Birkhoff’s center of compact dissipative dynamical systems

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Abstract

We introduce the notion of Birkhoff center for arbitrary dynamical systems admitting a compact global attractor. It is shown that Birkhoff center of dynamical system coincides with the closure of the set of all positively Poisson stable points of dynamical system.

Keywords: dynamical system, global attractor, Birkhoff center.

1 Introduction

Let $X$ be a metric space, $T = \mathbb{R}_+$ or $\mathbb{Z}_+$, $S = \mathbb{R}$ or $\mathbb{Z}$ and $(X, T, \pi)$ be a flow on $X$ and $M \subseteq X$ be a nonempty, compact and positively invariant subset of $X$. Denote by $\Omega(M) := \{x \in M : \text{there exist } \{x_n\} \subseteq M \text{ and } \{t_n\} \subseteq T \text{ such that } x_n \to x, \ t_n \to +\infty \text{ as } n \to \infty \text{ and } \pi(t_n, x_n) \to x\}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \cap \alpha_x$, where by $\omega_x$ (respectively, $\alpha_x$) is denoted the $\omega$ (respectively, $\alpha$)-limits set of $x$.

It is well known (see, for example, [1, 3]) the following result for two-sided ($T = S$) dynamical systems on the compact metric spaces.

**Theorem 1** (Birkhoff theorem) The following statements hold:

1. there exists a nonempty, compact and invariant subset $\mathcal{B}(\pi) \subseteq X$ with the properties:
   
   (i) $\Omega(\mathcal{B}(\pi)) = \mathcal{B}(\pi)$;
(ii) $\mathcal{B}(\pi)$ is the maximal compact invariant subset of $J$ with the property (i).

2. $\mathcal{B}(\pi) = \overline{\mathcal{P}(\pi)}$, i.e., the set of all Poisson stable points $\mathcal{P}(\pi)$ of the dynamical system $(X, \mathbb{R}, \pi)$ is dense in $\mathcal{B}(\pi)$.

The set $\mathcal{B}(\pi)$ is called the Bikhhoff center of dynamical system $(X, \mathbb{R}, \pi)$.

2 The set of non-wandering points of dynamical system

Denote by $\Phi_x$ the set of all entire trajectories $\gamma_x$ of $(X, T, \pi)$ passing through the point $x$ at the initial moment $t = 0$.

Definition 1. A point $p \in X$ is said to be positively (respectively, negatively) Poisson stable, if $x \in \omega_x$ (respectively, there exists an entire trajectory $\gamma_x \in \Phi_x$ such that $x \in \alpha_{\gamma_x}$, where $\alpha_{\gamma_x} := \{q \in X : \text{there exists } t_n \to -\infty \text{ such that } \gamma_x(t_n) \to q \text{ as } n \to \infty\}$).

Lemma 1. Let $M$ be a nonempty, compact and positively invariant set, then the following statements hold:

1. if $p \in M$ is positively (negatively) Poisson stable, then $p \in \Omega(M)$;
2. $\Omega(M)$ is a nonempty, compact and positively invariant subset of $M$;
3. if $(X, T, \pi)$ is a compactly dissipative dynamical system and $J$ is its Levinson center [2, ChI], then the set $\Omega(M)$ is nonempty, compact, positively invariant and $\Omega(M) \subseteq J$.

3 Birkhoff center of compact dissipative dynamical system

Let $(X, T, \pi)$ be a compact dissipative dynamical system [2, ChI] and $J$ be its Levinson center [2, ChI] and $M \subseteq X$ be a nonempty, closed
and positively invariant subset from $X$. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to $M$) points of $(X, T, \pi)$. By Lemma 1 the set $M_1$ is a nonempty, compact and positively invariant subset of $J$. We denote by $M_2 := \Omega(M_1) \subseteq M_1$. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$. We can continue this processus and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. Thus we have a sequence $\{M_n\}_{n \in \mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set $M_n$ is nonempty, compact and positively invariant;
2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote by $M_\lambda := \bigcap_{n=1}^{\infty} M_n$, then $M_\lambda$ is a nonempty, compact (since the set $J$ is compact) and invariant subset of $J$. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_{\mu} \supseteq \ldots.$$ 

Now construct the set $M_\mu := \bigcap_{k=1}^{\infty} M_{\mu+k}$ and we denote by $M_{\mu+1} := \Omega(M_\mu)$ and so on. Thus we will obtain a transfinite sequence of nonempty, compact and invariant subsets

$$J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots. \quad (1)$$

Since $J$ is a nonempty compact set, then in the sequence (1) there is at most a countable family of different elements, i.e., there exists a $\nu$ such that $M_{\nu+1} = M_\nu$.

**Definition 2.** The set $\mathcal{B}(M) := M_\nu$ is said to be the center of Birkhoff for the closed and positively invariant set $M$. If $M = X$, then the set $\mathcal{B}(\pi) := \mathcal{B}(X)$ is said to be the Birkhoff center of compact dissipative dynamical system $(X, T, \pi)$. 

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Theorem 2. Suppose that \((X, T, \pi)\) is a compact dissipative dynamical system and \(J\) be its Levinson center, then the following statements hold:

1. \(\mathcal{B}(\pi)\) is a nonempty, compact and invariant set;
2. \(\mathcal{B}(\pi)\) is a maximal compact invariant subset \(M\) of \(X\) such that \(\Omega(M) = M\);
3. if for all \(t > 0\) the map \(\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathcal{B}(\pi)}\) is open, then the set of all positively Poisson stable points of \((X, T, \pi)\) is dense in \(\mathcal{B}(\pi)\), i.e., \(\mathcal{B}(\pi) = \overline{P(\pi)}\), where \(P(\pi) := \{p \in X : p \in \omega_p\}\).

References

