GLOBAL ATTRACTORS OF NON-AUTONOMOUS DIFFERENCE EQUATIONS: A GROWTH MODEL WITH ENDOGENOUS POPULATION GROWTH.

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Abstract. The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations. The results obtained are applied to the study of a triangular economic growth model $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ recently developed by Brianzoni S., Mammana C. and Michetti E.

1. Introduction

The present paper is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n\omega)u_n + F(u_n, \sigma^n\omega), \quad (A \in C(\Omega, [E]),\ F \in C(E \times \Omega, E))$$

where Ω is a metric space, E is a finite-dimensional Banach space with the norm $|\cdot|$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time $\mathbb{Z}_+$, $[E]$ is the space of all linear operators acting on E equipped with operator norm, $C(\Omega, [E])$ (respectively, $C(E \times \Omega, E)$) is the space of all continuous functions defined on Ω (respectively, on $E \times \Omega$) with values in $[E]$ (respectively, E) equipped with compact-open topology and F is a "small" perturbation. Analogous problem it was studied by Cheban D. et al. [9], when Ω is a compact invariant set. In this work we consider more general case, when Ω is not invariant, but there exists a compact invariant subset $J \subseteq \Omega$ (Levinson center) which attracts every compact subset from Ω.

The obtained results are applied while studying a special class of triangular maps describing a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates and the population growth rate dynamic is described by the logistic equation (see Brianzoni S., Mammana C. and Michetti E. [3]).

We consider the Solow-Swan growth model in discrete time with differential saving and VES production function as proposed by Brianzoni et al. in [4] while assuming that the population growth rate evolves according to the logistic law as in Brianzoni et al. [3] and Cheban et al. [9]. Our main goal is to study the qualitative and quantitative long run dynamics of the economic model to show that complex

Date: October 18, 2012.

1991 *Mathematics Subject Classification.* primary:37B20, 37B55, 37C55, 37C60, 37C65, 37C70, 37C75.

Key words and phrases. global attractor, triangular map, quasi-linear difference equations, non-autonomous dynamical systems; cocycle, economical dynamics, growth model.

futures results, as the one reached while considering the CES (Constant Elasticity of Substitution) technology.

This paper is organized as follows.

In Section 2 we collect some notions and facts from the theory of dynamical systems (semi-group dynamical system, cocycle, full trajectory, non-autonomous dynamical system, compact global attractor) which we use in our paper.

In Section 3 we give a result of the existence of compact global attractors of quasi-linear dynamical systems.

Section 4 is dedicated to the study a special class of the triangular maps $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ describing a triangular growth model with logistic population growth rate as studied in Brianzoni S., Mammana C. and Michetti E. [3]. We apply here our general results for studying this concrete dynamical system.

2. Some Notions and Facts from Dynamical Systems

In this Section we collect some notions and facts from the theory of dynamical systems (both with continuous and discrete time) which we use in our paper.

2.1. **Triangular maps and non-autonomous dynamical systems.** Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ their Cartesian product. Recall [8, 17] that a continuous map $F : X \to X$ is called triangular if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e., $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\left\{ \begin{array}{l} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{array} \right. \tag{1}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}$ is the set of all integer numbers and $\mathbb{Z}_+ := \{n \in \mathbb{Z} : \ n \geq 0\}$.

Along with system (1) we consider the family of equations

$$u_{n+1} = f(u_n, g^n \omega) \ (\omega \in \Omega), \tag{2}$$

which is equivalent to system (1). Let $\varphi(n, u, \omega)$ be a solution of equation (2) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

(i) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
(ii) $\varphi(n + m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n \omega$;
(iii) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ is continuous.

Denote by (Ω, Z_+, σ) the semi-group dynamical system generated by positive powers of the map $g : \Omega \to \Omega$, i.e., $\sigma(n, \omega) := g^n \omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Recall [7, 16] that a triple $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ (or briefly φ) is called a cocycle over the semi-group dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W.

Let $X := W \times \Omega$ and $(X, \mathbb{Z}_+, \pi)$ be a semi-group dynamical system on X, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then $(X, \mathbb{Z}_+, \pi)$ is called [16] a skew-product dynamical system, generated by cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$.

Remark 2.1. *Thus, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa, i.e., having a cocycle* $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ *we can define a triangular map* $F : W \times \Omega \to W \times \Omega$ *by the equality*

$$F(u, \omega) := (f(u, \omega), g(\omega)),$$

where $f(u, \omega) := \varphi(1, u, \omega)$ *and* $g(\omega) := \sigma(1, \omega)$ *for all* $u \in W$ *and* $\omega \in \Omega$. *The semi-group dynamical system defined by the positive powers of the map* $F : X \to X$ $(X := W \times \Omega)$ *coincides with the skew-product dynamical system, generated by cocycle* $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$

Taking into consideration this remark we can study triangular maps in the framework of cocycles with discrete time.

Let $(X, \mathbb{Z}_+, \pi)$ (respectively, $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$) be a semi-group dynamical system (respectively, a cocycle). A map $\gamma : \mathbb{Z} \to X$ is called an entire trajectory of the semi-group dynamical system $(X, \mathbb{Z}_+, \sigma)$ passing through the point $x \in X$ if $\gamma(0) = x$ and $\gamma(n + m) = \pi(m, \gamma(n))$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$.

Let Ω be a complete metric space, $(X, \mathbb{Z}_+, \pi)$ (respectively, $(\Omega, \mathbb{Z}_+, \sigma)$) be a semi-group dynamical system on X (respectively, Ω), and $h : X \to \Omega$ be a homomorphism of $(X, \mathbb{Z}_+, \pi)$ onto $(\Omega, \mathbb{Z}_+, \sigma)$. Then the triple $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is called a non-autonomous dynamical system (NDS).

