MARKUS-YAMABE CONJECTURE FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is the study of the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles). In particularly, we present some new results for non-autonomous version of Markus-Yamabe conjecture.

1. Introduction

1.1. **Markus–Yamabe conjecture (MYC)** [33]**.** Consider the differential equation

$$u' = f(u) \tag{1}$$

and suppose that the Jacobian $f'(u)$ of f has only eigenvalues with negative real part for all u. The *Markus Yamabe conjecture* is that if $f(0) = 0$, then 0 is a globally asymptotically stable solution for (1).

It is easy to prove **MYC** for $n = 1$. In the two-dimensional case the affirmative answer to **MYC** was obtained in the works [15, 17, 16] (see also the references therein). In the work [9] (see also [10, 11] and the references therein) is given a polynomial counterexample to the Markus–Yamabe conjecture. If $n > 2$ there are also some additional conditions forcing the Markus–Yamabe conjecture. For example if $f'(u)$ is negative definite for all $u \in \mathbb{R}^n$ the conjecture was proved in [19, 20] (see also [26, 27, 33]). For triangular systems **MYC** was proved in [33].

1.2. **The discrete Markus–Yamabe conjecture (DMYC)** [12, 41]**.** Let f be a C^1 mapping from $\mathbb{R}^n$ into itself such that $f(0) = 0$ and for all $u \in \mathbb{R}^n$, $f'(u)$ has all its eigenvalues with modulus less than one. Then 0 is a globally asymptotically stable solution of the difference equation

$$u(n+1) = f(u(n)). \tag{2}$$

In his book [29] J. P. LaSalle proves the **DMYC** for $n = 1$. The discrete Markus–Yamabe conjecture is true only for planar maps (see [12] and also the references therein) and the answer to the question is yes only in the case of planar polynomial

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maps. The authors [12] prove that the **DMYC** is true for triangular maps defined on $\mathbb{R}^n$ and for polynomial maps defined on $\mathbb{R}^2$. In the works [8, 32] the **DMYC** is proved for gradient maps.

1.3. **Belitskii–Lyubich conjecture** [1]**.** Let $\mathfrak{B}$ be a Banach space, $\Omega \subset \mathfrak{B}$ an open subset and $f : \Omega \mapsto \mathfrak{B}$ be a compact and continuously differentiable in Ω. Suppose D is a nonempty bounded convex open subset of X such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup\limits_{x\in\overline{D}} r(f'(x)) < 1$ ($r(A)$ is the spectral radius of linear bounded operator A). Then the discrete dynamical system $(\overline{D}, f)$, generated by positive powers of $f : \overline{D} \mapsto \overline{D}$, admits a unique globally asymptotically stable fixed point.

The aim of this paper is the study the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles).

The idea of applying methods of the theory of dynamical systems to the study of non-autonomous differential equations is not new. It has been successfully applied to the resolution of different problems in the theory of linear and non-linear non-autonomous differential equations for more than forty years. First this approach to non-autonomous differential equations was introduced in the works of L. G. Deyseach and G. R. Sell [14], R. K. Miller [36], V. M. Millionshchikov [37]-[39], G. Seifert [53], G. R. Sell [54, 55, 56], B. A. Shcherbakov [59, 60], later in the works of I. U. Bronshtein [3, 4], R. A. Johnson [21, 22], B. M. Levitan and V. V. Zhikov [31], Sacker R. J. [42, 43], Sacker R. J and Sell G. R. [44, 45, 46, 47, 48, 49, 50, 51, 52], G. R.Sell, W. Shen and Y. Yi [57], B. A. Shcherbakov [61, 62], V. V. Zhikov [66, 67, 68] and many other authors. This approach consists of naturally linking with equation

$$x' = f(t, x) \tag{3}$$

a pair of dynamical systems and a homomorphism of the first onto the second. In one dynamical system is put the information about the right hand side of equation (3) and in the other about the solutions of equation (3).

This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of global attractors of dynamical systems which we use in our paper.

Section 3 is dedicated to the study of non-autonomous dynamical systems with convergence. We present some important tests of convergence (Theorems 3.6, 3.7 and 3.10) of non-autonomous dynamical systems with minimal base.

In section 4 we study the Markus–Yamabe problem for non-autonomous systems. In this section we prove the necessary and sufficient conditions of global asymptotic stability the trivial section of non-autonomous dynamical systems with continuous or discrete time (Theorem 4.4 – the main result of paper). We apply this result to the different classes of non-autonomous evolutions equations (finite-dimensional systems, gradient systems, triangular systems).

We give in section 5 some new results concerning the discrete Markus–Yamabe problem for non-autonomous systems. In particularly we present the affirmative answer to the **DMYC** for non-autonomous contractive, triangular and potential (gradient) maps.

2. Compact Global Attractors of Dynamical Systems

Let X be a topological space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of the nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of additive group $\mathbb{S}$.

Definition 2.1. *Triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions:*

$$\pi(0, x) = x; \tag{4}$$

$$\pi(s, \pi(t, x)) = \pi(s + t, x); \tag{5}$$

is called a dynamical system. If $\mathbb{T} = \mathbb{R}$ ($\mathbb{R}_+$) or $\mathbb{Z}$ ($\mathbb{Z}_+$), then the dynamical system $(X, \mathbb{T}, \pi)$ is called a group (semi-group). In the case, when $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a flow, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a cascade (discrete flow).

Sometimes, briefly, we will write xt instead of $\pi(t, x)$.

Below X will be a complete metric space with metric ρ.

Definition 2.2. *The function $\pi(\cdot, x) : \mathbb{T} \to X$ is called a motion passing through the point x at the moment $t = 0$ and the set $\Sigma_x := \pi(\mathbb{T}, x)$ is called a trajectory of this motion.*

Definition 2.3. *A nonempty set $M \subseteq X$ is called positively invariant (negatively invariant, invariant) with respect to dynamical system $(X, \mathbb{T}, \pi)$ or, simple, positively invariant (negatively invariant, invariant), if $\pi(t, M) \subseteq M$ ($M \supseteq \pi(t, M), \pi(t, M) = M$) for every $t \in \mathbb{T}$.*

Definition 2.4. *A closed positively invariant set, which does not contain its own closed positively invariant subset, is called minimal.*

It easy to see that every positively invariant minimal set is invariant.

Definition 2.5. *A closed positively invariant (invariant) set is called indecomposable, if it can not be represented in the form of union of two nonempty disjoint positively invariant (invariant) subsets.*

Definition 2.6. *Let $M \subseteq X$. The set*

$$\omega(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}$$

is called ω-limit for M.

Definition 2.7. *The set $W^s(\Lambda)$ ($W^u(\Lambda)$), defined by equality*

$$W^s(\Lambda) := \{x \in X |\lim_{t \to +\infty} \rho(xt, \Lambda) = 0\}$$

is called a stable manifold of the set $\Lambda \subseteq X$.

Definition 2.8. *The set M is called:*

- *orbital stable, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(xt, M) < \varepsilon$ for all $t \geq 0$;*
- *attracting, if there exists $\gamma > 0$ such that $B(M, \gamma) \subset W^s(M)$;*
- *asymptotic stable, if it is orbital stable and attracting;*
- *global asymptotic stable, if it is asymptotic stable and $W^s(M) = X$;*
- *uniform attracting, if there exists $\gamma > 0$ such that*

$$\lim_{t\to+\infty} \sup_{x\in B(M,\gamma)} \rho(xt, M) = 0.$$

Definition 2.9. *The system $(X, \mathbb{T}, \pi)$ is called:*

– *point dissipative if there exists a nonempty compact subset $K \subseteq X$ such that for every $x \in X$*

$$\lim_{t\to+\infty} \rho(xt, K) = 0; \tag{6}$$

– *compact dissipative if the equality (6) takes place uniformly w.r.t. x on the compacts from X;*
– *locally dissipative if for any point $p \in X$ there exist $\delta_p > 0$ such that the equality (6) takes place uniformly w.r.t. $x \in B(p, \delta_p)$;*
– *bounded dissipative if the equality (6) takes place uniformly w.r.t. x on every bounded subset from X.*
– *locally completely (compact) if for any point $p \in X$ there exist $\delta_p > 0$ and $l_p > 0$ such that the set $\pi^{l_p} B(p, \delta_p)$ is relatively compact, where $B(x, \delta) := \{x \in X \mid \rho(x, p) < \delta\}$.*

Theorem 2.10. [7, Ch.I] *For the locally completely (compact) dynamical systems the point, compact and local dissipativity are equivalent.*

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a compact set attracting every compact subset from X. Let us set

$$J := \omega(K) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi^\tau K}. \tag{7}$$

It can be shown [7, Ch.I] that the set J defined by equality (7) doesn't depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called a Levinson center of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Theorem 2.11. [7, 18, 58] *If $(X, \mathbb{T}, \pi)$ is a compactly dissipative dynamical system and J is its center of Levinson, then :*

(i) *J is invariant, i.e. $\pi^t J = J$ for all $t \in \mathbb{T}$;*
(ii) *J is orbitally stable, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\beta(xt, J) < \varepsilon$ for all $t \geq 0$;*
(iii) *J is an attractor of the family of all compact subsets of X;*
(iv) *J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$.*

Denote by

$$D^+(M) := \bigcap_{\varepsilon>0} \overline{\bigcup\{\pi^t B(M,\varepsilon) | t \geq 0\}},$$

$$J^+(M) := \bigcap_{\varepsilon>0} \bigcap_{t\geq 0} \overline{\bigcup\{\pi^\tau B(M,\varepsilon) | \tau \geq t\}},$$

$D^+_x := D^+(\{x\})$ and $J^+_x := J^+(\{x\})$.

Theorem 2.12. [7, Ch.I] *Let $(X, \mathbb{T}, \pi)$ be point dissipative. For $(X, \mathbb{T}, \pi)$ to be compact dissipative it is necessary and sufficient that the set $D^+(\Omega_X)$ $(J^+\Omega_X))$ be compact and orbital stable. In this case $J = D^+(\Omega_X)$ $(J = J^+(\Omega_X))$ where J is the center of Levinson of the dynamical system $(X, \mathbb{T}, \pi)$ and $\Omega_X := \overline{\cup\{\omega_x \mid x \in X\}}$.*

3. Non-Autonomous Dynamical Systems with Convergence

Definition 3.1. *$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be convergent if the following conditions are valid:*

(i) *the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;*
(ii) *the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$ where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and $J_X(J_Y)$ is the Levinson's center of the dynamical system $(X, \mathbb{T}_1, \pi)((Y, \mathbb{T}_2, \sigma))$.*

Let $X \dot{\times} X := \{(x_1, x_2) : x_1, x_2 \in X, h(x_1) = h(x_2)\}$. function $V : X \dot{\times} X \to \mathbb{R}_+$ is said to be continuous, if $x^i_n \to x^i$ $(i = 1, 2$ and $h(x^1_n) = h(x^2_n))$ implies $V(x^1_n, x^2_n) \to V(x^1, x^2)$.

