THE STRUCTURE OF GLOBAL ATTRACTORS FOR NON-AUTONOMOUS PERTURBATIONS OF DISCRETE GRADIENT-LIKE DYNAMICAL SYSTEMS

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Abstract. In this paper we give the complete description of the structure of compact global (forward) attractors for non-autonomous perturbations of discrete autonomous gradient-like dynamical systems under the assumption that the original discrete autonomous system has a finite number of hyperbolic stationary solutions. We prove that the perturbed non-autonomous (in particular τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) system has exactly the same number of invariant sections (in particular the perturbed systems has the same number of τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) solutions). It is shown the compact global (forward) attractor of non-autonomous perturbed system coincides with the union of unstable manifolds of this finite number of invariant sections.

1. Introduction

Denote by $\mathbb{R}$ the set of all real numbers and $\mathbb{R}_+ := \{t \in \mathbb{R} | \ t \geq 0\}$, $(\mathfrak{B}, |\cdot|)$ a Banach space with the norm $|\cdot|$ and $C(\mathfrak{B}, \mathfrak{B})$ the set of all continuous mappings $f : \mathfrak{B} \mapsto \mathfrak{B}$. Let $A : D(A) \mapsto \mathfrak{B}$ be the generator of C_0-semigroup of bounded linear operators $\{U(t) : \ t \in \mathbb{R}_+\}$ and $f_0 \in C(\mathfrak{B}, \mathfrak{B})$ a differentiable function that is Lipschitz continuous in bounded subsets of $\mathfrak{B}$.

We consider a semi-linear equation

$$x' = Ax + f_0(x) \quad (x \in \mathfrak{B}). \tag{1}$$

Suppose that the dynamical system $(\mathfrak{B}, \mathbb{R}_+, \pi)$, generated by equation (1),is gradient and admits a compact global attractor J_0. It is well known (see, for example, [2] and [3, Ch.V]) that if the equation

$$Ax + f_0(x) = 0$$

has a finite number of hyperbolic solutions

$$p_1, p_2, \ldots, p_k \tag{2}$$

then

$$J_0 = \bigcup_{i=1}^{k} W^u(p_i),$$

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where $W^u(p_i)$ is the unstable manifold of p_i. Along with equation (1) we consider its non-autonomous perturbation

$$u' = Au + f_0(u) + F(t,u,\lambda). \quad (\lambda \in \Lambda := [-\lambda_0, \lambda_0] \text{ and } \lambda_0 > 0) \tag{3}$$

Suppose that the following conditions are fulfilled:

(H.1) in the series of stationary points (2) does not exist any l-cycle ($1 \le l \le k$);
(H.2) for each $r > 0$

$$\lim_{|\lambda|\to 0} \sup_{t\in\mathbb{R}} \sup_{|x|\le r} |F(t,u,\lambda)| = 0;$$

(H.3) for each $\lambda \in \Lambda$ equation (3) has a compact global pullback attractor (see, for example, [6, Ch.I]) $I_\lambda(t)$ ($t \in \mathbb{R}$);
(H.4) there exists a compact subset $K \subset \mathfrak{B}$ such that, for all bounded subset $B \subset \mathfrak{B}$

$$\lim_{t\to+\infty} \sup_{\tau\in\mathbb{R}} \sup_{0\le\lambda\le\lambda_0} dist(U_\lambda(t,\tau)B, K) = 0,$$

where $U_\lambda(t,\tau)$ is the Cauchy operator of equation (3).

In the work [6] (see also [7, Ch.V] and [27]) it was proved that under conditions (H.1)-(H.4) the following statements hold:

(i) there are $r_0 > 0$ and $0 < \varepsilon_0 \le \lambda_0$ such that for each $i \in \{1, 2, \dots, k\}$ and $0 < \lambda < \varepsilon_0$ equation (3) admits a unique bounded on $\mathbb{R}$ solution ϕ^i_λ with the values from $B[0, r_0] := \{x \in \mathfrak{B} : \ |x| \le r_0\}$;
(ii) the pullback attractor of equation (3) coincides with the union of unstable manifolds for the solutions ϕ^i_λ ($i = 1, 2, \dots, k$);
(iii) for each $x \in \mathfrak{B}$ there exists a unique solution ϕ^i_λ of equation (3) such that

$$\lim_{t\to+\infty} |U_\lambda(t,\tau)x - \phi^i_\lambda(t)| = 0.$$

The aim of this paper is generalization and refinement of the above results from [6] for difference equations.

