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Compact Global Attractors of Non-Autonomous Gradient-Like Dynamical Systems

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Abstract. In this paper we study the asymptotic behavior of gradient-like non-autonomous dynamical systems. We give a description of the structure of the Levinson center (maximal compact invariant set) for this class of systems.

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1 Introduction

Denote by $\mathbb{S}$ the set of all real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers and $\mathbb{S}_+ := \{s \in \mathbb{S} : \ s \geq 0\}$. Let $\mathbb{T} := \mathbb{R}_+$ or $\mathbb{Z}_+$, X be a complete metric space and $(X, \mathbb{T}, \pi)$ be a dynamical system.

A continuous function $V : \ X \mapsto \mathbb{R}$ is said to be a (global) Lyapunov function for $(X, \mathbb{T}, \pi)$ if $V(\pi(t, x)) \leq V(x)$ for all $x \in X$ and $t \in \mathbb{T}$.

If $\pi(t, x) = x$ for all $t \geq 0$, then $x \in X$ is called a fixed (stationary) point of the dynamical system $(X, \mathbb{T}, \pi)$, and by $Fix(\pi)$ we will denote the set of all fixed points of $(X, \mathbb{T}, \pi)$.

A dynamical system $(X, \mathbb{T}, \pi)$ with the Lyapunov function V is called a gradient system if the equality $V(\pi(t, x)) = V(x)$ (for all $t \geq 0$) implies $x \in Fix(\pi)$.

The simplest example of gradient dynamical system is defined by the differential equation

$$x' = -\nabla V(x) \quad (x \in \mathbb{R}^n), \tag{1}$$

where $V : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuously differentiable function and $\nabla := \{\partial_{x_1}, \dots, \partial_{x_n}\}$. Indeed, if we suppose that equation (1) admits a unique solution $\pi(t, x)$ passing through the point $x \in \mathbb{R}^n$ at the initial moment $t = 0$ and defined on $\mathbb{R}_+$, then

$$\frac{d}{dt} V(\pi(t, x)) = -|\nabla V(\pi(t, x)|^2 \leq 0 \tag{2}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. From (2) we obtain $V(\pi(t, x)) \leq V(x)$ for all $t \geq 0$ and if $V(\pi(t, x)) = V(x)$ for all $t \geq 0$, then from (1) we have $x \in Fix(\pi)$, i. e., $(\mathbb{R}^n, \mathbb{R}_+, \pi)$ (the dynamical system generates by equation (1).

The asymptotic behavior of gradient dynamical systems is well studied (see, for example, [2], [3, ChIII], [9, ChV], [16, ChIII], [17], [24, ChIX] and the bibliography therein).

The aim of this paper is to study the asymptotic behavior of a class of abstract non-autonomous (gradient-like) dynamical systems. The paper is organized as follows. In the second section we give with the proof some (more or less) known results for general autonomous gradient-like semi-group dynamical systems (Lemma 1).

The third section is dedicated to the gradient systems. The main result (Theorem 2) of this section contains the sufficient conditions when a gradient dynamical system admits a compact global attractor.

In the fourth section we study the gradient dynamical systems with finite number of fixed points. For the compact dissipative gradient dynamical system we give a description of the structure of its Levinson center (Lemma 4).

The fifth section is dedicated to the study of the relation between the set of all fixed points and chain recurrent points (Theorem 4) for the gradient dynamical systems admitting a compact global attractor.

In the sixth section we introduce the notion of gradient-like non-autonomous dynamical systems. The main result of this section is Theorem 7 which contains a description of the structure of compact global attractor for a gradient-like non-autonomous dynamical system with minimal base.

2 Gradient-like systems

Denote by $\omega_x := \{y \in X :$ there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $t_n \to +\infty$ and $\pi(t_n, x) \to y$ as $n \to +\infty$ $\}$ and $J^+_x := \{y \in X :$ there exist sequences $\{t_n\} \subset \mathbb{T}$ and $\{x_n\} \subset X$ such that $t_n \to +\infty,\ x_n \to x$ and $\pi(t_n, x_n) \to y$ as $n \to +\infty$ $\}$.

Definition 1. A continuous function $\gamma : \mathbb{S} \mapsto X$ is called a full trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passing through the point $x \in X$ at the initial moment $t = 0$ if $\gamma(0) = x$ and $\pi(t, \gamma(s)) = \gamma(t + s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

By Φ_x we denote the set of all full trajectories of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment and $\alpha_\gamma := \{y \in X :$ there exists a sequence $t_n \to -\infty$ such that $\gamma(t_n) \to y\}$.

Definition 2. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be a gradient-like dynamical system if it has a global Lyapunov function.

Lemma 1. *Let $(X, \mathbb{T}, \pi)$ be a gradient-like dynamical system and V be its Lyapunov function, then the following statements hold:*

1) *for all $x \in X$ there exists a constant $C_x \in \mathbb{R}$ such that $V(p) = C_x$ for any $p \in \omega_x$ if the positive semi–trajectory $\pi(\mathbb{T}_+, x)$ is relatively compact;*

2) *if $\gamma \in \Phi_x$ and the negative semi–trajectory $\gamma(\mathbb{T}_-)$ is relatively compact, then there exists $c_\gamma \in \mathbb{R}$ such that $V(q) = c_\gamma$ for all $q \in \alpha_\gamma$;*

3) *if $x \in X$ is a non-wandering point (i.e., $x \in J^+_x$), then $V(\pi(t, x)) = V(x)$ for all $t \in \mathbb{T}_+$.*

Proof. Consider the continuous function $\psi : \mathbb{T}_+ \mapsto \mathbb{R}$ defined by the equality $\psi(t) := V(\pi(t,x))$ for all $t \geq 0$. Since $\psi(t_2) \leq \psi(t_1)$ for all $t_2 \geq t_1$ and V is upper-bounded along trajectories of $(X, \mathbb{T}, \pi)$, then there exists $\lim\limits_{t\to+\infty} V(\pi(t,x)) = C_x$. Let now $p \in \omega_x$, then there exist $t_n \to +\infty$ such that $p = \lim\limits_{t\to+\infty} \pi(t_n, x)$ and, consequently, $V(p) = \lim\limits_{n\to\infty} V(\pi(t_n, x)) = C_x$.

Consider the function $\psi : \mathbb{T}_- \mapsto \mathbb{R}$ defined by the equality $\psi(s) := V(\gamma(s))$ for all $s \in \mathbb{T}_-$, where $\gamma \in \Phi_x$. Since $\psi(s_1) \geq \psi(s_2)$ for all $s_1 \leq s_2$ $(s_1, s_2 \in \mathbb{T}_-)$ and ψ is upper-bounded on $\mathbb{T}_-$, then there exists $\lim\limits_{t\to+\infty} V(\gamma(t)) = c_\gamma$. If $q \in \alpha_\gamma$ then there exists a sequence $\{s_n\} \subseteq \mathbb{T}_-$ with $s_n \to -\infty$ such that $q = \lim\limits_{n\to\infty} \gamma(s_n)$ and $V(q) = \lim\limits_{n\to\infty} V(\gamma(s_n)) = c_\gamma$.