Lemma 3.2. [6] *Let X be a compact metric space and $\langle (X, \mathbb{T}_1, \pi),\ (\Omega, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(\Omega, \mathbb{T}_2, \sigma)$ are minimal;*
(ii) $\lim\limits_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ *for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.*

Then for all $y \in Y$ the fiber $X_y := h^{-1}(y) = \{x \in X \mid h(x) = y\}$ consists a single point.

Theorem 3.3. [7, Ch.II] *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system, $M \neq \emptyset$ be a compact positively invariant set. Suppose that the following conditions are fulfilled:*

(i) $h(M) = Y$;
(ii) *$M \bigcap X_y$ contains a single point for all $y \in Y$;*
(iii) *M is globally asymptotically stable, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, p) < \delta$ $(x \in X_y, p \in M_y := M \bigcap X_y)$ implies $\rho(xt, pt) < \varepsilon$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty} \rho(xt, M_{h(x)t}) = 0$ for all $x \in X$.*

Then the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent.

Theorem 3.4. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and Y be a compact minimal set, then the following conditions are equivalent:*

1. *$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent;*
2. *every semi-trajectory $\Sigma_x^+ (x \in X)$ is relatively compact and asymptotically stable;*
3. (a) *every semi-trajectory $\Sigma_x^+ (x \in X)$ is relatively compact;*
   (b) $\lim\limits_{t\to+\infty} \rho(x_1 t, x_2 t) = 0$ *for all* $(x_1, x_2) \in X\dot{\times}X$;
   (c) *for any $\varepsilon > 0$ and $K \in C(X)$ there exists $\delta(\varepsilon, K) > 0$ such that $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2); x_1, x_2 \in K)$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \geq 0$.*
4. *every semi-trajectory $\Sigma_x^+ (x \in X)$ is relatively compact and the equality*
$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in K\dot{\times}K} \rho(x_1 t x_2 t) = 0$$
holds for all $K \in C(X)$.

Proof. We will prove that condition 1. implies condition 2. If we suppose that it is not so, then there are $p_0 \in X, \varepsilon_0 > 0, p_n \to p_0$ $(h(p_n) = h(p_0))$ and $t_n \to +\infty$ such that

$$\rho(p_n t_n, p_0 t_n) \geq \varepsilon_0. \tag{8}$$

Since $(X, \mathbb{T}_1, \pi)$ is compactly dissipative we may suppose that the sequences $\{p_n t_n\}$ and $\{p_0 t_n\}$ are convergent. Letting $\bar{p} = \lim\limits_{n\to+\infty} p_n t_n$, $\bar{p}_0 = \lim\limits_{n\to+\infty} p_0 t_n$ and taking into consideration (8), we will have $\bar{p} \neq \bar{p}_0$. On the other hand, $h(\bar{p}) = \lim\limits_{n\to+\infty} h(p_n) t_n = \lim\limits_{n\to+\infty} h(p_0) t_n = h(\bar{p}_0) = \bar{y} \in J_Y$ and, consequently, $\bar{p}, \bar{p}_0 \in J_X \bigcap X_{\bar{y}}$, but by virtue of condition 1. we have $\bar{p} = \bar{p}_0$. The obtained contradiction proves the necessary assertion.

Now we will note that condition 2. implies condition 3.b. To prove this implication is sufficient to show that
$$\lim_{t\to+\infty} \rho(x_1 t, x_2 t) = 0$$
for all $(x_1, x_2) \in X\dot{\times}X$. Assuming the contrary we obtain

$$\rho(x_1^0 t_n, x_2^0 t_n) \geq \varepsilon_0. \tag{9}$$

We may assume that the sequences $\{x_i^0 t_n\} (i = 1, 2)$ and $\{y_0 t_n\}$ $(y_0 = h(x_1^0) = h(x_2^0))$ are convergent. We denote by $\bar{x}_i^0 := \lim\limits_{n\to+\infty} x_i^0 t_n$ and $\bar{y}_0 := \lim\limits_{n\to+\infty} y_0 t_n$, then $\bar{x}_1^0, \bar{x}_2^0 \in J_X \bigcap X_{\bar{y}_0}$ and according to the condition 1. $\bar{x}_1^0 = \bar{x}_2^0$, where J_X is the Levinson center of dynamical system $(X\mathbb{T}_1, \pi)$. The last equality and the inequality (9) are contradictory. This contradiction proves the necessary assertion.

We note that

$$\lim_{t\to+\infty} \rho(xt, pt) = 0 \tag{10}$$

for all $p \in X$ and $x \in X_q$ $(q = h(p))$. Let $K \in C(X)$ and $\varepsilon > 0$, then there exists $\delta(\varepsilon, K) > 0$ such that $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2); x_1, x_2 \in K)$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for any $t \geq 0$. Assuming the contrary, we obtain $K_0 \in C(X), \varepsilon_0 > 0, \delta_n \to 0$ $(\delta_n > 0), \{x_n^i\} \subseteq K_0$ $(i = 1, 2)$ and $t_n \to +\infty$ such that $\rho(x_n^1, x_n^2) < \delta_n$ and

$$\rho(x_n^1 t_n, x_n^2 t_n) \geq \varepsilon_0. \tag{11}$$

Since K_0 is a compact subset of X we may suppose that the sequences $\{x_n^i\}$ $(i = 1,2)$ are convergent and we denote by $\bar{x} := \lim_{n\to+\infty} x_n^1 = \lim_{n\to+\infty} x_n^2$ $(\bar{x} \in K_0)$. According to the condition 2., for $\varepsilon_0 > 0$ and $\bar{x} \in K_0$ there exists $\delta(\frac{\varepsilon_0}{3}, \bar{x}) > 0$ such that $\rho(x, \bar{x}) < \delta(\frac{\varepsilon_0}{3}, \bar{x})$ $(h(x) = h(\bar{x}))$ implies $\rho(xt, \bar{x}t) < \frac{\varepsilon_0}{3}$ for all $t \geq 0$. Since $x_n^i \to \bar{x}$ $(i = 1,2)$, then there exists $\bar{n}$ such that $\rho(x_n^i, \bar{x}) < \delta(\frac{\varepsilon_0}{3}, \bar{x})$ $(n \geq \bar{n})$ and, consequently,

$$\rho(x_n^1 t, x_n^2 t) \leq \frac{2\varepsilon_0}{3} \tag{12}$$

for all $t \geq 0$ and $n \geq \bar{n}$. But the inequalities (12) and (11) are contradictory. Thus, we showed that condition 2. implies condition 3.

We will prove that condition 3. implies condition 4. If we suppose the contrary, then there exist $\varepsilon_0 > 0$, $K_0 \in C(X)$, $t_n \to +\infty$ and $\{x_n^i\} \subseteq K_0$ $(i = 1,2; h(x_n^1) = h(x_n^2))$ such that the inequality (11) holds. We may assume without loss of generality that the sequences $\{x_n^i\}$ $(i = 1,2)$ are convergent, because K_0 is compact. Let $x^i := \lim_{n\to+\infty} x_n^i$, $0 < \varepsilon < \varepsilon_0$ and $\delta(\frac{\varepsilon}{3}, K_0) > 0$ be chosen according to condition 3.c. Since $h(x^1) = h(x^2)$ and $x^1, x^2 \in K_0$, then for $\frac{\varepsilon}{3}$ there exists $L(\frac{\varepsilon}{3}, x^1, x^2) > 0$ such that $\rho(x^1 t, x^2 t) < \frac{\varepsilon}{3}$ for all $t \geq L(\frac{\varepsilon}{3}, x^1, x^2)$ and, consequently,

$$\rho(x_n^1 t_n, x_n^2 t_n) \leq \rho(x_n^1 t_n, x^1 t_n) + \rho(x^1 t_n, x^2 t_n) + \rho(x^2 t_n, x_n^2 t_n) < \varepsilon \tag{13}$$

for sufficiently large n. The inequalities (12) and (13) are contradictory. Hence, the necessary assertion is proved.

Finally, we note that 4. implies 1. Let $x \in X$ be an arbitrary point. According to condition 4. the ω-limit set ω_x of point x contains a compact minimal set $M \subseteq \omega_x$. By Lemma 3.2 the set $M_y := M \cap X_y$ consists a single point m_y. Under the condition 4. we have $\lim_{t\to+\infty} \rho(xt, m_y t) = 0$ and, consequently, $\omega_x = M$. Now we will show that the dynamical system $(X, \mathbb{T}_1, \pi)$ has at most one compact minimal set. If we suppose that it is not true, then there are two different compact minimal subsets M^i $(i = 1,2)$ of X and, consequently, $M^1 \cap M^2 = \emptyset$. By Lemma 3.2 under the condition 4. M_y^i consists a single point m_y^i and $m_y^1 \neq m_y^2$ for all $y \in Y$. On the other hand by condition 4. we have $\lim_{t\to+\infty} \rho(m_y^1 t, m_y^2 t) = 0$ for all $y \in Y$ and, consequently, $M^1 = M^2$. The obtained contradiction proves our statement. Thus the dynamical system $(X, \mathbb{T}_1, \pi)$ has a unique compact minimal set $M \subseteq X$ and $\lim_{t\to+\infty} \rho(xt, m_{h(x)} t) = 0$ for all $x \in X$. Finally we will show that under the condition 4. the set M is stable, i.e for all $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon)$ such that $\rho(x, m_{h(x)}) < \delta$ implies $\rho(xt, m_{h(x)} t) < \varepsilon$ for all $t \geq 0$. In fact, if we suppose the contrary, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$, $\{x_n\}$ and $t_n \to +\infty$ such that

$$\rho(x_n t_n, m_{h(x_n)} t_n) \geq \varepsilon_0 \text{ and } \rho(x_n, m_{h(x_n)}) < \delta_n \tag{14}$$

for all $n \in \mathbb{N}$. Note that the set $K_0 := \overline{\{x_n \mid n \in \mathbb{N}\}} \cup M$ is compact and by condition 4. we have

$$\lim_{t\to+\infty} \sup_{n\in\mathbb{N}} \rho(x_n t, m_{h(x_n)} t) = 0. \tag{15}$$

The equality (15) contradicts to (14). The obtained contradiction proves the stability of the set M. Now to finish the proof of Theorem it is sufficient to apply Theorem 3.3 $\square$

Remark 3.5. *a. If the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is two-sided, i.e. $\mathbb{T}_2 = \mathbb{S}$, then Theorem 3.4 was proved in* [7, Ch.II].

b. Note that in the proof of Theorem 3.4 we use the minimality of $(Y, \mathbb{T}_2, \sigma)$ only to show the implication 4.$\Longrightarrow$ 1.