Denote by $\mathbb{Z}$ the set of all entire numbers and $\mathbb{Z}_+ := \{t \in \mathbb{Z} |\ t \ge 0\}$. Let $f \in C(\mathfrak{B}, \mathfrak{B})$ be a continuously differentiable function that is Lipschitz continuous in bounded subsets of $\mathfrak{B}$.

We consider a difference equation

$$x(n+1) = f(x(n)) \quad (x \in \mathfrak{B}). \tag{4}$$

Suppose that the dynamical system $(\mathfrak{B}, \mathbb{Z}_+, \pi)$, generated by equation (4), admits a compact global attractor J_0. It is well known (see, for example, [2] and [3, Ch.V]) that if the equation

$$f(x) = x$$

has a finite number of hyperbolic solutions $p_1, p_2, \dots, p_k$ then

$$J_0 = \bigcup_{i=1}^{k} W^u(p_i),$$

where $W^u(p_i)$ is the unstable manifold of p_i. Along with equation (4) we consider the non-autonomous perturbation of equation (4) in the forme

$$u'(n+1) = f(u(n)) + F(\sigma(n,y), u(n), \lambda) \quad (y \in Y), \tag{5}$$

where $(Y, \mathbb{Z}, \sigma)$ is a dynamical system on the metric space Y and λ is a small parameter (for example, $\lambda \in \Lambda := [-\lambda_0, \lambda_0]$ and λ_0 is a fixed positive number).

In this paper we give the complete description of the structure of compact global (forward) attractors for non-autonomous perturbations (5) of autonomous gradient-like dynamical systems (4) under the assumption that autonomous system (4) has a finite number of hyperbolic stationary solutions. We establish that non-autonomous system (5) (in particular τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) has exactly the same number of invariant sections (in particular the perturbed systems has the same number of τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff respectively) solutions. We prove that the compact global (forward) attractor of non-autonomous perturbed system (5) coincides with the union of unstable manifolds of this finite number of invariant sections.

Remark 1.1. *Note that every compact global (forward) attractor for non-autonomous system (5) is also a global pullback attractor (see* [10, Ch.II]*). The converse statement, generally speaking, is not true (see* [10, Ch.VIII]*).*

This paper is organized as follow.

In Section 2 we give some notions and facts from the theory of linear dynamical systems (exponential dichotomy, Green function, non-homogeneous (affine) dynamical systems, invariant sections) which we use in our paper. In this Section we also study the uniform compatible solutions in the sense of B. A. Shcherbakov (in particular, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) for linear non-homogeneous dynamical systems.

Section 3 is dedicated to the study the problem of existence at least one continuous and invariant section for semi-linear equations and its continuously dependence on parameter (Theorems 3.3 and 3.10).

We prove in Section 4 the main result (Theorem 4.16) of the paper. This statement contains the complete description of the structure of compact global (forward) attractor for non-autonomous perturbations of autonomous gradient-like dynamical systems under the assumption that the non-perturbed autonomous system has a finite number of hyperbolic stationary solutions.

2. Linear non-autonomous dynamical systems

2.1. Linear non-autonomous dynamical systems with exponential dichotomy.

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ (or shortly φ) be a linear cocycle [22, 24] over dynamical system $(Y, \mathbb{Z}, \sigma)$ with the fiber $\mathfrak{B}$, i.e., φ is a continuous mapping from $\mathbb{Z}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ satisfying the following conditions:

(i) $\varphi(0, u, y) = u$ for all $u \in \mathfrak{B}$ and $y \in Y$;
(ii) $\varphi(n+m, y) = \varphi(n, \varphi(m, u, y), \sigma(m, y))$ for all $n, m \in \mathbb{Z}_+$, $u \in \mathfrak{B}$ and $y \in Y$;

(iii) for all $(n,y)\in\mathbb{Z}_+\times Y$ the mapping $\varphi(n,\cdot,y):\mathfrak{B}\mapsto\mathfrak{B}$ is linear.

