \quad (\omega \in \Omega := C(\mathbb{Z}_+, \mathcal{M}))$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(j) := f_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$. We denote by $\varphi(n, x_0, \omega)$ the solution of equation (5) issuing from the point $x_0 \in E$ at the initial moment $n = 0$. Note that $\mathfrak{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathfrak{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in W, \omega \in \Omega\}$, i.e., $DI(\mathcal{M})$ (or inclusion (4)) is equivalent to the family of non-autonomous equations (5) ($\omega \in \Omega$).

From the general properties of difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \rightarrow W$ satisfies the following conditions:

- (i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in W \times \Omega$;
- (ii) $\varphi(n + \tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in W \times \Omega$;
- (iii) the mapping φ is continuous;
- (iv) for any $n, \tau \in \mathbb{Z}_+$ and $\omega_1, \omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that

$$(6) \quad U(n, \omega_2)U(\tau, \omega_1) = U(n + \tau, \omega_3),$$

where $\omega \in \Omega$, $U(n, \omega) := \varphi(n, \cdot, \omega) = \prod_{k=0}^{n-1} \omega(k)$, $\omega(k) := f_{i_k}$ ($k = 0, 1, \dots, n$) and $f_{i_0} := Id_W$.

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of group \mathbb{S} , X, Y be two metric (or topological) spaces and (X, \mathbb{T}_1, π) (respectively $(Y, \mathbb{T}_2, \sigma)$) be a semigroup dynamical system on X (respectively on Y). A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called a non-autonomous dynamical system, where $h : X \rightarrow Y$ is a homomorphism from (X, \mathbb{T}_1, π) onto $(Y, \mathbb{T}_2, \sigma)$, i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $x \in X$ and $t \in \mathbb{T}_1$.

Let W, Ω be two topological spaces and $(\Omega, \mathbb{T}_2, \sigma)$ be a semi-group dynamical system on Ω .

Recall [15] that a triplet $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma) \rangle$ (or briefly φ) is called a cocycle over $(\Omega, \mathbb{T}_2, \sigma)$ with the fiber W , if φ is a mapping from $\mathbb{T}_1 \times W \times \Omega$ to W satisfying the following conditions:

1. $\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;
2. $\varphi(t + \tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{T}_1$ and $(x, \omega) \in W \times \Omega$;
3. the mapping φ is continuous.

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T}_1 \rightarrow X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e., $\pi = (\varphi, \sigma)$). Then it is easy to check that (X, \mathbb{T}_1, π) is a dynamical system on X , which is called a skew-product dynamical system [1], [15]; but $h = pr_2 : X \rightarrow \Omega$ is a homomorphism of (X, \mathbb{T}_1, π) onto $(\Omega, \mathbb{T}_2, \sigma)$ and hence $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma) \rangle$ over the dynamical system $(\Omega, \mathbb{T}_2, \sigma)$ with the fiber W , then there can be constructed a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h \rangle$ ($X := W \times \Omega$), which we will call a non-autonomous dynamical system generated (associated) by cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma) \rangle$ over $(\Omega, \mathbb{T}_2, \sigma)$.

From the presented above it follows that every $DI(\mathcal{M})$ (respectively, inclusion (4)) in a natural way generates a cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is the solution of equation (5) issuing from the point $x \in W$ at the initial moment $n = 0$. Thus, we can study inclusion (4) (respectively, $DI(\mathcal{M})$) in the framework of the theory of cocycles with discrete time.

Below we need the following result.

Theorem 2.1. [9] *Let \mathcal{M} be a compact subset of $C(W, W)$ and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. Then*

- (i) $\Omega = \overline{Per(\sigma)}$, where $Per(\sigma)$ is the set of all periodic points of $(\Omega, \mathbb{Z}_+, \sigma)$ (i.e. $\omega \in Per(\sigma)$, if there exists $\tau \in \mathbb{N}$ such that $\sigma(\tau, \omega) = \omega$);
- (ii) the set Ω is compact;
- (iii) Ω is invariant, i.e., $\sigma^t \Omega = \Omega$ for all $t \in \mathbb{Z}_+$;
- (iv) φ satisfies the condition (6).

3. CHAOTIC ATTRACTORS OF DISCRETE CONTROL SYSTEMS

In Section 3 we give the conditions of existence of chaotic attractor for discrete control systems.

Denote by \mathfrak{A} the set of all mapping $\psi : \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties:

- (G1) ψ is continuous;
- (G2) there exists a positive number t_0 such that:
 - (a) $\psi(t_0, r) < r$ for all $r > 0$;
 - (b) the mapping $\psi(t_0, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotone increasing.
- (G3) $\psi(t + \tau, r) \leq \psi(t, \psi(\tau, r))$ for all $t, \tau \in \mathbb{Z}_+$ and $r \in \mathbb{R}_+$.

Remark 3.1. 1. *Note that the functions $\psi(t, r) = \mathcal{N}q^t r$ ($\mathcal{N} > 0$ and $q \in (0, 1)$) and $\psi(t, r) = \frac{r}{1+rt}$ belong to \mathfrak{A} , where $(t, r) \in \mathbb{Z}_+ \times \mathbb{R}_+$.*

2. *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function satisfying the conditions:*

- (i) $f(r) < r$ for all $r > 0$;
- (ii) f is monotone increasing.

Then the mapping $\psi : \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by equality

$$\psi(t, r) = x(t)$$

for all $(t, r) \in \mathbb{Z}_+ \times \mathbb{R}_+$, where $x(t)$ is a unique solution of difference equation $x_{t+1} = f(x_t)$ with initial data $x_0 = r$, belongs to \mathfrak{A} .