Let W and Ω be complete metric spaces, $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system on Y and $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber W (or, for short, φ). We denote by $X := W \times \Omega$ and define on X a skew product dynamical system $(X, \mathbb{Z}_+, \pi)$ ($\pi = (\varphi, \sigma)$, i.e., $\pi(t, (w, \omega)) = (\varphi(t, w, \omega), \sigma(t, \omega))$ for all $t \in \mathbb{Z}_+$ and $(w, \omega) \in W \times \Omega$). Then the triple $\langle (X, \mathbb{Z}_+, \pi), ((\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is a non-autonomous dynamical system generated by cocycle φ), where $h = pr_2 : X \mapsto \Omega$ is the projection on the second component.

2.2. **Global attractors of dynamical systems.** Let $\mathfrak{M}$ be a family of subsets from X.

A semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$ will be called $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(n, M) \subseteq B(K, \varepsilon)$ for any $n \geq L(\varepsilon, M)$, where K is a certain fixed subset from X depending only on $\mathfrak{M}$. In this case we will call K an attracting set for $\mathfrak{M}$.

For the applications the most important ones are the cases when K is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, or $\mathfrak{M} := B(X)$ where $C(X)$ (respectively, $B(X)$) is the family of all compact (respectively, bounded) subsets from X.

The system $(X, \mathbb{Z}_+, \pi)$ is called[7]:

– point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{n\to+\infty} \rho(\pi(n,x),K) = 0; \tag{3}$$

– compactly dissipative if the equality (3) takes place uniformly w.r.t. x on the compact subsets from X.

Let $(X, \mathbb{Z}_+, \pi)$ be a compactly dissipative semi-group dynamical system and K be an attracting set for $C(X)$. We denote by

$$J := \Omega(K) = \bigcap_{n\ge 0} \overline{\bigcup_{m\ge n} \pi(m,K)},$$

then the set J does not depend of the choice of K and is characterized by the properties of the semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$. The set J is called a Levinson center of the semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$.

Let $(X, \mathbb{Z}_+, \pi)$ be a dynamical system and $x \in X$. Denote by

$$\omega_x := \Omega(\{x\}) = \bigcap_{n\ge 0} \overline{\bigcup_{m\ge n} \pi(m,x)}$$

the ω-limit set of point x.

If $(X, \mathbb{Z}_+, \pi)$ is a two sided dynamical system (i.e., the map $\pi(1,\cdot) : X \mapsto X$ is an homeomorphism) then the set

$$\alpha_x = \bigcap_{n\le 0} \overline{\bigcup_{m\le n} \pi(m,x)}$$

is said to be α-limit set of x.

Let $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber E (or shortly φ).

A cocycle φ is called:

- dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t\to+\infty} |\varphi(t,u,\omega)| \le r$$

for all $\omega \in \Omega$ and $u \in E$;
- uniform dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t\to+\infty} \sup_{\omega\in\Omega', |u|\le R} |\varphi(t,u,\omega)| \le r$$

for all compact subsets $\Omega' \subseteq \Omega$ and $R > 0$.

Theorem 2.2. [9] *If the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and the cocycle φ is uniform dissipative, then the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compact dissipative.*

3. Global attractors of quasi-linear triangular systems

In Section 3 we present a result of the existence of compact global attractors of quasi-linear dynamical systems.

Let Ω be a complete metric space and $(\Omega, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system on Ω with discrete time.

If W is a complete metric space, then by $C(\Omega, W)$ we denote the space of all the continuous functions $f : \Omega \to W$ endowed with the compact-open topology, i.e., the uniform convergence on compact subsets in Ω.

Consider a linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n, \quad (\omega \in \Omega) \tag{4}$$

where $A \in C(\Omega, [E])$.

Let $U(n, \omega)$ be the Cauchy operator of linear equation (4).

We will say that equation (4) is uniformly exponential stable if there exist constants $0 < q < 1$ and $N > 0$ such that

$$\|U(n, \omega)\| \leq Nq^n$$

for all $\omega \in \Omega$ and $n \in \mathbb{Z}_+$.

Consider a difference equation

$$u_{n+1} = \mathcal{F}(u_n, \sigma(n, \omega)) \quad (\omega \in \Omega). \tag{5}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (5) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (5) is said to be dissipative (respectively, uniform dissipative), if there the cocycle φ generated by equation (5) is so, i.e., there exists a positive number r such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \leq r \quad \text{(respectively, } \limsup_{n \to +\infty} \sup_{\omega \in \Omega', |u| \leq R} |\varphi(n, u, \omega)| \leq r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all compact subset $\Omega' \subseteq \Omega$ and $R > 0$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n + F(u_n, \sigma(n, \omega)), \tag{6}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (condition (ii) in Theorem 3.1).

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n.$$

Theorem 3.1. [9] *Suppose that the following conditions hold:*

(i) *equation (4) is uniformly exponential stable, i.e., there are positive numbers N and $q < 1$ such that*

$$\|U(n, \omega)\| \leq Nq^n \qquad (n \in Z_+); \tag{7}$$

(ii) $|F(u,\omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1-q)N^{-1})$ *for all* $u \in E$ *and* $\omega \in \Omega$.

Then equation (6) is uniform dissipative.

Theorem 3.2. [8] *Let* (Ω, Z_+, σ) *be a compactly dissipative system and* φ *be a cocycle generated by equation (6). Under the conditions of Theorem 3.1 the skew-product system* (X, Z_+, π) *(*$X := E \times \Omega$ *and* $\pi := (\varphi, \sigma)$*), generated by cocycle* φ *admits a compact global attractor.*

Remark 3.3. *All the results of Section 3 remain true, if we replace the phase space* E *by positively invariant (with respect to cocycle* φ *generated by (5)) subset* $V \subset E$.