Denote by $[\mathfrak{B}]$ the Banach space of all linear bounded operators A acting on the space $\mathfrak{B}$ equipped with the operator norm $||A||:=\sup\limits_{|x|\le 1}|Ax|$.

Remark 2.1. *1. Let $\langle\mathfrak{B},\varphi,(Y,\mathbb{Z},\sigma)\rangle$ be a cocycle over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$ and U be the mapping from $\mathbb{Z}_+\times Y$ into $[\mathfrak{B}]$ defined by the equality*

$$U(n,y):=\varphi(n,\cdot,y), \tag{6}$$

then it possesses the following properties:

a. *$U(0,y)=Id_{\mathfrak{B}}$ for all $y\in Y$;*
b. *$U(n+m,y)=U(n,\sigma(m,y))U(m,y)$ for all $n,m\in\mathbb{Z}_+$;*
c. *the mapping $\varphi:\mathbb{Z}_+\times\mathfrak{B}\times Y\mapsto\mathfrak{B}$ defined by equality (6) is continuous.*

2. It is easy to show that if the space $\mathfrak{B}$ is finite-dimensional, then the mapping $U:\mathbb{Z}_+\times Y\mapsto[\mathfrak{B}]$ $((n,y)\to U(n,y))$ is continuous.

3. Let U be a mapping from $\mathbb{Z}_+\times Y$ into $[\mathfrak{B}]$ with properties a.-c., then $\langle\mathfrak{B},\varphi,(Y,\mathbb{Z},\sigma)\rangle$ is a cocycle over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$, where φ is a mapping from $\mathbb{Z}_+\times\mathfrak{B}\times Y$ into $\mathfrak{B}$ defined by equality (6).

Example 2.2. *Let Y be a complete metric space, $(Y,\mathbb{Z},\sigma)$ be a dynamical system on Y. Consider the following linear difference equation*

$$x(n+1)=A(\sigma(n,y))x(n),\quad (y\in Y) \tag{7}$$

where $A\in C(Y,[\mathfrak{B}])$. From the general properties the linear difference equations we have the following conditions:

a. *for any $u\in\mathfrak{B}$ and $y\in Y$ equation (7) has exactly one solution that is defined on $\mathbb{Z}_+$ and satisfies the condition $\varphi(0,u,y)=u$;*
b. *the mapping $\varphi:(n,u,y)\to\varphi(n,u,y)$ is continuous in the topology of $\mathbb{Z}_+\times\mathfrak{B}\times Y$.*

Under the above assumptions the equation (7) generates a linear cocycle $\langle\mathfrak{B},\varphi,(Y,\mathbb{Z},\sigma)\rangle$ over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$.

Example 2.3. *Consider the difference equation*

$$x(n+1)=A(n)x(n) \tag{8}$$

where $A\in C(\mathbb{Z},[\mathfrak{B}])$. Along this equation (8) consider its H-class, i.e., the following family of equations

$$x(n+1)=B(n)x(n), \tag{9}$$

where $B\in H(A):=\overline{\{A_m:\ m\in\mathbb{Z}\}}$, A_m is m-translation of A and by bar is denoted the closure in $C(\mathbb{Z},[\mathfrak{B}])$. Note that the following conditions are fulfilled for equation (8) and its H-class (9):

a. *for any $u\in\mathfrak{B}$ and $B\in H(A)$ equation (9) has exactly one solution $\varphi(n,u,B)$ with the condition $\varphi(0,u,B)=v$;*
b. *the mapping $\varphi:(n,u,B)\to\varphi(t,u,B)$ is continuous in the topology of $\mathbb{Z}_+\times\mathfrak{B}\times C(\mathbb{Z};[\mathfrak{B}])$.*

Denote by $(H(A), \mathbb{Z}, \sigma)$ *the shift dynamical system on* $H(A)$. *Under the above assumptions the equation (8) generates a linear cocycle* $\langle \mathfrak{B}, \varphi, (H(A), \mathbb{Z}, \sigma)\rangle$ *over dynamical system* $(H(A), \mathbb{Z}, \sigma)$ *with the fiber* $\mathfrak{B}$.