Theorem 3.6. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be a non-autonomous dynamical system and the following conditions hold:*

(i) *Y is a compact minimal set;*
(ii) *every point $x \in X$ is stable in the sense of Lagrange, i.e. the set $H^+(x) := \overline{\{\sigma(t, x) \mid t \in \mathbb{S}_+ \text{ and } t \geq 0\}}$ is compact;*
(iii) *there exists a continuous function $V : X\dot{\times}X \to \mathbb{R}_+$, satisfying the following conditions:*
   a. *V is positively defined;*
   b. *$V(x_1t, x_2t) < V(x_1, x_2)$ for all $t > 0$ and $(x_1, x_2) \in X\dot{\times}X\setminus\Delta_X$, where $\Delta_X := \{(x, x) \mid x \in X\}$.*

Then the following statements take place:

(i) *there exists a unique compact minimal set $M \subseteq X$;*
(ii) *$\omega_x = M$for all point $x \in X$;*
(iii) *the set $M_y := \{x \in M \mid h(x) = y\}$ consists a single point, i.e. $M_y = \{m_y\}$;*
(iv) *$\lim\limits_{t\to+\infty} \rho(\pi(t, x), \pi(t, m_y)) = 0$ for all $x \in X_y$ and $y \in Y$.*

Proof. Let $(X\dot{\times}X, \mathbb{S}_+, \pi\dot{\times}\pi) := (X, \mathbb{S}_+, \pi)\dot{\times}(X, \mathbb{S}_+, \pi)$ be a direct product of $(X, \mathbb{S}_+, \pi)$ and $(X, \mathbb{S}_+, \pi)$, i.e. $\pi\dot{\times}\pi(t, (x_1, x_2)) := (\pi(t, x_1), \pi(t, x_2))$ for all $(x_1, x_2) \in X\dot{\times}X$ and $t \in \mathbb{S}_+$. By condition (ii) the point $(\bar{x}_1, \bar{x}_2) \in X\dot{\times}X$ is L^+ stable in $(X\dot{\times}X, \mathbb{S}_+, \pi\dot{\times}\pi)$, and consequently from Condition b. of the theorem follows the existence of a finite limit

$$V_0 = \lim_{t\to+\infty} V(\bar{x}_1t, \bar{x}_2t). \tag{16}$$

Let $(p, q) \in \omega_{(\bar{x}_1,\bar{x}_2)}$, then from (16) it follows that $V(p, q) = V_0$. By the invariance of $\omega_{(\bar{x}_1,\bar{x}_2)}$ we have $V(pt, qt) = V(p, q)$ for all $t \in \mathbb{S}_+$, and, according to condition (iii) of Theorem, $p = q$, i.e.,

$$\omega_{(\bar{x}_1,\bar{x}_2)} \subseteq \Delta_X := \{(x, x) \mid x \in X\}. \tag{17}$$

We will show now that for all $(x_1, x_2) \in X\dot{\times}X$ $(h(x_1) = h(x_2))$ the equality

$$\lim_{t\to+\infty} \rho(x_1t, x_2t) = 0$$

holds. If we suppose that it is not true, then there exist $y_0 \in Y, \bar{x}_1, \bar{x}_2 \in X_{y_0}, \varepsilon_0 > 0$ and $t_n \to +\infty$ such that

$$\rho(\bar{x}_1t_n, \bar{x}_2t_n) \geq \varepsilon_0. \tag{18}$$

We may suppose that the sequences $\{\bar{x}_it_n\}$ $(i = 1, 2)$ are convergent. Let us put $\bar{p} = \lim\limits_{n\to+\infty} \bar{x}_1t_n$ and $\bar{q} = \lim\limits_{n\to+\infty} \bar{x}_2t_n$, then $(\bar{p}, \bar{q}) \in \omega_{(\bar{x}_1,\bar{x}_2)}$ and from (18) it follows that $\bar{p} \neq \bar{q}$. The last equality contradicts the inclusion (17). The obtained contradiction proves the required statement.

Let $x \in X$ and $X_1 := H^+(x)$. Consider the non-autonomous dynamical system $\langle (X_1, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ induced by $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$. We will show that $\langle (X_1, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ possesses the property of convergence. To this end by Theorem 3.4 it is sufficient to show that every semi-trajectory $\Sigma_x^+ (x \in X)$ is asymptotically stable. If we suppose that the last statement is not true, then there exist $x_n, x_0 \in X_1$, $\varepsilon_0 > 0$, $x_n \to x_0$ $(h(x_n) = h(x_0))$ and $t_n > 0$ such that

$$\rho(\pi(t_n, x_n), \pi(t_n, x_0)) \geq \varepsilon_0. \tag{19}$$

On the other hand we have

$$V(\pi(t_n, x_n), \pi(t_n, x_0)) < V(x_n, x_0) \to 0 \tag{20}$$

as $n \to +\infty$. We can suppose that the sequences $\{\pi(t_n, x_n)\}$ and $\{\pi(t_n, x_0)\}$ are convergent. Let $\bar{x} := \lim_{n\to+\infty} \pi(t_n, x_n)$ and $\bar{x_0} := \lim_{n\to+\infty} \pi(t_n, x_0)$, then by the inequality (20) we have $V(\bar{x}, \bar{x_0}) = 0$. Since the function V is positively defined, then we obtain $\bar{x} = \bar{x_0}$. But the last equality contradicts to inequality (19). The obtained contradiction proves our statement. Thus the non-autonomous dynamical system $\langle (X_1, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is convergent. Let M be the Levinson center of dynamical system $(X_1, \mathbb{S}_+, \pi)$, then $h(M) = Y$ and $M_y := M \cap X_y$ consists a single point $\{m_y\}$ for all $y \in Y$. This means, in particularly, that $h : M \mapsto Y$ is a dynamical homeomorphism. Thus M is a compact minimal set and $\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, m_y)) = 0$ for all $x \in X_{1y}$ and $y \in Y$.

Let now $y \in Y$ and $x_1, x_2 \in X_y$, then the sets $M_i := \omega_{x_i}$ $(i = 1, 2)$ are two minimal sets and $h : M_i \mapsto Y$ (i=1,2) is a dynamical homeomorphism. Let as show that $M_1 = M_2$. If we suppose the contrary, then there exists a point $y_0 \in Y$ and $x_i^0 \in M_i \cap X_{y_0}$ (i=1,2) such that $x_1^0 \neq x_2^0$. Consider the function $\psi(t) := V(\pi(t, x_1^0), \pi(t, x_1^0))$ $(t \in \mathbb{S}_+)$. Note that $\psi(t) < p(0)$ for all $t > 0$ and there exists a strict increasing sequence $\{t_n\} \to +\infty$ such that: $\sigma(t_n, y_0) \to y_0$, $\ldots < \psi(t_n) < \ldots < \psi(t_1) < \psi(0)$ and $\psi(0) = \lim_{n\to+\infty} \psi(t_n) < \psi(0)$. The last contradiction show that our assumption is not true, i.e. $M_1 = M_2$. □

Theorem 3.7. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and the following conditions hold:*

(i) *Y is a compact minimal set;*
(ii) *every point $x \in X$ is stable in the sense of Lagrange;*
(iii) *there exists a continuous function $V : X \dot{\times} X \to \mathbb{R}_+$, satisfying the following conditions:*
   a. *V is positively defined;*
   b. *$V(x_1 t, x_2 t) \leq V(x_1, x_2)$ for all $t \geq 0$ and $(x_1, x_2) \in X \dot{\times} X$;*
   c. *for any $(x_1, x_2) \in X \dot{\times} X \backslash \Delta_X$ there is a $t_0 > 0$ such that $V(x_1 t_0, x_2 t_0) < V(x_1, x_2)$.*

Then the following statements take place:

(i) *there exists a unique compact minimal set $M \subseteq X$;*
(ii) *$\omega_x = M$for all point $x \in X$;*
(iii) *the set $M_y := \{x \in M \mid h(x) = y\}$ consists a single point, i.e. $M_y = \{m_y\}$;*
(iv) *$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, m_y)) = 0$ for all $x \in X_y$ and $y \in Y$.*

Proof. Let $V : X\dot{\times}X \mapsto \mathbb{R}_+$ be the continuous function with properties a., b. and c. We put

$$\mathcal{V}(x_1,x_2) := \int_0^{+\infty} V(x_1t,x_2t)e^{-t}dt \text{ (if } \mathbb{T}_1 = \mathbb{R}_+) \tag{21}$$

and

$$\mathcal{V}(x_1,x_2) := \Sigma_{t=0}^{+\infty} V(x_1t,x_2t)e^{-t} \text{ (if } \mathbb{T}_1 = \mathbb{Z}_+). \tag{22}$$

From the definition of the function $\mathcal{V}$ follow its continuity and positive definiteness. The function $\mathcal{V}$ satisfies condition b. of Theorem 3.6. In fact, if we suppose the contrary, then there exit $(\bar{x}_1,\bar{x}_2) \in X\dot{\times}X$ and $t_0 > 0$ such that $V(\bar{x}_1t_0,\bar{x}_2t_0) = V(\bar{x}_1,\bar{x}_2)$ and $\bar{x}_1 \neq \bar{x}_2$. Then from (21) and from the last equality, it follows that

$$\mathcal{V}(\bar{x}_1(t_0+t),\bar{x}_2(t_0+t)) = \mathcal{V}(\bar{x}_1t,\bar{x}_2t) \tag{23}$$

for all $t \geq 0$. By virtue of (23), the function $\varphi(t) = \mathcal{V}(\bar{x}_1t,\bar{x}_2t)$ is t_0 periodic. It is obvious that φ is continuous and non-increasing. Therefore, φ is stationary and, hence,

$$V(\bar{x}_1t,\bar{x}_2t) = V(\bar{x}_1,\bar{x}_2) \tag{24}$$

for all $t \geq 0$. On the other hand according to condition c. of Theorem we have $V(\bar{x}_1t_0,\bar{x}_2t_0) < V(\bar{x}_1,\bar{x}_2)$. The last inequality contradicts to the equality (24). The obtained contradiction shows that our assumption is not true. Now to finish the proof of Theorem it is sufficient to refer Theorem 3.6. □

Corollary 3.8. *Let $(X,\mathbb{S}_+,\pi)$ be an autonomous dynamical system and the following conditions hold:*

(i) *every point $x \in X$ is stable in the sense of Lagrange;*
(ii) *there exists a continuous function $V : X\times X \to \mathbb{R}_+$, satisfying the following conditions:*
   a. *V is positively defined;*
   b. *$V(x_1t,x_2t) \leq V(x_1,x_2)$ for all $t \geq 0$ and $(x_1,x_2) \in X\times X$;*
   c. *for any $(x_1,x_2) \in X\times X\setminus\Delta_X$ there is a $t_0 > 0$ such that $V(x_1t_0,x_2t_0) < V(x_1,x_2)$.*

Then the following statements take place:

(i) *there exists a unique fixed point $p \in X$, i.e. $\pi(t,p) = p$ for all $t \in \mathbb{S}_+$;*
(ii) *$\lim\limits_{t\to+\infty} \rho(\pi(t,x),p) = 0$ for all $x \in X$.*

Proof. This statement directly follows from Theorem 3.6. In fact, let $Y = \{q\}$ be a singe point and $(Y,\mathbb{S},\sigma)$ be an autonomous dynamical system defined by equality $\sigma(t,q) = q$ for all $t \in \mathbb{S}$. Consider the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$, where $h(x) = q$ for all $x \in X$. Now to finish the proof of Corollary it is sufficient to apply Theorem 3.6. □

Remark 3.9. *1. For discrete dynamical systems ($\mathbb{S}_+ = \mathbb{Z}_+$) Corollary 3.8 was established by A. Lasota* [30].