4. Application: a growth model with VES technology

4.1. **The model.** The Solow-Swan growth model (see [18] and [19]) with VES (Variable Elasticity of Substitution) technology has been studied by Karagiannis et al. [13] while assuming continuous time: the authors show that the model can exhibit unbounded endogenous growth despite the absence of exogenous technical change and the presence of non-reproducible factors. Anyway their model is unable to produce economic fluctuations.

More recently in [4], Brianzoni et al. studied the discrete time Solow-Swan growth model, where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm and Kaas [6] and where the production function $f : R_+ \to R_+$, mapping capital per worker u into output per worker $f(u)$, is of the VES type. Following Kargiannis et al. [13], they considered the specification of the VES production function in intensive form given by Revamkar [14] as follows:

$$f(u) = Ak^{a\gamma}[1 + bau]^{(1-a)\gamma}\ (u \ge 0), \tag{8}$$

being $A > 0$, $a \in (0, 1]$, $b \ge -1$ and $1/u \ge -b$, while assuming that the production function exhibits constant return to scale, i.e., $\gamma = 1$.

Anyway, in their work the authors assume that the labor force grows at a constant rate. This last hypothesis is usually assumed in standard economic growth theory, however, this assumption is unable to explain possible fluctuations in the growth rate. For this reason a number of economic growth model with endogenous population growth has been proposed (see, for instance, Brianzoni et al. [1, 2, 3]). In particular Brianzoni et al. [3] and Cheban et al. [9] recently investigated the neoclassical growth model with differential saving and CES production function under the assumption that the labor force dynamics is described by the logistic equation. Such a law satisfies the following economic properties: (1) when population is small in proportion to the environmental carrying capacity, then it grows at a positive constant rate and (2) when population is larger in proportion to the environmental carrying capacity, the resources become relatively more scarce and, as result, this must affect the population growth rate negatively.

In the present work we consider the Solow-Swan growth model in discrete time with differential saving and VES production function as proposed by Brianzoni et al. in [4] while assuming that the population growth rate evolves according to the logistic law as in Brianzoni et al. [3] and Cheban et al. [9]. Our main goal is to study the qualitative and quantitative long run dynamics of the economic model to show that complex futures results, as the one reached while considering the CES (Constant Elasticity of Substitution) technology.

Let us consider the following equation describing the evolution of the capital per capita u in the standard neoclassical Solow-Swan growth model with differential saving (see [4]):

$$F(u,\omega) = \frac{1}{1+\omega}[(1-\delta)u + s_w w(u) + s_r u f'(u)], \tag{9}$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively. The wage rate equals the marginal product of labor which is $w(u) := f(u) - uf'(u)$, furthermore shareholders receive the marginal product of capital $f'(u)$ which implies that the total capital income per worker is $uf'(u)$.

Observe that $\omega \geq 0$ represents the labor force growth rate: in our formulation we let it vary with time. More precisely we add a further assumption, that is the population growth rate evolves according to the law

$$\omega' = \frac{rh\omega}{h+(r-1)\omega}$$

.

Consider the case $b \geq 0$. By substituting the VES production function given by (8) (with $\gamma = 1$) in (9) we obtain the following map describing the evolution of the capital accumulation:

$$H(u,\omega) = \frac{1}{1+\omega}\{(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\} \tag{10}$$

The resulting system, $T = (\omega', u')$, describing capital per worker (u) and population growth rate (ω) dynamics, is given by:

$$T := \begin{cases} u' = \frac{1}{1+\omega}\left[(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\right] \\ \\ \omega' = \frac{rh\omega}{h+(r-1)\omega}. \end{cases} \tag{11}$$

We get a discrete-time dynamical system described by the iteration of a map of the plane of triangular type. In fact the second component of the previous system does not depend on u, therefore the map is characterized by the triangular structure:

$$T := \begin{cases} u' = g(u,\omega) \\ \\ \omega' = f(\omega) \end{cases} . \tag{12}$$

As a consequence, the dynamics of the map T are influenced by the dynamics of the one–dimensional map f, that is the well-known Beverton-Holt map.

4.2. **Dynamics of the Beverton-Holt map** $f(\omega) = \frac{rh\omega}{h+(r-1)\omega}$**.** In this Subsection we study the dynamics of the one-dimensional monotone dissipative dynamical systems $(\mathbb{R}_+, \mathbb{Z}_+, \pi)$, generated by a strict monotone increasing map $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Consider a continuous mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

(i) $f(0) = 0$*;*
(ii) f *is strict monotone increasing;*
(iii) *the function* f *is bounded on* $\mathbb{R}_+$*;*
(iv) *there exists a number* $\alpha > 0$ *such that* $f(\alpha) > \alpha$.

Then the following statement hold:

(i) *there exists a number* $x_0 > \alpha$ *such that* $f(x_0) = x_0$*;*
(ii) *the dynamical system* $(\mathbb{R}_+, f)$ *is point dissipative and* $\omega_x \subseteq [0, b]$ *for all* $x \in \mathbb{R}_+$*, where* $b = \lim\limits_{x\to\infty} f(x)$*;*
(iii) *the dynamical system* $(\mathbb{R}_+, f)$ *admits a compact global attractor* $J \subset \mathbb{R}_+$*;*
(iv) $J = [0, x_0]$*, where* x_0 *is some fixed point of* f*;*
(v) $\omega_x = \{x_0\}$*or all* $x > x_0$*;*
(vi) *for any* $x \in (0, x_0)$ *there exists two fixed points* p *and* q *of the map* f *such that* $\lim\limits_{n\to\infty} f^n(x) = p$ *and* $\lim\limits_{n\to\infty} f^{-n}(x) = q$*;*
(vii) *if the mapping* f*, in addition, is strict convex (i.e., the set* $G_f := \{(x, y) : x \in \mathbb{R}_+ \text{ and } 0 \le y \le f(x)\}$ *is strict convex in* $\mathbb{R}^2$*), then*
  (a) x_0 *is a unique positive fixed point of the mapping* f*;*
  (b) $\lim\limits_{n\to\infty} f^{-n}(x) = 0$ *for all* $x \in [0, x_0)$*;*
  (c) $\lim\limits_{n\to\infty} f^n(x) = x_0$ *for all* $x > 0$*;*
  (d) *the fixed point* x_0 *is Lyapunov stable, i.e., for all* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that* $|x - x_0| < \delta$ *implies* $|f^n(x) - x_0| < \varepsilon$ *or all* $n \ge 0$*;*
  (e) *the point* 0 *is Lyapunov stable in the negative direction, i.e., for all* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that* $0 \le x < \delta$ *implies* $0 \le f^{-n}(x) < \varepsilon$ *or all* $n \ge 0$.