Note that equation (8) and its H*-class can be written in the form (7). In fact. We put* $Y := H(A)$ *and denote by* $\mathcal{A} \in C(Y, [\mathfrak{B}])$ *defined by equality* $\mathcal{A}(B) := B(0)$ *for all* $B \in H(A) = Y$, *then* $B(m) = \mathcal{A}(\sigma(B, m))$ $(\sigma(m, B) := B_m$, *where* $B_m(n) := B(n+m)$ *for all* $n \in \mathbb{Z}$). *Thus the equation (8) with its* H*-class can be rewrite as follow*

$$x(n+1) = \mathcal{A}(\sigma(n, B))x(n). \ (B \in H(A))$$

Definition 2.4. *Recall (see, for example,* [12, Ch.VI],[22]*) that the linear cocycle* $\langle \mathfrak{B}, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ *is hyperbolic (or equivalently, satisfies the condition of exponential dichotomy), if there exists a continuous projection valued function* $P : Y \to [\mathfrak{B}]$ *satisfying:*

(i) $P(\sigma(n, y))U(n, y) = U(n, y)P(y)$ *for all* $(n, y) \in \mathbb{Z}_+ \times Y$*:*
(ii) *for all* $(n, y) \in \mathbb{Z}_+ \times Y$ *the operator* $U_Q(n, y)$ *is invertible as an operator from* $ImQ(y)$ *to* $ImQ(\sigma(n, y))$, *where* $Q(y) := Id_{\mathfrak{B}} - P(y)$ *and* $U_Q(n, y) := U(n, y)Q(y)$;
(iii) *there exist constants* $0 < \nu < 1$ *and* $\mathcal{N} > 0$ *such that*

$$\|U_P(n, y)\| \le \mathcal{N}\nu^n \text{ and } \|U_Q(n, y)^{-1}\| \le \mathcal{N}\nu^n$$

for all $y \in Y$ *and* $n \in \mathbb{Z}_+$, *where* $U_P(n, y) := U(n, y)P(y)$ *and* $U(n, y) = \varphi(n, \cdot, y)$.

A Green's function $G(t, y)$ (see, for example, [12, Ch.VII],[16, Ch.VII]) for hyperbolic cocycle φ is defined by

$$G(n, y) := \begin{cases} U_P(n, y), \text{ if } n \ge 0 \text{ and } y \in Y \\ -U_Q(n, y), \text{ if } n < 0 \text{ and } y \in Y, \end{cases}$$

where

$$U_Q(n, y) := U_Q(-n, \sigma(n, y))Q(\sigma(n, y))$$

for all $n < 0$ and $y \in Y$.

Theorem 2.5. *Suppose that the following conditions are fulfilled:*

(i) *the linear cocycle* $\langle \mathfrak{B}, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ *over dynamical system* $(Y, \mathbb{Z}, \sigma)$ *with the fiber* $\mathfrak{B}$ *is hyperbolic;*
(ii) Y *be a compact metric space.*

Then the following statements hold:

(i) *there is positive constant* $\mathcal{N}$ *and* $0 < \nu < 1$ *such that* $||G(n, y)|| \le \mathcal{N}\nu^{|n|}$ *for all* $n \in \mathbb{Z}$;
(ii) *the mapping* $g : (\mathbb{Z} \setminus 0) \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$, *defined by equality* $g(n, u, y) := G(n, y)u$ *for all* $(n, u, y) \in (\mathbb{Z} \setminus 0) \times \mathfrak{B} \times Y$, *is continuous;*
(iii) *Green's operator* $\mathbb{G}$ *defined by*

$$(\mathbb{G}f)(y) := \sum_{m=-\infty}^{+\infty} G(-m, \sigma(-m, y))f(\sigma(-m, y)) \tag{10}$$

is a linear bounded operator on $C(Y,\mathfrak{B})$, where $C(Y,\mathfrak{B})$ is the Banach space of all continuous functions $f : Y \mapsto \mathfrak{B}$ equipped with the norm $||f|| := \max_{y\in Y} |f(y)|$;