Let $p \in J_x^+$. Since $J_x^+ \subseteq J_{\pi(t,x)}^+$ for all $t \in \mathbb{T}$, then we have $p = \lim\limits_{n\to\infty} \pi(t_n, \tilde{x}_n)$, where $t_n \to +\infty$ and $\tilde{x}_n \to \pi(t,x)$. Thus we obtain

$$V(p) = \lim_{n\to\infty} V(\pi(t_n, \tilde{x}_n)) \leq \lim_{n\to\infty} V(\tilde{x}_n) = V(\pi(t,x)).$$

In particular, $V(x) \leq V(\pi(t,x))$ since $x \in J_x^+$. On the other hand $V(\pi(t,x)) \leq V(x)$ for all $x \in X$ and $t \geq 0$ and, consequently, we have $V(\pi(t,x)) = V(x)$ for all $t \geq 0$. □

Let M be a subset of X. Denote by $W^u(M) := \{x \in X :$ there exists $\gamma \in \Phi_x$ such that $\lim\limits_{t\to-\infty} \rho(\gamma(t), M) = 0\}$.

Remark 1. The first and second statements of Lemma 1 are well known (LaSalle's invariance principle).

2. The third statement is a slight modification of a result from [4, p.131].

3 Gradient systems

Definition 3. $x \in X$ is called a stationary point (fixed point, singular point) if $\pi(t,x) = x$ for all $t \in \mathbb{T}$.

Denote by $Fix(\pi)$ the set of all fixed points of dynamical system $(X, \mathbb{T}, \pi)$ and $J_x^+ := \{p \in X :$ there are $x_n \to x$ and $t_n \to +\infty$ such that $\pi(t_n, x_n) \to p\}$.

Definition 4. $(X, \mathbb{T}, \pi)$ is called a gradient dynamical system if there exists a Lyapunov function $V : X \mapsto \mathbb{R}$ with the following property: if $V(\pi(t,x)) = V(x)$ for all $t \geq 0$ then $x \in Fix(\pi)$.

Lemma 2. *Let $(X, \mathbb{T}, \pi)$ be a gradient dynamical system, then $\Omega_X = \Omega(\pi) = Fix(\pi)$, where $\Omega_X := \overline{\bigcup\{\omega_x : \ x \in X\}}$ and $\Omega(\pi) = \{x \in X : \ x \in J_x^+\}$.*

Proof. The inclusions $Fix(\pi) \subseteq \Omega_X \subseteq \Omega(\pi)$ are evident and take place for arbitrary dynamical systems (including gradient systems too). To finish the proof of Lemma it is sufficient to establish the inclusion $\Omega(\pi) \subseteq Fix(\pi)$ for gradient systems. Let $x \in \Omega(\pi)$, then $x \in J_x^+$ and by Lemma 1 (item 3) we have $V(\pi(t,x)) = V(x)$ for all $t \geq 0$ and, consequently, $x \in Fix(\pi)$. □

Definition 5. A subset $M \subseteq X$ is called bounded if its diameter $d(M) := \sup\{d(p,q) :\ p,q \in M\}$ is finite.

Remark 2. 1. A subset $M \subseteq X$ is bounded if and only if for every $x_0 \in X$ there exists a number $C_{x_0} \geq 0$ such that $\rho(x_0, x) \leq C_{x_0}$ for all $x \in M$.

2. A subset $M \subseteq X$ is unbounded if and only if there exist a point $x_0 \in X$ and a sequence $\{x_n\} \subseteq M$ such that $\rho(x_n, x_0) \to +\infty$ as $n \to \infty$.

Let A and B be two bounded subsets from X. Denote by $\beta(A,B) := \sup\{\rho(a,B) :\ a \in A\}$, where $\rho(a,B) := \inf\{\rho(a,b) :\ b \in B\}$.

Definition 6. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

– point dissipative if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t \to \infty} \rho(\pi(t,x), K) = 0 \tag{3}$$

for all $x \in X$;

– compact dissipative if it is point dissipative and equality (3) takes pace uniformly with respect to $x \in X$ on every compact subset from X.

Remark 3. If the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative, then in X there exists a maximal compact invariant set J (Levinson center [10, ChI]) which attracts every compact subset from X.

Theorem 1 (see [10, ChI])**.** *Suppose that $(X, \mathbb{T}, \pi)$ is a point dissipative dynamical system, then it will be compact dissipative if and only if for any compact subset $K \subseteq X$ the set $\Sigma_K^+ := \{\pi(t,x) :\ t \geq 0,\ x \in K\}$ is relatively compact.*

Theorem 2. *Suppose that the following conditions hold:*

1) $(X, \mathbb{T}, \pi)$ is asymptotically compact;

2) $(X, \mathbb{T}, \pi)$ is a gradient dynamical system and $V : X \mapsto \mathbb{R}$ is its Lyapunov function;

3) the set $Fix(\pi)$ is bounded;

4) for any sequence $\{x_n\}$ with the property $\rho(x_n, x_0) \to +\infty$ as $n \to \infty$ we have $V(x_n) \to +\infty$, where x_0 is some point from X.

Then the following statements hold:

1) the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative;

2) if the Lyapunov function V is bounded on every bounded subset from X, then the Levinson center J of $(X, \mathbb{T}, \pi)$ attracts every bounded subset M from X, i.e.,

$$\lim_{t \to +\infty} \beta(\pi^t M, J) = 0.$$

Proof. Let $x \in X$ be an arbitrary point. Note that the positive semi-trajectory Σ_x^+ of point x is a bounded set. In fact, if we suppose that it is not so, then there exist a point $x_0 \in X$ and a sequence $t_n \to +\infty$ such that $\rho(\pi(t_n, x), x_0) \to +\infty$ as $n \to \infty$. Under the conditions of Theorem we have $V(\pi(t_n, x)) \to +\infty$. On the other hand we have $V(\pi(t_n, x)) \leq V(x)$ for all $n \in \mathbb{N}$. The obtained contradiction proves our statement. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact, then the semi-trajectory Σ_x^+ is relatively compact, and consequently, ω_x is a nonempty compact set. According to Lemma 2 we have $\omega_x \subseteq Fix(\pi)$. Note that the set $Fix(\pi)$ is closed and invariant. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact and $Fix(\pi)$ is bounded, then it is compact. Thus every semi-trajectory Σ_x^+ of $(X, \mathbb{T}, \pi)$ is relatively compact and there exists a nonempty compact subset $K := Fix(\pi) \subset X$ such that $\Omega_X \subseteq K$. This means that the dynamical system $(X, \mathbb{T}, \pi)$ is point-wise dissipative.