Let $\psi \in \mathfrak{A}$. A set \mathcal{M} of operators from $C(W, W)$ is said to be ψ -contracting, if

$$\rho(f_{i_t} f_{i_{t-1}} \dots f_{i_1}(x_1), f_{i_t} f_{i_{t-1}} \dots f_{i_1}(x_2)) \leq \psi(t, \rho(x_1, x_2))$$

for all $x_1, x_2 \in W$ and $t \in \mathbb{N}$, where $f_{i_1}, f_{i_2}, \dots, f_{i_t} \in C(W)$ and $i_1, i_2, \dots, i_t \in \mathbb{N}$.

The set $S \subset W$ is

- (i) nowhere dense, provided the interior of the closure of S is empty set, $\text{int}(\text{cl}(S)) = \emptyset$;
- (ii) totally disconnected, provided the connected components are single points;
- (iii) perfect, provided it is closed and every point $p \in S$ is the limit of points $q_n \in S$ with $q_n \neq p$.

The set $S \subset W$ is called a Cantor set, provided it is totally disconnected, perfect and compact.

The subset M of (X, \mathbb{T}, π) is called (see, for example, [14]) chaotic, if the following conditions hold:

- (i) the set M is transitive, i.e. there exists a point $x_0 \in X$ such that $M = \overline{H(x_0)} := \overline{\{\pi(t, x_0) : t \in \mathbb{T}\}}$;
- (ii) $M = \text{Per}(\pi)$, where $\text{Per}(\pi)$ is the set of all periodic points of (X, \mathbb{T}, π) .

Recall that a point $x \in X$ of the dynamical system (X, \mathbb{T}, π) is called Poisson's stable, if x belongs to its ω -limit set $\omega_x := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, x)}$.

Theorem 3.2. *Suppose that the following conditions are fulfilled:*

- a. \mathcal{M} is a finite subset of $C(W, W)$, i.e., $\mathcal{M} := \{f_1, f_2, \dots, f_m\}$ and $m \geq 2$;
- b. \mathcal{M} is ψ -contracting for some $\psi \in \mathfrak{A}$.

Then the following statement hold:

- (i) the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ ($\Omega := C(\mathbb{Z}_+, \mathcal{M})$) generated by $DI(\mathcal{M})$ is compactly dissipative;
- (ii) the skew-product dynamical system (X, \mathbb{Z}_+, π) generated by $DI(\mathcal{M})$ is compactly dissipative;
- (iii) $I = \overline{\text{Per}(\varphi)}$, where $\text{Per}(\varphi) := \{u \in W : \exists \tau \in \mathbb{N} \text{ and } \omega \in \Omega \text{ such that } \sigma(\tau, \omega) = \omega \text{ and } \varphi(\tau, u, \omega) = u\}$;
- (iv) if every map $f \in \mathcal{M}$ is invertible, then
 1. Levinson's center J of the skew-product dynamical system (X, \mathbb{Z}_+, π) is a chaotic Cantor set;
 2. there exists a residual subset $J_0 \subseteq J$ (large in the sense of Baire category), consisting from Poisson's stable points, such that the positive semi-trajectory of every point $x_0 \in J_0$ is dense on J ;
 3. $I = \text{pr}_1(J)$ ($\text{pr}_1 : X \rightarrow \Omega$, where I is the Levinson's center of cocycle φ and $X := W \times \Omega$), i.e., I is a continuous image of the Cantor set J .

Proof. Let $Y = \Omega := C(\mathbb{Z}_+, Q)$ and $(Y, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system of shifts on Y . Then Y is compact. By Theorem 2.1, $\overline{\text{Per}(\sigma)} = \Omega$ and Ω is compact and invariant.

Let $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$ (i.e. $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=0}^{n-1} \omega(k)$ ($\omega \in \Omega$)), (X, \mathbb{Z}_+, π) be a skew-product system associated by the cocycle φ (i.e., $X := W \times \Omega$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}_+, \pi)$,

$(Y, \mathbb{Z}_+, \sigma), h)$ ($h := pr_2 : X \rightarrow Y$) be a non-autonomous dynamical system generated by cocycle φ . Under the conditions of Theorem 2.1 we have

$$\rho(\varphi(n, u_1, \omega), \varphi(n, u_2, \omega)) \leq \psi(n, \rho(u_1, u_2))$$

for all $n \in \mathbb{Z}_+$, $u_1, u_2 \in W$ and $\omega \in \Omega$. Now to finish the proof of the theorem it is sufficient to apply Theorem 6.1.3 [8, Ch.VI] and Theorem 2.1 to the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h \rangle$ and denote by $I_\omega := pr_1(J_\omega)$, where J is Levinson center of the dynamical system (X, \mathbb{Z}_+, π) , $J_\omega := J \cap h^{-1}(\omega)$ and $h := pr_2$. \square

Let $\mathcal{M} \subset C(W)$, $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \pi) \rangle$ (respectively (X, \mathbb{Z}_+, π)) be a cocycle (a skew-product dynamical system) generated by $DI(\mathcal{M})$ and let I (J) be Levinson center of the cocycle φ (respectively, skew-product dynamical system (X, \mathbb{Z}_+, π)).

The set I is said to be a chaotic attractor of $DI(\mathcal{M})$, if

- (i) the set J is chaotic, i.e. J is transitive and $J = \overline{Per(\sigma)}$, where J is the Levinson center of the skew-product dynamical system (X, \mathbb{Z}_+, π) generated by $DI(\mathcal{M})$;
- (ii) $I = pr_1(J)$.

Remark 3.3. 1. Theorem 3.2 it was proved in [9] for the special case, when $\psi(t, r) = \mathcal{N}q^t$ ($(t, r) \in \mathbb{Z}_+ \times \mathbb{R}_+$, $\mathcal{N} > 0$ and $q \in (0, 1)$).