2. For dynamical systems on compact metric space Corollary 3.8 improves the well known theorem of Nemytskii-Edelstein (see, for example, [41] *and also* [2]*).*

Theorem 3.10. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system and the following conditions hold:*

(i) *Y is a compact minimal set;*
(ii) *the dynamical system $(X,\mathbb{T}_1,\pi)$ is locally compact;*
(iii) *every point $x\in X$ is stable in the sense of Lagrange;*
(iv) *there exists a continuous function $V : X\dot{\times}X\to\mathbb{R}_+$, satisfying the following conditions:*
  a. *V is positively defined;*
  b. *$V(x_1t,x_2t)\le V(x_1,x_2)$ for all $t\ge 0$ and $(x_1,x_2)\in X\dot{\times}X\setminus\Delta_X$;*
  c. *for any $(x_1,x_2)\in X\dot{\times}X\setminus\Delta_X$ there is a $t_0>0$ such that $V(x_1t_0,x_2t_0)< V(x_1,x_2)$.*

Then the following statements take place:

(i) *the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent;*
(ii) *there exists a unique compact minimal set $M\subseteq X$ such that $J_X=M$, where J_X is the Levisnon center of dynamical system $(X,\mathbb{S}_+,\pi)$;*
(iii) *$\lim\limits_{t\to+\infty}\rho(\pi(t,x),\pi(t,m_y))=0$ for all $x\in X_y$ and $y\in Y$, where $\{m_y\}=M_y$.*

Proof. By Theorem 3.7 the non-autonomous dynamical system $\langle (X,\ \mathbb{S}_+,\ \pi),\ (Y,\ \mathbb{S},\ \sigma),\ h\rangle$ is point dissipative and $\Omega_X=M$ is a compact minimal set which is dynamical homeomorphic to Y. Since the dynamical system $(X,\mathbb{S}_+,\pi)$ is locally compact, then according to Theorem 2.10 it is compactly dissipative. Let J_X be its Levinson center, then $M\subseteq J_X$. To finish the proof of Theorem it is sufficient to show that $J_X=M$ or equivalently, that the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is convergent. To this end, according to Theorem 3.4 it is sufficient to show that for any $\varepsilon>0$ and $K\in C(X)$ there exists $\delta(\varepsilon,K)>0$ such that $\rho(x_1,x_2)<\delta(h(x_1)=h(x_2);x_1,x_2\in K)$ implies $\rho(x_1t,x_2t)<\varepsilon$ for all $t\ge 0$. If we suppose that it is not true, then there are $\varepsilon_0>0$, $K_0\in C(X)$, $x_n^1,x_n^2\in K_0$ $(h(x_n^1)=h(x_n^2))$ and $t_n>0$ such that

$$\rho(x_n^1,x_n^1)<\frac{1}{n}\ \text{and}\ \rho(\pi(t_n,x_n^1),\pi(t_n,x_n^2))\ge\varepsilon_0. \tag{25}$$

Since the set K_0 is compact, then we can suppose that the sequences $\{x_n^i\}$ $(i=1,2)$ are convergent. Let $x^i:=\lim\limits_{n\to\infty}x_n^i$, by inequality (25) we have $x^1=x^2$. On the other hand

$$V(\pi(t_n,x_n^1),\pi(t_n,x_n^2))<V(x_n^1,x_n^2)\to 0 \tag{26}$$

as $n\to+\infty$. We can suppose that the sequences $\{\pi(t_n,x_n^i)\}$ $(i=1,2)$ are convergent. Let $\bar{x}^i:=\lim\limits_{n\to+\infty}\pi(t_n,x_n^i)$, then by the inequality (26) we have $V(\bar{x}^1,\bar{x}^2)=0$. Since the function V is positively defined, then we obtain $\bar{x}^1=\bar{x}^2$. But the last equality contradicts to inequality (25). The obtained contradiction proves our statement. $\square$

Corollary 3.11. *Let $(X,\mathbb{S}_+,\pi)$ be an autonomous dynamical system and the following conditions hold:*

(i) *the dynamical system* $(X, \mathbb{S}_+, \pi)$ *is locally compact;*
(ii) *every point* $x \in X$ *is stable in the sense of Lagrange;*
(iii) *there exists a continuous function* $V : X \dot{\times} X \to \mathbb{R}_+$, *satisfying the following conditions:*
  a. V *is positively defined;*
  b. $V(x_1 t, x_2 t) < V(x_1, x_2)$ *for all* $t > 0$ *and* $(x_1, x_2) \in X \times X \setminus \Delta_X$.

Then the following statements take place:

(i) *there exists a unique fixed point* $p \in X$, *i.e.* $\pi(t, p) = p$ *for all* $t \in \mathbb{S}_+$*;*
(ii) $\lim\limits_{t\to+\infty} \rho(\pi(t, x), p) = 0$ *for all* $x \in X$*;*
(iii) *the fixed point* $p \in X$ *is uniformly contracting, i.e. there exists a positive number* γ *such that* $\lim\limits_{t\to+\infty} \sup\limits_{\rho(x,p)\le\gamma} \rho(\pi(t, x), p) = 0.$

Proof. This statement follows from Theorem 3.7 and can be proved using the same reasoning as well as in the proof of Corollary 3.8. □

Recall [7] that the dynamical systems $(X, \mathbb{T}, \pi)$ satisfies the condition of Ladyzhenskaya, if for all bounded subset $M \in B(X)$ there exists a compact subset $K \in C(X)$ such that the equality

$$\lim_{t\to+\infty} \beta(\pi(t, M), K) = 0. \tag{27}$$

Theorem 3.12. [7, ChI] *Suppose that the dynamical system* $(X, \mathbb{T}, \pi)$ *satisfies the condition of Ladyzhenskaya, then the following conditions are equivalent:*

(i) *the dynamical system* $(X, \mathbb{T}, \pi)$ *is pontwice dissipative i.e. there exists a nonempty compact subset* $K_1 \in C(X)$ *which attracts every point from* X*;*
(ii) *the dynamical system* $(X, \mathbb{T}, \pi)$ *is boundedly dissipative i.e. there exists a nonempty compact subset* $K_2 \in C(X)$ *which attracts every point bounded subset from* X.

Theorem 3.13. *Let* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *be a non-autonomous dynamical system and the following conditions hold:*

(i) Y *is a compact minimal set;*
(ii) *the dynamical system* $(X, \mathbb{T}_1, \pi)$ *satisfies the condition of Ladyzhenskaya;*
(iii) *every point* $x \in X$ *is stable in the sense of Lagrange;*
(iv) *there exists a continuous function* $V : X \dot{\times} X \to \mathbb{R}_+$, *satisfying the following conditions:*
  a. V *is positively defined;*
  b. $V(x_1 t, x_2 t) \le V(x_1, x_2)$ *for all* $t > 0$ *and* $(x_1, x_2) \in X \dot{\times} X \setminus \Delta_X$, *where* $\Delta_X := \{(x, x) \mid x \in X\}$*;*
  c. *for any* $(x_1, x_2) \in X \dot{\times} X \setminus \Delta_X$ *there is a* $t_0 > 0$ *such that* $V(x_1 t_0, x_2 t_0) < V(x_1, x_2)$.

Then the following statements take place:

(i) *the non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is convergent;*

(ii) *there exists a unique compact minimal set $M \subseteq X$ such that $J_X = M$, where J_X is the Levisnon center of dynamical system $(X, \mathbb{T}_1, \pi)$;*

(iii)

$$\lim_{t\to+\infty} \sup_{x\in B} \rho(\pi(t,x), \pi(t, m_{h(x)})) = 0 \tag{28}$$

for every bounded subset B of X.

Proof. Since the dynamical system $(X, \mathbb{T}_1, \pi)$ satisfies the condition of Ladyzhenskaya, then every point $x \in X$ is stable in the sense of Lagrange. By Theorem 3.7 the non-autonomous dynamical system $\langle (X,\ \mathbb{T}_1,\ \pi),\ (Y,\ \mathbb{T}_2,\ \sigma),\ h\rangle$ is point dissipative and $\Omega_X = M$ is a compact minimal set which is dynamical homeomorphic to Y. Since the dynamical system $(X, \mathbb{T}_1, \pi)$ possesses the properties of Ladyzhenskaya, then according to Theorem 3.12 it is compactly dissipative. Let J_X be its Levinson center, then $M \subseteq J_X$. Now we will show that $J_X = M$ or equivalently, that the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent. To this end, according to Theorem 3.4 it is sufficient to show that for any $\varepsilon > 0$ and $K \in C(X)$ there exists $\delta(\varepsilon, K) > 0$ such that $\rho(x_1, x_2) < \delta(h(x_1) = h(x_2); x_1, x_2 \in K)$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \geq 0$. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $K_0 \in C(X)$, $x_n^1, x_n^2 \in K_0$ $(h(x_n^1) = h(x_n^2))$ and $t_n > 0$ such that

$$\rho(x_n^1, x_n^1) < \frac{1}{n} \text{ and } \rho(\pi(t_n, x_n^1), \pi(t_n, x_n^2)) \geq \varepsilon_0. \tag{29}$$

On the other hand we have

$$V(\pi(t_n, x_n^1), \pi(t_n, x_n^2)) \leq V(x_n^1, x_n^2) \to 0 \tag{30}$$

as $n \to +\infty$. We can suppose that the sequences $\{\pi(t_n, x_n^i)\}$ $(i = 1, 2)$ are convergent. Let $\bar{x^i} := \lim_{n\to+\infty} \pi(t_n, x_n^i)$, then by the inequality (30) we have $V(\bar{x^1}, \bar{x^2}) = 0$. Since the function V is positively defined, then we obtain $\bar{x^1} = \bar{x^2}$. But the last equality contradicts to inequality (29). The obtained contradiction proves our statement.