(iv)
$$||\mathbb{G}f|| \le \mathcal{N}\frac{1+\nu}{1-\nu}||f||$$
for all $f \in C(Y,\mathfrak{B})$;

(v) *if $\gamma = \mathbb{G}(f)$, then*
$$\gamma(\sigma(n,y)) = U(n,y)\gamma(y) + \sum_{m=0}^{n} U(n-m,\sigma(m,y))f(\sigma(m,y)).$$

Proof. This statement can be proved using the same argument as in case of linear cocycles with the continuous times (for more details see [12, Ch.VII],[15, Ch.IV],[23, Ch.III]). □

2.2. **Linear non-homogeneous (affine) dynamical systems.** Let $\langle\mathfrak{B},\varphi, (Y, \mathbb{Z}\ \sigma)\rangle$ be a linear cocycle over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$, $f \in C(Y,\mathfrak{B})$ and ψ be a mapping from $\mathbb{Z}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ defined by equality

$$\psi(n,u,y) := U(n,y)u + \sum_{m=0}^{n} U(n-m,\sigma(m,y))f(\sigma(m,y)). \tag{11}$$

From the definition of the mapping ψ it follows that ψ possesses the following properties:

1. $\psi(0,u,y) = u$ for all $(u,y) \in \mathfrak{B}\times Y$;
2. $\psi(n+m,u,y) = \psi(n,\psi(m,u,y),\sigma(m,y))$ for all $n,m \in \mathbb{Z}_+$ and $(u,y) \in \mathfrak{B}\times Y$;
3. the mapping $\psi : \mathbb{Z}_+ \times \mathfrak{B}\times Y \mapsto \mathfrak{B}$ is continuous;
4. $\psi(n,\lambda u + \mu v,y) = \lambda\psi(n,u,y) + \mu\psi(n,v,y)$ for all $n \in \mathbb{Z}_+$, $u,v \in \mathfrak{B}$, $y \in Y$ and $\lambda,\mu \in \mathbb{Z}$ (or $\mathbb{C}$) with the condition $\lambda+\mu = 1$, i.e., the mapping $\psi(n,\cdot,y) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(n,y) \in \mathbb{Z}_+ \times Y$.

Definition 2.6. *A triplet $\langle\mathfrak{B},\psi,(Y,\mathbb{R},Y)\rangle$is called an affine (a linear non-homogeneous) cocycle over dynamical system $(Y,\mathbb{Z},Y)$ with the fiber $\mathfrak{B}$, if ψ is a mapping from $\mathbb{Z}_+ \times \mathfrak{B}\times Y$ into $\mathfrak{B}$ possessing the properties 1.-4.*

Remark 2.7. *If we have a linear cocycle $\langle\mathfrak{B},\varphi,(Y,\mathbb{Z},\sigma)\rangle$ over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$, $U(n,y) := \varphi(n,\cdot,y)$ for all $(n,y) \in \mathbb{Z}_+ \times Y$ and $f \in C(Y,\mathfrak{B})$, then by equality (11) is defined an affine cocycle $\langle\mathfrak{B},\psi,(Y,\mathbb{Z},\sigma)\rangle$ over dynamical system $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathfrak{B}$ which is called an affine (a linear non-homogeneous) cocycle associated by linear cocycle φ and the function $f \in C(Y,\mathfrak{B})$.*

Example 2.8. *Let Y be a complete metric space, $(Y,\mathbb{Z},\sigma)$ be a dynamical system and $f \in C(Y,\mathfrak{B})$. Consider the following linear non-homogeneous difference equation*

$$x(n+1) = A(\sigma(n,y))x(n) + f(\sigma(n,y)), \quad (y \in Y) \tag{12}$$

where $A \in C(Y,[\mathfrak{B}])$.