2. The problem of the existence of compact global attractors for $DI(\mathcal{M})$ with finite \mathcal{M} (collage or iterated function system (IFS)) was studied before in works [2, 3, 4] (see also the bibliography therein). In [2, 3, 4] the statement close to Theorem 3.2 was proved. Namely:

- (i) in [2] it was announced the first and proved the second statement of Theorem 3.2, if $\psi(t, r) = q^t r$ ($t \in \mathbb{Z}_+$ and $q \in (0, 1)$);
- (ii) in [3, 4] they considered the case when W is a compact metric space and every map $f \in \mathcal{M} = \{f_1, f_2, \dots, f_m\}$ ($i = 1, \dots, m$) is contracting (not obligatory invertible). For this type of $DI(\mathcal{M})$ it was proved the existence of a compact global attractor \mathcal{A} such that for all $u \in \mathcal{A}$ and almost all $\omega \in \Omega$ (with respect to certain measure on Ω) the trajectory $\varphi(n, u, \omega) = U(n, \omega)u$ ($U(n, \omega) := \prod_{k=0}^{n-1} f_{i_k}$, ($i_k \in \{1, \dots, m\}$) and $f_{i_0} := Id_W$) was dense in \mathcal{A} .

4. COMPACT GLOBAL CHAOTIC ATTRACTORS OF DISCRETE CONTROL SYSTEMS: GENERAL CASE

In Section 3 it was given (Theorem 3.2) a description the structure of the attractor J (respectively, I) of $DI(\mathcal{M})$. The problem of description of the structure of the attractor I of $DI(\mathcal{M})$ in general case (when the maps $f \in \mathcal{M}$ are not invertible) is more complicated. We study this problem in this section.

Theorem 4.1. Let (W, ρ) be a complete metric space and $f : W \mapsto W$ be a ψ -contraction, i.e., $\rho(f^t(x_1), f^t(x_2)) \leq \psi(t, \rho(x_1, x_2))$ for all $t \in \mathbb{Z}_+$ and $x_1, x_2 \in W$, where $f^t := f^{t-1} \circ f$ for all $t \in \mathbb{N}$. Then the following statements hold:

- (i) there exists a unique fix point $p \in W$ of f , i.e., $f(p) = p$;

(ii) the point p is uniformly attracting, i.e.,

$$(7) \quad \lim_{t \rightarrow +\infty} \rho(f^t(x), p) = 0$$

for all $x \in W$ and (7) takes place uniformly with respect to x on every bounded subset from W .

Proof. This statement is a particular case of Theorem 6.1.3 from [8] (Chapter VI, page 177). \square

Let (W, ρ) be a complete metric space and \mathcal{M} be a compact subset from $C(W)$. Denote by \mathcal{F} the set-valued mapping defined by the equality

$$\mathcal{F}(u) := \{f(u) : f \in \mathcal{M}, u \in W\}.$$

Let $\psi \in \mathfrak{A}$. A set-valued mapping $F : X \rightarrow C(X)$ is called ψ -contracting, if

$$\alpha(F^t(x_1), F^t(x_2)) \leq \psi(t, \rho(x_1, x_2))$$

for all $x_1, x_2 \in X$ and $t \in \mathbb{Z}_+$, where $\alpha : C(X) \times C(X) \mapsto \mathbb{R}_+$ is the Hausdorff distance and $F^t := F^{t-1} \circ F$ for all $t \in \mathbb{N}$.

Theorem 4.2. *Let $F : X \rightarrow C(X)$ be a set-valued ψ -contracting mapping. Then the discrete dynamical system (X, F) , generated by positive powers of F , is compactly dissipative.*

Proof. Let $\psi \in \mathfrak{A}$ and $F : X \rightarrow C(X)$ be ψ -contracting. By the set-valued mapping $F : X \rightarrow C(X)$ we define a single-valued mapping $\tilde{F} : C(X) \rightarrow C(X)$ by the formula $\tilde{F}(A) := F(A)$, where $F(A) := \bigcup\{F(x) \mid x \in A\}$ for all $A \in C(X)$. Then

$$\alpha(\tilde{F}^t(A), \tilde{F}^t(B)) \leq \psi(t, \alpha(A, B))$$

for all $A, B \in C(X)$, i.e., $\tilde{F} : (C(X), \alpha) \rightarrow (C(X), \alpha)$ is a single-valued ψ -contracting mapping from the complete metric space $(C(X), \alpha)$ into itself. By Theorem 4.1 the mapping \tilde{F} has a single stationary point K (i.e., $\tilde{F}(K) = K$) such that

$$(8) \quad \lim_{t \rightarrow +\infty} \alpha(\tilde{F}^t(A), K) = 0$$

for all $A \in C(X)$. Note that $K = F(K)$, i.e., $K \subseteq X$ is a compact invariant set of dynamical system (X, F) and by (8) we get

$$(9) \quad \lim_{t \rightarrow +\infty} \sup_{x \in A} \alpha(\tilde{F}^t(x), K) = 0$$

for all $A \in C(X)$. Therefore, (X, F) is compactly dissipative and from (9) and the invariance of K it follows that the set K is Levinson center of (X, F) . \square

Lemma 4.3. [8, Ch.II] *Let \mathfrak{M} be some family of bounded subsets from X . A set-valued dynamical system (X, \mathbb{T}, π) is \mathfrak{M} dissipative, if and only if there exists $t_0 > 0$ such that would be \mathfrak{M} dissipative the cascade (X, P) , where $P(x) := \pi(t_0, x)$ ($x \in X$).*

Theorem 4.4. *Let (X, \mathbb{T}, π) be a set-valued dynamical system and let exist $\psi \in \mathfrak{A}$ such that*

$$(10) \quad \alpha(\pi(t, x_1), \pi(t, x_2)) \leq \psi(t, \rho(x_1, x_2))$$

for all $x_1, x_2 \in X$ and $t \in \mathbb{T}$. Then (X, \mathbb{T}, π) is compactly dissipative.