To finish the proof it is sufficient to prove the equality (28). Suppose that it is not true, then the exists a bounded subset B_0 of X, $\varepsilon_0 > 0$, $\{x_n\} \subseteq B_0$ and $t_n \to +\infty$ such that

$$\rho(x_n t_n, m_{h(x_n)} t_n) \geq \varepsilon_0 \tag{31}$$

for all $n \in \mathbb{N}$. Without loss of generality we can suppose that the sequence $\{x_n t_n\}$ is convergent. Let $\bar{x} := \lim_{n\to+\infty} x_n t_n$ and $\bar{y} := h(\bar{x}) = \lim_{n\to+\infty} h(x_n) t_n$. Since $m_{h(x_n)} t_n = m_{h(x_n) t_n} \to m_{\bar{y}}$ as $n \to +\infty$, then we have $\bar{x} \in J_X \bigcap X_{\bar{y}} = \{m_{\bar{y}}\}$, i.e. $\bar{x} = m_{\bar{y}}$. But the last equality contradicts to the inequality (31). The obtained contradiction completes the proof of Theorem. □

4. Non-Autonomous Version of Markus-Yamabe Conjecture

4.1. **Global Asymptotic Stability.** Let (X, h, Y) be a vector bundle, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma),\ h\rangle$ be non-autonomous dynamical system, $\theta_y \in X_y := h^{-1}(y)$ be a null element $(|\theta_y| = 0)$ and $\Theta := \{\theta_y \mid y \in Y\}$ be a null section of the vector bundle (X, h, Y). Below we will suppose that $\mathbb{T}_2 = \mathbb{S}$ and the null section Θ is invariant, i.e. $\Theta \subseteq X$ is an invariant set of the dynamical system $(X, \mathbb{T}_1, \pi)$.

Definition 4.1. *the null section Θ is called uniformly stable if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $y \in Y$, $x \in X_y$ and $|x| < \delta$ implies $|xt| < \varepsilon$ for all $t \geq 0$.*

Definition 4.2. *If Θ is uniformly stable and $\lim\limits_{t\to+\infty} |xt| = 0$ for all $x \in X$, then the null section is called globally uniformly asymptotically stable.*

Denote by $\mathfrak{A} := \{a \mid a : \mathbb{R}_+ \to \mathbb{R}_+, a$ is continuous, strict increasing and $a(0) = 0\ \}$.

Theorem 4.3. [7, Ch.II] *Let Y be compact and (X, h, Y) be a finite-dimensional vectorial bundle fiber. For the null section Θ to be globally uniformly asymptotically stable it is necessary and sufficient that there would exist a continuous function $V : X \to \mathbb{R}_+$ satisfying the following conditions:*

1. *$V(x) \geq a(|x|)$ for all $x \in X$, $V(\theta_y) = 0$ for all $y \in Y$ and $Im\, a = Im\, V$, where $a \in \mathfrak{A}$.*
2. *$V(xt) \leq V(x)$ for all $x \in X$ and $t \geq 0$.*
3. *the level lines of V do not contain non-null ω-limit points of the dynamical system $(X, \mathbb{T}_1, \pi)$.*

Theorem 4.4. *Let Y be compact and (X, h, Y) be a finite-dimensional vectorial bundle fiber. For the null section Θ to be globally uniformly asymptotically stable it is necessary and sufficient that there would exist a continuous function $V : X \to \mathbb{R}_+$ satisfying the following conditions:*

1. *$V(x) \geq a(|x|)$ for all $x \in X$, $V(\theta_y) = 0$ for all $y \in Y$ and $Im\, a = Im\, V$, where $a \in \mathfrak{A}$.*
2. *$V(xt) < V(x)$ for all $x \in X \setminus \Theta$ and $t > 0$.*

Proof. Sufficiency. Let the conditions of the theorem be satisfied. Show that the null section Θ is uniformly stable. Suppose that it is not true. Then there exist $\varepsilon_0 > 0$, $|x_n| < \delta$, $\delta_n \downarrow 0$ and $t_n \geq 0$ such that

$$|x_n t_n| \geq \varepsilon_0. \tag{32}$$

On the other hand, $0 \leq a(|x_n t_n|) \leq V(x_n t_n) \leq V(x_n) \to 0$ as $n \to +\infty$ and, consequently, $|x_n t_n| \to 0$. The last contradicts to the equality (32).

Now let us show that

$$\lim_{t\to+\infty} |xt| = 0 \tag{33}$$

for all $x \in X$. In fact, if we suppose the contrary then there exists $x_0 \in X$ ($|x_0| \neq 0$) such that $\limsup\limits_{t\to+\infty} |x_0 t| > 0$, i.e. there exist $\varepsilon_0 > 0$ and $t_n \to +\infty$ for which

$$|x_0 t_n| \geq \varepsilon_0. \tag{34}$$

Note that $\Sigma^+_{x_0}$ is relatively compact. In fact, $a(|x_0 t|) \leq V(x_0 t) \leq V(x_0)$ and, consequently, $|x_0 t| \leq a^{-1}(V(x_0))$ for all $t \geq 0$. So, the sequence $\{x_0 t_n\}$ can be considered convergent. Assume $\tilde{x} := \lim\limits_{n\to+\infty} x_0 t_n$, then $\tilde{x} \in \omega_{x_0}$. We will show that there exists $c \geq 0$ for which $V(x) = c$ for all $x \in \omega_{x_0}$. In fact. Consider the function $\varphi(t) := V(x_0 t)$ (for all $t \in \mathbb{T}_1$). Under the conditions of Theorem the function φ is bounded ($\varphi(t) \leq \varphi(0)$ for all $t \geq 0$) and strictly decreasing ($\varphi(t_1) < \varphi(t_2)$ for all $t_1 < t_2$) and, consequently, there exists

$$\lim_{t \to +\infty} V(x_0 t) := c \geq 0. \tag{35}$$

Since the function V is continuous, then from the equality (35) it follows that $V(x) = c$ for all $x \in \omega_{x_0}$.

Now we will show that $c = 0$, where c is the constant from the equality (35). If we suppose that $c > 0$, then $\omega_{x_0} \cap \Theta = \emptyset$ because if $\bar{x} \in \omega_{x_0} \cap \Theta$, then $c = V(\bar{x}) = 0$. We have $V(xt) = c$ for all $x \in \omega_{x_0}$ and $t \in \mathbb{T}_1$. On the other hand $c = V(xt) < V(x) = c$ for all $t > 0$ and $x \in \omega_{x_0}$. The obtained contradiction proves our statement. Thus by equality (35) we obtain the equality (33) and, consequently, the global uniform asymptotic stability of the null section is proved.

Necessity. Let the null section Θ be globally uniformly asymptotically stable. By Theorem 4.3 there exists the continuous function $\mathcal{V} : X \mapsto \mathbb{R}_+$ satisfying the conditions 1.-3. of Theorem 4.3. Let us put

$$V(x) := \int_0^{+\infty} \mathcal{V}(xt)e^{-t}dt \ (\text{if} \mathbb{T}_1 = \mathbb{R}_+) \tag{36}$$

and

$$V(x) := \Sigma_{t=0}^{+\infty} \mathcal{V}(xt)e^{-t} \ (\text{if} \mathbb{T}_1 = \mathbb{Z}_+). \tag{37}$$

From the definition of the function V follows its continuity and positive definiteness ($V(x) \geq a(|x|)$ for all $x \in X$). The function V satisfies to condition 2. of Theorem. In fact, if we suppose the contrary, then there exists $x_0 \in X \setminus \Theta$ and $t_0 > 0$ such that $V(x_0 t_0) = V(x_0)$. Then from (36) and the last equality, it follows that

$$V(x_0(t_0 + t)) = V(x_0 t) \tag{38}$$

for all $t \geq 0$. By virtu of (38), the function $\varphi(t) := \mathcal{V}(x_0 t)$ (for all $t \geq 0$) is t_0 periodic. It is obvious that φ is continuous and non-increasing. Therefore, φ is stationary and, hence, $\mathcal{V}(x_0 t) = \mathcal{V}(x_0) > 0$ for all $t \geq 0$. Reasoning as above we can prove that ω_{x_0} is a nonempty compact invariant set and $\mathcal{V}(x) = \mathcal{V}(x_0) > 0$ for all $x \in \omega_{x_0}$. The last equality contradicts to condition 3. of Theorem 4.3. The obtained contradiction completes the proof of the theorem. □

Corollary 4.5. *Let Y be compact and (X, h, Y) be a finite-dimensional vectorial bundle fiber. For the null section Θ to be globally uniformly asymptotically stable it is necessary and sufficient that there would exist a continuous function $V : X \to \mathbb{R}_+$ satisfying the following conditions:*

1. *$V(x) \geq a(|x|)$ for all $x \in X$, $V(\theta_y) = 0$ for all $y \in Y$ and $Im\, a = Im\, V$, where $a \in \mathfrak{A}$;*
2. *$V(xt) \leq V(x)$ for all $x \in X$ and $t \geq 0$;*
3. *$V(xt) < V(x)$ if $xs \notin \Theta$ for all $s \in [0, t]$.*

Proof. This statement can by proved with slight modification of the proof of Theorem 4.4. □

Remark 4.6. *Note that Theorems 4.3, 4.4 and Corollary 4.5 remain true and for the infinite-dimensional vectorial fibers (X, h, Y) if we suppose that the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.*

4.2. **Finite-dimensional systems.** Denote by E^n a n-dimensional Euclidean space with the scalar product $\langle , \rangle$ and the norm $|\cdot|$ generated by the scalar product. Let $[E^n]$ be a space of all the linear mappings $A : E^n \mapsto E^n$ equipped with the operator norm.

The function $F \in C(Y \times E^n, E^n)$ is called regular if for any $(y, u) \in Y \times E^n$ there exists a unique solution $\varphi(t, u, y)$ of the equation

$$u' = F(\sigma(t, y), u) \tag{39}$$

with initial condition $\varphi(0, u, y) = u$ defined on $\mathbb{R}_+$, i.e. (39) generates a cocycle φ on E^n.

Theorem 4.7. *Let Y be a compact metric space, $F \in C(Y \times E^n, E^n)$, $\mathcal{W} \in C(Y, [E^n])$ and the following conditions hold:*

(i) *the operator-function $\mathcal{W}$ is positively defined, i.e. $\langle \mathcal{W}(y)u, u\rangle \in \mathbb{R}$ for all $y \in Y$, $u \in \mathbb{E}^n$, and there exists a positive constant a such that $\langle \mathcal{W}(y)u, u\rangle \geq a|u|^2$ for all $y \in Y$ and $u \in E^n$;*
(ii) *the function $t \to \mathcal{W}(\sigma_t y)$ is differentiable for every $y \in Y$ and $\dot{\mathcal{W}}(y) \in C(Y, [E^n])$, where $\dot{\mathcal{W}}(y) := \frac{d}{dt}\mathcal{W}(\sigma(t, y))|_{t=0}$;*
(iii) *$\langle \dot{\mathcal{W}}(y)(u-v) + (\mathcal{W}(y) + \mathcal{W}^*(y))(F(y, u) - F(y, v)), u - v\rangle < 0$ for all $y \in Y$ and $u, v \in E^n$ $(u \neq v)$, where $\mathcal{W}^*(y)$ is an adjoint operator;*
(iv) *$F(y, 0) = 0$ for all $y \in Y$;*
(v) *the function $F \in C(Y \times E^n, E^n)$ is regular.*

Then the trivial solution of equation (39)is global asymptotically stable.

