ON THE STABILITY OF GRADIENT-LIKE SYSTEMS

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Abstract. This paper is dedicated to the study the problem of stability of some classes of gradient-like system of differential equations (both autonomous and non-autonomous cases). We present two main results. The first is a generalization of Absil & Kurdyka theorem about stability of gradient systems with analytic potential for non-gradient systems. Secondly we generalize for some classes of gradient-like non-autonomous systems the well-known Lagrange-Dirichlet theorem.

1. Introduction

Let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}^n$ be the real n-dimensional Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$. Let $r > 0$ denote by $B(0, r) := \{x \in \mathbb{R}^n : \ ||x|| < r\}$ (respectively, $B[0, r] := \{x \in \mathbb{R}^n : \ ||x|| \le r\}$). If $U \subset \mathbb{R}^n$ (respectively, $V \subset \mathbb{R}^m$) is an open subset, the by $C^k(U, V)$ we denote the set of all k-time continuously differentiable functions $f : U \mapsto V$ equipped with the compact-open topology and by $D^l f(x)$ the Fréchet derivative of function f of order l at the point $x \in U$. In particular, if $f \in C^1(U, \mathbb{R})$ then we put

$$\nabla f(x) := Df(x) = (\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x)).$$

A point $a \in U$ is called critical for $f \in C^1(U, \mathbb{R})$, if $\nabla f(a) = 0$.

Given a real-analytic function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and a critical point $a \in \mathbb{R}^n$ and U be neighborhood of a. By S. Lojasiewicz was established the following remarkable result.

Lemma 1.1. [12, 13] ***(Lojasiewicz's inequality)*** *Let $f : U \mapsto \mathbb{R}$ be real analytic function. Then, there exists an open neighborhood W of a in U, and $c, \rho \in \mathbb{R}$ such that $c > 0$, $0 < \rho < 1$ and*

$$||\nabla f(x)|| \ge c|f(x) - f(a)|^\rho \tag{1}$$

for all $x \in W$, where $\nabla f(x)$ is the gradient vector.

Consider a system of differential equations

$$x' = f(x), \ (x \in U \subset \mathbb{R}^n), \tag{2}$$

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where U is an open set.

Suppose that the function $f \in C(U, \mathbb{R}^n)$ and satisfies the conditions that ensure the uniqueness of solutions in U. The solution of (2) passing through the point $x_0 \in U$ at the initial moment $t = 0$ will be denoted by $x(x_0, t)$, i.e., $x(x_0, 0) = x_0$.

Let $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and we consider the gradient-system

$$x' = -\nabla V(x), \tag{3}$$

where $\nabla V(x)$ denotes the gradient of V at $x \in \mathbb{R}^n$.

We assume that $V : \mathbb{R}^n \mapsto \mathbb{R}$ is analytic on an open set $U \subset \mathbb{R}^n$ containing the origin, that $V(0) = 0$ and that $\nabla V(0) = 0$.

The Lojasiewicz inequality implies that if $V : \mathbb{R}^n \mapsto \mathbb{R}$ is real analytic function, then any bounded solution $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}^n)$ of the gradient flow (3) converges to a critical point of V as t tends to infinity. Namely, by S. Lojasiewicz was established the following theorem.

Theorem 1.2. [14] *Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a real analytic function and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}^n$ be a bounded solution of equation (3). Then there exists a critical point $a \in \mathbb{R}^n$ of (3) such that $\lim\limits_{t\to+\infty} \varphi(t) = a$.*

Definition 1.3. *Let U be a neighborhood of the origin $0 \in \mathbb{R}^n$, $V \in C(U, \mathbb{R})$ and $V(0) = 0$. The function $V \in C(U, \mathbb{R})$ has:*

- *a local minimum at the point $x = 0$, if there exists a positive number ε such that $V(x) \geq 0$ for all $x \in B(0, \varepsilon)$;*
- *a strict minimum at the point $x = 0$, if $V(x) > 0$ for all $x \in B(0, \varepsilon) \setminus \{0\}$.*

Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin $0 \in \mathbb{R}^n$, let $\varepsilon_0 > 0$ be a positive number such that $B(0, \varepsilon_0) \subset U$ and $f(0) = 0$.

Definition 1.4. *The trivial solution of equation (2) is said to be:*

(i) *Lyapunov stable, if for arbitrary $0 < \varepsilon < \varepsilon_0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $||x_0|| < \delta$ implies $||x(x_0, t)|| < \varepsilon$ for all $t \in \mathbb{R}_+$;*
(ii) *attractive, if there exists a number $0 < \nu < \varepsilon_0$ such that*
$$\lim_{t\to+\infty} ||x(x_0, t)|| = 0$$
for all $x \in B(0, \nu)$;
(iii) *asymptotically stable, if it is stable and attractive.*

In the work of Absil P.-A. and Kurdyka K. [1] it was investigated the problem of Lyapunov stability of stationary solutions for gradient system (3). They are proved that, if $x_0 \in \mathbb{R}^n$ is a local minimizer of V, then x_0 is a stable equilibrium point of equation (3). More exactly the following statement takes place.