Let now M be an arbitrary nonempty compact subset from X. We will prove that the positive semi-trajectory Σ_M^+ of the set M is relatively compact. To this end under the conditions of Theorem it is sufficient to establish that it is bounded. Denote by $c := \max\{V(x) : \ x \in M\}$ and $M_c := \{x \in X : \ V(x) \leq c\}$. Note that M_c is a bounded subset of X. Indeed, if we suppose that it is not true, then there exists a point $x_0 \in X$ and sequence $\{x_n\} \subseteq M_c$ such that $\rho(x_n, x_0) \to +\infty$ and, consequently, $V(x_n) \to +\infty$ as $n \to +\infty$. On the other hand $x_n \in M_c$ and, consequently, $V(x_n) \leq c$ for all $n \in \mathbb{N}$. The obtained contradiction proves our statement. Thus the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative and semi-trajectory Σ_M^+ is relatively compact for any compact subset $M \subseteq X$. By Theorem 1 $(X, \mathbb{T}, \pi)$ is compact dissipative.

Denote by J its Levinson center and consider an arbitrary bounded subset M from X then, under the conditions of Theorem 2, the set $V(M) \subset \mathbb{R}$ is bounded. Now we will prove that the semi-trajectory Σ_M^+ is a bounded subset of X for every bounded set $M \subseteq X$. Indeed, denote by $c := \sup\{V(x) : \ x \in M\}$, then we have $V(\pi(t, x)) \leq V(x) \leq c$ for all $x \in M$ and $t \geq 0$ and, consequently $\Sigma_M^+ \subseteq M_c$. Thus the set Σ_M^+ is bounded and positive invariant. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact, then the set $\Omega(M)$ is nonempty, compact and it attracts the set M, i.e.,

$$\lim_{t \to +\infty} \beta(\pi^t M, \Omega(M)) = 0,$$

where

$$\Omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}.$$

Since J is a maximal compact invariant of the dynamical system $(X, \mathbb{T}, \pi)$, then $\Omega(M) \subseteq J$ and, consequently, J attracts M. The theorem is proved. $\square$

Remark 4. Note that the second statement of Theorem 2 remains true (see Theorem 3.8.5 [16, ChIII] and [21]) if we replace the condition of boundedness of the function V on every bounded subset from X by the following: for every bounded subset M

from X there exists a number $\tau \geq 0$ such that the set $\{\pi(t,x) : \ x \in M, \ t \geq \tau\}$ is bounded.

4 Gradient systems with finite number of fixed points

In this section we will study the gradient dynamical systems $(X, \mathbb{T}, \pi)$ with finite set $Fix(\pi)$ of fixed points.

Lemma 3. *Let $(X, \mathbb{T}, \pi)$ be a dynamical system and the following conditions hold:*

1) *every positive semi-trajectory Σ_x^+ is relatively compact;*

2) *$\Omega_X \subseteq Fix(\pi)$;*

3) *the set $Fix(\pi)$ is finite, i.e., there are a finite number of stationary points $p_1, p_2, \ldots, p_m$ of dynamical system $(X, \mathbb{T}, \pi)$ such that $\Omega_X = \{p_1, p_2, \ldots, p_m\}$.*

Then every point $x \in X$ is asymptotically stationary, i.e., there exists a point $p_i \in Fix(\pi)$ $(1 \leq i \leq m)$ such that

$$\lim_{t \to +\infty} \rho(\pi(t,x), p_i) = 0. \tag{4}$$

Proof. Let $x \in X$. Since Σ_x^+ is relatively compact, then the set ω_x is nonempty, compact, invariant, and it attracts the point x. On the other hand $\omega_x \subseteq \Omega_X = \{p_1, p_2, \ldots, p_m\}$. Thus there are $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ such that $\omega_x = \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$. Taking into account that the set ω_x is dynamically undecomposable we conclude that the set ω contains a single stationary point p_i and, consequently, we have the equality (4). $\square$

We put $A_x := \bigcup\{\alpha_\gamma : \ \gamma \in \Phi_x\}$, $A_X := \overline{\bigcup\{\alpha_x : \ x \in X\}}$ and $\Delta_X := \Omega_X \bigcup A_X$. If $p \in Fix(\pi)$, then by $W^u(p) := \{y \in X : \lim\limits_{t \to -\infty} \rho(\gamma(t), p) = 0$ for certain $\gamma \in \Phi_y\}$ we denote the unstable "manifold" of p.

Lemma 4. *Suppose that the following conditions are fulfilled:*

1) *the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and J is its Levinson center;*

2) *$\Delta_X \subseteq Fix(\pi)$;*

3) *the set $Fix(\pi)$ is finite, i.e., $Fix(\pi) = \{p_1, p_2, \ldots, p_m\}$.*

Then the following equality

$$J = \bigcup\{W^u(p) : \ p \in Fix(\pi)\}$$

takes place.

Proof. Since J is a maximal compact invariant set of $(X,\mathbb{T},\pi)$, then $Fix(\pi)\subseteq J$ and $W^u(p)\subseteq J$ for all $p\in Fix(\pi)$. Thus to finish the proof it is sufficient to establish that $J\subseteq\bigcup\{W^u(p):\ p\in Fix(\pi)\}$. Since $J\subseteq X$ is a compact invariant set, then every motion $\gamma\in\Phi:=\bigcup\{\Phi_x:\ x\in J\}$ is defined on $\mathbb{S}$ and Φ is compact with respect to compact-open topology. Consider a two-sided shift dynamical system (with uniqueness) $(\Phi,\mathbb{S},\sigma)$. We note that $Fix(\sigma)=\{p_1,p_2,\dots,p_m\}$. The inclusion $\{p_1,p_2,\dots,p_m\}\subseteq Fix(\sigma)$ is evident. Thus to prove our statement it is sufficient to show the inclusion $Fix(\sigma)\subseteq\{p_1,p_2,\dots,p_m\}$. Let $\psi\in Fix(\sigma)$, then $\psi(t)=\psi(0)$ for all $t\in\mathbb{S}$ and, consequently, $\psi\in\Phi_{\psi(0)}$ and $\psi(0)\in\alpha_\psi\subseteq\Delta_X\subseteq Fix(\pi)=\{p_1,p_2,\dots,p_m\}$. Thus there exists a number $1\le i\le m$ such that $\psi(0)=p_i$ and, consequently, $\psi(t)=p_i$ for all $t\in\mathbb{S}$, i.e., $\psi=p_i\in\{p_1,p_2,\dots,p_m\}$. Let $x\in J$ and $\gamma\in\Phi_x$. Denote by $\tilde{\alpha}_\gamma$ the α–limit set of the point $\gamma\in\Phi$ with respect to shift dynamical system $(\Phi,\mathbb{S},\sigma)$. If $\psi\in\tilde{\alpha}_\gamma$, then there exists a sequence $t_n\to-\infty$ such that $\psi(t)=\lim\limits_{n\to\infty}\gamma(t+t_n)$ and the last equality takes place uniformly on every compact from $\mathbb{S}$. Thus $\psi(0)\in\alpha_\gamma\subseteq\{p_1,p_2,\dots,p_m\}$ and $\psi\in\Phi_{\psi(0)}$. Thus there exists a number i such that $\psi(0)=p_i$ and, consequently, $\psi=p_i$, i.e., $\tilde{\alpha}_\gamma\subseteq\{p_1,p_2,\dots,p_m\}$. Taking into account that the set $\tilde{\alpha}_\gamma$ is dynamically undecomposable we conclude that the set $\tilde{\alpha}_\gamma$ consists of a single stationary point p_j and, consequently, $\lim\limits_{t\to-\infty}\rho(\gamma(t),p_j)=0$. Thus we have $x\in W^u(p_j)$ and, consequently, $J\subseteq\bigcup\{W^u(p):\ p\in Fix(\pi)\}$. □

Remark 5. Lemma 4 remains true if we replace the condition $\Delta_X\subseteq Per(\pi)$ by $A_X\subseteq Per(\pi)$, where $A_X:=\bigcup\{\alpha_\gamma:\gamma\in\Phi(\pi)\}$.

