BELITSKII–LYUBICH CONJECTURE FOR C-ANALYTIC DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is study the problem of global asymptotic stability of solutions for $\mathbb{C}$-analytical dynamical systems (both with continuous and discrete time). In particularly, we present some new results for C-analytical version of Belitskii–Lyubich conjecture. Some applications this results for periodic $\mathbb{C}$-analytical differential/difference equations and the equations with impulse are given.

1. Introduction

1.1. **Markus–Yamabe conjecture (MYC)** [29]**.** Consider the differential equation

$$u' = f(u) \tag{1}$$

and suppose that the Jacobian $f'(u)$ of f has only eigenvalues with negative real part for all u. The *Markus Yamabe conjecture* is that if $f(0) = 0$, then 0 is a globally asymptotically stable solution for (1).

It is easy to prove **MYC** for $n = 1$. In the two-dimensional case the affirmative answer to **MYC** was obtained in the works [14, 16, 15] (see also the references therein). In the work [7] (see also [8, 9] and the references therein) is given a polynomial counterexample to the Markus–Yamabe conjecture. If $n > 2$ there are also some additional conditions forcing the Markus–Yamabe conjecture. For example if $f'(u)$ is negative definite for all $u \in \mathbb{R}^n$ the conjecture was proved in [21, 22] (see also [24, 25, 29]). For triangular systems **MYC** was proved in [29].

1.2. **The discrete Markus–Yamabe conjecture (DMYC)** [10, 33]**.** Let f be a C^1 mapping from $\mathbb{R}^n$ into itself such that $f(0) = 0$ and for all $u \in \mathbb{R}^n$, $f'(u)$ has all its eigenvalues with modulus less than one. Then 0 is a globally asymptotically stable solution of the difference equation

$$u(n+1) = f(u(n)).$$

In his book [27] J. P. LaSalle proves the **DMYC** for $n = 1$. The discrete Markus–Yamabe conjecture is true only for planar maps (see [10] and also the references therein) and the answer to the question is yes only in the case of planar polynomial

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maps. The authors [10] prove that the **DMYC** is true for triangular maps defined on $\mathbb{R}^n$ and for polynomial maps defined on $\mathbb{R}^2$. In the works [4, 28] the **DMYC** is proved for gradient maps.

1.3. **Belitskii–Lyubich conjecture** [2]**.** Let E be a Banach space, $\Omega \subset E$ an open subset and $f : \Omega \mapsto E$ be a compact and continuously differentiable in Ω. Suppose D is a nonempty bounded convex open subset of X such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup\limits_{x \in \overline{D}} r(f'(x)) < 1$ ($r(A)$ is the spectral radius of linear bounded operator A). Then the discrete dynamical system $(\overline{D}, f)$, generated by positive powers of $f : \overline{D} \mapsto \overline{D}$, admits a unique globally asymptotically stable fixed point.

In generale case the answer to Belitskii-Lyubich conjecture is negative. Namely by Slyusarchuk V.E. [37] and Shih Mau-Hsiang and Wu, Jinn-Wen [38] was proved that even in the two-dimensional case this statement is not true.

In the work [38] was given a positive answer to Bielitskii-Lyubich conjecture for compact holomorphic mappings. We will present in this paper answer to this problem for asymptotically compact holomorphic maps.

The aim of this paper is study the problem of global asymptotic stability of solutions for holomorphic dynamical systems (both with continuous and discrete time). We present some new results for $\mathbb{C}$-analytical version of Belitskii–Lyubich conjecture. Some applications this results for periodic $\mathbb{C}$-analytical differential/difference equations and the equations with impulse are given.

This paper is organized as follows.

In Section 2 we give a positive answer to Belitskii–Lyubich conjecture for asymptotically compact holomorphic dynamical systems with discrete time.

Section 3 is dedicated to the study of Belitskii–Lyubich problem for asymptotically compact holomorphic flows.

In section 4 we study the holomorphic dissipative dynamical systems (both with continuous and discrete times).

We give in section 5 some applications of obtained general results for periodical holomorphic differential/difference equations and differential equations with impulse.

2. Belitskii–Lyubich conjecture

Let E be a Banach space. If $B \subset E$ is bounded, we define the set measure of noncompactness of B, $\alpha(B)$, by $\alpha(B) := \inf\{\varepsilon > 0 : B$ has a finite cover by sets whose diameters do not exceed $\varepsilon\}$. Clearly, B is precompact iff $\alpha(B) = 0$.

Definition 2.1. *A function F whose domain is a subset of E is called* [1, Ch.I] *a k-set-contraction operator, if there is a nonnegative constant k such that $\alpha(F(B)) \leq k\alpha(B)$ for every bounded subset B of the domain of F.*

It is known [1, Ch.III],[12] that the Schauder Fixed Point Theorem extends to the class of k-set-contractions for which $k < 1$.

Definition 2.2. *An operator $F : E \mapsto E$ is called Fréchet differentiable at the point $x_0 \in E$ if there exists a linear bounded operator $A : E \mapsto E$ such that for all $h \in E$ we have $F(x_0 + h) - F(x_0) = Ah + \omega(x_0, h)$, where $\omega(x_0, h)$ satisfies the condition $\lim\limits_{|h|\to 0} \frac{|\omega(x_0,h)|}{|h|} = 0$. In this case the expression Ah is called the Frechet differential of F at x_0 and is denoted by $Ah = dF(x_0, h)$.*

It is known [1, Ch.I] that a Fréchet derivative of a k-set-contraction is a k-set-contraction.