Proof. Consider the function $g(x) := f(x) - x$ and note that $g(\alpha) > 0$ and $g(\beta) < 0$ for all sufficiently large β ($\beta > b$) and, consequently, there exists $x_0 \in (\alpha, \beta)$ such that $g(x_0) = 0$ or $f(x_0) = x_0$.

Let $x \in \mathbb{R}_+$ be an arbitrary point. Since the semi-trajectory $\Sigma_x^+ := \{x, f(x), \dots, f^n(x), \dots\} \subseteq \{x\} \bigcup [0, b]$ is relatively compact, then the set ω_x is nonempty, compact and invariant. Let $q \in \omega_x$, then there exists a sequence $\{n_k\} \subset \mathbb{Z}_+$ such that $q = \lim\limits_{k\to+\infty} f^{n_k}(x)$ and $n_k \to \infty$ as $k \to \infty$ and, consequently, $q \in [0, b]$. Thus the dynamical system $(\mathbb{R}_+, f)$ is point dissipative.

Since the phase space $\mathbb{R}_+$ of the dynamical system $(\mathbb{R}_+, f)$ is local compact, then by Theorem 1.10 [7, ChI] it is compactly dissipative and by Theorem 1.6 [7, ChI] $(\mathbb{R}_+, f)$ admits a compact global attractor J (J is its maximal compact invariant set).

According to Theorem 1.32 [7, ChI] the global attractor J (Levinson center) of $(\mathbb{R}_+, f)$ is connected because the phase space $\mathbb{R}_+$ is so. On the other hand 0 is a fixed point and, consequently, $0 \in J$. Note that $\alpha \in J$. In fact, since $f(\alpha) > \alpha$, then

$$\alpha < f(\alpha) < f^2(\alpha) < \ldots < f^n(\alpha) < \ldots \ .$$

Thus the sequence $\{f^n(\alpha)\}$ is bounded and strict monotone decreasing and, consequently, it is convergent. Let $p := \lim_{n\to\infty} f^n(\alpha)$, then $p \geq \alpha$ and $f(p) = p$. From the last equality it follows that $p \in J$ and,consequently, $\alpha \in [0,p] \subseteq J$. Thus $0, \alpha \in J$ and, consequently, there exists a number $x_0 \geq \alpha > 0$ such that $J = [0, x_0]$. To finish this statement it is sufficient to show that $f(x_0) = x_0$. Note that the boundary $\partial J = \{0, x_0\}$ of the invariant set J is also invariant. In particular this means that $f(x_0) = x_0$ or $f(x_0) = 0$. Since the mapping f is strictly monotone decreasing and $x_0 > 0$, then the equality $f(x_0) = 0$ is not possible and, consequently, $f(x_0) = x_0$.

Let $x > x_0$. Since The set $J = [0, x_0]$ is invariant, then $f^n(x) > x_0$ for all $n \in \mathbb{Z}_+$ and, consequently, $\omega_x \subset [x_0, +\infty)$. On the other hand $\omega_x \subseteq J = [0, x_0]$ and, consequently, $\omega_x \subseteq [x_0, +\infty) \bigcap [0, x_0] = \{x_0\}$.

Let $x \in (0, x_0)$ be an arbitrary number. Since the function f is strict monotone increasing and bounded on $\mathbb{R}_+$, then there exists a limit $b := \lim_{x\to\infty} f(x)$ and the reverse function $f^{-1} : [0, b) \mapsto \mathbb{R}_+$ is also strict monotone increasing. Consider the sequence $\{f^n(x)\}$. We will show that the sequence $\{f^n(x)\}$ is monotone. In fact, if $f(x) > x$ (respectively $f(x) < x$), then $f^{n+1}(x) > f^n(x)$ (respectively, $f^{n+1}(x) < f^n(x)$) for all $n \in \mathbb{N}$. Since J is invariant, then $f^n(x) \in J$ for all $n \in \mathbb{N}$. Thus the sequence $\{f^n(x)\}$ is bounded and monotone and, consequently, there exists $\lim_{n\to\infty} f^n(x) = p$. It easy to check that $f(p) = p$. Taking into account that the mapping f is an homeomorphism on the set J and it is invariant. Reasoning as above it easy to show that the sequence $\{f^{-n}(x)\}$ is monotone and bounded and, consequently, there exists $\lim_{n\to\infty} f^{-n}(x) = q$ and $f(q) = q$.

Suppose now that the function f is also strict convex. We will show that in this case x_0 is a unique positive fixed point of f. Suppose that it is not so, then there exists a fixed point $\bar{x} \in (0, x_0)$. Note that the points $(0,0), (\bar{x}, \bar{x})$ and (x_0, x_0) belong to G_f and $\Delta^+ := \{(x,x) : \ x \in \mathbb{R}_+\}$. Thus $(\bar{x}, \bar{x}) \in G_f \bigcap \Delta^+$ and $\bar{x} = \lambda x_0$, where λ is some number from $(0,1)$. The last inclusion contradicts to strict convexity of the set G_f. The obtained contradiction proves our statement.