Theorem 1.5. *(P.-A. Absila and K. Kurdyka* [1]*). Let V be real analytic in a neighborhood of $z \in \mathbb{R}^n$. Then the following statements hold:*

(i) *z is a stable equilibrium point of (3) if and only if it is a local minimum of V;*

(ii) *z is an asymptotically stable equilibrium point of (3) if it is a strict local minimum of V.*

This result, which may fail if V is only assumed C^∞ [1, 2], can be easily extended to any gradient system (3) with C^1 function V satisfying to Lojasiewicz's inequality (1).

One of the aim of this paper is generalization of Absila & Kurdyka's result for arbitrary (non-gradient) system (2) which possesses a Lyapunov function satisfying some special condition. To formulate this result we need some notation and notions.

Let $U \subset \mathbb{R}^n$, $x(x_0,t)$ be the solution of equation (2) passing through the point $x_0 \in \mathbb{R}^n$ at the initial moment $t=0$ such that $x(x_0,t) \in U$ for all $t \geq 0$ and $V: U \mapsto \mathbb{R}$. Denote by

$$\frac{dV(x_0)}{dt}\Big|_{(2)} := \frac{dV(x(x_0,t))}{dt}\Big|_{t=0} = \lim_{t\to 0^+} \frac{V(x(x_0,t)) - V(x_0)}{t}$$

(or shortly $\dot{V}(x_0)$) the time derivative of function V by virtue of system (2) at the point x_0.

Remark 1.6. *If $V \in C^1(U,\mathbb{R})$, then*

$$\frac{dV(x)}{dt}\Big|_{(2)} = \langle \nabla V(x), f(x)\rangle.$$

Let $\mathfrak{C}$ be the set of all continuous functions $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties:

(i) $\phi(0) = 0$;
(ii) $\phi(u) > 0$ for all $u > 0$;
(iii) ϕ is continuously differentiable on $(0, +\infty)$;
(iv) $\phi'(u) > 0$ for all $u > 0$, where ϕ' is the derivative of ϕ.

Remark 1.7. *Not that the function $\phi(u) := c\frac{u^\alpha}{\alpha}$ $(c, \alpha > 0)$ belongs to $\mathfrak{C}$.In particular, if $\alpha = 1$, then $\phi(u) = cu$ belongs to $\mathfrak{C}$. The function $\phi(u) = \ln(1+u)$ also belongs to $\mathfrak{C}$ and so on.*

Suppose that for (2) there exists a neighborhood U of $x = 0$ and continuously differentiable function $V : U \to \mathbb{R}_+$ such that

1) $V(0) = 0$;
2) there exists a function $\phi \in \mathfrak{C}$ such that

$$\frac{d\phi(V(x))}{dt}\Big|_{(2)} \leq -||f(x)||$$

for all $x \in U$.

Then the trivial solution of (2) is stable.

We give the proof of this statement in the second section (Theorem 2.3).

The other important object of this paper is some class of second order systems (so called gradient-like second order systems on the torus).

Let $U \subset \mathbb{R}^n$ be neighborhood of the origin $0 \in \mathbb{R}^n$, $F \in C^1(U,\mathbb{R})$ and $\nabla F(0) = 0$. The problem of stability of trivial solution of equation

$$x'' + \nabla F(x) = 0 \tag{4}$$

is a classical problem. Below we give a well-known result in this direction.

Theorem 1.8. *(**Lagrange-Dirichet** theorem) Suppose that following conditions hold:*

(i) $F(0) = 0$;
(ii) $F(x) > 0$ *for all* $x \in U \setminus \{0\}$.

Then the trivial solution of equation (4) is Lyapunov stable.

In the second part of paper we generalize of this result for two class of systems:

(i) for the second order non-autonomous systems

$$x'' + G(\sigma(t,\theta), x') + \nabla F(x) = 0, \ \ ((\theta, x) \in \mathcal{T}^m \times \mathbb{R}^n) \tag{5}$$

where $G \in C(\mathcal{T}^m \times \mathbb{R}^n, \mathbb{R}^n)$ with the condition $\langle G(\theta,x), x\rangle \geq 0$ for all $(\theta,x) \in \mathcal{T}^m \times \mathbb{R}^n$ and $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is the dynamical system on the torus $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$ generated by equation

$$\theta' = \Phi(\theta). \tag{6}$$

In particular, if $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is an irrational winding of $\mathbb{T}^m$, i.e., $\Phi(\theta) = \nu = (\nu_1, \nu_2, \dots, \nu_m)$ and $\nu_1, \nu_2, \dots, \nu_m$ are some real rationally independent number, then equation (7) is a non-autonomous equation with the quasi-periodic (with respect to time) coefficients;

(ii)

$$x'' + \nabla_x F(\sigma(t,\theta), x) = 0, \ \ (x \in \mathbb{R}^n, \ \theta \in \mathcal{T}^m) \tag{7}$$

where $\Phi \in C^(\mathcal{T}^m, \mathbb{R}^m)$ and $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is the dynamical system on the torus $\mathcal{T}^m$ generated by equation (6).

We present the sufficient conditions for the uniform Lyapunov stability of trivial solution for equation (5) (Theorem 3.3) and also for equation (7) (Theorem 3.8).