Definition 2.3. *Let E be a complex Banach space and $U \subset E$ bean open set. The mapping $f : U \mapsto E$ is called:*

(i) *G-holomorphic at the point $x_0 \in U$, if there exists a positive number δ such that $B(x_0, \delta) \subset U$ and the mapping $\lambda \mapsto f(x_0 + \lambda u)$ is holomorphic for every $u \in E \setminus \{0\}$, where $|\lambda| < \frac{\delta}{|u|}$;*

(ii) *holomorphic, if it is continuous and G-holomorphic at every point in U.*

Definition 2.4. *A fixed point $x_0 \in U$ of the mapping $f : U \mapsto E$ is said to be:*

- *stable, if for arbitrary positive number ε the exists a positive number $\delta = \delta(\varepsilon)$ such that $|x - x_0| < \delta$ implies $|f^n(x) - x_0| < \varepsilon$ for all $n \in \mathbb{Z}_+$;*
- *attracting, if there exists a positive number γ such that*

$$\lim_{n\to\infty} |f^n(x) - x_0| = 0 \tag{2}$$

  for all $x \in B(0, \gamma) := \{x \in E : \ |x - x_0| < \gamma\}$;
- *asymptotically stable, if x_0 is stable and attracting;*
- *uniformly asymptotically stable, if x_0 is attracting and equality (2) takes place uniformly with respect to $x \in B(0, \gamma)$.*

Lemma 2.5. *Let $x_0 \in U$ be a fixed point of the map $f : U \mapsto E$. If x_0 is uniformly asymptotically stable, then it is stable.*

Proof. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_n \to 0$, $|x_n - x_0| < \delta_n$ and $\{k_n\} \subset Z_+$ such that $k_n \to \infty$ as $n \to \infty$ and

$$|f^{k_n}(x_n) - x_0| \geq \varepsilon_0 \tag{3}$$

for all $n \in \mathbb{Z}_+$.

Since x_0 is uniformly asymptotically stable, then there exists a positive number γ such that equality (2) holds uniformly with respect to $x \in B(x_0, \gamma)$. In particular, for arbitrary $\varepsilon \in (0, \varepsilon_0)$ there exists a natural number $N_1 = N_1(\varepsilon) \in \mathbb{N}$ such that

$$|f^n(x) - x_0| < \varepsilon$$

for all $n \geq N_1$ and $x \in B(x_0, \varepsilon)$. There exists a natural number N_2 such that $\delta_n < \gamma$ for all $n \geq N_2$. Denote by $N := \max\{N_1, N_2\}$, then $|x_n - x_0| < \delta_n < \gamma$ for all $n \geq N$ and, consequently,

$$|f^{k_n}(x_n) - x_0| < \varepsilon \tag{4}$$

for all n sufficiently large. Inequalities (3) and (4) are contradictory. The obtained contradiction proves our statement. □

Corollary 2.6. *Let x_0 be a fixed point of the map $f : U \mapsto E$. If x_0 is uniformly asymptotically stable, then it is asymptotically stable.*

Lemma 2.7. [3, Ch.I] *Let x_0 be an asymptotically stable fixed point of the map $f : U \mapsto E$. Then for any compact subset $K \subset B(0, \gamma)$, where γ is a positive number figuring in the definition of asymptotic stability of x_0, we have*

$$\lim_{n\to\infty} \max_{x\in K} |f^n(x) - x_0| = 0.$$

Remark 2.8. *1. If the Banach space E is finite-dimensional, then every asymptotically stable fixed point is uniformly asymptotically stable. This statement follows, for example, from the Lemma 2.7 and the fact that finite-dimensional Banach space E is locally compact.*

2. If the Banach space E is infinite-dimensional, then from asymptotic stability of the fixed point x_0, generally speaking, it does not follow uniform asymptotic stability of x_0. The corresponding example can be find, for example, in [6] *(Example 3.1).*

Theorem 2.9. *Let U be a non-empty bounded domain in a complex Banach space E, $f : U \mapsto E$ be holomorphic and $f(x_0) = x_0$. If x_0 is attracting (respectively, uniformly asymptotically stable), then $\lim_{n\to\infty} A^n u = 0$ for all $u \in E$ (respectively, $\lim_{n\to\infty} A^n u = 0$ uniformly with respect to u on every bounded subset from E), where $A := f'(x_0)$.*

Proof. Let x_0 be an attracting (respectively, uniformly asymptotically stable) fixed point of the map $f : U \mapsto E$. Then there exists a positive number γ such that $B(x_0, \gamma) \subset U$ and

$$\lim_{n\to\infty} f^n(x) = x_0 \tag{5}$$

for all $x \in B(x_0, \gamma)$ and according to Lemma 2.7 equality (5) takes place uniformly in x on every compact subset K from $B(x_0, \gamma)$ (respectively, uniformly with respect to $x \in B(x_0, \gamma)$). Thus the sequence $\{f^n\}$ of functions converges uniformly to the constant function $g(x) \equiv x_0$ on each compact subset of $B(x_0, \gamma)$ (respectively, uniformly with respect to x on $B(x_0, \gamma)$). Fixe $u \in E$. Then the map $\lambda \mapsto f(x_0+\lambda u)$ is holomorphic in $\Delta(u, \gamma) := \{\lambda \in \mathbb{C} : \ |\lambda| < \gamma/|u|\}$. Let $\nu \in (0, \gamma/|u|)$ and $A := f'(x_0)$. By Cauchy's integral formula

$$Au = \frac{df(x_0 + \lambda u)}{d\lambda}\Big|_{\lambda=0} = \frac{1}{2\pi i} \int_{|\lambda|=\nu} \frac{f(x_0 + \lambda u)}{\lambda^2} d\lambda.$$

Since $f(x_0) = x_0$ and $(f^n(x_0))' = f'(f^{n-1}(x_0)) \circ f'(f^{n-2}(x_0)) \circ \ldots f'(f(x_0) \circ f'(x_0) = A^n$, then we obtain

$$A^n(u) = \frac{df^n(x_0 + \lambda u)}{d\lambda}\Big|_{\lambda=0} = \frac{1}{2\pi i} \int_{|\lambda|=\nu} \frac{f^n(x_0 + \lambda u)}{\lambda^2} d\lambda \quad (n = 1, 2, \ldots).$$

Since $\{x_0 + \lambda u : \ |\lambda| = \nu\}$ is a compact subset of $B(x_0, \gamma)$, then we have

$$\lim_{n\to\infty} A^n(u) = \frac{1}{2\pi i} \int_{|\lambda|=\nu} \lim_{n\to\infty} \frac{f^n(x_0 + \lambda u)}{\lambda^2} d\lambda = \tag{6}$$

$$\frac{1}{2\pi i}\int_{|\lambda|=\nu}\frac{1}{\lambda^2}d\lambda\ x_0 = 0 \quad (n = 1, 2, \cdots)$$