To finish the proof of Theorem it is sufficient to show that the point x_0 (respectively, point 0) is Lyapunov stable in the positive (respectively, negative) direction. Note that the set $A := \{x_0\}$ (respectively, $B := \{0\}$) is local maximal compact invariant set of the map f. Now Lyapunov stability in the positive direction (respectively, in the negative direction) of the point x_0 (respectively, 0) it follows from Theorem 8.2 [7, ChVIII]. □

Remark 4.2. *Note that the item (iv) of Theorem 4.1 remain true without the assumption that the mapping f is bounded. It is sufficient to suppose that the dynamical system $(f, \mathbb{R}_+)$ is compactly dissipative and f is strictly monotone increasing.*

Lemma 4.3. *Let* $f(x) := \frac{hrx}{h+(r-1)x}$ *for all* $x \in \mathbb{R}_+$*,* $h > 0$ *and* $r > 1$*, then the following statements hold:*

(i) $f'(x) = \frac{rh^2}{(h+(r-1)x)^2}$ *for all* $x > 0$*;*
(ii) $f''(x) = \frac{-2r(r-1)h^2}{(h+(r-1)x)^3}$ *for all* $x > 0$*;*
(iii) $f(\alpha) > \alpha$*, where* $\alpha := h/2$*.*

Proof. This statement is evident. □

Corollary 4.4. *Let* $f(x) := \frac{hrx}{h+(r-1)x}$ *for all* $x \in \mathbb{R}_+$*,* $h > 0$ *and* $r > 1$*, then the following statements hold:*

(i) *the mapping* f *is strict monotone increasing and convex;*
(ii) f *admits two fixed points* $x = 0$ *and* $x = h$*;*
(iii) *the fixed point* 0 *is asymptotically stable in the negative direction and* $W^u(0) := \{x \in \mathbb{R}_+ : \lim\limits_{n\to\infty} f^{-n}(x) = 0\} = [0, h)$*;*
(iv) *the fixed point* h *is asymptotically Lyapunov stable in the positive direction and* $W^s(0) := \{x \in \mathbb{R}_+ : \lim\limits_{n\to\infty} f^n(x) = h\} = (0, +\infty)$*;*
(v) *the dynamical system* $(\mathbb{R}_+, f)$ *is compact dissipative and its Levinson center (compact global attractor)* $J = [0, h]$*.*

Proof. This statement it follows from Theorem 4.1 and Lemma 4.3. □

One say that the mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ admits an holomorphic extension, if there exists $\delta > 0$ and a holomorphic function $\tilde{f} : B_\delta \mapsto \mathbb{C}$ such that $\tilde{f}\big|_{\mathbb{R}_+} = f$, where $B_\delta := \bigcup\limits_{r\geq 0} \{(x, y) : (x - r)^2 + y^2 < \delta^2\}$.

Theorem 4.5. *Under the conditions of Theorem 4.1, if the function* f *admits an holomorphic extension, then* f *has a finite number of fixed points.*

Proof. Consider the holomorphic function $F(z) := \tilde{f}(z) - z$ defined on B_δ. Denote by $Fix(f) := \{x \in \mathbb{R}_+ : f(x) = x\}$ and note that $Fix(f) \subset J = [0, x_0]$. On the other hand every point $z \in Fix(f)$ is a null of the holomorphic function F. Since holomorphic function admits at most a finite number of nulls on every compact subset, then the set $Fix(f)$ contains at most a finite number of points. □

Thus, the dynamical system $(\mathbb{R}_+, f)$, generated by Beverton-Holt map f admits a compact global attractor J for all $r > 1$. In addition J possesses the following property: if $h > 0$ and $r > 1$, then $J = [0, h]$ and in this case the fixed point $\omega = 0$ is a repeller (i.e., $\omega = 0$ is an asymptotically stable in the negative direction fixed point), but $\omega = h$ is an attractor with domain of attraction $(0, +\infty)$.

4.3. Existence of an attractor for $b \in (0, +\infty)$.

Lemma 4.6. *The function* $H(u, \omega)$ *can be presented in the following form*

$$H(u, \omega) = \frac{1}{1+\omega}\{(1 - \delta + s_r ab\frac{A}{(ab)^a})u\} + R(u, \omega), \tag{13}$$

where $R(u,\omega)$ is bounded, i.e., there exists a positive constant C such that $|R(u,\omega)| \leq C$ for all $\omega \in [0,+\infty)$ and $u \in [0,+\infty)$.

Proof. This statement can be proved with slight modification of the proof of Lemma 6.1 from [10]. □

Theorem 4.7. *Let $b > 0$ and $\delta > s_r ab\frac{A}{(ab)^a}$, then the dynamical system $(\mathbb{R}^2_+, T)$, generated by map (11), admits a compact global attractor $J \subset \mathbb{R}^2_+$ which possesses the following properties:*

(i) *the set J is connected;*
(ii) *for every $\omega \in [0,h]$ the set $J_\omega := \{(x,\omega) : (x,\omega) \in J\}$ is connected;*
(iii) *for all $\omega \in (0,h)$ there exists a positive number b_ω such that $I_\omega = [0,b_\omega]$, where $I_\omega := pr_1(J_\omega)$;*
(iv) *for all $x := (u,\omega) \in \mathbb{R}^2_+$ we have:*
  (a) *$\omega_x \subseteq J_h$ if $\omega \neq 0$ (i.e., if $p \in \omega_x$, then $pr_2(p) = h$);*
  (b) *$\alpha_x \subseteq J_0$ if $x \in J$ and $\omega \neq h$ (this means that $pr_2(q) = 0$ for all $q \in \alpha_x$).*

Proof. Let $b > 0$, then by Lemma 4.6 the function T_1 can be written in the form

$$T_1(u,\omega) = \frac{1}{1+\omega}\{(1-\delta + s_r ab\frac{A}{(ab)^a}u\} + R(u,\omega), \tag{14}$$

where $R(u,\omega)$ is bounded, i.e., there exists a positive constant M such that $|R(u,\omega)| \leq M$ for all $(u,\omega) \in \mathbb{R}^2_+$.