Under the above assumptions equation (7) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{Z}, \sigma)$ with the fiber $\mathfrak{B}$. According to Remark 2.7 by equality (11) is defined a linear non-homogeneous cocycle $\langle B, \psi, (Y, \mathbb{Z}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{Z}, \sigma)$ with the fiber $\mathfrak{B}$. Thus every non-homogeneous linear difference equations (12) generates a linear non-homogeneous cocycle ψ.

2.3. **Invariant sections of linear non-homogeneous difference equation.** Consider a linear non-homogeneous difference equation

$$x(n+) = A(\sigma(n, y))x(n) + f(\sigma(n, y)), \quad (y \in Y) \tag{13}$$

where $A \in C(Y, [\mathfrak{B}])$. Denote by $\psi(n, u, y)$ a unique solution of equation (13) defined on $\mathbb{Z}_+$ with initial data $\psi(0, u, y) = u$.

Definition 2.9. *A function $\nu \in C(Y, \mathfrak{B})$ is said to be an invariant section of equation (13) if the function $\varphi(n) := \nu(\sigma(n, y))$ (for all $n \in \mathbb{Z}_+$) is a solution of equation (13), i.e., $\nu(\sigma(n, y)) = \psi(n, \nu(y), y)$ for all $(n, y) \in \mathbb{Z}_+ \times Y$ or equivalently*

$$\nu(\sigma(n, y)) = U(n, y)\nu(y) + \sum_{m=0}^{n} U(n-m, \sigma(m, y))f(\sigma(m, y))$$

for all $(n, y) \in \mathbb{Z}_+ \times Y$.

Let $\langle \mathfrak{B}, \psi, (Y, \mathbb{Z}, \sigma)\rangle$ be a cocycle over dynamical system $(Y, \mathbb{Z}, \sigma)$ with the fiber $\mathfrak{B}$. Denote by $(X, \mathbb{Z}_+, \pi)$ the skew-product dynamical system, generated by cocycle ψ and dynamical system $(Y, \mathbb{Z}, \sigma)$. If ψ is a cocycle generated by equation (13), then we will say that the corresponding skew-product dynamical system is associated by equation (13).

Remark 2.10. *A function $\nu \in C(Y, \mathfrak{B})$ is an invariant section for equation (13) if and only if the set $M := \{(\nu(y), y) : \ y \in Y\}$ is a positively invariant subset of skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ associated by equation (13).*

2.4. **Uniform compatible solutions in the sense of B. A. Shcherbakov.** Let $(Y, \mathbb{Z}, \sigma)$ be a dynamical system on the metric space Y. Consider the difference equation

$$x(n+1) = A(\sigma(n, y))x(n) + f(\sigma(n, y)) \quad (y \in Y), \tag{14}$$

where $A \in C(\mathbb{Z}, [\mathfrak{B}])$ and $f \in C(Y, \mathfrak{B})$.

Note that

(i) equation
$$x(n+1) = A(\sigma(n, y))x(n) \tag{15}$$
admits a unique solution $\varphi(n, u, y)$ with the initial data $\varphi(0, u, y) = u$ for all $(u, y) \in \mathfrak{B} \times Y$ and defined on $\mathbb{Z}_+$;

(ii) the mapping $\varphi : \mathbb{Z}_+ \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$ is continuous.

Remark 2.11. *The following statements hold:*

(i) *for all $(u, y) \in \mathfrak{B} \times Y$ equation (14) admits a unique solution $\psi(t, u, y)$ with the initial data $\psi(0, u, y) = u$;*

(ii)

$$\psi(n,u,y) = U(n,y)u + \sum_{m=0}^{n} U(n-m,\sigma(m,y))f(\sigma(m,y));$$

(iii) *the triplet* $\langle \mathfrak{B}, \psi, (Y,\mathbb{Z},\sigma)\rangle$ *is a (affine) cocycle over* $(Y,\mathbb{Z},\sigma)$ *with the fiber* $\mathfrak{B}$.