Remark 6. The statements close to Lemma 4 (see also Remark 5) were published in the works [3, ChIII] and [23].

Lemma 5 (see [1])**.** *Let $x\in X$ and $y\in\omega_x$, then $J^+_x\subseteq J^+_y=D^+_y$.*

Lemma 6. *Suppose that $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J is its Levinson center, then the following statements hold:*

1) *$\omega_x\subseteq\Omega(\pi)$ for all $x\in X$;*

2) *$\alpha_\gamma=\emptyset$ for all $\gamma\in\Phi_x$ and $x\notin J$;*

3) *$\alpha_\gamma\subseteq\Omega(\pi)$ for all $\gamma\in\Phi_x$ and $x\in J$.*

Proof. Let $x\in X$. Since the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative, then ω_x is a nonempty, compact and invariant set. If $y\in\omega_x$, then according to Lemma 5 we have $J^+_y=D^+_y$. Since $y\in D^+_y$, then $y\in J^+_y$, i.e., $\omega_x\subseteq\Omega(\pi)$.

Let now $x\notin J$, $\gamma\in\Phi_x$ and $p\in\alpha_\gamma$, then there exists a sequence $t_n\to-\infty$ such that $p=\lim\limits_{n\to\infty}\gamma(t_n)$. Denote by $K:=\{\gamma(t_n)\}\bigcup\{p\}$, then the set K is compact. Since the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J its Levinson center, then $\Omega(K)$ is a nonempty compact set and $\Omega(K)\subseteq J$. Note that $x=\gamma(0)=\pi(-t_n,\gamma(t_n))$ and, consequently, $x\in\Omega(K)\subseteq J$. The obtained contradiction proves our statement.

If $x \in J$, $\gamma \in \Phi_x$ and $p \in \alpha_\gamma$, then there exists a sequence $t_n \to -\infty$ such that $p = \lim_{n\to\infty} \gamma(t_n)$. Consider the sequence $\{\sigma(t_n, \gamma)\} \subseteq \Phi$. Since the space Φ is compact (with respect to compact-open topology) without loss of generality we may suppose that the sequence $\{\sigma(t_n, \gamma)\}$ is convergent. Denote by ψ its limit, then $\psi \in \tilde{\alpha}_\gamma$ and, consequently, $\psi \in J^+_\psi$. This means that there are sequence $\{\psi_n\} \to \psi$ and $t_n \to +\infty$ such that $\psi = \lim_{n\to\infty} \sigma(t_n, \psi_n)$. In particular, we have $p = \psi(0) = \lim_{n\to\infty} \sigma(t_n, \psi_n)(0) = \lim_{n\to\infty} \psi_n(t_n) = \lim_{n\to\infty} \pi(t_n, \psi_n(0))$. Since $\psi_n(0) \to \psi(0) = p$ as $n \to \infty$, then we have $p \in J^+_p$. $\square$

Corollary 1. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J is its Levinson center, then we have $\Delta_X \subseteq \Omega(\pi)$.*

Definition 7. Let $p, q \in Fix(\pi)$. The point p is said to be chained to q, written $q \mapsto p$, if there exists a full trajectory $\gamma \in \Phi_x$ for some $x \notin \{p, q\}$ such that $\alpha_\gamma = q$ and $\omega_x = \{p\}$. A finite sequence $\{p_1, p_2, \dots, p_k\} \subseteq Fix(\pi)$ is called a k-chain if $p_1 \mapsto p_2 \mapsto \dots \mapsto p_k$. The k-chain is called a k-cycle if $p_k = p_1$.

Definition 8. Recall that:

– a nonempty compact invariant subset M of X is said to be locally maximal for $(X, \mathbb{T}, \pi)$ if it is the maximal compact invariant set in some neighborhood of itself;

– a dynamical system $(X, \mathbb{T}, \pi)$ is called asymptotically compact if for every bounded positive invariant subset $B \subseteq X$ its ω-limit set $\Omega(B)$ is nonempty, compact and

$$\lim_{t\to\infty} \beta(\pi(t, B), \Omega(B)) = 0.$$

Lemma 7. *Suppose that the following conditions hold:*

1) $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J its Levinson center;

2) $(X, \mathbb{T}, \pi)$ is a gradient system and $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$;

3) the set $Fix(\pi)$ does not contain any 1-cycles.

Then every set $M_i := \{p_i\}$ $(i = 1, 2, \dots, m)$ is locally maximal.

Proof. We will show that every subset M_i $(i = 1, 2, \dots, m)$ is locally maximal. Denote by $d := \min\{d_{ij} : \ i, j = 1, 2, \dots, m \text{ and } i \neq j\}$, where $d_{ij} = \rho(p_i, p_j)$ is the distance between p_i and p_j. We will show that M_i is the maximal invariant set in $B(p_i, \delta) := \{x \in X : \ \rho(x, p_i) < \delta\}$, where $0 < \delta < d/3$. Indeed, suppose that it is not true, then there exists a compact invariant set $M \subset B(p_i, \delta)$ such that $M \neq M_i$. Let $x \in M \setminus M_i$, then there exists a full trajectory $\gamma \in \Phi_x$ such that $\gamma(\mathbb{S}) \subseteq M$. By Lemma 3 and Lemma 4 there exist points $p, q \in Fix(\pi)$ such that $\alpha_\gamma = \{p\}$ and $\omega_x = \{q\}$. Since $p, q \in M \subset B(p_i, \delta)$, then according to the choice of δ we have $p = q = p_i$, i.e., we obtain a 1-cycle $p_i \mapsto p_i$. Thus the obtained contradiction completes the proof of Lemma. $\square$

5 Chain-recurrent motions

Let $\Sigma \subseteq X$ be a compact positive invariant set, $\varepsilon > 0$ and $t > 0$.

Definition 9. The collection $\{x = x_0, x_1, x_2, \dots, x_k = y; t_0, t_1, \dots, t_k\}$ of the points $x_i \in \Sigma$ and the numbers $t_i \in \mathbb{T}$ such that $t_i \geq t$ and $\rho(x_i t_i, x_{i+1}) < \varepsilon$ $(i = 0, 1, \dots, k-1)$ is called (see, for example, [7,8], [14,15] and [24]) a (ε, t, π)–chain joining the points x and y.

We denote by $P(\Sigma)$ the set $\{(x, y) : x, y \in \Sigma, \forall\ \varepsilon > 0\ \forall\ t > 0\ \exists\ (\varepsilon, t, \pi)$-chain joining x and $y\}$. The relation $P(\Sigma)$ is closed, invariant and transitive [7,14,19,22,24].