2. Stability by the semi-definite Lyapunov functions

In this section we will study the problem of Lyapunov stability of trivial solution for some class of gradient-like systems using so-called semi-definite Lyapunov functions.

Definition 2.1. *Let* $U \subset \mathbb{R}^n$ *be a neighborhood of origin* $0 \in \mathbb{R}^n$. *A function* $V : U \mapsto \mathbb{R}$ *is called* [3, 9] *(see also the bibliography therein) a semi-definite Lyapunov function for*

$$x' = f(x), \tag{8}$$

where $f \in C(U, \mathbb{R}^n)$ *and* $f(0) = 0$, *if the following conditions are fulfilled:*

(i) $V(0) = 0$ *and* $V(x) \geq 0$ *for all* $x \in U$;
(ii) $\dot{V}(x) \leq 0$ *for all* $x \in U$, *where* $\dot{V}(x) = \frac{dV(x)}{dt}\big|_{(8)}$.

Remark 2.2. *1. It is well-known that if equation (8) admits a positive-definite Lyapunov function V with $\dot{V}(x) \leq 0$ for all $x \in U$, then the trivial solution of equation (8) is stable (Lyapunov stability theorem).*

2. If equation (8) admits a semi-definite Lyapunov function, then generally speaking, the trivial solution is not Lyapunov stable (see, for example, [7]-[9]*).*

3. In the works [7]-[9] *it was indicated some additional conditions which assure Lyapunov stability (respectively, asymptotic stability or instability) of trivial solution for (8), if it admits a semi-definite Lyapunov function.*

4. In this section we will give some new conditions which guarantee stability (respectively, asymptotic stability or instability) which can not be obtained from results of B. Kalitine [7]-[9].

2.1. **Stability.**

Theorem 2.3. *Suppose that for (2) there exists a neighborhood U of $x = 0$ and continuously differentiable function $V : U \to \mathbb{R}_+$ such that*

1) $V(0) = 0$;
2) *there exists a function $\phi \in \mathfrak{E}$ such that*
$$\frac{d\phi(V(x))}{dt}\Big|_{(2)} \leq -||f(x)||$$
for all $x \in U$.

Then the trivial solution of (2) is stable.

Proof. Let $\varepsilon > 0$ be an arbitrary number $\varepsilon > 0$ so that the closure of the ball $\overline{B(0,\varepsilon)}$ is contained in U. By continuity of V and assumption 1), we can choose the number of $\delta \in]0, \varepsilon/2[$ such that

$$V(x) < \phi^{-1}(\varepsilon/2) \ \forall x \in B(0,\delta), \tag{9}$$

where by ϕ^{-1} is denoted the inverse function for ϕ. Let $x_0 \in B(0,\delta)$ and $x(x_0,t)$ be the corresponding solution of system (2). Note that
$$\frac{d\phi(V(x(x_0,t)))}{dt} \leq -||f(x(x_0,t))||$$
and, consequently,
$$||f(x(x_0,t))|| \leq -\frac{d\phi(V(x(x_0,t)))}{dt}. \tag{10}$$
Denote by $l(x_0,t)$ the length of trajectory $x(x_0,t)$ on the segment $[0,t]$, then from (10) we obtain
$$l(x_0,t) = \int_0^t ||x'(x_0,\tau)||d\tau = \int_0^t ||f(x(x_0,\tau))||d\tau \leq \tag{11}$$
$$-(\phi(V(x(x_0,t))) - \phi(V(x_0)))) \leq \phi(V(x_0)).$$
Thus, under the conditions of Theorem, the length of trajectory $x(x_0,t)$ is finite and, consequently, its ω-limit set ω_{x_0} consists a single point $p = p(x_0)$, i.e., $\lim\limits_{t\to+\infty} x(x_0,t) = p$.

Therefore, by (11) as long as $x(x_0,t) \in B(0,\varepsilon)$, the arc trajectory $x(x_0,t)$ of solution will be in the ball $B(0,\varepsilon)$ and its distance from the surface of the bull is less than $\phi(V(x_0))$. Taking into account (9) from the last inequality we conclude that the distance of the arc trajectory $x(x_0,t)$ is less than $\varepsilon/2$. But this means that $x(x_0,t) \in B(0,\varepsilon)$ for all $t > 0$, i.e., the trivial solution $x = 0$ is stable in the sense of Lyapunov. □

Remark 2.4. *1. If there exist a neighborhood U of $x = 0$ and continuously differentiable function $V : U \to \mathbb{R}_+$, $c > 0$ and $\rho \in (0,1)$ such that $V(0) = 0$, $V(x) \geq 0$ for all $x \in U$ and $\frac{dV(x)}{dt}\big|_{(2)} \leq -c||f(x)||V(x)^\rho$, then condition 2) of Theorem 2.3 holds. In fact, if we take $\phi(u) := \frac{u^{1-\rho}}{c(1-\rho)}$, then $\phi \in \mathfrak{C}$ and $\frac{d\phi(V(x))}{dt}\big|_{(2)} \leq -||f(x)||$ for all $x \in U$.*

2. If there exist a neighborhood U of $x = 0$ and continuously differentiable function $V : U \to \mathbb{R}_+$, $c > 0$ and $\rho \in (0,1)$ such that $V(0) = 0$, $V(x) \geq 0$ for all $x \in U$ and $\frac{dV(x)}{dt}\big|_{(2)} \leq -||f(x)||(1 + V(x))$, then condition 2) of Theorem 2.3 is fulfilled. To see this fact it is sufficient to take in quality of ϕ the function $\phi(u) = \ln(1 + u)$.