Since $0 \leq \frac{1}{1+\omega} \leq 1$ for all $\omega \in \mathbb{R}_+$, then from (14) we obtain

$$0 \leq T_1(u,\omega) \leq \alpha u + M \tag{15}$$

for all $(u,\omega) \in \mathbb{R}^2_+$, where $\alpha := 1 - \delta + s_r ab\frac{A}{(ab)^a} < 1$.

Since the map T is triangular, to prove the existence of compact global attractor J it is sufficient to apply Theorem 3.2 (see also Remark 3.3).

According to Theorem 1.32 from [7, ChI] the set J is connected. To prove the connectedness of the set J_ω we note that the map $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ ($f(\omega) := \frac{Hr\omega}{H+(r-1)\omega}$ for all $\omega \in \mathbb{R}_+$) is reversible, then by Theorem 2.25 [7, ChII] the set I_ω and, consequently, the set J_ω is also connected because $J_\omega = I_\omega \times \{\omega\}$. Thus for all $\omega \in [0,\omega]$ there are two numbers $a_\omega, b_\omega \in \mathbb{R}_+$ such that $I_\omega = [a_\omega, b_\omega]$. It easy to see that $a_\omega = 0$ for all $\omega \in [0,h]$ because $(0,\omega) \in J$ for all $\omega \in [0,H]$. Now to finish the proof of Theorem it is sufficient to note that $b_\omega > 0$ for all $\omega \in (0,H)$. If we suppose that it is not so then there exists an $\bar{\omega} \in (0,H)$ such that $b_{\bar{\omega}} = 0$. From the last equality it follows that $\bar{\omega}$ is a fixed point of f (i.e., $f(\bar{\omega}) = \bar{\omega}$). The obtained contradiction proves our statement.

Let $x = (u,\omega) \in \mathbb{R}^2_+$ with the condition $\omega \neq 0$. Note that $\pi(t,x) = (\varphi(t,u,\omega), f^t\omega)$. Since $\omega \neq 0$, then $f^t\omega \to H$ as $t \to +\infty$. Let $p \in \omega_x$, then there exists a sequence $t_k \to +\infty$ ($t_k \in \mathbb{Z}_+$) such that $\pi(t_k,x) = (\varphi(t_k,u,\omega), f^{t_k}\omega) \to p$ as $k \to \infty$, i.e., $pr_2(p) = \lim_{k\to\infty} f^{t_k}\omega = H$. If $x \in J$ and $q \in \alpha_x$, then reasoning as above and taking into consideration that $\lim_{t\to\infty} f^{-t}\omega = 0$ (for all $\omega \in (0, H)$) we prove that $pr_2(q) = 0$. □

4.4. **Structure of the attractor.** In this subsection we suppose that $b > 0$. Let $H(u) = (1-\delta)u + f(u)[s_w(1-a) + s_r(a + abu)]$ and $H(u,\omega) = \frac{1}{1+\omega}H(u)$.

Lemma 4.8. *The following statements hold:*

(i) *let* $f(u) := Au^a(1+abu)^{-a}$, *then*

$$f'(u) = \frac{af(u)}{u(1+abu)}; \tag{16}$$

(ii) *if* $H(u) = (1-\delta)u + f(u)[s_w(1-a) + s_r(a+abu)]$, *then*

$$H'(u) = 1-\delta + \big(\frac{a}{u(1+abu)}[s_w(1-a) + s_r(a+abu)] + s_r ab\big)f(u). \tag{17}$$

Proof. This statement is evident. □

Lemma 4.9. *The following statements hold:*

(i) $H'(u) \geq 1-\delta > 0$ *for all* $u \in (0,+\infty)$;

(ii)

$$\lim_{u\to\infty} H'(u) = 1-\delta \quad \textit{and} \quad \lim_{u\to+0} H'(u) = +\infty; \tag{18}$$

(iii) *there exists* $u_0 > 0$ *such that* $H(u) \geq (h+2)u$ *for all* $u \in [0, u_0]$.

Proof. The first and second statements are evident. To prove the third statement we note that from (18) it follows that for given $h > 0$ there exists a positive number u_0 such that $H'(u) \geq h+2$ for all $u \in (0, u_0]$. Let now $\xi \in (0, u_0)$, then we have

$$H(u) - H(\xi) = H'(\theta)(u-\xi) \geq (h+2)(u-\xi) \tag{19}$$

for all $u \in (0, u_0)$, where $\theta \in (\xi, u)$. Passing into limit in (19) as $\xi \to 0$ and taking into account the continuity of $H(u)$ at the point $u = 0$ and the equality $H(0) = 0$ we obtain $H(u) \geq (h+2)u$ for all $u \in (0, u_0)$. Lemma is proved. □

Lemma 4.10. *Let* $(\mathbb{R}^2_+, T)$*be a dynamical system generated by map (11) (i.e.,* $T^t(u,\omega) = (\varphi(t,u,\omega), f^t(\omega))$*) and* $\varphi(t,u,\omega) \in [0, u_0]$ *for all* $t \in \mathbb{Z}_+$, *then* $\varphi(t,u,\omega) \geq u$ *for all* $t \in \mathbb{Z}_+$, $u \in (0, u_0]$ *and* $\omega \in [0, h+1]$.