Proof. Let (X, \mathbb{T}, π) be a set-valued dynamical system, $\psi \in \mathfrak{A}$ and the condition (10) be fulfilled. Let us choose $t_0 > 0$ ($t_0 \in \mathbb{T}$) so that $\psi(t_0, r) < r$ for all $r > 0$ and $\psi(t_0, \cdot)$ be monotone increasing, and consider the cascade (X, P) generated by positive powers of the set-valued mapping $P(x) := \pi(t_0, x)$ ($x \in X$). Denote by $\tilde{\psi} : \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ the mapping defined by equality $\tilde{\psi}(t, r) := \psi(t_0 t, r)$ for all $(t, r) \in \mathbb{Z}_+ \times \mathbb{R}_+$. It easy to check that $\tilde{\psi} \in \mathfrak{A}$ and that the mapping P is a $\tilde{\psi}$ -contraction and by Theorem 4.2 the dynamical system (X, P) is compactly dissipative. From Lemma 4.3 the compact dissipativity of the dynamical system (X, \mathbb{Z}_+, π) follows. The theorem is proved. \square

Theorem 4.5. *Suppose the following conditions are fulfilled:*

- (i) $\mathcal{M} := \{f_i : i \in I\}$ is a compact subset from $C(W, W)$;
- (ii) the set \mathcal{M} of operators is ψ -contracting for some $\psi \in \mathfrak{A}$.

Then the following statements hold:

- (i) $\alpha(\mathcal{F}^t(A), \mathcal{F}^t(B)) \leq \psi(t, \alpha(A, B))$ for all $A, B \in C(W)$ and $t \in \mathbb{Z}_+$;
- (ii) the set-valued cascade (W, \mathcal{F}) is compactly dissipative;
- (iii) $\lim_{t \rightarrow +\infty} \alpha(\mathcal{F}^t(M), I) = 0$ for all $M \in C(W)$, where I is the Levinson center of set-valued dynamical system (W, \mathcal{F}) .

Proof. It is easy to check that

$$F^n = \{f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1} : i_k \in I (k = 1, 2, \dots, n)\} = \{U(n, \omega) : \omega \in \Omega\},$$

where $\Omega := C(\mathbb{Z}_+, \mathcal{M})$, $U(n, \omega) := \prod_{k=0}^{n-1} f_{i_k} = f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1} \circ f_{i_0}$, $\omega(k) := f_{i_k}$ ($k = 1, 2, \dots, n$) and $f_{i_0} := Id_W$.

Let $\psi \in \mathfrak{A}$ from the definition of ψ -contraction of the family \mathcal{M} . We will prove that

$$(11) \quad \alpha(F^t(A), F^t(B)) \leq \psi(t, \alpha(A, B))$$

for all $A, B \in C(W)$. Indeed, let $v \in F^t(B)$. Since $F^t(B) = U(t, \Omega)(B)$, then there exist $x_2 \in B$ and $\omega \in \Omega$ such that $v = U(t, \omega)x_2$. We choose a point $x_1 \in A$ such that $\rho(x_1, x_2) \leq \alpha(A, B)$. Then we have

$$\rho(U(t, \omega)x_1, v) = \rho(U(t, \omega)x_1, U(t, \omega)x_2) \leq \psi(t, \rho(x_2, x_1)) \leq \psi(t, \alpha(A, B)).$$

Thus, for an arbitrary point $v \in F^t(B)$ there is a point $u := U(t, \omega)x_1 \in F^t(A)$ such that

$$\rho(u, v) \leq \psi(t, \alpha(A, B))$$

and, hence,

$$(12) \quad \beta(F^t(A), F^t(B)) \leq \psi(t, \alpha(A, B)).$$

Similarly we have the inequality

$$(13) \quad \beta(F^t(B), F^t(A)) \leq \psi(t, \alpha(A, B)).$$

Inequality (11) follows from inequalities (12) and (13). Now to finish the proof of theorem it is enough to cite Theorem 4.4. \square

Lemma 4.6. *Let \mathcal{M} be a compact subset of $C(W)$, \mathcal{M} be ψ -contracting and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. For each $\omega \in \Omega$, $n \in \mathbb{N}$, and $x \in W$, define*

$$\phi(\omega, n, x) := \omega(1) \circ \omega(2) \circ \dots \circ \omega(n)x.$$

Let K denote a nonempty compact subset from W . Then there exists a real number $C \geq 0$ such that

$$\rho(\phi(\omega, m, x_1), \phi(\omega, n, x_2)) \leq \psi(m \wedge n, C)$$

for all $\omega \in \Omega$, all $m, n \in \mathbb{Z}_+$, and all $x_1, x_2 \in K$, where $m \wedge n := \min(n, m)$.

Proof. Let ω, m, n, x_1 , and x_2 be as stated in the lemma. By Theorem 4.5 the set-valued cascade (W, \mathcal{F}) is compactly dissipative, where $\mathcal{F}(x) := \{f(x) : f \in \mathcal{M}\}$. Then the set $\tilde{K} := \overline{\cup_{n=0}^{\infty} \mathcal{F}^n(K)}$ is compact and positively invariant. Without any loss of generality we can suppose that $m < n$. Then observe that

$$\phi(\omega, n, x_2) = \phi(\omega, m, \phi(\sigma(m, \omega), n - m, x_2)),$$

where $\sigma(m, \omega) = \{\omega(m + i)\}_{i \in \mathbb{Z}_+} \in \Omega$. Let $x_3 := \phi(\sigma(m, \omega), n - m, x_2)$. Then x_3 belongs to \tilde{K} . Hence we can write

$$\begin{aligned} \rho(\phi(\omega, m, x_1), \phi(\omega, n, x_2)) &= \rho(\phi(\omega, m, x_1), \phi(\omega, m, x_3)) \leq \\ &\psi(m, \rho(x_1, x_3)) \leq \psi(m, C), \end{aligned}$$

where $C := \max\{\rho(x_1, x_3) : x_1, x_3 \in \tilde{K}\}$. C is finite because \tilde{K} is a compact subset. \square