Definition 10. The point $x \in \Sigma$ is called chain-recurrent (in Σ) if $(x, x) \in P(\Sigma)$.

We denote by $\mathfrak{R}(\Sigma)$ the set of all chain-recurrent (in Σ) points from Σ.

Definition 11. Let $A \subseteq X$ be a nonempty positive invariant set. The set A is called (see, for example, [18]) internally chain recurrent if $\mathfrak{R}(A) = A$, and internally chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ and $t > 0$, there is an (ε, t, π)-chain in A connecting a and b.

The set of all chain recurrent points for $(X, \mathbb{T}, \pi)$ is denoted by $\mathfrak{R}(\Sigma)$, i.e., $\mathfrak{R}(\Sigma) := \{x \in \Sigma : (x, x) \in P(\Sigma)\}$. On $\mathfrak{R}(\Sigma)$ we will introduce a relation $\sim$ as follows: $x \sim y$ if and only if $(x, y) \in P(\Sigma)$ and $(y, x) \in P(\Sigma)$. It is easy to check that the introduced relation $\sim$ on $\mathfrak{R}(\Sigma)$ is a relation of equivalence and, consequently, it is easy to decompose it into the classes of equivalence $\{\mathfrak{R}_\lambda : \lambda \in \mathcal{L}\}$ (internally chain transitive subsets), i.e., $\mathfrak{R}(\Sigma) = \sqcup\{\mathfrak{R}_\lambda :\ \lambda \in \mathcal{L}\}$. By Proposition 2.6 from [7] (see also [14] and [19,22,24] for the semi-group dynamical systems) the defined above components of the decomposition of the set $\mathfrak{R}(\Sigma)$ are closed and positive invariant.

Lemma 8 (see [18])**.** *Let $x \in X$, $\gamma \in \Phi_x$ and the positive (respectively, negative) semi-trajectory of the point $x \in X$ is relatively compact. Then the ω (respectively, α)-limit set of the point x is internally chain-transitive, i. e., $\mathfrak{R}(\omega_x) = \omega_x$ (respectively, $\mathfrak{R}(\alpha_\gamma) = \alpha_\gamma$).*

Theorem 3 (see [18])**.** *Assume that each fixed point of $(X, \mathbb{T}, \pi)$ is a locally maximal invariant set, that there is no k-cycle ($k \geq 1$) of fixed points, and that every pre-compact orbit converges to some fixed point of $(X, \mathbb{T}, \pi)$. Then any compact internally chain-transitive set is a fixed point of $(X, \mathbb{T}, \pi)$.*

Remark 7. 1. Theorem 3 was established in [18] for the dynamical systems with discrete time, i. e., $\mathbb{T} \subseteq \mathbb{Z}$.

2. Theorem 3 for the dynamical systems with continuous time (i.e., $\mathbb{R}_+ \subseteq \mathbb{T}$) may be established with slight modifications of the proof of Theorem 3.2 [18] and using some results from [19].

Theorem 4. *Suppose that the following conditions hold:*

1) $(X,\mathbb{T},\pi)$ *is a compact dissipative dynamical system and* J *its Levinson center;*

2) $(X,\mathbb{T},\pi)$ *is a gradient system and* $Fix(\pi)=\{p_1,p_2,\dots,p_m\}$*;*

2) the set $Fix(\pi)$ *does not contain any* k*-cycles* $(k\ge 1)$*.*

Then $\mathfrak{R}(J)=\{p_1,p_2,\dots,p_m\}$*.*

Proof. By Lemma 7 the set $M_i:=\{p_i\}$ $(i=\overline{1,m})$ is locally maximal.

Since $Fix(\pi)\subseteq\mathfrak{R}(\pi)$, then to prove this statement it is sufficient to show that $\mathfrak{R}(\pi)\subseteq Fix(\pi)$. Indeed, let $\{\mathfrak{R}_\lambda(\pi):\ \lambda\in\Lambda\}$ be the family of all chain transitive components of $\mathfrak{R}(\pi)$, then $\mathfrak{R}(\pi)=\coprod\{\mathfrak{R}_\lambda(\pi):\ \lambda\in\Lambda\}$. By Lemma 3 and Theorem 3 for any $\lambda\in\Lambda$ there exists a number $i\in\{1,\dots,m\}$ such that $\mathfrak{R}_\lambda(\pi)=\{p_i\}$ and consequently $\mathfrak{R}(\pi)\subseteq Fix(\pi)$. Theorem is proved. □

Remark 8. Note that using the same arguments as in the proof of Theorem 4 and some properties of the chain current set we can establish a more general statement. Namely, the following statement holds.

Suppose that the following conditions hold:

1) $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J its Levinson center;

2) M is a closed and invariant subset of $\mathfrak{R}(\pi)$;

3) $M=\bigcup_{k=1}^{m}M_k$, $M_i\bigcap M_j=\emptyset$ (for all $i\ne j$) and M_k $(k=1,\dots,m)$ is closed and invariant;

3) $\Delta_X\subseteq M$;

4) the family $M_1,\dots,M_m$ of subsets does not contain any l-cycles $(l\ge 1)$.

Then $\mathfrak{R}(J)=M$.

6 Non-autonomous gradient-like dynamical systems

Definition 12. Let $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ be two dynamical systems and $\mathbb{T}_1\subseteq\mathbb{T}_2$. A triplet $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is called a non-autonomous dynamical system, where $h:X\mapsto Y$ is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$, i. e., $h(\pi(t,x))=\sigma(h(x),t)$ for all $x\in X$ and $t\in\mathbb{T}_1$.

Definition 13. A mapping $\phi:Y\mapsto X$ is called:

1) a section of bundle space (X,h,Y) if $h\circ\phi=Id_Y$, i.e., $h(\phi(y))=y$ for all $y\in Y$;

2) an invariant section of non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ if $h\circ\phi=Id_Y$ and $\phi(\sigma(t,y))=\pi(t,\phi(y))$ for all $y\in Y$ and $t\in\mathbb{T}_1$.

Definition 14. A positive invariant (respectively, negative invariant or invariant) subset M of dynamical system $(X,\mathbb{T},\pi)$ is called dynamically decomposable if there are two positive invariant (respectively, negative invariant or invariant) subsets M_i $(i=1,2)$ of M such that:

1) $M_1\bigcap M_2=\emptyset$;

2) $M=M_1\bigcup M_2$.

Otherwise M is called dynamically indecomposable.

Definition 15. A dynamical system $(X,\mathbb{T},\pi)$ is called minimal if $H(x)=X$ for all $x\in X$, where $H(x):=\overline{\{\pi(t,x):\ t\in\mathbb{T}\}}$.

Definition 16. The non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is called gradient-like if the following conditions hold:

1) the space Y is compact and $(Y,\mathbb{T}_2,\sigma)$ is minimal;

2) the dynamical system $(X,\mathbb{T}_1,\pi)$ is compact dissipative and J is its Levinson center;

3) there are a finite number of invariant sections $\phi_1,\phi_2,\dots,\phi_m$ of non-autonomous dynamical system $(X,\mathbb{T}_1,\pi)$ such that $\Delta_X=\coprod_{i=1}^m\phi_i(Y)$.