Proof. Let $\langle E^n, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be the cocycle generated by equation (39).

Denote by $\mathcal{V} : E^n \times Y \mapsto \mathbb{R}^+$ the function defined by the equality $\mathcal{V}(u, y) := \langle W(y)u, u\rangle$ for all $(u, y) \in X := E^n \times Y$. If $|\varphi(s, u, y)| > 0$ for all $s \in [0, t] \subset \mathbb{R}_+$, then

$$\frac{d}{dt}\mathcal{V}(\sigma(t, y), \varphi(t, u, y)) = \langle \dot{W}(\sigma(t, y))\varphi(t, u, y), \varphi(t, u, y)\rangle +$$
$$\langle (W(\sigma(t, y)) + W_*(\sigma(t, y)))F(\sigma(t, y), \varphi(t, u, y)), \varphi(t, u, y)\rangle < 0.$$

Now to finish the proof it is sufficient to apply Corollary 4.5. □

Example 4.8. *As an example that illustrates this theorem we can consider the following equation*

$$u' = g(\sigma(t, \omega), u),$$

where $g \in C(E^n \times \Omega, E^n)$ and $\langle Ag(u, \omega), u\rangle < 0$ for all $u \in E^n$ $(u \neq 0)$ and $\omega \in \Omega$, where $A \in [E^n]$ is a self-adjoint positive definite matrix and $g(0, \omega) = 0$ for all $\omega \in \Omega$.

Theorem 4.9. *Let Y be a compact metric space, $F \in C(Y \times E^n, E^n)$, $\mathcal{W} \in [E^n]$ and the following conditions hold:*

(i) *the operator* $\mathcal{W}$ *is positively defined, i.e.* $\langle \mathcal{W}u, u\rangle \in \mathbb{R}$ *for all* $u \in E^n$, *and there exists a positive constant* a *such that* $\langle \mathcal{W}u, u\rangle \ge a|u|^2$ *for all* $u \in E^n$;

(ii) $\langle (\mathcal{W}+\mathcal{W}^*)F(y,u), u\rangle < 0$ *for all* $y \in Y$ *and* $u \in E^n$ $(u \neq 0)$, *where* $\mathcal{W}^*$ *is an adjoint operator;*

(iii) $F(y,0) = 0$ *for all* $y \in Y$;

(iv) *the function* $F \in C(Y \times E^n, E^n)$ *is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. Denote by $\mathcal{V} : E^n \times Y \mapsto \mathbb{R}^+$ the function defined by the equality $\mathcal{V}(u,y) := \langle Wu, u\rangle$ for all $(u,y) \in X := E^n \times Y$. If $|\varphi(s,u,y)| > 0$ for all $s \in [0,t] \subset \mathbb{R}_+$, then

$$\frac{d}{dt}\mathcal{V}(\varphi(t,u,y), \sigma(t,y)) = \langle (W+W_*)F(\sigma(t,y), \varphi(t,u,y)), \varphi(t,u,y)\rangle < 0.$$

Now to finish the proof it is sufficient to apply Corollary 4.5. □

4.3. **Gradient Systems.** Let $F \in C(Y \times E^n, E^n)$ be continuously differentiable in $u \in E^n$ and denote by $F'_u(y,u)$ its derivative (Jacobian matrix) with respect to u.

Definition 4.10. *The continuously differentiable function* $V \in C(Y \times E^n, \mathbb{P})$ $(\mathbb{P} = \mathbb{C}$ *or* $\mathbb{R})$ *is called a potential for* $F \in C(Y \times E^n, E^n)$ *if* $F(y,u) = V'_u(y,u)$ *for all* $(y,u) \in Y \times E^n$.

Remark 4.11. *If the function* $F \in C(Y \times E^n, E^n)$ *is potential (i.e.* $F(y,u) = V'_u(y,u)$*) and its potential* $V \in C(Y \times E^n, \mathbb{P})$ *is twice differentiable, then* $F'_u(y,u) = V''_{u^2}(y,u) = (\frac{\partial^2 V}{\partial u_i \partial u_j})^n_{i,j=1}$.

Definition 4.12. *The equation (39) is called gradient if its right hand side* $F \in C(Y \times E^n)$ *is a potential function, i.e. there exists a continuously differentiable function* $V \in C(Y \times E^n, \mathbb{P})$ *such that*

$$F(y,u) = V'_u(y,u)$$

for all $(y,u) \in Y \times E^n$.

Theorem 4.13. *Let* Y *be a compact metric space,* $F \in C(Y \times E^n, E^n)$, $\mathcal{W} \in [E^n]$ *and the following conditions hold:*

(i) *the function* $F \in C(Y \times E^n, E^n)$ *is continuously differentiable in* $u \in E^n$;

(ii) *the operator* $\mathcal{W}$ *is positively defined, i.e.* $\langle \mathcal{W}u, u\rangle \in \mathbb{R}$ *for all* $u \in E^n$, *and there exists a positive constant* a *such that* $\langle \mathcal{W}u, u\rangle \ge a|u|^2$ *for all* $u \in E^n$;

(iii) $\langle (\mathcal{W}+\mathcal{W}^*)F'_u(y,u)u, u\rangle < 0$ *for all* $y \in Y$ *and* $u \in E^n$ $(u \neq 0)$, *where* $\mathcal{W}^*$ *is an adjoint operator;*

(iv) $F(y,0) = 0$ *for all* $y \in Y$;

(v) *the function* $F \in C(Y \times E^n, E^n)$ *is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. Denote by $\mathcal{V} : E^n \times Y \mapsto \mathbb{R}^+$ the function defined by equality $\mathcal{V}(u,y) := \langle W(u,u\rangle$ for all $(u,y) \in X := E^n \times Y$. If $|\varphi(s,u,y)| > 0$ for all $s \in [0,t] \subset \mathbb{R}_+$, then

$$\begin{aligned} &\tfrac{d}{dt}\mathcal{V}(\varphi(t,u,y), \sigma(t,y)) = \langle (W+W_*)F(\sigma(t,y), \varphi(t,u,y)), \varphi(t,u,y)\rangle = \\ &\langle (W+W_*)F_u(\sigma(t,y), \varphi(t,u,y))\varphi(t,u,y)), \varphi(t,u,y)\rangle < 0, \qquad (40) \end{aligned}$$

where $\theta(t,u,y) \in [0,1]$ for all $(t,u,y) \in \mathbb{R}_+ \times E^n \times Y$.

Now to finish the proof it is sufficient to apply Theorem 4.9. $\square$

Corollary 4.14. *Let Y be a compact metric space, $\mathbb{R}^n$ be a real n-dimensional Euclidean space, $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{W} \in [\mathbb{R}^n]$ and the following conditions hold:*

(i) *the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is continuously differentiable in $x \in \mathbb{R}^n$;*
(ii) *the operator $\mathcal{W}$ is positively defined, i.e. $\langle \mathcal{W}u, u\rangle \in \mathbb{R}$ for all $u \in \mathbb{R}^n$, and there exists a positive constant a such that $\langle \mathcal{W}u, u\rangle \geq a|u|^2$ for all $u \in \mathbb{R}^n$;*
(iii) *$\langle (\mathcal{W}F'_u(y,u) + F'^*_u(y,u)W)u, u\rangle < 0$ for all $y \in Y$ and $u \in \mathbb{R}^n$ $(u \neq 0)$, where $F'^*_u(y,u)$ is an adjoint operator;*
(iv) *$F(y,0) = 0$ for all $y \in Y$;*
(v) *the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. This statement follows from Theorem 4.13. In fact, if $\mathbb{R}^n$ is a real finite-dimensional Euclidean space, then we have

$$\langle(((W + W^*)F'_u(y,u)u, u\rangle = \langle(\mathcal{W}F'_u(y,u) + F'^*_u(y,u)W)u, u\rangle < 0$$

for all $(y,u) \in Y \times \mathbb{R}^n$ $(u \neq 0)$. Now it is sufficient to apply Theorem 4.13. $\square$

Corollary 4.15. *Let Y be a compact metric space, $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{W} \in [\mathbb{R}^n]$ and the following conditions hold:*

(i) *the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is continuously differentiable in $x \in \mathbb{R}^n$;*
(ii) *the operator $F'_u(y,u)+F'^*_u(y,u)$ has only negative eigenvalues for all $(y,u) \in Y \times \mathbb{R}^n$;*
(iii) *$F(y,0) = 0$ for all $y \in Y$;*
(iv) *the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. This statement follows from Corollary 4.15. In fact, since $\mathbb{R}^n$ is a real finite-dimensional space and the operator $F'_u(y,u) + F'^*_u(y,u)$ is auto-adjoint, then we have

$$2\langle F'_u(y,u)u, u\rangle = \langle(F'_u(y,u) + F'^*_u(y,u))u, u\rangle < 0$$

for all $(y,u) \in Y \times \mathbb{R}^n$ $(u \neq 0)$. Now it is sufficient to apply Corollary 4.14 with $W = Id_{\mathbb{R}^n}$. $\square$

Corollary 4.16. *Let Y be a compact metric space, $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{W} \in [\mathbb{R}^n]$ and the following conditions hold:*

(i) *the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is continuously differentiable in $x \in \mathbb{R}^n$;*
(ii) *the equation (39) is gradient, i.e. there exists a continuously differentiable function $V \in C(Y \times \mathbb{R}^n, \mathbb{R})$ such that $F(y,u) = V'_u(y,u)$ for all $(y,u) \in Y \times \mathbb{R}^n$;*
(iii) *the function $V \in C(Y \times E, \mathbb{R})$ is twice continuously differentiable;*
(iv) *the Jacobian $F'_u(y,u) = V''_{uu}(y,u)$ of F has only negative eigenvalues for all $(y,u) \in Y \times \mathbb{R}^n$;*

(v) $F(y,0)=0$ *for all* $y\in Y$*;*
(vi) *the function* $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ *is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. This statement follows from Corollary 4.15. In fact, since $\mathbb{R}^n$ is a real finite-dimensional space and the operator $F'_u(y,u)=V''_{uu}(y,u)$ is auto-adjoint, then we have

$$\langle F'_u(y,u)u,u\rangle=\langle V''_{uu}(y,u)u,u\rangle<0$$

for all $(y,u)\in Y\times\mathbb{R}^n$ $(u\neq 0)$. Now it is sufficient to apply Corollary 4.14. □

Remark 4.17. *For autonomous system this statement was proved in* [12].