Example 2.5. Let $V \in C^1(U,\mathbb{R})$ be a real analytic function with the conditions $V(0) = 0$ and $\nabla V(0) = 0$. Consider a gradient system of differential equations (3).

Suppose that the potential V of gradient system (3) has a local minimum at the origin. We take V as a Lyapunov function, then

$$V(x) \geq 0 \text{ and } V(0) = 0 \tag{12}$$

and, consequently, condition 1) of Theorem 2.3 holds. We will show that condition 2) of Theorem 2.3 is also fulfilled. In fact. Then by Lojasiewicz Lemma there exist numbers $c > 0$ and $0 < \rho < 1$ such that

$$||\nabla V(x)|| \geq cV(x)^\rho \tag{13}$$

in some neighborhood $W \subset U$ of 0 and, consequently, we have

$$\dot{V}(x) = -||\nabla V(x)||^2 \leq -c||f(x)||(V(x))^\rho$$

for all $x \in W$. Thus, all the conditions of Theorem 2.3 (see also Remark 2.4, item 1.) are fulfilled, which means that if a function $f(x)$ is analytic in a neighborhood of the origin and has a local minimum at this point, then the trivial solution of the gradient system is stable. In the work [1] it was proved that for real analytic system this statement is reversible.

Remark 2.6. *Let $f : U \mapsto \mathbb{R}_+$ be a C^1 function, satisfying the Lojaciewicz inequality (13) in some neighborhood $W \subset U$. Then it easy to check that the function V defined by (12) satisfies condition 2) (if we take $\phi(u) = \frac{u^{1-\rho}}{c(1-\rho)}$) of Theorem 2.3 and, consequently, the trivial solution of gradient system (3) is Lyapunov stable.*

Below we will give an example of system of differential equation which is not gradient, but for this system Theorem 2.3 is applicable.

Example 2.7. Consider a system of differential equation

$$\dot{x} = \varphi^\sigma(x)f(x),\ x \in B(0,h) \subset \mathbb{R}^n, \tag{14}$$

where h and σ are some positive numbers, $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$. In addition, $\varphi(0) = 0$ and the system (14) satisfies the conditions of existence, uniqueness and non-local extendability on the right in the ball $B(0, h)$.

Suppose that the following conditions are fulfilled:

(i)

$$\langle \nabla\varphi(0), f(0)\rangle < 0; \tag{15}$$

(ii)

$$\varphi(x) \geq 0, \text{ for all } \ x \in B(0, h).$$

We put $V(x) := \varphi(x) \geq 0$, then

$$\frac{dV(x)}{dt}\Big|_{(14)} = \varphi^{\sigma}(x)\langle \nabla\varphi(x), f(x)\rangle. \tag{16}$$

We will show that there are $h > 0$ and $0 < \rho < 1$ such that

$$\frac{dV(x)}{dt}\Big|_{(14)} \leq -||f(x)||V^{\rho}(x) \tag{17}$$

for all $x \in B(0, h)$. In fact, by inequality (15) there exists a positive number β such that

$$\langle \nabla\varphi(x), f(x)\rangle < -\beta \tag{18}$$

for all $x \in B(0, h)$, if the number $h > 0$ is small enough.

On the other hand taking into account that $\varphi(0) = 0$ and the continuity of the functions φ and f we can chose h such that

$$||f(x)||\varphi^{\rho-\sigma}(x) \leq \beta \tag{19}$$

for all $x \in B[0, h]$, where $\rho > \sigma$.

From (16)-(19) we obtain

$$\begin{gathered} \tfrac{dV(x)}{dt}\Big|_{(14)} = \varphi^{\sigma}(x)\langle \nabla\varphi(x), f(x)\rangle \leq \\ -\varphi^{\sigma}(x)\beta \leq -||f(x)||\varphi^{\rho}(x) = -||f(x)||V^{\rho}(x). \end{gathered}$$

If $0 < \sigma < \rho < 1$, then by Theorem 2.3 we conclude that the trivial solution of system (14) is Lyapunov stable.

Example 2.8. ***(Lienard's equation)*** Consider a differential equation

$$\ddot{y} + a\dot{y} + g(y, \dot{y}) = 0, \tag{20}$$

where g is a function defined by equality

$$g(y, \dot{y}) = bsign(\dot{y} + ay)|\dot{y} + ay|^{\sigma} \quad (0 < \sigma < 1)$$

and $a, b \in \mathbb{R}$ are some parameters.

If $\sigma = 0$, then the function g is piecewise constant and in this case equation (20) describes, in particular, with the dynamics of the servo system backlash in the contact device and the gearing [10].

