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Birkhoff's center of compact dissipative dynamical systems

David Cheban

Abstract

We introduce the notion of Birkhoff center for arbitrary dynamical systems admitting a compact global attractor. It is shown that Birkhoff center of dynamical system coincides with the closure of the set of all positively Poisson stable points of dynamical system.

Keywords: dynamical system, global attractor, Birkhoff center.

1 Introduction

Let X be a metric space, $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$ and $(X, \mathbb{T}, \pi)$ be a flow on X and $M \subseteq X$ be a nonempty, compact and positively invariant subset of X. Denote by $\Omega(M) := \{x \in M :$ there exis $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{T}$ such that $x_n \to x$, $t_n \to +\infty$ as $n \to \infty$ and $\pi(t_n, x_n) \to x\}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \bigcap \alpha_x$, where by ω_x (respectively, α_x) is denoted the ω (respectively, α)-limits set of x.

It is well known (see, for example, [1, 3]) the following result for two-sided ($\mathbb{T} = \mathbb{S}$) dynamical systems on the compact metric spaces.

Theorem 1 *(Birkhoff theorem) The following statements hold:*

1. *there exists a nonempty, compact and invariant subset $\mathfrak{B}(\pi) \subseteq X$ with the properties:*

   (i) $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$;

  (ii) $\mathfrak{B}(\pi)$ *is the maximal compact invariant subset of* J *with the property* (i).

2. $\mathfrak{B}(\pi) = \overline{\mathcal{P}(\pi)}$, *i.e., the set of all Poisson stable points* $\mathcal{P}(\pi)$ *of the dynamical system* $(X, \mathbb{R}, \pi)$ *is dense in* $\mathfrak{B}(\pi)$.

The set $\mathfrak{B}(\pi)$ is called the Bikkhoff center of dynamical system $(X, \mathbb{R}, \pi)$.

2 The set of non-wandering points of dynamical system

Denote by Φ_x the set of all entire trajectories γ_x of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment $t = 0$..

Definition 1. *A point* $p \in X$ *is said to be positively (respectively, negatively) Poisson stable, if* $x \in \omega_x$ *(respectively, there exists an entire trajectory* $\gamma_x \in \Phi_x$ *such that* $x \in \alpha_{\gamma_x}$*, where* $\alpha_{\gamma_x} := \{q \in X :$ *there exists* $t_n \to -\infty$ *such that* $\gamma_x(t_n) \to q$ *as* $n \to \infty\}$*).*

Lemma 1. *Let* M *be a nonempty, compct and positively invariant set, then the following statements hold:*

1. *if* $p \in M$ *is positively (negatively) Poisson stable, then* $p \in \Omega(M)$*;*
2. $\Omega(M)$ *is a nonempty, compact and positively invariant subset of* M*;*
3. *if* $(X, \mathbb{T}, \pi)$ *is a compactly dissipative dynamical system and* J *is its Levinson center [2, ChI], then the set* $\Omega(M)$ *is nonempty, compact, positively invariant and* $\Omega(M) \subseteq J$.

3 Birkhoff center of compact dissipative dynamical system

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system [2, ChI] and J be its Levinson center [2, ChI] and $M \subseteq X$ be a nonempty, closed and positively invariant subset from X. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to M) points of $(X, \mathbb{T}, \pi)$. By Lemma 1 the set M_1 is a nonempty, compact and positively invariant subset of J. We denote by $M_2 := \Omega(M_1) \subseteq M_1$. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$. We can continue this processus and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. Thus we have a sequence $\{M_n\}_{n\in\mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set M_n is nonempty, compact and positively invariant;

2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote by $M_\lambda := \bigcap\limits_{n=1}^{\infty} M_n$, then M_λ is a nonempty, compact (since the set J is compact) and invariant subset of J. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$
\begin{gathered}
J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\
M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq M_{\lambda+1} \supseteq \ldots \supseteq M_{\lambda+k} \supseteq \ldots .
\end{gathered}
$$

Now construct the set $M_\mu := \bigcap\limits_{k=1}^{\infty} M_{\mu+k}$ and we denote by $M_{\mu+1} := \Omega(M_\mu)$ and so on. Thus we will obtain a transfinite sequence of non-empty, compact and invariant subsets

$$
\begin{gathered}
J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\
M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots .
\end{gathered} \tag{1}
$$

Since J is a nonempty compact set, then in the sequence (1) there is at most a countable family of different elements, i.e., there exists a ν such that $M_{\nu+1} = M_\nu$.

Definition 2. *The set $\mathfrak{B}(M) := M_\nu$ is said to be the center of Birkhoff for the closed and positively invariant set M. If $M = X$, then the set $\mathfrak{B}(\pi) := \mathfrak{B}(X)$ is said to be the Birkhoff center of compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.*

Theorem 2. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J be its Levinson center, then the following statements hold:*

1. *$\mathfrak{B}(\pi)$ is a nonempty, compact and invariant set;*
2. *$\mathfrak{B}(\pi)$ is a maximal compact invariant subset M of X such that $\Omega(M) = M$;*
3. *if for all $t > 0$ the map $\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathfrak{B}(\pi)}$ is open, then the set of all positively Poisson stable points of $(X, \mathbb{T}, \pi)$ is dense in $\mathfrak{B}(\pi)$, i.e., $\mathfrak{B}(\pi) = \overline{P(\pi)}$, where $P(\pi) := \{p \in X : \ p \in \omega_p\}$.*

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David Cheban[1]

[1] State University of Moldova
Email: cheban@usm.md, davidcheban@yahoo.com