(respectively, equality (6) takes place uniformly with respect to u on $S(0,\nu) := \{u \in E : \ |u| = \nu\} \subset E$ for all $\nu > 0$). Since $u \in E$ is arbitrary (respectively, $\nu > 0$ is arbitrary) we conclude that $\lim\limits_{n\to\infty} A^n u = 0$ for all $u \in E$ (respectively, $\lim\limits_{n\to\infty} A^n u = 0$ uniformly with respect to u on every bounded subset from E), where $A : f'(x_0)$. Theorem is completely proved. □

Corollary 2.10. *Let U be a non-empty bounded domain in a complex Banach space E, $f : U \mapsto E$ be holomorphic and $f(x_0) = x_0$. If the point x_0 is uniformly asymptotically stable, then $r(f'(x_0)) < 1$.*

Proof. This statements follows from Theorem 2.9 and the fact that for every linear bounded operator $A : E \mapsto E$ the following two statements are equivalent (the proof see, for example, in [5, Ch.IV, Theorem 4.3.13] or [6, Theorem 3.5]):

(i) $\lim\limits_{n\to\infty} ||A^n|| = 0$;
(ii) $r(A) < 1$,

where $r(A)$ is the spectral radius of A. □

Remark 2.11. *Note that Theorem 2.9 (respectively, Corollary 2.10) is not true for the real analytic functions. For example the mapping $f : \mathbb{R} \mapsto \mathbb{R}$ defined by equality*

$$f(x) = \frac{x}{(1 + 2x^2)^{1/2}}$$

is real analytic on $U := (-\varepsilon, \varepsilon) \subset \mathbb{R}$, where ε is a small enough positive number. It easy to check that $f(0) = 0$, $\lim\limits_{t\to\infty} f^n(x) = 0$ for all $x \in U$ and $f'(0) = 1$, i.e., the point x_0 is an asymptotically stable fixed point for mapping f, but $r(f'(0)) = 1$. The required example is constructed.

Definition 2.12. *A continuous mapping $f : E \mapsto E$ is called:*

(i) *condensing (see, for example,* [1, 36]*) if $\alpha(f(A)) < \alpha(A)$ for all bounded subset $A \subset E$ with $\alpha(A) > 0$;*
(ii) *asymptotically compact (see, for example,* [17, 26]*) if for any bounded positively invariant subset $M \subset E$ there exists a nonempty compact subset $K \subset E$ such that*

$$\lim_{n\to\infty} \beta(f^n(M), K) = 0.$$

Remark 2.13. *1. Every k-set contraction is a condensing operator* [17].

2. A condensing operator is asymptotically compact [17].

Let M be a subset of E. Denote by $V_0 := \overline{co} f(M)$, where by $\overline{co}(A)$ is denoted the closed convex hull of the set A, and for $\nu > 0$

$$V_\nu = \begin{cases} \overline{co} f(M \bigcap V_{\nu-1}), & \text{if } \nu - 1 \text{ exists} \\ \bigcap\limits_{\beta<\nu} V_\beta, & \text{otherwise.} \end{cases} \tag{7}$$

There is an ordinal number δ such that $V_\nu = V_\beta$ for all $\nu \geq \beta$.

Definition 2.14. *The limit set V_β of the transfinite sequence (7) is called* [1, Ch.I],[36] *ultimated range (or limit range) of the operator f on the set M and is denoted by $f^\infty(M)$. The operator $f : M \mapsto E$ is said to be ultimately compact (or limit compact)* [1, Ch.I],[36] *if the set $f(M \bigcap f^\infty(M))$ is relatively compact.*

Theorem 2.15. [18],[35] *Suppose that the operator $f : E \mapsto E$ maps a nonempty convex closed subset M into itself. If f is asymptotically compact, then it has at least one fixed point in M.*

Remark 2.16. *In the works* [18],[35] *it was established Theorem 2.15 for the condensing operators. In general case this statement can be proved using absolutely the same arguments as in the work* [35] *(see also* [1, Ch.I,p.26]*).*

Denote by $Hol(U, E)$ the set of all holomorphic functions $f : U \mapsto E$ equipped with the compact-open topology.

Denote by $Fix(f, D) := \{x \in D : \ f(x) = x\}$, where $D \subseteq E$ and $W^s(p) := \{x \in D : \lim\limits_{n\to\infty} \rho(f^n(x), p) = 0\}$ for all $p \in Fix(f, D)$.

Theorem 2.17. *Let E be a complex Banach space, let U be a non-empty bounded domain in a Banach space E, $f \in Hol(U, E)$ be an asymptotically compact operator. Suppose that the following conditions hold:*

- (i) *D is a non-empty bounded convex open subset of U such that $\overline{D} \subset U$;*
- (ii) *$f(\overline{D}) \subseteq \overline{D}$;*
- (iii) *$r(f'(x)) < 1$ for all $x \in Fix(f, \overline{D})$.*

Then

- (i) *f has a unique fixed point $x_0 \in \overline{D}$;*
- (ii) *x_0 is globally asymptotically stable, i.e., $W^s(x_0) = D$.*

Proof. By Theorem 2.15 the set $M := Fix(f, \overline{D})$ is a nonempty subset of $\overline{D}$. It easy to see that M is closed and invariant. Since f is asymptotically compact, then M is a compact set. Let $x \in M$. Since $r(f'(x)) < 1$, then by Theorem 5.2 [32, Ch.V] there exists a positive number δ_x such that

- (i) $f(U_x) \subseteq U_x$;
- (ii)
$$\lim_{n\to\infty} f^n(y) = x$$
for all $y \in U_x$ and (2) holds uniformly with respect to $y \in U_x$, where $U_x := B(x, \delta_x)$.

Thus $M \bigcap U_x = \{x\}$ for all $x \in M$, i.e., every point $x \in M$ is isolated and taking into account compactness of M we concludes that M contains a finite number of points, i.e., $M = \{p_1, p_2, \dots, p_m\}$. Denote by $\tilde{D} := D \bigcup U_{p_1} \bigcup \dots \bigcup U_{p_m}$ and $\mathcal{E} := \{f^n : \ n \in \mathbb{Z}_+\}$. Under the conditions of Theorem 2.17 there exists a positive constant M such that
$$|f^n(x)| \le M$$
for all $x \in \tilde{D}$ and $n \in \mathbb{Z}_+$. Then by Montel's theorem [30, Ch.II] the family $\mathcal{E}$ is a relatively compact subset of $C(\tilde{D}, E)$.