Differential equation (20) can be rewrite as a system of equations on the phase plane $x = (x_1, x_2) \in \mathbb{R}^2$

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -bsign(x_2 + ax_1)|x_2 + ax_1|^\sigma - ax_2 \end{pmatrix} \tag{21}$$

Consider the Lyapunov function

$$V(x) = \frac{1}{2}(x_2 + ax_1)^2, \tag{22}$$

for which the time derivative obtained by virtue of (21) is given by

$$\dot{V}(x) = -b|x_2 + ax_1|^{1+\sigma}. \tag{23}$$

Assume that

$$b > 0 \tag{24}$$

and we will show that condition 2) of Theorem 2.3 is fulfilled.

Indeed, for the right-hand side of (21) we have

$$||f(x)|| = \sqrt{x_2^2 + (ax_2 + bsign(x_2 + ax_1)|x_2 + ax_1|^\sigma)^2}. \tag{25}$$

Let c be a positive constant figuring in condition 2) of Theorem 2.3. It is clear that for the number c there is a neighborhood U of the origin such that

$$b \ge c\sqrt{x_2^2 + (ax_2 + bsign(x_2 + ax_1)|x_2 + ax_1|^\sigma)^2}$$

for all $(x_1, x_2) \in U$.

Therefore, in view of (23) and (25) we have:

$$\dot{V}(x) = -b|x_2 + ax_1| \le$$
$$-c\sqrt{x_2^2 + (ax_2 + bsign(x_2 + ax_1)|x_2 + ax_1|^\sigma)^2}|x_2 + ax_1| =$$
$$-c||f(x)||\sqrt{2V(x)}.$$

Thus condition 2) of Theorem 2.3 holds and, consequently, under condition (24) the solution $y = \dot{y} = 0$ of equation (20) is stable.

3. Non-autonomous gradient-like systems

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$ with the frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m)$, i.e., $\sigma(t, \theta) := (\nu_1 t + \theta_1, \nu_2 t + \theta_2, \dots, \nu_m t + \theta_m)$ for all $t \in \mathbb{R}$ and $\theta := (\theta_1, \theta_2, \dots, \theta_m) \in \mathcal{T}^m$, where $\nu_1, \nu_2, \dots, \nu_m$ are some irrationally independent real numbers.

In this section we will study some gradient-like second order non-autonomous systems on the torus $\mathcal{T}^m$.

Let $F \in C^1(\mathbb{R}^n \times \mathcal{T}^m, \mathbb{R})$ and $\nabla F(0, \theta) = 0$ for all $\theta \in \mathcal{T}^m$. Below we will study the problem of stability of trivial solution for system

$$\begin{cases} x'' + \nabla_x F(\theta, x) = 0 \quad (x \in \mathbb{R}^n) \\ \theta' = \Phi(\theta) \quad (\theta \in \mathcal{T}^m), \end{cases} \tag{26}$$

where $\Phi \in C(\mathcal{T}^m, \mathbb{R}^m)$.

Everywhere below we will suppose that the functions F and Φ are regular (see, for example, [4, 16]), i.e., for every $(x_0, x_0', \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$ system (26) admits a unique solution $(\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta), \sigma(t, \theta))$ defined on $\mathbb{R}_+$. This means that system (26) generates a semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ on the space $X := \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$, where $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is a dynamical system associated by equation

$$\theta' = \Phi(\theta), \tag{27}$$

$(\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta))$ is a unique solution of equation

$$x'' + \nabla F(\sigma(t, \theta), x) = 0 \quad (\theta \in \mathcal{T}^m) \tag{28}$$

passing through the point (x_0, x_0') at the initial moment $t = 0$, $\pi(t, (x_0, x_0', \theta)) := (\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta), \sigma(t, \theta))$ for all $(t, x_0, x_0', \theta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$.

Remark 3.1. *1. By arguments above autonomous system (26) and non-autonomous equation (28) (in fact the family of non-autonomous equations depending on parameter $\theta \in \mathcal{T}^m$) are equivalent.*

2. If equation (28) admits a trivial solution, then the set $\{(0, 0)\} \times \mathcal{T}^m \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$ is an invariant subset (invariant torus) of system (26).

Definition 3.2. *Recall (see, for example,* [4, Ch.II]*) that the trivial solution of equation (28) (or equivalently, the invariant torus of system (26)) is said to be uniformly (with respect to $\theta \in \mathcal{T}^m$) Lyapunov stable, if for arbitrary $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $||x_0||^2 + ||x_0'||^2 < \delta^2$ implies $||\varphi(t, x_0, x_0', \theta)||^2 + ||\varphi'(t, x_0, x_0', \theta)||^2 < \varepsilon^2$ for all $t \in \mathbb{R}_+$ and $\theta \in \mathcal{T}^m$.*

Denote by $\mathcal{K}$ the set of all continuous functions $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties:

(i) $a(0) = 0$;
(ii) a is monotonically strictly increasing.

Theorem 3.3. *Suppose that following conditions hold:*

(i) *$F(\theta, 0) = 0$ and $\nabla_x F(\theta, 0) = 0$ for all $\theta \in \mathcal{T}^m$;*
(ii) *there exists a function $a \in \mathcal{K}$ such that $F(\theta, x) \geq a(||x||)$ for all $x \in \mathbb{R}^n$ and $\theta \in \mathcal{T}^m$;*
(iii) *$\langle \nabla_\theta F(\theta, x), \Phi(\theta) \rangle \leq 0$ for all $(x, \theta) \in \mathbb{R}^n \times \mathcal{T}^m$.*

Then the trivial solution of equation (28) is uniformly Lyapunov stable.