Let $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system, $X_y:=h^{-1}(y)=\{x\in X:\ h(x)=y\}$ $(y\in Y)$ and $p\in X$. Denote by $W^s_y(p):=\{x\in X_y:\ \lim\limits_{t\to+\infty}\rho(\pi(t,x),\pi(t,p))=0\}$ and $W^u_y(p):=\{x\in X_y:\ \lim\limits_{t\to-\infty}\rho(\gamma(t),\pi(t,p))=0$ for certain $\gamma\in\Phi_x$ $\}$.

Theorem 5. *Let $\langle(X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a gradient-like non-autonomous dynamical system, then the following statements take place:*

1) for all $y\in Y$ and $x\in X_y$ there exists a unique invariant section ϕ_i such that $x\in W^s_y(\phi_i(y))$, i.e.,

$$\lim_{t\to+\infty}\rho(\pi(t,x),\phi_i(\sigma(t,y)))=0; \tag{5}$$

2) $J_y=\bigcup_{i=1}^m W^u_y(\phi_i(y))$ for all $y\in Y$.

Proof. Let $y\in Y$ and $x\in X_y$. Since $(X,\mathbb{T},\pi)$ is compact dissipative, then the positive semi-trajectory $\Sigma^+_x:=\{\pi(t,x):\ t\geq 0\}$ of point x is relatively compact and, consequently, its ω–limit set ω_x is a nonempty, compact, invariant and dynamically indecomposable set. Under the conditions of Theorem 5 we have $\omega_x\subseteq\Delta_X=\bigcup_{i=1}^m\phi_i(Y)$. Note that

$$\phi_i(Y)\bigcap\phi_j(Y)=\emptyset$$

for all $i\neq j$ $(1\leq i,j\leq m)$. In fact, if we suppose the contrary, then there are $i_0\neq j_0$ $(1\leq i_0,j_0\leq m)$ and y_0 such that $\phi_{i_0}(y_0)=\phi_{j_0}(y_0)$. Since Y is minimal, then for any $y \in Y$ there exists a sequence $\{t_n\} \subseteq \mathbb{T}$ such that $y = \lim_{t\to+\infty} \sigma(t_n, y_0)$ and, consequently, $\phi_{i_0}(y) = \lim_{t\to+\infty} \phi_{i_0}(\sigma(t_n, y_0)) = \lim_{t\to+\infty} \phi_{j_0}(\sigma(t_n, y_0)) = \phi_{j_0}(y)$. The obtained contradiction proves our statement. Thus there exists a unique natural number $1 \le i \le m$ such that $\omega_x = \phi_i(Y)$ because $\phi_i(Y)$ is a minimal set of the dynamical system $(X, \mathbb{T}, \pi)$ and, consequently,

$$\omega_x \bigcap X_y = \phi_i(y)$$

for all $y \in Y$. Now we will establish the equality (5). If we suppose that it is not true, then there are $y_0 \in Y$, $x_0 \in X_{y_0}$, $\varepsilon_0 > 0$, and $t_n \to +\infty$ such that

$$\rho(\pi(t_n, x_0), \phi_i(\sigma(t_n, y_0))) \ge \varepsilon_0. \tag{6}$$

Under the conditions of Theorem 5 we may suppose that the sequences $\{\sigma(t_n, y_0)\}$ and $\{\pi(t_n, x_0)\}$ are convergent. Denote by $\bar{y}$ and $\bar{x}$ their limits respectively then from (6) we obtain $\bar{x} \ne \phi_i(\bar{y})$. On the other hand $\bar{x} \in X_{\bar{y}} \bigcap \omega_{x_0} = \{\phi_i(\bar{y})\}$, i.e., $\bar{x} = \phi_i(\bar{y})$. The obtained contradiction completes the proof of the first statement of Theorem 5.

Now we will prove the second statement of Theorem 5. Let $y \in Y$, $x \in W^u_y(\phi_i(y))$ and $\gamma \in \Phi_x$, then $x \in J$. In fact, $\gamma(\mathbb{S})$ is relatively compact. Since $\alpha_\gamma \subseteq \Delta_X \subseteq J$, then there exists a sequence $\tau_n \to -\infty$ such that $\gamma(\tau_n) \to p \in J$. Since the Levinson center J of dynamical system $(X, \mathbb{T}, \pi)$ attracts every compact subset from X, then in particular it attracts also the compact subset $\overline{\gamma(\mathbb{S})}$ and, consequently, we have $\rho(x, J) = \lim_{n\to\infty} \rho(\pi(-\tau_n, \gamma(\tau_n)), J) = 0$. This means that $x \in J$. Thus to finish the proof it is sufficient to show that $J_y \subseteq \bigcup_{i=1}^m W^u_y(\phi_i(y))$ for all $y \in Y$. Let $y \in Y$, $x \in J_y$ and $\gamma \in \Phi_x$. Note that $\alpha_\gamma \bigcap X_y \ne \emptyset$. In fact, let $\tau_n \to -\infty$ such that $\sigma(\tau_n, y) \to y$. Since $\gamma(\mathbb{S}) \subseteq J$ we may suppose that the sequence $\{\gamma(\tau_n)\}$ is convergent, then its limit p belongs to $\alpha_\gamma \bigcap X_y$. Evidently, under the conditions of Theorem 5 we have $\alpha_\gamma \bigcap X_y \subseteq \{\gamma_1(y), \gamma_2(y), \ldots, \gamma_m(y)\}$. Thus there are natural numbers $1 \le i_1 < i_2 < \ldots < i_k \le m$ such that $\alpha_\gamma \bigcap X_y = \{\gamma_{i_1}(y), \gamma_{i_2}(y), \ldots, \gamma_{i_k}(y)\}$. Note that the following equality

$$\lim_{t\to-\infty} \inf_{1\le l\le k} \rho(\gamma(t), \phi_{i_l}(\sigma(t, y))) = 0$$

takes place. Suppose that it is not true, then there are $\varepsilon_0 > 0$ and $\tau_n \to -\infty$ such that

$$\rho(\gamma(\tau_n), \phi_{i_l}(\sigma(\tau_n, y))) \ge \varepsilon_0 \tag{7}$$

for all $l = 1, 2, \ldots, k$. Under the conditions of Theorem 5 we may suppose that the sequences $\{\sigma(\tau_n, y)\}$ and $\{\gamma(\tau_n)\}$ converge. Denote by $\bar{y}$ (respectively, $\bar{x}$) the limit of the sequence $\{\sigma(\tau_n, y)\}$ (respectively, $\{\gamma(\tau_n)\}$), then $\bar{x} \in X_{\bar{y}}$ and $\bar{x} \in \alpha_\gamma \bigcap X_{\bar{y}} = \{\gamma_{i_1}(\bar{y}), \gamma_{i_2}(\bar{y}), \ldots, \gamma_{i_k}(\bar{y})\}$. On the other hand, passing to the limit in (7) as $n \to \infty$ we obtain $\bar{x} \notin \{\gamma_{i_1}(\bar{y}), \gamma_{i_2}(\bar{y}), \ldots, \gamma_{i_k}(\bar{y})\}$. The obtained contradiction proves our statement. Let us show that there exists a number $1 \le i_0 \le k$ such that