Denote by $\mathfrak{E} := \{g :$ there exists a sequence $\{n_k\} \subset \mathbb{Z}_+$ such that $n_k \to \infty$ and $f^{n_k} \to g$ as $k \to \infty\}$, where $f^{n_k} \to g$ means convergence in $C(\tilde{D}, E)$. Since $\mathcal{E}$ is a relatively compact subset of $C(\tilde{D}, E)$, then $\mathfrak{E}$ is a nonempty and compact subset of $C(\tilde{D}, E)$. Let $\xi \in \mathfrak{E}$, then there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \to \infty$ and $f^{n_k} \to \xi$ in $C(\tilde{D}, E)$. If $p \in M$, then ξ possess the following properties:

(i) $\xi \in Hol(\tilde{D}, E)$;
(ii) $\xi(p) = p$;
(iii) $\xi(x) = \lim\limits_{k\to\infty} f^{n_k}(x) = p$ for all $x \in U_p$.

Since U_p is a connected open subset of E, then by identity theorem [30, Ch.II] we have $\xi(x) = p$ for all $x \in \tilde{D}$. Since $\xi \in Hol(\tilde{D}, E)$, then f admits at most one fixed point in $\overline{D}$, i.e., M consists a single point $M = \{p\}$. Thus we obtain the following equality

$$\lim_{k\to\infty} f^{n_k}(x) = p \tag{8}$$

for all $x \in \tilde{D}$. Now we will prove that

$$\lim_{n\to\infty} f^{n}(x) = p$$

for all $x \in \tilde{D}$. Let $x \in \tilde{D}$ and we consider the sequence $\{f^n(x)\}$. According to (8) there exists a number $k_0 \in \mathbb{N}$ such that $f^{n_k}(x) \in U$ for all $k \geq k_0$ and consequently, we obtain

$$\lim_{n\to\infty} f^{n}(x) = \lim_{n\to\infty} f^{n-n_{k_0}}(f^{n_{k_0}}(x)) = p.$$

Theorem is proved. □

Remark 2.18. *Note that Theorem 2.17 remains true if we replace the condition "$r(f'(x)) < 1$ for all $x \in Fix(f, \overline{D})$" by the following: there exists a fixed point $p \in Fix(f, \overline{D})$ such that $r(f'(p)) < 1$. This statement can be proved by slight change of the proof of Theorem 2.17.*

3. Belitskii–Lyubich conjecture for holomorphic flows

Let $(E, \mathbb{R}_+, \pi)$ be a flow (semi-group dynamical system with continuous time $\mathbb{R}_+$). Everywhere in this section we will suppose that E is a complex Banach space and the mapping $\pi(t, \cdot) : E \mapsto E$ is holomorphic for all $t \in \mathbb{R}_+$.

Definition 3.1. *A dynamical system $(E, \mathbb{R}_+, \pi)$ is called asymptotically compact (see, for example,* [17, 26]*) if for any bounded positively invariant subset $M \subset E$ there exists a nonempty compact subset $K \subset E$ such that*

$$\lim_{t\to\infty} \beta(\pi(t, M), K) = 0.$$

Theorem 3.2. *Suppose that the following conditions hold:*

(i) *$M \subset E$ is a nonempty, bounded, convex and closed subset;*
(ii) *the set M is positively invariant, i.e., $\pi(t, M) \subseteq M$ for all $t \in \mathbb{R}_+$;*
(iii) *the dynamical system $(E, \mathbb{R}_+, \pi)$ is asymptotically compact.*

The $(E, \mathbb{R}_+, \pi)$ admits at least one fixed point (stationary point) in M, i.e., there exists a point $p \in M$ such that $\pi(t, p) = p$ for all $t \in \mathbb{R}_+$.

Proof. Let $\{t_n\}$ be a decreasing sequence such that $\lim\limits_{n\to\infty} t_n = 0$ and t be an arbitrary number from $\mathbb{R}_+$. Denote by $f_n := \pi(t_n, \cdot) : E \mapsto E$. Under the conditions of Theorem 3.2 according to Theorem 2.15 the mapping f_n admits at least one fixed point $p_n \in M$. We will show that the set $A := \{p_n |\ n \in \mathbb{N}\}$ is relatively compact. To this end we denote by $k_n \in \mathbb{N}$ a number such that $t_n k_n \geq n$. Note that the set $A' := \{\pi(t, A) |\ t \geq 0\}$ is positively invariant and bounded. Since the dynamical system $(E, \mathbb{R}_+, \pi)$ is asymptotically compact, then there exists a nonempty compact subset $K \subseteq M$ such that

$$\lim_{t\to\infty} \beta(\pi(t, A), K) = 0.$$

On the other hand we have $p_n = \pi(t_n k_n, p_n)$ and, consequently,

$$\rho(p_n, K) = \rho(\pi(t_n k_n, pn), K) \leq \beta(\pi(t_n k_n, A), K) \to 0 \tag{9}$$

as $n \to \infty$. Taking into account the compactness of K we concludes from (9) the sequence $\{p_n\}$ is relatively compact. Thus without loss of generality we can suppose that the sequence $\{p_n\}$ s convergent. Denote by p its limit. Let $t \in \mathbb{R}_+$, then there are $m_n \in \mathbb{N}$ and $\tau_n \in [0, t_n)$ such that $t = t_n m_n + \tau_n$. Thus we have

$$\pi(t, p_n) = \pi(t_n m_n + \tau_n, p_n) = \pi(\tau_n, p_n) \tag{10}$$

for all $n \in \mathbb{N}$. Passing into limit in (10) as $n \to \infty$ we obtain $\pi(t, p) = p$ for all $t \in \mathbb{R}_+$ because $0 \leq \tau_n < t_n$, $t_n \to 0$ and $p_n \to p$ as $n \to \infty$. Theorem is completely proved. □

Remark 3.3. *Theorem 3.2 was proved by Jones G. S.* [23] *in the case when the dynamically system $(E, \mathbb{R}_+, \pi)$ is completely continuous, i.e., there exists a number $t_0 > 0$ such that $\pi(t_0, \cdot) : E \mapsto E$ is completely continuous.*

Let $(E, \mathbb{R}_+, \pi)$ be a semi-flow on E and $M \subseteq E$. Denote by $Fix(\pi, M) := \{x \in M :$ $\pi(t, x) = x$ for all $t \in \mathbb{R}_+\}$ and $W^s(p, M) := \{x \in M :$ such that $\lim\limits_{t\to+\infty} \rho(\pi(t, x), p) = 0\}$.