Theorem 4.7. *Let (W, ρ) be a complete metric space. Let \mathcal{M} be a compact subset of $C(W, W)$, \mathcal{M} be ψ -contracting and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. For each $\omega \in \Omega$, $n \in \mathbb{N}$, and $x \in W$ define*

$$\phi(\omega, n, x) := \omega(1) \circ \omega(2) \circ \dots \circ \omega(n)x.$$

Then

$$\phi(\omega) := \lim_{n \rightarrow +\infty} \phi(\omega, n, x)$$

exists, belong to I (I is Levinson center of set-valued cascade (W, \mathcal{F})), and it is independent of $x \in W$. If K is a compact subset of W then the convergence is uniform over $(\omega, x) \in \Omega \times K$. The function $\phi : \Omega \rightarrow I$ thus provided is continuous and onto.

Proof. Let $x \in W$. Let $K \in C(W)$ be such that $x \in K$. Construct \tilde{K} as in Lemma 4.6. Define $\mathcal{F} : C(W) \rightarrow C(W)$ by equality $\mathcal{F}(B) := \{\mathcal{F}(x) : x \in B\}$. Under the conditions of Theorem 4.7 by Theorem 4.5 we have

$$\alpha(\mathcal{F}^t(B), \mathcal{F}^t(A)) \leq \psi(t, \alpha(A, B))$$

for all $A, B \in C(W)$ and $n \in \mathbb{Z}_+$. Thus the mapping $\mathcal{F} : C(W) \rightarrow C(W)$ is a ψ -contraction on the metric space $(C(W), \alpha)$; and we have

$$I = \lim_{n \rightarrow +\infty} \mathcal{F}^n(K),$$

where $\mathcal{F}^t := \mathcal{F}^{t-1} \circ \mathcal{F}$ ($t \geq 2$). In particular $\{\mathcal{F}^t(K)\}$ is a Cauchy sequence in $(C(W), \alpha)$. Notice that $\phi(\omega, n, x) \in \mathcal{F}^n(K)$. It follows from Theorem 4.5, that if $\lim_{n \rightarrow +\infty} \phi(\omega, n, x)$ exists, then it belongs to I .

That the later limit does exist follows from the fact that, for fixed $\omega \in \Omega$ and $x \in W$, $\{\phi(\omega, n, x)\}_{n \in \mathbb{Z}_+}$ is a Cauchy sequence: by Lemma 4.6

$$\rho(\phi(\omega, m, x), \phi(\omega, n, x)) \leq \psi(m \wedge n, C)$$

for all $x \in K$ and $\omega \in \Omega$. Notice that the right hand site here tends to zero as m and n tend to infinity. The uniformity (with respect to $x \in K$ and $\omega \in \Omega$) of the convergence follows from the fact that the constant C is independent of $(\omega, x) \in \Omega \times K$. Since the map $\phi(\cdot, n, \cdot) : \Omega \times W \rightarrow W$ ($n \in \mathbb{Z}_+$) is continuous then $\phi : \Omega \rightarrow W$ is continuous too.

Finally, we prove that ϕ is onto. Let $a \in I$. Then, since by Theorem 4.5 $I = \lim_{n \rightarrow +\infty} \mathcal{F}^n(\{x\})$ it follows that there exists a sequence $\{\omega_n\} \subseteq \Omega$ such that

$$\lim_{n \rightarrow \infty} \phi(\omega_n, n, x) = a.$$

Since Ω is compact, it follows that $\{\omega_n\}$ possesses a convergent subsequence with limit $\omega \in \Omega$. Without loss of generality assume $\lim_{n \rightarrow \infty} \omega_n = \omega$. Then the number of successive initial agreements between the components of ω_n and ω increases without limit. That is, if $l_n := \text{number of elements in } \{j \in \mathbb{Z}_+ : \omega_n(k) = \omega(k) \text{ for } 1 \leq k \leq j\}$, then $l_n \rightarrow +\infty$ as $n \rightarrow +\infty$. It follows that

$$\rho(\phi(\omega, n, x), \phi(\omega_n, n, x)) \leq \psi(l_n, C).$$

By taking the limit on both sides as $n \rightarrow +\infty$ we find $\rho(\phi(\omega), a) = 0$ which implies $\phi(\omega) = a$. Hence $\phi : \Omega \rightarrow W$ is onto. \square

Let \mathcal{M} be a compact subset of $C(W, W)$ and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. A point $a \in W$ is called m -periodic ($m \in \mathbb{N}$) point of $DI(\mathcal{M})$ if there exists an m -periodic point $\omega \in \Omega$ such that $\varphi(m, a, \omega) = a$.

Lemma 4.8. *The following statement hold:*

- (i) $\phi(\omega, n, \phi(\sigma(n, \omega))) = \phi(\omega)$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$;
- (ii) Let $\omega \in \Omega$ be an m -periodic point of dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$, i.e., $\omega(k+m) = \omega(k)$ for all $k \in \mathbb{Z}_+$. Then $\varphi(m, a, \tilde{\omega}) = a$, where $a := \phi(\omega)$ and $\tilde{\omega}$ is an m -periodic point of dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ defined by condition $\tilde{\omega}_k := \omega_{m-k}$ for all $k = 0, 1, \dots, m-1$.

Proof. Notice that

$$\phi(\omega, n, \phi(\sigma(n, \omega))) = \prod_{k=0}^n \omega(k) \left(\lim_{s \rightarrow +\infty} \prod_{k=0}^s \omega(k+n)x \right) = \lim_{s \rightarrow +\infty} \prod_{k=0}^{s+n} \omega(k)x = \phi(\omega),$$

for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$, where $\omega(0) := Id_W$.