$$\lim_{t\to-\infty} \rho(\gamma(t), \phi_{i_0}(\sigma(t, y)) = 0.$$

Denote by $r := \inf\{\rho(\phi_i(y), \phi_j(y)) : \ y \in Y, 1 \le i, j \le m; \ i \ne j\}$. From (6) it follows that the number r is positive. For a number ε, $0 < \varepsilon < r/3$, we will find $L(\varepsilon) > 0$ such that

$$\inf\{\rho(\gamma(t), \phi_{i_l}(\sigma(t, y)) : 1 \le l \le k\} < \varepsilon$$

for all $t < -L(\varepsilon)$. Let $t_0 < -L(\varepsilon)$, then there exists $1 \le i_{l_0} \le k$ such that

$$\rho(\gamma(t_0), \phi_{i_{l_0}}(\sigma(t_0, y))) < \varepsilon.$$

Assume $\delta(t_0) := \sup\{\tilde{\delta} : \ \rho(\gamma(t_0), \phi_{i_{l_0}}(\sigma(t_0, y))) < \varepsilon$ for all $t \in [t_0 - \tilde{\delta}, t_0]\}$. Let us show that $\delta(t_0) = +\infty$. Suppose the contrary, then

$$\rho(\gamma(t_0'), \phi_{i_{l_0}}(\sigma(t_0', y))) \ge \varepsilon$$

where $t_0' = t_0 - \delta(t_0)$, and there exists $k_0 \ne i_0$ $(1 \le k_0 \le k)$ such that

$$\rho(\gamma(t_0'), \phi_{i_{l_0}}(\sigma(t_0', y))) < \varepsilon.$$

On the other hand,

$$\rho(\gamma(t_0'), \phi_{i_{k_0}}(\sigma(t_0', y))) \ge \rho(\phi_{i_{l_0}}(\sigma(t_0', y)), \phi_{i_{k_0}}(\sigma(t_0', y))) - \rho(\phi_{i_0}(\sigma(t_0', y), \gamma(t_0')) > r - \varepsilon > 2\varepsilon. \tag{8}$$

Inequality (8) contradicts the assumption. So, we found $L(\varepsilon) > 0$ and $1 \le i_0 \le k$ such that

$$\rho(\gamma(t), \phi_{i_0}(\sigma(t, y))) < \varepsilon$$

for all $t \le -L(\varepsilon)$. Let us show that the number i_0 does not depend on the choice of ε. In fact, if we suppose the contrary, then we can find numbers ε_1 and ε_2, natural numbers i_1 and i_2 $(1 \le i_1 \ne i_2 \le k)$, and $L(\varepsilon_1) > 0$ and $L(\varepsilon_2) > 0$ satisfying the conditions mentioned above. Assume $L := \max(L(\varepsilon_1), L(\varepsilon_2))$, then

$$\rho(\phi_{i_1}(\sigma(t, y)), \phi_{i_2}(\sigma(t, y))) \le \rho(\phi_{i_1}(\sigma(t, y)), \gamma(t)) + \rho(\gamma(t), \phi_{i_2}(\sigma(t, y))) \le \varepsilon_1 + \varepsilon_2 < 2r/3 < r. \tag{9}$$

Inequality (9) contradicts the choice of r. Theorem is completely proved. □

Definition 17. Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a non-autonomous dynamical system and Y be a compact minimal set. A compact minimal set $M \subseteq X$ is called [20, 25] an m–fold covering of Y if *card* $h^{-1}(y) = m$ for all $y \in Y$.

Theorem 6 (see [6, 20, 25, 26]). *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a non-autonomous dynamical system and Y be a compact minimal set, then the following statements are equivalent:*

1) a compact minimal set $M \subseteq X$ is an m–fold covering of Y;

2) (a) there exists a $y_0 \in Y$ such that card $h^{-1}(y_0) = m$;

(b) the minimal set M is distal with respect to $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$, i.e., $\inf\limits_{t\in\mathbb{T}} \rho(\pi(t,x_1),\pi(t,x_2)) > 0$ for all $(x_1,x_2) \in X \bigotimes X := \{(x_1,x_2) : x_1,x_2 \in X$ with condition $h(x_1)=h(x_2)\}$.

Theorem 7. *Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

1) *Y is a compact minimal set;*
2) *the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center;*
3) *there are a finite number of minimal compact subsets $M^1, M^2,\dots,M^k \subseteq X$ such that $\Delta_X \subseteq \bigcup_{i=1}^k M^i$;*
4) *for every $i=1,2,\dots,k$ there exists a natural number m_i such that $card M^i_y = m_i$ for all $y\in Y$, where $M_y := h^{-1}(y)$.*

Then the following statements take place:

1) *for all $y\in Y$ and $x\in X_y$ there exist a unique natural number $1\le i\le k$ and a point $p_i\in M^i_y$ such that $x\in W^s_y(p_i)$, i.e.,*

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,p_i)) = 0; \tag{10}$$

2) *$J_y = \bigcup_{i=1}^m \bigcup\limits_{p\in M^i_y} W^u_y(p)$ for all $y\in Y$.*

Proof. This statement can be proved with slight modification of the proof of Theorem 5. □

Definition 18. The point x of dynamical system $(X,\mathbb{T},\pi)$ is called τ ($\tau\in\mathbb{T}$) periodic if Φ_x contains a motion γ such that $\gamma(t+\tau)=\gamma(t)$ for all $t\in\mathbb{S}$.

Denote by $Per(\pi)$ the set of all periodic points of $(X,\mathbb{T},\pi)$, $W^s(p) := \{x\in X : \lim\limits_{t\to+\infty}\rho(\pi(t,x),\pi(t,p)) = 0\}$ and $W^u(p) := \{x\in X : \lim\limits_{t\to-\infty}\rho(\gamma(t),\pi(t,p)) = 0$ for certain $\gamma\in\Phi_x\}$.

Corollary 2. *Suppose that $(X,\mathbb{T},\pi)$ is an autonomous dynamical system with discrete time (i.e., $\mathbb{T}\subseteq\mathbb{Z}$) and the following conditions are fulfilled:*

1) *the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center;*
2) *the set $Per(\pi)$ contains a finite number of points, i. e., $Per(\pi)=\{p_1,p_2,\dots,p_m\}$;*
3) *$\Delta_X \subseteq Per(\pi)$.*

Then the following statements hold:

1) for all $x \in X$ *there exists a unique natural number* $1 \le i \le m$ *such that* $x \in W^s_y(p_i)$, *i. e.,*

$$\lim_{t \to +\infty} \rho(\pi(t,x), \pi(t,p_i)) = 0;$$

2) $J = \bigcup_{i=1}^m W^u(p_i)$.