Theorem 3.4. *Let E be a complex Banach space and $(E, \mathbb{R}_+, \pi)$ be a semi-flow on E, let U be a non-empty bounded domain in a Banach space E, for every $t \in \mathbb{R}_+$ the mapping $\pi(t\cdot) \in Hol(U, E)$ and the dynamical system $(E, \mathbb{R}_+, \pi)$ be asymptotically compact. Suppose that the following conditions hold:*

(i) *D is a non-empty bounded convex open subset of U such that $\overline{D} \subset U$;*
(ii) *$\pi(t, \overline{D}) \subseteq \overline{D}$ for all $t \in \mathbb{R}_+$;*
(iii) *$r(\pi'(t, x)) < 1$ for all $t > 0$ and $x \in Fix(\pi, \overline{D})$, where $\pi'(t, x)$ is the Fréchet derivative of $\pi(t, \cdot) : E \mapsto E$ at the point x.*

Then

(i) *$(E, \mathbb{R}_+, \pi)$ has a unique fixed point p in $\overline{D}$;*
(ii) *p is globally asymptotically stable, i.e., $W^s(p, D) = D$.*

Proof. At first we note that under the conditions of Theorem 3.4 the set $Fix(\pi, \overline{D})$ by Theorem 3.2 is not empty. Let t_0 be an arbitrary positive number and $f := \pi(t_0, \cdot)$, then it easy to check that under the conditions of Theorem 3.4 we can apply Theorem 2.17 to map f. Thus $f = \pi(t_0, \cdot)$ admits a unique fixed point $p \in \overline{D}$ which is globally asymptotically stable.

Let now $x \in Fix(\pi, \overline{D})$ be an arbitrary fixed point of $(E, \mathbb{R}_+, \pi)$ belonging to $\overline{D}$. Since

$$f(x) = \pi(t_0, x) = \pi(t_0, \pi(t, x)) = \pi(t, \pi(t_0, x)) = \pi(t, x) = x$$

for all $t \in \mathbb{R}_+$ and, consequently, $x \in Fix(f, \overline{D}) = \{p\}$. Thus we have $Fix(\pi, \overline{D}) = \{p\}$.

We will show that the unique fixed point p of $(E, \mathbb{R}_+, \pi)$ is Lyapunov stable. In fact. Let ε be an arbitrary positive number, then by integral continuity of $(E, \mathbb{R}_+, \pi)$ at the point p there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$\rho(x, p) < \delta \ \Rightarrow \ \rho(\pi(\tau, x), p) < \varepsilon \tag{11}$$

for all $\tau \in [0, 1]$. Since p is a globally asymptotically stable fixed point of the map $f = \pi(t_0, \cdot)$, then for the number $\delta(\varepsilon) > 0$ there exists a number $\gamma = \gamma(\varepsilon) > 0$ such that

$$\rho(x, p) < \gamma \ \Rightarrow \ \rho(f^n(x), p) < \delta \tag{12}$$

for all $n \in \mathbb{Z}_+$. Let now $t \in \mathbb{R}_+$ and $t = n_t + \tau_t$, where $n_t := [t]$ and $\tau_t := \{t\} \in [0, 1)$. Note that $\pi(t, x) = \pi(n_t + \tau_t, x) = \pi(\tau_t, f^{n_t}(x))$ and taking into account (11) and (12) we obtain

$$\rho(\pi(t, x), p) = \rho(\pi(\tau_t, f^{n_t}(x)), p) < \varepsilon$$

if $\rho(x, p) < \gamma$.

To finish the proof of Theorem it is sufficient to show that $W^s(\pi, D) = D$. Let ε be an arbitrary positive number and $x \in D$. Since $\lim_{n \to \infty} f^n(x) = p$, then there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f^n(x), p) < \delta \tag{13}$$

for all $n \geq N$, where $\delta = \delta(\varepsilon) > 0$ is chosen from the integral continuity of $(E, \mathbb{R}_+, \pi)$ at the point p. From (12) and (13) we obtain

$$\rho(\pi(t, x), p) = \rho(\pi(\tau_t, f^{n_t}(x)), p) < \varepsilon$$

for all $t \geq N(\varepsilon)$, i.e., $\lim_{t \to \infty} \pi(t, x) = p$. □

4. Holomorphic dissipative dynamical systems

Denote by $\mathbb{T}$ the set $\mathbb{Z}_+$ or $\mathbb{R}_+$ and by $(E, \mathbb{T}, \pi)$ a dynamical system on E. In this section we will suppose that the mapping $\pi(t, \cdot) : E \mapsto E$ is holomorphic for every $t \in \mathbb{T}$.

Definition 4.1. *A dynamical system $(E, \mathbb{T}, \pi)$ on the Banach space E is said to be dissipative if there exists a positive number R_0 such that for all $r > 0$ we can find a positive number $L = L(r)$ such that*

$$|\pi(t, x)| \leq R_0 \tag{14}$$

for all $|x| \leq r$ and $t \geq L(r)$.