Proof. It is clear that second order system (26) is equivalent to the first order system

$$\begin{cases} x' = y \\ y' = -\nabla F_x(\theta, x) \quad ((x, y, \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m). \\ \theta' = \Phi(\theta) \end{cases} \tag{29}$$

Consider the function $V : \mathbb{R}^n \times \mathbb{R}^n \times T^m \mapsto \mathbb{R}$ defined by equality

$$V(x, y, \theta) := \frac{1}{2}||y||^2 + F(\theta, x). \tag{30}$$

It easy to check that under the conditions of Theorem 3.3 the function V possesses properties:

(i) $V \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m, \mathbb{R})$;
(ii) $V(0,0,\theta) = 0$ for all $\theta \in \mathcal{T}^m$;
(iii)
$$V(x,y,\theta) \geq \frac{1}{2}||y||^2 + a(||x||)$$
for all $(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$;
(iv)
$$\begin{aligned} \frac{dV(x,y,\theta)}{dt}\Big|_{(29)} &= \langle \nabla_x V(x,y,\theta), y\rangle + \langle \nabla_y V(x,y,\theta), -\nabla_x F(x,\theta)\rangle + \\ &\langle \nabla_\theta V(x,y,\theta), \Phi(\theta)\rangle = \langle \nabla_x F(x,\theta)), y\rangle + \langle y, -\nabla_x F(\theta,x)\rangle + \\ &\langle \nabla_\theta F(x,\theta), \Phi(\theta)\rangle = \langle \nabla_\theta F(x,\theta), \Phi(\theta)\rangle \leq 0 \end{aligned} \tag{31}$$

for all $(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$.

From (31) we obtain

$$V(\varphi(t,x,y,\theta), \varphi'(t,x,y,\theta), \sigma(t,\theta)) \leq V(x,y,\theta) \tag{32}$$

for all $t \in \mathbb{R}_+$ and $(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$.

Now we will show that the trivial solution of equation(28) is uniformly Lyapunov stable. If we suppose that it is not so, then there are $\varepsilon_0 > 0$, $\theta_k \in \mathcal{T}^m$, $\delta_k \to 0$ ($\delta_k > 0$), $\{x_k\}, \{y_k\} \subset \mathbb{R}^n$ and $t_k \to +\infty$ as $k \to \infty$ such that

$$||x_k||^2 + ||y_k||^2 < \delta_k^2 \tag{33}$$

and

$$||\varphi(t_k, x_k, y_k, \theta_k)||^2 + ||\varphi'(t_k, x_k, y_k, \theta_k)||^2 \geq \varepsilon_0^2 \tag{34}$$

for all $k \in \mathbb{N}$. Since $\mathcal{T}^m$ is compact, then without loss of generality we can suppose that the sequence $\{\theta_k\}$ is convergent and denote its limit by θ_0. From (32) and (33) we have

$$\begin{aligned} &V(\varphi(t_k, x_k, y_k, \theta_k), \varphi'(t_k, x_k, y_k, \theta_k), \sigma(t_k, \theta_k)) \leq \\ &V(x_k, y_k, \theta_k) \to V(0,0,\theta_0) = 0. \end{aligned} \tag{35}$$

Taking into account (30) from (35) we obtain

$$||\varphi(t_k, x_k, y_k, \theta_k)||^2 + ||\varphi'(t_k, x_k, y_k, \theta_k)||^2 \to 0 \tag{36}$$

as $k \to \infty$. Inequality (34) contradicts to (36). The obtained contradiction completes the proof of Theorem. □

Example 3.4. Consider the following second order differential equation

$$x'' + p(\sigma(t,\theta))\nabla F(x) = 0, \quad (\theta \in \mathcal{T}^m) \tag{37}$$

where $p \in C^1(\mathcal{T}^m, \mathbb{R})$ and $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is a dynamical system generated by autonomous equation (27) or the system

$$\begin{cases} x' = y \\ y' = -p(\theta)\nabla F(x) \quad ((x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m) \\ \theta' = \Phi(\theta) \end{cases} \tag{38}$$

which is equivalent to equation (37).

Suppose that the following conditions are fulfilled:

(i) $p(\theta) > 0$ for all $\theta \in \mathcal{T}^m$;
(ii) $\langle \nabla p(\theta), \Phi(\theta) \rangle \leq 0$ for all $\theta \in \mathcal{T}^m$;
(iii) $F(0) = 0$ and $\nabla F(0) = 0$;
(iv) there exists a function $a \in \mathcal{K}$ such that $F(x) \geq a(||x||)$ for all $x \in \mathbb{R}^n$.

Consider the function $V : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m \mapsto \mathbb{R}_+$ defined by equality

$$V(x, y, \theta) = \frac{1}{2}||y||^2 + p(\theta)F(x).$$

Note the function V possesses the following properties:

(i) $V \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m, \mathbb{R})$;
(ii)
$$\frac{dV(x, y, \theta)}{dt}\Big|_{(38)} = F(x)\langle \nabla p(\theta), \Phi(\theta) \rangle \leq 0$$
for all $(x, y, \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$;
(iii)
$$V(x, y, \theta) \geq \frac{1}{2}||y||^2 + \alpha a(||x||)$$
for all $(x, y, \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$, where $\alpha := \min\limits_{\theta \in \mathcal{T}^m} p(\theta) > 0$;

then by Theorem 3.3 the trivial solution of equation (37) is uniformly Lyapunov stable.