Proof. Formulated statements directly follow from Theorem 7. □

Remark 9. If the dynamical system $(X, \mathbb{T}, \pi)$ with continuous time (i. e., $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}_+$) admits at least one nontrivial periodic point, then:

1. $Per(\pi)$ contains a continuum subset. In fact, if $p \in Per(\pi)$ is a τ–periodic point, then $\pi(t,t_1) \neq \pi(t_2,p)$ for all $t_1, t_2 \in (0,\tau)$ $(t_1 \neq t_2)$ and, consequently, $\{\pi(t,p) : \ t \in [0,\tau]\} \subseteq Per(\pi)$ and $\{\pi(t,p) : \ t \in [0,\tau]\}$ is homeomorphic to $[0,\tau]$.

2. Corollary 2 does not take place.

The last statement (second item of Remark 9) can be confirmed by the following example.

Example 1. Let us consider the dynamical system defined on the plane $\mathbb{R}^2$ by the following rule. Let the origin $O(0,0)$ be a stationary point, the unit circumference S^1 be the trajectory of the periodic motion with the period $\tau = 1$. The rest of motions will be not singular. And besides we assume that every semi-trajectory Σ^+_x is not un. st. $\mathcal{L}^+\Sigma^+_x$ for every point $x \in \mathbb{R}^2 \setminus (S^1 \cup O)$. The described dynamical system is given by the system of differential equations which in polar coordinates looks as the following:

$$\begin{cases} \dot{r} = (r-1)^2 \\ \dot{\varphi} = r. \end{cases}$$

It is easy to see that ω–limit set of the point x is a trajectory of 1–periodic point for all $x \in \mathbb{R}^2 \setminus O$, but the point x itself is not asymptotically 1–periodic, since Σ^+_x is not un.st. $\mathcal{L}^+\Sigma^+_x$ (see Theorem 1.3.1 from [11]). In this example we have $\Delta_X = S^1 \cup \{O\}$.

References

[1] Auslander J., Seibert P. *Prolongations and stability in dynamical systems.* Ann. Inst. Fourier (Grenoble), 1964, **14**, 237–268.

[2] Babin A. V., Vishik M. I. *Regular Attractor of Semi-Groups and Evolutional Equations.* Journal Math. Pures et Appl., 1983, **62**, 172–189.

[3] Babin A. V., Vishik M. I. *Attractors of Evolutionary Equations.* Moscow, Nauka, 1989 (in Russian). [English translation in North–Holland, Amsterdam, 1992.]

[4] Bhatia N. P., Szegö G. P. *Stability Theory of Dynamical Systems.* Lecture Notes in Mathematics. Springer, Berlin–Heidelberg–New York, 1970.

[5] Bondarchuk V. S., Dobrynskii V. A. *Dynamical Systems with Hyperbolic Center.* Functional and Differential-Difference Equations. Institute of Mathematics, Academy of Sciences of Ukraine, 1974, 13–40 (in Russian).

[6] Bronsteyn I. U. *Extensions of Minimal Transformation Group.* Noordhoff, 1979.

[7] Bronstein I. U. *Nonautonomous Dynamical Systems.* Ştiintsa, Chişinău, 1984 (in Russian).

[8] Bronstein I. U., Burdaev B. P. *Chain Recurrence and Extensions of Dynamical Systems.* Mat. Issled., 1980, No. 55 (1980), 3–11 (in Russian).

[9] Carvalho Alexandre N., Langa Jose A., Robinson James C. *Attractors for infinite-dimensional non-autonomous dynamical systems.* Applied Mathematical Sciences, vol. 182. Springer, New York, 2013.

[10] Cheban David N. *Global Attractors of Nonautonomous Dissipative Dynamical Systemst.* Interdisciplinary Mathematical Sciences, vol. 1, River Edge, NJ: World Scientific, 2004, xxvi+502 pp.

[11] Cheban David N. *Asymptotically Almost Periodic Solutions of Differential Equations.* Hindawi Puidblishing Corporation, 2009, xvii+186 pp.

[12] Cheban David N. *Global Attractors of Set-Valued Dynamical and Control Systems.* Nova Science Publishers Inc, New York, 2010, xvii+269 pp.

[13] Choi S. K., Chu C. K., Park J. S. *Chain Recurrent Sets for Flows on Non-Compact Spaces.* J. Dyn. Diff.Eqns, 2002, **14**, No. 4, 597–610.

[14] Conley C. *Isolated Invariant Sets and the Morse Index.* Region. Conf., Ser. Math., 1978, No. 38, Am. Math. Soc., Providence, RI.

[15] Conley C. *The gradient structure of a flows: I.* Ergodic Theory & Dynamical Systems, 1988, **8***, 11–26.

[16] Hale J. K. *Asymptotic Behaviour of Dissipative Systems.* Amer. Math. Soc., Providence, RI, 1988.

[17] Hale J. K. *Stability and Gradient Dynamical Systems.* Rev. Math. Complut., 2004, **17** (10), 7–57.

[18] Hirsch M. W., Smith H. L., Zhao X.-Q. *Chain Transitivity, Attractivity, and Strong Repellors for Semidynamical Systems.* J. Dyn. Diff. Eqns, 2001, **13**, No. 1, 107–131.

[19] Hurley Mike. *Chain recurrence, semiflows, and gradients.* Journal of Dynamics and Differential Equations, 1995, **7**, No. 3, 437–456.

[20] Levitan B. M., Zhikov V. V. *Almost Periodic Functions and Differential Equations.* Cambridge Univ. Press, Cambridge, 1982.

[21] Pata Vittorino. *Gradient systems of closed operators.* Central European Journal of Mathematics, 2009, **7**(3), 487–492.

[22] Patrao M. *Morse Decomposition of Semiflows on Topologicl Spaces.* Journal of Dinamics and Differential Equations, 2007, **19**, No. 1, 215–241.

[23] Raugel G. *Global Attractors in Partial Differential Equations.* Handbook of Dynamical Systems, vol. 2 (Edited by B. Fiedler), Elsevier 2002, Chapter VII, 885–982.

[24] Robinson C. *Dynamical Systems: Stabilty, Symbolic Dynamics and Chaos (Studies in Advanced Mathematics),* Boca Raton Florida: CRC Press, 1995.

[25] Sacker R. J., Sell G. R. *Finite Extensions of Minimal Transformation Groups.* Memoirs of the American Math. Soc., 1974, **190**, 325–334.

[26] Sacker R. J., Sell G. R. *Lifting Properties in Skew–Product Flows with Applications to Differential Equations.* Memoirs of the American Math. Soc., vol. 190, Providence, R. I., 1977.

[27] SHARKOVSKY A. N., DOBRYNSKY V. A. *Nonwandering points of dynamical systems.* Dynamical Systems and Stability Problems of the Solutions to Differential Equations. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1973, 165–174 (in Russian).

[28] SIBIRSKY K. S. *Introduction to Topological Dynamics.* Kishinev, RIA AN MSSR, 1970 (in Russian). [English translationn: Introduction to Topological Dynamics. Noordhoff, Leyden, 1975]

[29] STUART A., HUMPHRIES A. R. *Dynamical Systems and Numerical Analysis.* CUP, 1996.

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