Definition 4.2. *A set S is said to lie strictly inside a subset D of a Banach space E if there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset D$ whenever $x \in S$.*

Theorem 4.3. *(Earle-Hamilton* [13]*, see also Harris* [19]*). Let D be a nonempty domain in a complex Banach space E and let $h : D \mapsto D$ be a bounded holomorphic function. If $h(D)$ lies strictly inside D, then the following statement hold:*

(i) *there exists a metric d on D and a number $\alpha \in (0,1)$ such that $d(h(x),h(y)) \leq \alpha d(x,y)$ for all $x,y \in D$, i.e., h is a d-contraction;*
(ii) *there exists a unique fixed point $p \in D$ of h;*
(iii)

$$\lim_{n\to\infty} h^n(x) = p \tag{15}$$

for all $x \in D$;
(iv) *there exists a positive number C such that $\rho(x,y) \leq Cd(x,y)$ for all $x,y \in D$, where $\rho(x,y) := |x-y|$.*

Remark 4.4. *Note that under the conditions of Theorem 4.3 equality (15) takes place uniformly with respect to $x \in D$ because*

$$\rho(h^n(x),p) \leq Cd(h^n(x),p) \leq C\frac{\alpha^n}{1-\alpha} diam_d D$$

for all $x \in D$, where $diam_d D := \sup\{d(x,y) : \ x,y \in D\}$.

Theorem 4.5. *Suppose that $(E,\mathbb{T},\pi)$ is a dynamical system on the complex Banach space E and the following conditions hold:*

(i) *the dynamical system $(E,\mathbb{T},\pi)$ is holomorphic;*
(ii) *$(E,\mathbb{T},\pi)$ is dissipative.*

Then there exists a unique fixed point $p \in E$ of dynamical system $(E,\mathbb{T},\pi)$ in E such that

(i)

$$\lim_{t\to+\infty} \pi(t,x) = p \tag{16}$$

for all $x \in E$;
(ii) *equality (16) takes place uniformly in x on every bounded subset D from E.*

Proof. Let $(E,\mathbb{T},\pi)$ be dissipative, R_0 be a positive number figuring in the definition of dissipativity and $r > R_0$, then there exists a number $L = L(r) > 0$ such that inequality (14) holds. Denote by $D = B(0,r) := \{x \in E : \ |x| < r\}$ and $f := \pi(t_0,\cdot)$, where $t_0 \geq L(r)$, then $f(D) \subseteq B[0,R_0] := \{x \in E : \ |x| \leq R_0\}$. Thus all conditions of Theorem 4.3 hold and, consequently, there exists a unique fixed point $p \in D$ (in fact $p \in B[0,R_0]$) such that equality (16) holds uniformly with respect to $x \in B(0,r)$.

Now we will establish that p is a unique fixed point of $(E,\mathbb{T},\pi)$. In fact. Let $t_0 \geq L(r)$ and $t \in \mathbb{T}$. Note that $\pi(t_0,\pi(t,p)) = \pi(t,\pi(t_0,p)) = \pi(t,p)$ for all $t \in \mathbb{T}$. Since p is a unique fixed point of the map $f = \pi(t_0,\cdot)$, then $\pi(t,p) = p$ for all $t \in \mathbb{T}$. Since $\lim\limits_{n\to\infty} \pi(nt_0,x) = f^n(x) = p$for all $x \in B[0,r]$, then reasoning as in the proof of Theorem 3.4 we obtain (16).

To finish the proof of Theorem it is sufficient to show that (16) takes place uniformly with respect to $x \in B[0,r]$. In fact. Let $\varepsilon > 0$ be an arbitrary positive number and $x \in B[0,r]$. Since $\lim_{n\to\infty} f^n(x) = p$ uniformly with respect to $x \in B[0,r]$, then there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f^n(x), p) < \delta \tag{17}$$

for all $n \geq N$ and $x \in B[0,r]$, where $\delta = \delta(\varepsilon) > 0$ is chosen from the integral continuity of $(E, \mathbb{R}_+, \pi)$ at the point p, i.e.,

$$\rho(\pi(t,x), p) < \varepsilon \text{ for all } t \in [0,1], \tag{18}$$

if $\rho(x,p) < \delta$. From (17) and (18) we obtain

$$\rho(\pi(t,x),p) = \rho(\pi(\tau_t, f^{n_t}(x)), p) < \varepsilon$$

for all $t \geq N(\varepsilon)$ and $x \in B[0,r]$, i.e.,

$$\lim_{t\to\infty} \sup_{x\in B[0,r]} |\pi(t,x) - p| = 0.$$

□

Lemma 4.6. *Let $(E, \mathbb{T}, \pi)$ be a holomorphic dynamical system on the complex Banach space E. If*

$$\lim_{t\to+\infty} \pi(t,x) = p \tag{19}$$

for all $x \in E$ and equality (19) takes place uniformly in x on every bounded subset D from E, then p is a stable fixed point of $(E, \mathbb{T}, \pi)$.

Proof. If we suppose that this statement is not true, then there exist $\varepsilon_0 > 0$, $0 < \delta_n \to 0$ as $n \to \infty$, $|x_n| < \delta$ and $t_n \to +\infty$ ($t_n \in \mathbb{T}$) such that

$$|\pi(t_n, x_n) - p| \geq \varepsilon_0 \tag{20}$$

for all $n \in \mathbb{N}$. By equality (19) we have

$$\lim_{n\to\infty} \pi(t_n, x_n) = p. \tag{21}$$

Passing into limit in (20) as $n \to \infty$ and taking into account (21) we obtain $0 \geq \varepsilon_0$. The last inequality contradicts to the choice of the number ε_0. The obtained contradiction proves our statement. □

5. Some applications

5.1. **Periodic dissipative differential equations.** Let E be a complex Banach space. Consider a differential equation

$$x' = f(t,x), \tag{22}$$

where $f \in C(\mathbb{R} \times E, E)$, $f(t+\omega, x) = f(t,x)$ for all $(t,x) \in \mathbb{R} \times E$ ($\omega > 0$) and , $f(t,\cdot) \in Hol(E)$ for all $t \in [0,\omega)$. In this subsection we suppose that for all $x \in E$ the equation (22) (or the function f) is regular, i.e., it admits a unique solution $\varphi(t,x,f)$ passing through x at the initial moment $t = 0$ and defined on $\mathbb{R}_+$ and the mapping $\varphi(\cdot,\cdot,f) : \mathbb{R}_+ \times E \mapsto E$ is continuous.

Remark 5.1. *It easy to show that under the conditions above the mapping $\varphi(t,\cdot,f) : E \mapsto E$ is holomorphic for all $t \in [0,\omega)$.*

Recall that the mapping $P : E \mapsto E$ defined by $P(x) := \varphi(\omega, x, f)$ is called Poincaré mapping for (22).