Let $\langle \mathfrak{B}, \psi, (Y,\mathbb{Z},\sigma)\rangle$ be a cocycle generated by equation (14) and $(X,\mathbb{Z}_+,\sigma)$ be the skew-product dynamical system associated by cocycle ψ, i.e., $X := \mathfrak{B}\times Y$ and $\pi := (\psi,\sigma)$.

Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{Z}_+$ and $(X,\mathbb{T},\pi)$ be a dynamical system.

If $x \in X$, then by $\mathfrak{N}_x$ (respectively, $\mathfrak{M}_x$) we denote the set of all sequences $\{n_l\} \subset \mathbb{T}$ such that $\pi(n_l,x)$ converges to x (respectively, $\{\pi(n_l,x)\}$ converges in X).

Definition 2.12. *A point* $x \in X$ *is called* [25, 26] *comparable (respectively, uniformly comparable) by the character of recurrence with* $y \in Y$ *if* $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ *(respectively,* $\mathfrak{M}_y \subseteq \mathfrak{M}_x$*).*

For $x \in X$ we denote by $\Sigma_x := \{\pi(n,x):\ n \in \mathbb{Z}\}$ and $H(x) := \overline{\Sigma}_x$.

Definition 2.13. *The point* $x \in X$ *is said to be stable in the sense of Lagrange if the set* Σ_x *is relatively compact.*

Theorem 2.14. [25, 26] *The following statements hold:*

(i) *the point* $x \in X$ *is comparable by character of recurrence with* $y \in Y$ *if and only if there exists a continuous mapping* $h : \Sigma_y \mapsto \Sigma_x$ *such that* $h(\pi(n,y)) = \pi(n,x)$ *for all* $n \in \mathbb{T}$;

(ii) *if the point* $y \in Y$ *is stable in the sense of Lagrange, then the* $x \in X$ *is uniformly comparable by character of recurrence with* $y \in Y$ *if and only if there exists a continuous mapping* $h : H(y) \mapsto H(x)$ *such that* $h(y) = x$ *and* $h(\pi(n,q)) = \pi(n,h(q))$ *for all* $n \in \mathbb{T}$ *and* $q \in H(y)$.

Definition 2.15. *A number* $\tau \in \mathbb{T}$ *is called an* $\varepsilon > 0$ *shift of* x *(respectively, almost period of* x*), if* $\rho(x\tau, x) < \varepsilon$ *(respectively,* $\rho(x(\tau+t), xt) < \varepsilon$ *for all* $t \in \mathbb{T}$*).*

Definition 2.16. *A point* $x \in X$ *is called*

- *almost recurrent (respectively, Bohr almost periodic), if for any* $\varepsilon > 0$ *there exists a positive number* l *such that at any segment of length* l *there is an* ε *shift (respectively, almost period) of point* $x \in X$;
- *recurrent, if it almost recurrent and the set* $H(x) := \overline{\{xt \mid t \in \mathbb{T}\}}$ *is compact;*
- *Levitan almost periodic* [19], *if there exists a dynamical system* $(Y,\mathbb{T},\sigma)$ *and a Bohr almost periodic point* $y \in Y$ *such that* $\mathfrak{N}_y \subseteq \mathfrak{N}_x$;
- *almost automorphic in the dynamical system* $(X,\mathbb{T},\pi)$, *if the following conditions hold:*
  (a) x *is st.L;*
  (b) *there exists a dynamical system* $(Y,\mathbb{T},\sigma)$, *a homomorphism* h *from* $(X,\mathbb{T},\pi)$ *onto* $(Y,\mathbb{T},\sigma)$ *and an almost periodic in the sense of Bohr point* $y \in Y$ *such that* $h^{-1}(y) = \{x\}$.

Remark 2.17. *1. Every almost automorphic point $x \in X$ is also Levitan almost periodic. The converse statement, generally speaking, is not true* [8].

2. A Levitan almost periodic point x is almost automorphic if and only if its trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is relatively compact [8].