To prove the second statement we note that $\phi(\sigma(m, \omega)) = \phi(\omega)$ and $\varphi(m, a, \tilde{\omega}) = a$, if the point ω is m -periodic (i.e., $\sigma(m, \omega) = \omega$). \square

Theorem 4.9. *Let \mathcal{M} be a compact subset of $C(W, W)$, \mathcal{M} be ψ -contracting and $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$. Then the attractor I of $DI(\mathcal{M})$ is the closure of its periodic points.*

Proof. The space Ω is the closure of the set of periodic points. Lift this statement to I using the map $\phi : \Omega \rightarrow I$. From Lemma 4.8 it follows that if $\omega \in \Omega$ is m -periodic point of the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ then the point $\phi(\omega)$ will be m -periodic point of $DI(\mathcal{M})$. Now it is sufficient to note that if $S \subset \Omega$ is such that its closure equals Ω , then the closure of $\phi(S)$ equals I . \square

A point $x \in X$ of dynamical system (X, \mathbb{T}, π) is said to be almost recurrent (respectively, almost periodic), if for arbitrary $\varepsilon > 0$ there exists a positive number $l = l(\varepsilon)$ such that on every segment $[a, a + l] \subseteq \mathbb{T}$ there exists at least one number $\tau \in [a, a + l]$ such that $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called recurrent if it is almost recurrent and its trajectory is relatively compact.

Theorem 4.10. [8, Ch.VI] *Let $\omega \in \Omega$ be a stationary (τ -periodic, recurrent, Poisson stable) point and $\lim_{t \rightarrow +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ for all $x \in X_\omega$. Then there exists a unique stationary (τ -periodic, recurrent, Poisson stable) point $x_\omega \in X_\omega$ such that*

$$\lim_{t \rightarrow +\infty} \rho(\pi(t, x), \pi(t, x_\omega)) = 0$$

for all $x \in X_\omega$.

A point $x \in X$ is called almost periodic if for arbitrary $\varepsilon > 0$ there exists a positive number $l(\varepsilon)$ such that $[a, a + l] \cap \mathcal{T}(\varepsilon, x) \neq \emptyset$ for all $a \in \mathbb{T}_1$, where $[a, a + l] := \{t \in \mathbb{T}_1 : a \leq t \leq a + l\}$ and $\mathcal{T}(\varepsilon, x) := \{\tau \in \mathbb{T}_1 : \rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon \text{ for all } t \in \mathbb{T}_1\}$.

Theorem 4.11. [8, Ch.VI] *Let $\omega \in \Omega$ be a stationary (τ -periodic, almost periodic, recurrent, Poisson stable) point and $\lim_{t \rightarrow +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ for all $x \in X$ such that $h(x_1) = h(x_2)$. Then there exists a unique stationary (τ -periodic, almost periodic, recurrent, Poisson stable) point $x_\omega \in X_\omega$ such that*

$$\lim_{t \rightarrow +\infty} \rho(\pi(t, x), \pi(t, x_\omega)) = 0$$

for all $x \in X_\omega$.

Theorem 4.12. *Suppose that the following conditions hold:*

- (i) \mathcal{M} is a compact subset of $C(W, W)$ and $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ is the cocycle generated by $DI(\mathcal{M})$;
- (ii) \mathcal{M} is ψ -contracting, where $\psi \in \mathfrak{A}$;
- (iii) $\omega \in \Omega$ is a stationary (τ -periodic, almost periodic, recurrent, Poisson stable) point of $(\Omega, \mathbb{Z}_+, \sigma)$.

Then the equation

$$x_{n+1} = \omega(n)x_n$$

admits a unique stationary (τ -periodic, almost periodic, recurrent, Poisson stable) solution $\varphi(n, x_\omega, \omega)$ such that

$$|\varphi(t, x, \omega) - \varphi(t, x_\omega, \omega)| \leq \psi(t, |x - x_\omega|)$$

for all $t \in \mathbb{Z}_+$ and $x \in W$.

Proof. Let $\Omega := C(\mathbb{Z}_+, Q)$ and $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system of shifts on Ω . Then Ω is compact. By Theorem 2.1, $\overline{Per(\sigma)} = \Omega$ and Ω is compact and invariant.

Let $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle generated by $DI(\mathcal{M})$ (i.e., $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=0}^{n-1} \omega(k)$ ($\omega \in \Omega$)), (X, \mathbb{Z}_+, π) be a skew-product system associated by the cocycle φ (i.e., $X := W \times \Omega$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$ ($h := pr_2 : X \rightarrow Y$) be a non-autonomous dynamical system generated by the cocycle φ . By Theorem 3.2 the dynamical system (X, \mathbb{Z}_+, π) is compactly dissipative. Denote by J its Levinson center and (J, \mathbb{Z}_+, π) the dynamical system on J induced by (X, \mathbb{Z}_+, π) . Under the conditions of Theorem we have

$$\rho(\pi(t, x_1, \omega), \pi(t, x_2, \omega)) \leq \psi(t, \rho(x_1, x_2))$$

for all $t \in \mathbb{Z}_+$, $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$). Now to finish the proof of the theorem it is sufficient to apply Theorems 4.10 and 4.11 to non-autonomous dynamical system $\langle (J, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h \rangle$. \square

5. SOME APPLICATIONS

Consider a control dynamical system governed by the differential equation

$$(14) \quad x' = f(x; u) \quad (x \in \mathcal{E}, u \in U \subset \mathfrak{B}),$$

where \mathcal{E} and \mathfrak{B} are some Banach spaces.