Definition 5.2. *Differential equation (22)is called dissipative if there exists a positive number R_0 such that for all $r > o$ there exists a positive number $L = L(r)$ with the property $|\varphi(t, x, f)| \le R_0$ for all $|x| \le r$ and $t \ge L(r)$.*

Theorem 5.3. *Suppose that the following conditions hold:*

(i) *E is a complex Banach space;*
(ii) *the function f is ω-periodic in $t \in \mathbb{R}$;*
(iii) *the function $f \in C(\mathbb{R} \times E, E)$ is regular and $f(t, \cdot) \in Hol(E)$ for all $t \in [0, \omega)$;*
(iv) *equation (22) is dissipative.*

Then the following statements hold:

(i) *there exists a unique ω periodic solution $p(t)$ of equation (22);*
(ii) *the solution $p(t)$ of equation (22) is globally asymptotically stable.*

Proof. Denote by $P : E \mapsto E$ the Poincaré mapping for equation (22), i.e., $P(x) := \varphi(\omega, x, f)$. Since equation (22) is dissipative, then it is not difficult to check the discrete dynamical system (E, P) generated by powers of the map P is also dissipative. By Theorem 4.5 there exists a unique fixed point $x_0 \in E$ of the map P such that

$$\lim_{n\to\infty} |P^n(x) - x_0| = 0 \tag{23}$$

for all $x \in E$. Additionally, equality (23) takes place uniformly with respect to x on every bounded subset of E. By Theorem 4.5 (see also Lemma 4.6) the fixed point x_0 of the mapping P is stable, i.e., for arbitrary $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $|x - x_0| < \delta$ implies the inequality

$$|P^n(x) - x_0| < \varepsilon$$

for all $n \in \mathbb{Z}_+$. To finish the proof of Theorem it is sufficient to note that $p(t) := \varphi(t, x_0, f)$ is a unique ω-periodic solution of equation (22) and its stability results from the stability of x_0 with respect to discrete dynamical system (E, P). In fact, if we suppose that it is not so, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$ as $n \to \infty$, $|x_n| < \delta$ and $t_n \to +\infty$ ($t_n \in \mathbb{T}$) such that

$$|\varphi(t_n, x_n, f) - p(t_n)| \ge \varepsilon_0$$

for all $n \in \mathbb{N}$. Note that $t_n = k_n\omega + \tau_n$, where $k_n \in \mathbb{N}$ and $\tau_n \in [0, \omega)$. Without loss of generality we can suppose that the sequence $\{\tau_n\}$ is convergent. Denote its limit by τ_0, then we have

$$\varepsilon_0 \le |\varphi(k_n\omega + \tau_n, x_n, f) - p(k_n\omega + \tau_n)| = |\varphi(\tau_n, P^{k_n}(x_n), f) - p(\tau_n)| \tag{24}$$

for all $n \in \mathbb{N}$. Passing into limit in (24) as $n \to \infty$ and taking into account (21) we obtain $0 \ge \varepsilon_0$. The obtained contradiction completes the proof of Theorem. □

5.2. **Periodic difference equation.** All the results about differential equations, which are presented above, for difference equations hold too, because they were formulated and proved for general dynamical systems, both for dynamical systems with continuous time and those with discrete time. Below we will give some of them that we need to study periodical systems with impulse.

Consider a difference equation

$$u(k+1) = \Phi(k, u(k)) \quad (\Phi \in C(\mathbb{Z} \times E, E)). \tag{25}$$

Denote by $\varphi(\cdot, u, \Phi)$ the solution of equation (25) passing through the point u for $k = 0$. Suppose that Φ is p–periodic in $k \in \mathbb{Z}$, where $p \in \mathbb{Z}$. From general properties of solutions of the difference equation (25) follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times C(\mathbb{Z} \times E, E) \mapsto E$ possesses the following properties:

(i) $\varphi(0, u, \Phi) = u$ for all $(u, \Phi) \in E \times C(\mathbb{Z} \times E, E)$;
(ii) $\varphi(k + p, u, \Phi) = \varphi(k, \varphi(p, u, \Phi), \Phi)$ for all $k \in \mathbb{Z}_+$ and $u \in E$;
(iii) φ is continuous.

Definition 5.4. *By analogy with the case of differential equations, the difference equation (25) is said to be dissipative, if there is a positive number R_0 such that for all $R > 0$ there exist a positive number $L = L(R)$ such that*

$$|\varphi(k, v, \tilde{\Phi}| < R_0$$

for all $|v| \leq R$ and $k \geq L(R)$.

We define a mapping $P : E \to E$ in the following way: $P(v) := \varphi(p, v, \Phi)$ and denote by (E, P) the cascade generated by positive powers of P.

Theorem 5.5. *Suppose that the following conditions hold:*

(i) *E is a complex Banach space;*
(ii) *the function Φ is p-periodic in $tk \in \mathbb{Z}$;*
(iii) *$f(k, \cdot) \in Hol(E)$ for all $k \in \{0, \dots, p-1\}$;*
(iv) *equation (25) is dissipative.*

Then the following statements hold:

(i) *there exists a unique p-periodic solution $p(t)$ of equation (25);*
(ii) *the solution $p(t)$ of equation (25) is globally asymptotically stable.*

Proof. This statement can be proved by slight modification of the proof of Theorem 5.3. □

5.3. **Periodic equations with impulse.** Consider the following differential equation:

$$\dot{u} = f(t, u) \qquad (f \in C(\mathbb{R} \times E, E)).$$

Suppose that $f(t + \tau, u) = f(t, u)$ $(t \in \mathbb{R}, u \in E^n)$ and f is regular. Note, that

$$\varphi(t + \tau, u, f) = \varphi(t, \varphi(\tau, u, f), f)$$

for all $t \in \mathbb{R}_+$ and $u \in E$.

Let $\{t_k\} \subset \mathbb{R}$ $(t_0 = 0)$ and $\{s_k\} \subset E$ be such that for certain $p > 0$ $(p \in \mathbb{Z})$ $t_{k+p} - t_k = \tau$ and $s_{k+p} = s_k$ for all $k \in \mathbb{Z}$. It is well known [20] that these conditions are necessary and sufficient for τ–periodicity of the distribution $\sum\limits_{k=-\infty}^{+\infty} s_k\delta_{t_k}$.