Denote by $[\mathbb{R}^n]$ the set of all square matrix of order n, i.e., $A \in [\mathbb{R}^n]$ if $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \mathbb{R}$ for all $i, j = 1, 2, \ldots, n$.

Definition 3.5. *A matrix-function $P \in C(\mathcal{T}^m, [\mathbb{R}^n])$ is said to be:*

(i) *symmetric, if $P(\theta) = P^t(\theta)$ for all $\theta \in \mathcal{T}^m$, where $P^t(\theta)$ is the transposed matrix;*
(ii) *non-negative, if it is symmetric and $\langle P(\theta)x, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$ and $\theta \in \mathcal{T}^m$;*
(iii) *definite-positive, if $P(\theta)$ is symmetric and there exists a positive number c such that*

$$\langle P(\theta)x, x \rangle \geq c||x||^2 \tag{39}$$

for all $x \in \mathbb{R}^n$ and $\theta \in \mathcal{T}^m$.

Example 3.6. Consider the differential equation

$$x'' + \nabla F(x) + P(\sigma(t, \theta))x = 0 \quad (\theta \in \mathcal{T}^m) \tag{40}$$

or the system

$$\begin{cases} x' = y \\ y' = -\nabla F(x) - P(\theta)x \quad ((x, y, \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m) \\ \theta' = \Phi(\theta) \end{cases}$$

which is equivalent to equation (40), where $P \in C^1(\mathcal{T}^m, [\mathbb{R}^n])$ and $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is a dynamical system generated by autonomous equation (27).

Suppose that the following conditions are fulfilled:

(i) the matrix-function $P(\theta) = (p_{ij}(\theta))_{i,j=1}^n$ is definite-positive;
(ii) $\Sigma_{i,j=1}^n \langle \nabla p_{ij}(\theta), \Phi(\theta)\rangle x_i x_j \le 0$ for all $\theta \in \mathcal{T}^m$ and $x \in \mathbb{R}^n$;
(iii) $F(0) = 0$, $\nabla F(0) = 0$ and $F(x) \ge 0$ for all $x \in \mathbb{R}^n$.

Consider the function $V : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m \mapsto \mathbb{R}_+$ defined by equality

$$V(x,y,\theta) = \frac{1}{2}||y||^2 + F(x) + \frac{1}{2}\langle P(\theta)x, x\rangle.$$

Function V satisfies the following conditions:

(i) $V \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m, \mathbb{R})$;
(ii)

$$\frac{dV(x,y,\theta)}{dt}\Big|_{(38)} = \frac{1}{2}\sum_{i,j=1}^n \langle \nabla p_{ij}(\theta), \Phi(\theta)\rangle x_i x_j \le 0$$

for all $(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$;
(iii)

$$V(x,y,\theta) \ge \frac{1}{2}||y||^2 + c||x||^2$$

for all $(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$, where c is the positive constant figuring in (39).

Then by Theorem 3.3 the trivial solution of equation (40) is uniformly Lyapunov stable.

Remark 3.7. *Note that the trivial solution of equation (40) remains uniformly Lyapunov stable if the matrix-function $P(\theta)$ is only non-negative, but the function $F(x)$ has at origin $0 \in \mathbb{R}^n$ a strict minimum.*

Let Θ be a compact metric space, $(\Theta, \mathbb{R}, \sigma)$ be a dynamical system on Θ and $G \in C(\Theta \times \mathbb{R}^n, \mathbb{R}^n)$. Consider the second order equation

$$x'' + G(\sigma(t,\theta), x') + \nabla F(x) = 0 \quad (x \in \mathbb{R}^n,\ \theta \in \Theta) \tag{41}$$

or equivalently

$$\begin{cases} x' = y \\ y' = -G(\sigma(t,\theta), y) - \nabla F(x) \quad ((x,y) \in \mathbb{R}^n \times \mathbb{R}^n,\ \theta \in \Theta). \end{cases} \tag{42}$$

Theorem 3.8. *Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin $0 \in \mathbb{R}^n$. Suppose that the following conditions are fulfilled:*

(i) $G(\theta, 0) = 0$ *for all* $\theta \in \Theta$;
(ii) $\langle G(\theta, x), x\rangle \ge 0$ *for all* $(\theta, x) \in \Theta \times \mathbb{R}^n$;
(iii) $F \in C^1(U, \mathbb{R}_+)$;
(iv) $F(x) > 0$ *for all* $x \in U \setminus \{0\}$;
(v) $F(0) = 0$ *and* $\nabla F(0) = 0$.

Then the trivial solution of system (42) is uniformly Lyapunov stable.