4.4. **Triangular Systems.**

Theorem 4.18. *Let* W,Y *be two finite-dimensional Banach spaces,* $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)$ *be a cocycle on* W *and the following conditions be held:*

(i) 0 *is a unique fixed point of dynamical system* $(Y,\mathbb{T}_2,\sigma)$ *which is globally asymptotically stable;*
(ii) $\varphi(t,0,y)=0$ *for all* $t\in\mathbb{T}_1$ *and* $y\in Y$*;*
(iii) *there exist a continuous function* $V:Y\times W\to\mathbb{R}_+$ *satisfying the following conditions:*
   1. $V(y,u)\geq a(|u|)$ *for all* $(y,u)\in Y\times W,$ $V(y,0)=0$ *for all* $y\in Y$ *and* $Im\,a=Im\,V,$ *where* $a\in\mathfrak{A}$*;*
   2. $V(\sigma(t,y),\varphi(t,u,y))\leq V(y,u)$ *for all* $(y,u)\in Y\times W$ *and* $t\geq 0$*;*
   3. $V(\sigma(t,y),\varphi(t,u,y))<V(y,u)$ *if* $|\varphi(s,u,y)|>0$ *for all* $s\in[0,t]$.

Then the trivial motion of skew-product dynamical system $(X,\mathbb{T}_1,\pi)$ *(*$X:=Y\times W,$ $\pi=(\varphi,\sigma)$*) is globally asymptotically stable.*

Proof. Let $x_0=(y_0,u_0)\in X$ be an arbitrary point. From the Conditions a. and b. it follows that $\varphi(\cdot,u_0,y_0)$, i.e. $\sup_{t\geq 0}|\varphi(\cdot,u_0,y_0)|<+\infty$. Thus the motion $\pi(t,x_0)$ is relatively compact on $\mathbb{T}_+$ and, consequently, ω_{x_0} is a nonempty, compact invariant set of dynamical system $(X,\mathbb{T}_1,\pi)$. Consider the non-autonomous dynamical system $\langle(\tilde{X},\mathbb{T}_1,\pi),(\tilde{Y},\mathbb{T}_2,\sigma),h\rangle$, where $\tilde{Y}=H^+(y_0):=\overline{\{y_0t\mid t\geq 0\}}$, $\tilde{X}:=\tilde{Y}\times W$ and $h:=pr_2:\tilde{X}\mapsto\tilde{Y}$. By Theorem 4.4 the trivial motion of dynamical system $(\tilde{X},\mathbb{T}_1,\pi)$ is globally asymptotically stable and, in particularly, $\omega_{x_0}=\{0\}$. Thus the dynamical system $(X,\mathbb{T}_1,\pi)$ is point-wise dissipative and $\Omega_X=\{0\}$. Since X is a finite-dimensional space, then by Theorem 2.10 the dynamical system $(X,\mathbb{T}_1,\pi)$ is also compactly dissipative. Let J be the Levinson center of dynamical system $(X,\mathbb{T}_1,\pi)$, then by Theorem 2.12 $J=J^+(\Omega_X)$. To finish the proof of Theorem it is sufficient to show that $J=\{0\}$. Let $p\in J$, then there are $x_n\to 0$ and $t_n\to+\infty$ such that $p=\lim\limits_{t\to+\infty}x_nt_n$. Since $a(|x_nt_n|)\leq V(\pi(t_n,x_n))\leq V(x_n)\to 0$ as $n\to+\infty$, then $x_nt_n\to 0$ and, consequently, $p=0$. □

Let E^n be a n-dimensional Euclidean space, $E^n = E^{m_1} \times E^{m_2} \times \ldots \times E^{m_k}$ ($n = m_1 + m_2 + \ldots + m_k$). Consider the system

$$\begin{cases} u_1' = f_1(\sigma(t,y), u_1) \\ u_2' = f_2(\sigma(t,y), u_1, u_2) \\ \ldots \\ u_m' = f_m(\sigma(t,y), u_1, u_2, \ldots, u_m), \end{cases} \tag{41}$$

where $f_i \in C(Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}; E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i})$ ($i = 1, 2, \ldots, k$).

Theorem 4.19. *Let Y be a compact metric space, $f_i \in C(Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}; E^{m_i})$ ($i = 1, 2, \ldots, k$) and the following conditions hold:*

(i) *the function f_i is continuously differentiable in $u_i \in E^{m_i}$;*
(ii) *for all $i = 1, 2, \ldots, k$ there exists a continuously differentiable function $V_i \in C(Y \times, \mathbb{R})$ such that $f_i(y, u) = V'_{i\,u_i}(y, u)$ for all $(y, u) \in Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}$;*
(iii) *for all $i = 1, 2, \ldots, k$ the function $V_i \in C(Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}, \mathbb{R})$ is twice continuously differentiable in u_i;*
(iv) *the Jacobian $F_u'(y, u)$ ($(y, u) \in Y \times E^n$) of right hand side $F := (f_1, f_2, \ldots, f_m)$ of system (41) has only negative eigenvalues for all $(y, u) \in Y \times E$;*
(v) *$F(y, 0) = 0$ for all $y \in Y$;*
(vi) *the function $F \in C(Y \times E, E)$ is regular.*

Then the trivial solution of equation (39) is global asymptotically stable.

Proof. Let $F = (f_1, f_2, \ldots, f_k)$, then

$$\begin{gathered} det(F_u'(y,u) - \lambda I) = det(\tfrac{\partial f_1}{\partial u_1}(y, u_1) - \lambda I) \times \\ det(\tfrac{\partial f_2}{\partial u_2}(y, u_1, u_2) - \lambda I) \times \ldots \times det(\tfrac{\partial f_k}{\partial u_k}(y, u_1, u_2, \ldots, u_k) - \lambda I) \end{gathered} \tag{42}$$

and, consequently, $\sigma(F_u') = \cup_{i=1}^k \sigma(\frac{\partial f_i}{\partial u_i})$, where $\sigma(A)$ is the spectrum of operator A. We will prove this statement by induction with respect to k. If $k = 1$, then this statement coincides with Corollary 4.16. Assume that it is true for all $1 < i \le k-1$ and we will prove it for $i = k$. Denote by $M := E^{m_1} \times E^{m_2} \times \ldots \times E^{m_{k-1}}$ and $(M, \mathbb{R}_+, \pi)$ the dynamical system, generated by equation $x' = \tilde{F}(x)$ ($x \in M$), where $\tilde{F} = (f_1, f_2, \ldots, f_{k-1})$. Finally, let $\langle E^{m_k}, \varphi, (M, \mathbb{R}_+, \sigma)\rangle$ be a cocycle, generated by equation

$$u_k' = f_{m_k}(\sigma(t, y), u_k) \ (u_k \in E^{m_k},\ y \in M).$$

Let $V : Y \times E^{m_k} \mapsto \mathbb{R}_+$ be the function defined by equality $V(y, u_k) := \frac{1}{2}\langle u_k, u_k\rangle$. Reasoning as in the proof of Corollary 4.16 we can show that the function V possesses the properties 1.–3. from Theorem 4.18. Now to finish the proof it is sufficient to apply Theorem 4.18. $\square$

Remark 4.20. *For autonomous systems, when $m_1 = m_2 = \ldots = m_k = 1$ this statement was established in* [33] *(see also* [12]*).*

5. The discrete Markus-Yamabe problem for non-autonomous systems

5.1. **Triangular maps and non-autonomous dynamical systems.** Let W and Y be two complete metric spaces and denote by $X := W \times Y$ its cartesian product.

Recall (see, for example, [23, 24]) that a continuous map $F : X \mapsto X$ is called triangular if there are two continuous maps $f : W \times Y \mapsto W$ and $g : Y \mapsto Y$ such that $F = (f, g)$, i.e. $F(x) = F(y, u) = (f(y, u), g(y))$ for all $x =: (y, u) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(y_n, u_n) \\ y_{n+1} = g(y_n), \end{cases} \tag{43}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all non-negative integer numbers.

Along with system (43) we consider the family of equations

$$u_{n+1} = f(g^n y, u_n) \ (y \in Y), \tag{44}$$

which is equivalent to system (43). Let $\varphi(n, u, y)$ be a solution of equation (44) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times Y \mapsto W$ $((n, u, y) \mapsto \varphi(n, u, y)$) satisfies the following conditions:

(i) $\varphi(0, u, y) = u$ for all $u \in W$;
(ii) $\varphi(n + m, u, y) = \varphi(n, \varphi(m, u, y), \sigma(m, y))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $y \in Y$, where $\sigma(n, y) := g^n y$;
(iii) the map $\varphi : \mathbb{Z}_+ \times W \times Y \mapsto W$ is continuous.

Denote by $(Y, \mathbb{Z}_+, \sigma)$ the semi-group dynamical system generated by positive powers of the map $g : Y \mapsto Y$, i.e. $\sigma(n, y) := g^n y$ for all $n \in \mathbb{Z}_+$ and $y \in Y$.

Definition 5.1. *Recall* [7] *that a triplet $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ (or briefly φ) is called a cocycle (or non-autonomous dynamical system) over the dynamical system $(Y, \mathbb{Z}_+, \sigma)$ with fiber W.*

Thus, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa. Taking into consideration this remark we can study triangular maps in the framework of non-autonomous dynamical systems (cocycles) with discrete time.

5.2. **Contractive dynamical systems.** The mapping $F : X \mapsto X$ is called asymptotically compact, if the discrete dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by the positive powers of F (i.e. $\pi(n, x) := F^n(x)$ for all $(n, x) \in \mathbb{Z}_+ \times X$) is so.

Theorem 5.2. *Suppose that the following conditions hold:*

(i) *(Y, g) is a compact minimal dynamical system;*
(ii) *W is a Banach space;*
(iii) *the function $f \in C(Y \times W, W)$ is continuously differentiable in $u \in W$;*
(iv) *$||f'_u(y, u)|| < 1$ for all $(y, u) \in Y \times W$;*
(v) *$f(y, 0) = 0$ for all $y \in Y$;*
(vi) *the mapping $F : Y \times W \mapsto Y \times W$, where $F(y, u) = (g(y), f(y, u))$ for all $(y, u) \in Y \times W$, is asymptotically compact.*

Then the trivial solution of equation (44) is global asymptotically stable.