Let $S(\mathbb{R}_+, \mathcal{P})$ denote the set of the piecewise constant functions $u(t)$ defined on \mathbb{R}_+ that assume values of the set $\mathcal{P} := \{c_1, c_2, \dots, c_m\} \subseteq \mathfrak{B}$ and are continuous on $\mathbb{R}_+ \setminus \mathbb{Z}_+$. The functions $u(t)$ in the class $S(\mathbb{R}_+, \mathcal{P})$ are open-loop controls of system (14). Consider the set of control systems (14) with the open-loop control of the class $S(\mathbb{R}_+, \mathcal{P})$. These systems constitute a continual set. Particularly important among all systems of this set are m systems

$$x' = f(x; c_i) \quad (x \in \mathcal{E}, i = 1, 2, \dots, m).$$

Below we will consider some examples of this type.

5.1. Monotone ODEs.

Lemma 5.1. [10] *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a function satisfying the following conditions:*

- (H1) $\theta(0) = 0$;
- (H2) $\theta(t) > 0$ for all $t > 0$;
- (H3) θ is locally Lipschitz;
- (H4) f satisfy the condition of Osgud, i.e., $\int_0^\varepsilon \frac{dt}{\theta(t)} = +\infty$ for all $\varepsilon > 0$.

Then the equation

$$(15) \quad u' = -\theta(u)$$

admits a unique solution $\psi(t, r)$ with initial condition $\psi(0, r) = r$ and the mapping $\omega : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ possesses the following properties:

- (i) the mapping $\psi : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is continuous;
- (ii) $\psi(t, r) < r$ for all $r > 0$ and $t > 0$;
- (iii) for all $t \in \mathbb{R}_+$ the mapping $\psi(t, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is increasing;
- (iv) $\psi(0, t) = 0$ for all $t \in \mathbb{R}_+$;
- (v) $\lim_{t \rightarrow +\infty} \sup_{0 \leq r \leq r_0} \psi(t, r) = 0$ for all $r_0 > 0$.

Remark 5.2. 1. Lemma 5.1 gives us an algorithm of construction of function with properties (G1)-(G3). For example, the function $\omega(t, r) := \ln \frac{e^{t+r}}{e^{t+r} + 1 - e^r}$ ($t, r \in \mathbb{R}_+$) possesses the mentioned above properties, because it is a unique solution of differential equation (15) ($\theta(u) = e^u - 1$ for all $u \in \mathbb{R}_+$) with initial condition $\theta(0, r) = r$.

2. The function $\psi(t, r) := \frac{r}{(1+(\alpha-1)tr^{\alpha-1})^{1/(\alpha-1)}}$ is the solution of equation (15) ($\theta(u) = u^\alpha$ for all $u \in \mathbb{R}_+$ and $\alpha > 1$) with initial condition $\psi(0, r) = r$.

3. If the function $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies the conditions (H1)-(H4), then the function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$, defined by equality: $g(t) := a\theta(bt)$ for all $t \in \mathbb{R}_+$ (a and b are some positive numbers), possesses the same property.

Let H be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and $f \in C(H, H)$ be a function satisfying

$$(16) \quad \operatorname{Re}\langle f(u_1) - f(u_2), u_1 - u_2 \rangle \leq -\theta(t, |u_1 - u_2|)$$

for all $u_1, u_2 \in H$, where $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is some function with properties (H1)-(H4).

Theorem 5.3. [10] *If the function f verifies the condition (16), then*

- (i) the equation

$$(17) \quad u' = f(u)$$

generates a semigroup dynamical system (H, \mathbb{R}_+, π) , where $\pi(t, u)$ is a unique solution of equation (17) defined on \mathbb{R}_+ with the initial condition $\pi(0, u) = u$;

- (ii) the following inequality holds

$$|\pi(t, u_1) - \pi(t, u_2)|^2 \leq \psi(t, |u_1 - u_2|^2)$$

for all $u_1, u_2 \in H$ and $t \in \mathbb{R}_+$, where $\psi(t, r)$ is a unique solution of equation (15) with initial data $\psi(0, r) = r$ for all $r \in \mathbb{R}_+$, $|\cdot|$ is the norm generated by the scalar product $\langle \cdot, \cdot \rangle$ in the space H .

Consider a finite set of differential equations

$$(18) \quad u' = f_i(u) \quad (i = 1, 2, \dots, m)$$

with the right-hand sites $f_i \in C(H, H)$ satisfying the condition (16) with the function θ . Let (H, \mathbb{R}_+, π_i) ($i = 1, 2, \dots, m$) be the dynamical system, generated by (18) and (H, P_i) ($i = 1, 2, \dots, m$) be the cascade (discrete dynamical system), where $P_i(u) := \pi_i(1, u)$ for all $u \in H$ and $i = 1, 2, \dots, m$.

A point $x_0 \in H$ is said to be m -periodic ($m \in \mathbb{N}$) for control system (14), if there exist an m -periodic control $\omega \in S(\mathbb{R}_+, \mathcal{P})$ ($\omega(t+m) = \omega(t)$ for all $t \in \mathbb{R}_+$) such that the unique solution $x(t)$ of equation

$$(19) \quad x' = f(x, \omega(t))$$

with initial data $x(0) = x_0$ is m -periodic, i.e., $x(t+m) = x(t)$ for all $t \in \mathbb{R}_+$.