Proof. Since $F(0) = 0$ and $F(x) > 0$ for all $x \in U \setminus \{0\}$, then it easy to check that the function $V : U \times U \mapsto \mathbb{R}$ defined by equality

$$V(x,y) := \frac{1}{2}||y||^2 + F(x) \ ((x,y) \in U \times U) \tag{43}$$

possesses the following properties:

(i) $V \in C^1(U \times U, \mathbb{R})$;
(ii) $V(x, y) \geq 0$ for all $(x, y) \in U \times U$ and $V(0, 0) = 0$;
(iii) $V(x, y) > 0$ for all $(x, y) \in U \times U \setminus \{(0, 0)\}$;
(iv)
$$\frac{dV(x,y)}{dt}\Big|_{(42)} = \lim_{t \to 0^+} \frac{V(\varphi(t,x,y,\theta),\varphi'(t,x,y,\theta)) - V(x,y)}{t} = -\langle y, G(\theta, y)\rangle.$$

Using the same arguments as in the proof of Theorem 3.3 we can establish the uniform Lyapunov stability of trivial solution of equation (41). $\square$

Example 3.9. Let $p \in C(\Theta, \mathbb{R})$. Consider a second order differential equation

$$x'' + p(\sigma(t, \theta))x' + \nabla F(x) = 0, \quad ((\theta, x) \in \Theta \times \mathbb{R}^n) \tag{44}$$

where $(\Theta, \mathbb{R}, \sigma)$ is a dynamical system on the space Θ or equivalent system

$$\begin{cases} x' = y \\ y' = -p(\sigma(t, \theta))y - \nabla F(x) \quad (\theta \in \Theta). \end{cases} \tag{45}$$

Let $V : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ be a function defined by equality (43). Since

$$\frac{dV(x, y)}{dt}\Big|_{(45)} = -p(\theta)||y||^2,$$

then by Theorem 3.8 the trivial solution of equation (44) is uniformly Lyapunov stable, if the following conditions are fulfilled:

(i) $p(\theta) \geq 0$ for all $\theta \in \Theta$;
(ii) $F(0) = 0$ and $\nabla F(0) = 0$;
(iii) F has a strict minimum at origin $0 \in \mathbb{R}^n$.

Example 3.10. Let $p \in C(\mathbb{R}, \mathbb{R})$. Consider a second order differential equation

$$x'' + p(t)x' + \nabla F(x) = 0, \quad (((t, x) \in \mathbb{R} \times \mathbb{R}^n) \tag{46}$$

or equivalent system

$$\begin{cases} x' = y \\ y' = -p(t)y - \nabla F(x)\ . \end{cases} \tag{47}$$

If $V : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is a function defined by equality (43), then we have

$$\frac{dV(x, y)}{dt}\Big|_{(45)} = -p(t)||y||^2.$$

Suppose that the following conditions hold:

(i) the function $p \in C(\mathbb{R}, \mathbb{R})$ is bounded and uniformly continuous;
(ii) $p(t) \geq 0$ for all $t \in \mathbb{R}$;
(iii) $F(0) = 0$ and $\nabla F(0) = 0$;
(iv) $F(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

We will show that under the conditions listed above the trivial solution of equation (46) (or system (47)) is uniformly Lyapunov stable. Denote by $(C(\mathbb{R}, \mathbb{R}), \mathbb{R}, \sigma)$ the shift dynamical system [4, 16] (Bebutov's dynamical system) on $C(\mathbb{R}, \mathbb{R})$. Since $C(\mathbb{R}, \mathbb{R})$ is equipped with the compact-open topology and the function $p \in C(\mathbb{R}, \mathbb{R})$

is bounded and uniformly continuous, then its hull $H(p) := \overline{\{p_\tau : \tau \in \mathbb{R}\}}$ is a compact subset of $C(\mathbb{R},\mathbb{R})$. Where by p_τ is denoted the τ-shift of p, i.e., $p_\tau(t) := p(t+\tau)$ for all $t \in \mathbb{R}$, and by bar the closure in the space $C(\mathbb{R},\mathbb{R})$. Denote by $\Theta := H(p)$. Since the set $H(p)$ is invariant (with respect to shifts) and closed, then on $H(p)$ is induced a shift dynamical system $(H(p),\mathbb{R},\sigma)$. Consider a second order differential equation

$$x'' + \mathcal{P}(\sigma(t,\theta))x' + \nabla F(x) = 0, \quad ((\theta,x) \in H(p) \times \mathbb{R}^n) \tag{48}$$

where $\mathcal{P} : H(p) \mapsto \mathbb{R}$ is defined by equality $\mathcal{P}(\theta) := \theta(0)$ for all $\theta \in H(p)$. Along with equation (48) consider a system

$$\begin{cases} x' = y \\ y' = -\mathcal{P}(\sigma(t,\theta))y - \nabla F(x) \ . \end{cases}$$

Note that $\mathcal{P}(\theta) = \theta(0) = \lim_{n\to\infty} p(t_n)$, where $\{t_n\}$ is some sequence of real numbers. Since $p(t) \geq 0$ for all $t \in \mathbb{R}$, then $\mathcal{P}(\theta) \geq 0$ for all $\theta \in H(p)$. Finally, it easy to see that the mapping $\mathcal{P} : H(p) \mapsto \mathbb{R}$ is continuous. Reasoning as in the Example 3.9 we obtain the uniform stability of trivial solution for (46).

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