Proof. Let $u_1, u_2 \in W$, then by formula of finite difference we have

$$f(y, u_2) - f(y, u_1) = f'_u(y, u_1 + \theta(u_2 - u_1))(u_2 - u_1), \tag{45}$$

where $\theta = \theta(y, u_1, u_2) \in [0, 1]$. Since $||f'_u(y, u)|| < 1$ for all $(y, u) \in Y \times W$ and taking into account the equality (45) we obtain

$$|f(y, u_2) - f(y, u_1)| = |f'_u(u_1 + \theta(u_2 - u_1))(u_2 - u_1)| \leq$$
$$||f_u(y, u_1 + \theta(u_2 - u_1))|||u_2 - u_1| < |u_2 - u_1| \tag{46}$$

for all $u_1, u_2 \in W$ ($u_1 \neq u_2$). Now we will show that the discrete dynamical system $(X, \mathbb{Z}_+, \pi)$, generated by positive powers of $F : X \mapsto X$ is boundedly dissipative (in particular, compactly dissipative). In fact, let r be an arbitrary positive number, then the set $M = B[0, r] := \{x \in X \mid |x| \leq r\}$ is positively invariant, because $|F^n(y, u)| < |u| \leq r$ for all $n \in \mathbb{Z}_+$, where $|(y, u)| := |u|$. Since the map F is asymptotically compact, then there exists a nonempty compact subset $K_r \subset X$ such that

$$\lim_{n \to +\infty} \beta(F^n(M), K_r) = 0. \tag{47}$$

Thus the dynamical system $(X, \mathbb{Z}_+, \pi)$ satisfies the condition of Ladyzhenskaya. By Theorem 3.12 the dynamical system $(X, \mathbb{Z}_+, \pi)$ is boundedly dissipative. In particularly it is compactly dissipative and by Theorem 3.7 its Levinson center J is a compact minimal set. On the other hand the set $\Theta := \{(y, 0) \mid y \in Y\}$ is a compact invariant set because Y is so and, consequently, $\Theta \subseteq J$. Since J is minimal, then we have $J = \Theta$. Consider the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$, where $\sigma(n, y) := g^n(y)$ (for all ($y \in Y$ and $n \in \mathbb{Z}_+$) and $h := pr_2 : X \mapsto Y$. We define the function $V : X \dot{\times} X \mapsto \mathbb{R}_+$ by the equality $V((y, u_1), (y, u_2)) := |u_1 - u_2|$ for all $(y, u_1), (y, u_2) \in X$. From the inequality (46) we obtain $V(\pi(n, x_1), \pi(n, x_1)) < V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X \setminus \Delta_X$ and $n \in \mathbb{N}$. Since the dynamical system $(X, \mathbb{Z}_+, \pi)$ is asymptotically compact, to finish the proof it is sufficient to apply Theorem 3.13. □

Definition 5.3. *The mapping $F : X \mapsto X$ is called locally compact, if the discrete dynamical system $(X, \mathbb{Z}_+, \pi)$, generated by the positive powers of F, is so.*

Theorem 5.4. *Suppose that the following conditions hold:*

(i) *(Y, g) is a compact minimal dynamical system;*
(ii) *W is a Banach space;*
(iii) *the function $f \in C(Y \times W, W)$ is continuously differentiable in $u \in W$;*
(iv) *$||f'_u(y, u)|| < 1$ for all $(y, u) \in Y \times W$;*
(v) *$f(y, 0) = 0$ for all $y \in Y$;*
(vi) *the mapping $F := (g, f) : Y \times W \mapsto Y \times W$ is locally compact.*

Then the trivial solution of equation (44) is globally asymptotically stable.

Proof. Let $\langle W, \varphi, (Y, \mathbb{Z}_+, \sigma) \rangle$ be the cocycle, generated by mapping $F := (f, g)$ and $\sigma(n, y) := g^n y$ for all $y \in Y$ and $n \in \mathbb{Z}_+$. Denote by $(X, \mathbb{Z}_+, \pi)$ the skew-product dynamical system, where $X := Y \times W$ and $\pi(n, x) := F^n x$ for all $x = (y, u) \in X$ and $n \in \mathbb{Z}_+$. Consider the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h \rangle$, where $h := pr_2 : X \mapsto Y$ and the function $V : X \dot{\times} X \mapsto \mathbb{R}_+$ defined by equality $V((y, u_1), (y, u_2)) := |u_1 - u_2|$ for all $y \in Y$ and $u_1, u_2 \in W$. From the inequality (46) it follows that

$$V(\pi(n, x_1), \pi(n, x_2)) < V(x_1, x_2) \tag{48}$$

for all $n > 0$ and $(x_1, x_2) \in X\dot{\times}X \setminus \Delta_X$. Since $f(y,0) = 0$ for all $y \in Y$, then $\pi(n,(y,0)) = (y,0)$ for all $y \in Y$ and $n \in \mathbb{Z}_+$ and taking into account the inequality (48) we obtain that every positive semi-trajectory of $(X, \mathbb{Z}_+, \pi)$ is relatively compact. Now to finish the proof it is sufficient to apply Theorem 3.10. $\square$

5.3. **Potential mappings.**

Theorem 5.5. *Let $f \in C(Y \times W, W)$ be a potential mapping, i.e. there exists a continuously differentiable in $u \in W$ mapping $V \in C(Y \times W, \mathbb{R})$ such that $f(y,u) = V'_u(y,u)$ for all $(y,u) \in Y \times W$. Suppose that the following conditions hold:*

(i) *(Y, g) is a compact minimal dynamical system;*
(ii) *W is a finite-dimensional Banach space;*
(iii) *the function $V \in C(Y \times W, \mathbb{R})$ is twice continuously differentiable in $u \in W$;*
(iv) *$r(f'_u(y,u)) < 1$ for all $(y,u) \in Y \times W$, where $r(A)$ denote the spectral radius of the operator A;*
(v) *$f(y,0) = 0$ for all $y \in Y$.*

Then the trivial solution of equation (44) is globally asymptotically stable.

Proof. Note that $f'_u(y,u) = V''_{uu}(y,u)$ for all $(y,u) \in Y \times W$. Since the operator $V''_{uu}(y,u)$ is symmetric, then $||f'_u(y,u)|| = r(f'_u(y,u)) < 1$ for all $(y,u) \in Y \times W$. Now it is sufficient to refer Theorem 5.2. $\square$

Example 5.6. *Let $\Phi : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous function defined by equality $f(u) := \Phi'(u)$ for all $u \in \mathbb{R}^n$. It easy to note that $r(f'(u)) = ||f'(u)||$ for all $u \in \mathbb{R}^n$, where $r(f'(u))$ is the spectral radius of the operator $f'(u)$, because $f''(u)$ is an auto-adjoint operator. Thus, if $f(0) = 0$ and $r(f'(u)) < 1$ for all $u \in \mathbb{R}^n$, then the mapping $f = \Phi'$ has a unique fixed point 0 and it is globally asymptotically stable. In particularly, if f is a continuously differentiable mapping from $\mathbb{R}$ into itself, $f(0) = 0$ and $|f'(u)| < 1$ for all $u \in \mathbb{R}$, then 0 is a unique globally asymptotically stable point of f.*

Remark 5.7. *This result (Example 5.6) was established in the works* [8, 12].

Let E^n be a n-dimensional Banach space, $E^n = E^{m_1} \times E^{m_2} \times \ldots \times E^{m_k}$ $(n = m_1 + m_2 + \ldots + m_k)$. Consider the system

$$(49) \qquad \begin{cases} u_1(n+1) = f_1(\sigma(n,y), u_1(n)) \\ u_2(n+1) = f_2(\sigma(n,y), u_1(n), u_2(n)) \\ \ldots \\ u_m(n+1) = f_m(\sigma(n,y), u_1(n), u_2(n), \ldots, u_m(n)), \end{cases}$$

where $f_i \in C(Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}; E^{m_i})$ $(i = 1, 2, \ldots, k)$.

Theorem 5.8. *Let Y be a compact metric space, $f_i \in C(Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}; E^{m_i})$ $(i = 1, 2, \ldots, k)$ and the following conditions hold:*

(i) *the function f_i is continuously differentiable in $u_i \in E^{m_i}$;*
(ii) *for all $i = 1, 2, \ldots, k$ there exists a continuously differentiable function $V_i \in C(Y \times, \mathbb{R})$ such that $f_i(y,u) = \frac{\partial V_i}{\partial u_i}(y,u)$ for all $(y,u) \in Y \times E^{m_1} \times E^{m_2} \times \ldots \times E^{m_i}$;*

(iii) *for all* $i = 1, 2, \dots, k$ *the function* $V_i \in C(Y \times E^{m_1} \times E^{m_2} \times \dots \times E^{m_i}, \mathbb{R})$ *is twice continuously differentiable in* u_i*;*

(iv) *the Jacobian* $F'_u(y,u)$ *(*$(y,u) \in Y \times E^n$*) of right hand side* $F := (f_1, f_2, \dots, f_m)$ *of system (49) has all its eigenvalues with modulus less than one for all* $(y,u) \in Y \times E$*;*

(v) $F(y,0) = 0$ *for all* $y \in Y$*.*

Then the trivial solution of equation (49) is global asymptotically stable.

Proof. Let $F = (f_1, f_2, \dots, f_k)$, then

$$det(F'_u(y,u) - \lambda I) = det(\frac{\partial f_1}{\partial u_1}(y,u_1) - \lambda I) \times$$

$$det(\frac{\partial f_2}{\partial u_2}(y,u_1,u_2) - \lambda I) \times \dots \times det(\frac{\partial f_k}{\partial u_k}(y,u_1,u_2,\dots,u_k) - \lambda I) \tag{50}$$

and, consequently, $\sigma(F'_u) = \cup_{i=1}^{k} \sigma(\frac{\partial f_i}{\partial u_i})$, where $\sigma(A)$ is the spectrum of operator A. We will prove this statement by induction with respect to k. If $k = 1$, then this statement coincides with Theorem 5.5. Assume that it is true for all $1 < i \le k-1$ and we will prove it for $i = k$. Denote by $M := E^{m_1} \times E^{m_2} \times \dots \times E^{m_{k-1}}$ and $(M, \mathbb{Z}_+, \pi)$ the dynamical system, generated by equation $u(n+1) = \tilde{F}(u(n))$ ($x \in M$), where $\tilde{F} = (f_1, f_2, \dots, f_{k-1})$. Finally, let $\langle E^{m_k}, \varphi, (M, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle, generated by equation

$$u_k(n+1) = f_{m_k}(\sigma(n,y), u_k(n)) \ (u \in E^{m_k}, \ y \in M).$$

Let $V : Y \times E^{m_k} \mapsto \mathbb{R}_+$ be the function defined by equality $V(y,u_k) := |u_k|$. Reasoning as in the proof of Theorem 5.5 we can check that the function V possesses the properties 1.–3. from Theorem 4.18. Now to finish the proof it is sufficient to apply Theorem 4.18. □

Remark 5.9. *For autonomous discrete systems, when* $m_1 = m_2 = \dots = m_k = 1$ *this statement was established in* [12].

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