Consider a nonlinear τ–periodic equation with impulse

$$\dot{u} = f(t,u) + \sum_{k=-\infty}^{+\infty} s_k\delta_{t_k} = \mathfrak{F}. \tag{26}$$

It is known (see, for example, [20]) that, under the conditions above, the equation (26) has a unique generalized solution (which is piecewise continuous) passing through the point u when $t = 0$ for every $u \in E$. This solution we denote by $\varphi(\cdot, u, \mathfrak{F})$. Thus, $\lim\limits_{t\downarrow 0} \varphi(t,u,\mathfrak{F}) = u$. In addition, on every segment $]t_k, t_{k+1}[$ the equation (26) coincides with (26). Therefore, the equality

$$\varphi(t,u,\mathfrak{F}) = \varphi(t - t_k, c_k, f_{t_k})$$

holds, where the sequence $\{c_k\} \subset E$ satisfies the difference equation

$$c_{k+1} = \Phi(k, c_k),$$

where $\Phi(k,u) = \varphi(t_{k+1} - t_k, u, f_{t_k}) + s_k$ for all $(k,u)\mathbb{Z} \times E$.

Definition 5.6. *The equation with impulse (26) is said to be dissipative if there exists $R_0 > 0$ such that for all $R > 0$ there exists a positive number $L = L(R)$ such that*

$$|\varphi(t,u,\mathfrak{F})| < R_0$$

for all $|u| \leq R$.

Lemma 5.7. *If equation with impulse (26) is dissipative, then difference equation (27) is also dissipative.*

Proof. Let (26) be dissipative and $R_0 > 0$, $R > 0$ and $L = L(R)$ be the positive number from the dissipativity of (26). If $u \in E$, then

$$\varphi(k,u,\Phi) = \varphi(t_k,u,\mathfrak{F})$$

and, hence,

$$|\varphi(k,u,\Phi)| < R_0$$

for all $|u| \leq R$ and $k \geq k_0(R)$, where $k_0(R) := \min\{k \in \mathbb{N} : \ t_k \geq L(R)\}$. Lemma is proved. ☐

Define a mapping $P : E \to E$ by the equality

$$P(u) := \varphi(\tau,u,\mathfrak{F}) = \varphi(t_p,u,\mathfrak{F}) = \varphi(p,u,\Phi). \tag{27}$$

From equality (27) and general properties of solutions of difference equations follows the continuity of the mapping P.

Corollary 5.8. *If equation with impulse (26) is dissipative, then the cascade (E,P), where P is defined by equality (27), is dissipative.*

Theorem 5.9. *Suppose that the following conditions hold:*

(i) *E is a complex Banach space;*

(ii) *the function* f *is* ω*-periodic in* $t \in \mathbb{R}$*;*
(iii) *the function* $f \in C(\mathbb{R} \times E, E)$ *is regular and* $f(t, \cdot) \in Hol(E)$ *for all* $t \in [0, \omega)$*;*
(iv) *equation (26) is dissipative.*

Then the following statements hold:

(i) *there exists a unique* ω *periodic solution* $p(t)$ *of equation (26);*
(ii) *the solution* $p(t)$ *of equation (26) is globally asymptotically stable.*

Proof. Denote by $P : E \mapsto E$ the Poincare mapping for equation (26), i.e., $P(x) := \varphi(\omega, x, \mathfrak{F})$. Since equation (26) is dissipative, then by Corollary 5.8 the discrete dynamical system (E, P) generated by powers of the map P is also dissipative. By Theorem 4.5 there exists a unique fixed point $x_0 \in E$ of the map P such that

$$\lim_{n\to\infty} |P^n(x) - x_0| = 0$$

for all $x \in E$. Additionally, equality (23) takes place uniformly with respect to x on every bounded subset of E. By Theorem 4.5 (see also Lemma 4.6) the fixed point x_0 of the mapping P is stable. To finish the proof of Theorem it is sufficient to note that $p(t) := \varphi(t, x_0, \mathfrak{F})$ is a unique ω-periodic solution of equation (26) and its stability results from the stability of x_0 with respect to discrete dynamical system (E, P).

In fact, if we suppose that it is not so, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$ as $n \to \infty$, $|x_n| < \delta$ and $t_n \to +\infty$ ($t_n \in \mathbb{R}$) such that

$$|\varphi(t_n, x_n, \mathfrak{F}) - p(t_n)| \ge \varepsilon_0 \tag{28}$$

for all $n \in \mathbb{N}$. Note that $t_n = k_n\omega + \tau_n$, where $k_n \in \mathbb{N}$ and $\tau_n \in [0, \omega)$. Since $0 = t_0 < t_1 \ldots < t_{p-1} < t_p = \omega$, then there exists a number $m_n \in \mathbb{N}$ such that $\tau_n \in [t_{m_n}, t_{m_n+1})$ and, consequently,

$$\varphi(\tau_n, u, \mathfrak{F}) = \varphi(\tau_n - t_{m_n}, P^{m_n}(u), f_{t_{m_n}}) \tag{29}$$

for all $n \in \mathbb{N}$ and $u \in E$. From (28) and (29) we obtain

$$\varepsilon_0 \le |\varphi(\tau_n - t_{m_n}, P^{m_n}(u), f_{t_{m_n}}) - \varphi(\tau_n - t_{m_n}, x_0, f_{t_{m_n}})| \tag{30}$$

for all $m \in \mathbb{N}$. Since $m_n \in \{0, 1, \ldots, p-1\}$ for all $n \in \mathbb{N}$, then without loss of generality we can suppose that the sequence $\{t_{m_n}\}$ converges to $t_{\bar{m}}$, where $\bar{m} \in \{0, 1, \ldots, p-1\}$. Analogically, we can suppose that the sequence $\{t_n\}$ is convergent. Denote its limit by τ_0, then passing into limit in (30) as $n \to \infty$ and taking into account (28) we will obtain $\varepsilon_0 \le 0$. The obtained contradiction completes the proof of Theorem. □

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