Proof. Note that $H(u,\omega) = \frac{1}{1+\omega}H(u)$ and $\varphi(t,u,\omega)$ is a unique solution of equation

$$u_{t+1} = H(u_t, f^t(\omega)) \tag{20}$$

with initial data $\varphi(0,u,\omega) = u$. Let $u \in [0, u_0]$, then by Lemma 4.9 we have $\varphi(1,u,\omega) = H(u,\omega) \geq u$ for all $u \in [0, u_0]$ because $\frac{1}{1+\omega} \in [\frac{1}{h+2}, 1]$ for all $\omega \in [0, h+1]$. Note that $f^t[0, h+1] \subseteq [0, h+1]$ for all $t \in \mathbb{Z}_+$. If we suppose that $\varphi(t,u,\omega) \geq u$ for all $t = 1, 2, \ldots, n$, then we obtain $\varphi(t+1,u,\omega) = \varphi(1, \varphi(t,u,\omega), f^t(\omega)) \geq \varphi(t,u,\omega) \geq u$. Lemma is proved. □

Lemma 4.11. [4] *The following statements hold:*

(i) *if* $\frac{\omega+\delta}{A} > (ab)^{-a}abs_r$, *then* $H(\omega,u)$ *has two fixed points:* $u_1 = 0$ *and* $u_2 = k^* > 0$;
(ii) *the fixed point* u_1 *(respectively,* u_2*) is locally unstable (respectively, stable).*

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. The subset $A \subseteq X$ is said to be *chain transitive* (see [11, 12]) if for any $a, b \in A$, and any $\varepsilon > 0$ and $L > 0$, there are finite sequences $x_1, x_2, \dots, x_m \in A$ with $a = x_1, b = x_m$, and $t_1, t_2, \dots, t_m \geq L$ such that $\rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon$ $(1 \leq i \leq m-1)$. The sequence $\{x_1, x_2, \dots, x_m\}$ is called an ε–chain in A connecting a and b.

Recall that the invariant set $M \subset X$ of dynamical system $(X, \mathbb{Z}_+, \pi)$ is said to be dynamically decomposable if there are two nonempty invariant subsets $M_i \subset M$ $(i = 1, 2)$ such that $M_1 \bigcap M_2 = \emptyset$ and $M = M_1 \bigcup M_2$. In other case the set M is said to be dynamically undecomposable.

Remark 4.12. *1. If the positive semi-trajectory* $\Sigma_x^+ := \bigcup_{t\geq 0} \pi(t,x)$ *is relatively compact, then its* ω*-limit set* ω_x *is chain transitive* [5, 11, 15] *(respectively, dynamically undecomposable* [5]*).*

2. If the dynamical system $(X, \mathbb{Z}_+, \pi)$ *is two-sided and the negative semi-trajectory* $\Sigma_x^- := \bigcup_{t\leq 0} \pi(t,x)$ *is relatively compact, then its* α*-limit set* α_x *is chain transitive* [5, 11, 15] *(respectively, dynamically undecomposable* [5]*).*

Theorem 4.13. *Let* $\delta > s_r ab \frac{A}{(ab)^a}$, *then the following statements hold:*

(i) *the dynamical system* $(f_0, \mathbb{Z}_+)$ *(respectively,* $(f_1, \mathbb{Z}_+)$*) is compactly dissipative, where* $f_0(u) := H(0,u)$ *(respectively,* $f_1(u) := H(h,u)$*) for all* $u \in \mathbb{R}_+$;
(ii) $J_0 = [0, k_0^*]$ *(respectively,* $J_1 = [0, k_1^*]$*), where* J_0 *(respectively,* J_1*) is the Levinson center of the dynamical system* $(f_0, \mathbb{Z}_+)$ *(respectively,* $(f_1, \mathbb{Z}_+)$*) and* k_0^* *(respectively,* k_1^**) is its positive fixed point;*
(iii) *for all* $x = (\omega, u) \in \mathbb{R}_+^2$ *with* $\omega \neq 0, h$ *and* $u > 0$ *we have* $\omega_x = \{(h, k_1^*)\}$;
(iv) *for all* $x = (\omega, u) \in J$ *with* $\omega \in [0, h)$ *we have* $\alpha_x = \{(0,0)\}$ *or* $\alpha_x = \{(0, k_0^*)\}$.

Proof. The first statement follows from Theorem 4.7 because the set $\{(0,u) : u \in \mathbb{R}_+\}$ (respectively, $\{(h,u) : u \in \mathbb{R}_+\}$) is an invariant subset of the dynamical system $(\mathbb{R}_+^2, T)$.

Let J_0 (respectively, J_1) be the Levinson center of $(f_0, \mathbb{R}_+)$ (respectively, $(f_1, \mathbb{R}_+)$). Note the function f_0 (respectively, f_1) is strict monotone increasing because

$$\partial_u H(\omega, u) > 0$$

for all $(\omega, u) \in \mathbb{R}_+^2$. Now the second statement of Theorem it follows from Lemma 4.11, Theorem 4.1 (item (iv)) and Remark 4.2.