Theorem 5.4. *Suppose that $\mathcal{M} := \{P_i : i = 1, 2, \dots, m\}$. Under the conditions listed above the following statement hold:*

- (i) *the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ ($\Omega := C(\mathbb{Z}_+, \mathcal{M})$) generated by $DI(\mathcal{M})$ is compactly dissipative;*
- (ii) *the Levinson center I of $DI(\mathcal{M})$ is the closure of the periodic points of control system (14);*
- (iii) *the skew-product dynamical system (X, \mathbb{Z}_+, π) generated by $DI(\mathcal{M})$ is compactly dissipative;*
- (iv) *if every map $P_i \in \mathcal{M}$ is invertible, then*
 - (a) *Levinson's center J of the skew-product dynamical system (X, \mathbb{Z}_+, π) is a chaotic Cantor set;*
 - (b) *there exists a residual subset $J_0 \subseteq J$ (large in the sense of Baire category), consisting from Poisson's stable points, such that the positive semi-trajectory of every point $x_0 \in J_0$ is dense on J ;*
 - (c) *if I is the Levinson center for $DI(\mathcal{M})$, then $I = pr_1(J)$ ($pr_1 : X \rightarrow \Omega$ and $X := W \times \Omega$), i.e., I is a continuous image of the Cantor set J .*

Proof. This statement directly it follows from Theorems 3.2 and 4.9. □

5.2. Monotone evolution equations. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle, |\cdot| := \sqrt{\langle \cdot, \cdot \rangle}$ and E be a reflexive Banach space contained in H algebraically and topologically. Furthermore, let E be dense in H , and here H can be identified with a subspace of the dual E' of E and $\langle \cdot, \cdot \rangle$ can be extended by continuity to $E' \times E$.

Let $D(A) \subseteq H$ be the domain of definition of operator $A : H \rightarrow H$.

Remind [5, 6, 12] that an operator A is called:

- monotone, if for every $u_1, u_2 \in D(A) : \langle Au_1 - Au_2, u_1 - u_2 \rangle \geq 0$;
- semi-continuous, if the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by the equality $\varphi(\lambda) := \langle A(u + \lambda v), w \rangle$ is continuous.

Note that the family of monotone operators can be partially ordered by including graphics.

A monotone operator is called maximal, if it is maximal among the monotone operators.

Let us consider an evolutionary equation

$$(20) \quad \begin{aligned} \frac{dx}{dt} + Ax &= h \\ u(0) &= u, \end{aligned}$$

where $h \in H$ and A is a maximal monotone operator with the domain of definition $D(A)$,

$$|Au|_{E'} \leq C|u|_E^{p-1} + K, u \in E, p > 1,$$

coercive,

$$\langle Au, u \rangle \geq a|u|_E^p, u \in E, a > 0,$$

uniformly monotone,

$$(21) \quad \langle Au_1 - Au_2, u_1 - u_2 \rangle \geq \alpha|u_1 - u_2|^\beta \quad (\forall u_1, u_2 \in E, \text{ where } \beta \geq 2),$$

and semi-continuous (see [13]).

A nonlinear "elliptic" operator

$$\begin{aligned} Au &= - \sum_{i=1}^n \frac{\partial}{\partial x_i} \phi \left(\frac{\partial u}{\partial x_i} \right) \quad \text{in } D \subset \mathbb{R}^n \\ u &= 0 \text{ on } \partial D, \end{aligned}$$

where D is a bounded domain in \mathbb{R}^n , $\phi(\cdot)$ is an increasing function satisfying, $\phi(0) = 0$,

$$c|\xi - \eta|^p \leq \sum_{i=1}^n (\xi_i - \eta_i)(\phi(\xi_i) - \phi(\eta_i)) \leq C|\xi - \eta|^p \quad (\text{for all } \xi, \eta \in \mathbb{R}^n),$$

and provides an example with

$$H = L^2(D), E = W_0^{1,p}(D), E' = W^{-1,p'}(D), p' = \frac{p}{p-1}.$$

The following result is established in [13] (Ch.II and Ch.IV). If $x \in H$ and $p' = \frac{p}{p-1}$, then there exists a unique solution $\varphi(t, u) \in C(\mathbb{R}_+, H)$ of (20).

Theorem 5.5. *Suppose that the operator A satisfies the conditions above. Then equation(20) generates on the space H a semi-group dynamical system (H, \mathbb{R}_+, π) satisfying the following condition:*

$$|\pi(t, u_1) - \pi(t, u_2)|^2 \leq \psi(t, |u_1 - u_2|^2)$$

for all $t \in \mathbb{R}_+$ and $u_1, u_2 \in H$, where $\psi(t, r)$ is a unique solution of equation $y' = -2\alpha y^{\beta/2}$ with initial data $\psi(0, r) = r$.

Proof. This statement directly it follows from Theorem 7.10 [10]. □

Consider a finite set of differential equations

$$(22) \quad \frac{dx}{dt} + A_i x = h_i, \quad (i = 1, 2, \dots, m)$$

with the right-hand sites $h_i \in H$ and monotone operators A_i satisfying condition (21) with constant $\alpha_i > 0$. Let (H, \mathbb{R}_+, π_i) ($i = 1, 2, \dots, m$) be the dynamical system, generated by (22) and (H, P_i) ($i = 1, 2, \dots, m$) be the cascade (discrete dynamical system), where $P_i(u) := \pi(1, u)$ for all $u \in H$.

Theorem 5.6. *Suppose that $\mathcal{M} := \{P_i : i = 1, 2, \dots, m\}$. Under the conditions listed above the following statement hold:*

- (i) *the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ ($\Omega := C(\mathbb{Z}_+, \mathcal{M})$) generated by $DI(\mathcal{M})$ is compactly dissipative;*
- (ii) *the skew-product dynamical system (X, \mathbb{Z}_+, π) generated by $DI(\mathcal{M})$ is compactly dissipative;*
- (iii) *the Levinson's center I of cocycle φ possesses the following property: $I = \overline{Per(\varphi)}$, where $Per(\varphi) := \{u \in W : \exists \tau \in \mathbb{N} \text{ and } \omega \in \Omega \text{ such that } \sigma(\tau, \omega) = \omega \text{ and } \varphi(\tau, u, \omega) = u\}$.*

Proof. This statement it follows from Theorems 3.2 and 4.9. □

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