Let $x = (u, \omega) \in \mathbb{R}_+^2$ with $u > 0$ and $\omega \neq 0$ and $(\mathbb{R}_+^2, \pi)$ be a dynamical system generated by triangular map T (see (12)), i.e., $\pi(t,x) = (\varphi(t,u,\omega), f^t\omega)$ for all $t \in \mathbb{Z}_+$ and $(u, \omega) \in \mathbb{R}_+^2$, where $\varphi(t,u,\omega)$ is a unique solution of equation

$$u(t+1) = H(f^t\omega, u(t))$$

passing through the point u at the initial moment $t = 0$. Since the function f is strictly monotone and $\partial_u H(u, \omega) > 0$, then the semi-group dynamical system $(\mathbb{R}^2_+, \pi)$ in fact is two sided, i.e., every motion can be extended uniquely on $\mathbb{Z}$. Taking into account that the dynamical system $(\mathbb{R}^2_+, \pi)$ is compactly dissipative, then the positively semi-trajectory Σ^+_x is relatively compact, ω_x is a nonempty, compact, invariant and dynamically undecomposable set. Since the set ω_x is chain transitive, then $\omega_x = \{(k^*_1, h)\}$ or $\omega_x = \{(0, h)\}$. We will establish that the last equality is not possible. Suppose that $\omega_x = \{(0, h)\}$, then there exists a moment $t_0 \in \mathbb{Z}_+$ such that

$$f^t\omega \in [0, h + 1] \quad \text{and} \quad \varphi(t, u, \omega) \in (0, u_0) \tag{21}$$

for all $t \geq t_0$. Taking into account (21) without loss of generality we can suppose that $t_0 = 0$ (if it is necessary we can take in the quality of $x = (u, \omega)$ the point $x_0 := \pi(t_0, x)$, because $\omega_x = \omega_{x_0}$). Since $\omega_x = \{(h, 0)\}$, then we have

$$\lim_{t\to+\infty} \varphi(t, u, \omega) = 0. \tag{22}$$

On the other hand by Lemma 4.10 we have

$$\varphi(t, u, \omega) \geq u \tag{23}$$

for all $t \in \mathbb{Z}_+$. The conditions (22) and (23) are contradictory. The obtained contradiction proves our statement.

Let now $x = (u, \omega) \in J$ with $\omega \in [0, h)$, then by Theorem 4.7 we have $\alpha_x \subseteq J_0$. Note that the set α_x is chain transitive. On the other hand α_x is dynamically undecomposable and, consequently, $\alpha_x = \{(0, 0)\}$ or $\alpha_x = \{(0, k^*_0)\}$. Theorem is completely proved. □

References

[1] Brianzoni S., Mammana C. and Michetti E., Complex dynamics in the neoclassical growth model with differential savings and non-constant labor force growth. *Studies in Nonlinear Dynamics and Econometrics*, vol.11, No.1., 2007, pp.1-17.

[2] Brianzoni S., Mammana C. and Michetti E., Global attractor in Solow growth model with differential savings and endogenic labor force growth. *AMSE Periodicals, Modelling Measurement and Control, Series D,* 29(2), 2008, pp.19-37.

[3] Brianzoni S., Mammana C. and Michetti E., Non-linear dynamics in a business-cycle model with logistic population *Chaos Solitons and Fractals,* 40, 2009, pp 717-730

[4] Brianzoni S., Mammana C. and Michetti E., Variable elasticity of substitution in a discrete time Salow-Swan growth model with differential. *Chaos Solitons and Fractals,* vol. 45-1, 2012, pp 98-108

[5] Bronsteyn I. U., *ynamical Systems,* Kishinev, Shtiintsa, 1984. (in Russian)

[6] Bohm V. and Kaas L., Differential savings, factor shares, and endogenous growth cycles. *Journal of Economic Dynamics and Control* 24, 2000, pp. 965-980.

[7] Cheban D. N., *Global Attractors of Nonautonomous Dissipative Dynamical Systems. Interdisciplinary Mathematical Sciences 1.* River Edge, NJ: World Scientific, 2004, xxiv+502 pp.

[8] Cheban D. N. and Mammana C., Invariant Manifolds, Global Attractors and Almost Periodic Solutions of Non-autonomous Difference equations. *Nonlinear Analysis,* serie A, v. 56, No. 4, 2004, pp.465-484.

[9] Cheban D. N., Mammana C. and Michetti E., Global Attractors of Quasi-Linear Non-Autonomous Difference Equations. *Bulletinul Academiei de Stiinte a Republicii Moldova. Matematica,* No.1 (56), 2008, pp.84–104.

[10] Cheban D. N., Mammana C. and Michetti E., Global Attractors of. . .]Global attractors of quasi-linear non-autonomous difference equations: A growth model with endogenous population growth. *Submitted*

[11] Conley C., *Isolated Invariant Sets and the Morse Index.* Region. Conf. Ser. Math., No.38, 1978. Am. Math. Soc., Providence, RI.

[12] Hirsch M. W., Smith H. L. and Zhao X.-Q., Chain Transitivity, Attractivity, and Strong Repellers for Semidynamical Systems. *J. Dyn. Diff. Eqns* 13 (1) (2001), 107-131.

[13] Karagiannis, G., Palivos, T., and Papageorgiou C., Variable Elasticity of Substitution and Economic Growth, In Diebolt, Claude and Catherine Kyrtsou (eds), New Trends in Macroeconomics, 2005, 21-37. Berlin: Springer Verlag.

[14] Revankar N. S., A Class of Variable Elasticity of Substitution Production Function *Econometrica,* 39, 1971, 61-71.

[15] Robinson C., *Dynamical Systems: Stabilty, Symbolic Dynamics and Chaos (Studies in Advanced Mathematics).* Boca Raton Florida: CRC Press, 1995.

[16] Sell G. R., *Topological Dynamics and Ordinary Differential Equations.* Van Nostrand-Reinhold, London, 1971.

[17] Sharkovsky A. N., Kolyada S. F., Sivak A. G. and Fedorenko V. V., *Dynamics of One-Dimensional Maps.* Kluwer Academic Publishers Group, Mathematics and its Applications, 407, 1997.

[18] Solow R. M., A contribution to the theory of economic growth. *Quarterly Journal of Economics,* 70, 1956, pp. 65-94.

[19] Swan, T. W., Growth and Capital Accumulation. *Economic Record,* 32, 1956, pp. 334-361.

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