Theorem 2.18. [25, 26] *The following statements hold:*

(i) *If the point $x \in X$ is comparable by the character of recurrence with $y \in Y$ and y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point x is also so;*
(ii) *if the point $x \in X$ is uniformly comparable by the character of recurrence with $y \in Y$ and y is stationary (respectively, τ-periodic, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then the point x is also so.*

Definition 2.19. *A solution $\psi(n,u,y)$ of equation (14) is said to be uniformly compatible* [25, 26] *in the sense of B. A. Shcherbakov, if the point $x := (u,y) \in X$ of skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is uniformly comparable by character of recurrence with the point $y \in Y$.*

Lemma 2.20. *Let $\langle \mathfrak{B}, \psi, (Y, \mathbb{Z}, \sigma)\rangle$ be a cocycle and $\nu : Y \mapsto \mathfrak{B}$ be a continuous mapping with the property $\nu(\sigma(n,y)) = \psi(n, \nu(y), y)$ for all $(n,y) \in \mathbb{Z}_+ \times Y$. Then the following statements hold:*

(i) *the continuous mapping $\gamma := (\nu, Id_Y)$ from $(Y, \mathbb{Z}, \sigma)$ into $(X, \mathbb{Z}_+, \pi)$ is an homomorphism, i.e., $\gamma(\sigma(n,y)) = \pi(n, \gamma(y))$ for all $(n,y) \in \mathbb{Z}_+ \times Y$;*
(ii) *the point $x = (u,y)$ is comparable by character of recurrence with the point y;*
(iii) *if the space Y is compact, then the point $x = (u,y)$ is uniformly comparable by character of recurrence with the point $y \in Y$.*

Proof. The first statement of Lemma is evident. The second and third statements follow from Theorem 2.14. □

Corollary 2.21. *Suppose that the following conditions are fulfilled:*

(i) *$\langle \mathfrak{B}, \psi, (Y, \mathbb{Z}, \sigma)\rangle$ is a cocycle;*
(ii) *$\nu : Y \mapsto \mathfrak{B}$ is a continuous mapping with the property $\nu(\sigma(n,y)) = \psi(n, \nu(y), y)$ for all $(n,y) \in \mathbb{Z}_+ \times Y$.*

Then the following statements take hold:

(i) *if the point $y \in Y$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point $x = \gamma(y) = (\nu(y), y)$ is so;*
(ii) *if the space Y is compact and the point y is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, uniformly Poisson stable), then the point $x := \gamma(y) = (\nu(y), y)$ is so.*

Proof. The first (respectively, second) statement of Corollary it follows from Lemma 2.20 and Theorem 2.18. □

Theorem 2.22. *Suppose that the following conditions are fulfilled:*

(i) *the linear difference equation (7) is hyperbolic;*
(ii) *Y is a compact metric space.*

Then the following statements hold:

(i) *if $G(n,y)$ is the Green function of equation (7), then there are positive constants M and $0<\nu<1$ such that $||G(n,y)|| \le M\nu^{|n|}$ for all $n \in \mathbb{Z}$;*
(ii) *the mapping $g : \mathbb{Z}_* \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$, defined by equality $g(n,u,y) := G(n,y)u$ for all $(n,u,y) \in \mathbb{Z}_* \times \mathfrak{B} \times Y$, is continuous, where $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$;*
(iii) *Green's operator $\mathbb{G}$ defined by*
$$(\mathbb{G}f)(y) := \sum_{m=-\infty}^{+\infty} G(-m,\sigma(-m,y))f(\sigma(-m,y))$$
is a bounded operator acting on the Banach space $C(Y,\mathfrak{B})$;
(iv)
$$||\mathbb{G}f|| \le \mathcal{N}\frac{1+\nu}{1-\nu}||f||$$
for all $f \in C(Y,\mathfrak{B})$;
(v) *if $\nu = \mathbb{G}(f)$, then*
$$\nu(\sigma(n,y)) = U(n,y)\nu(y) + \sum_{m=0}^{n} U(n-m,\sigma(m,y))f(\sigma(m,y)),$$
i.e., ν is an invariant manifold (invariant section) of equation (13);
(vi) *the solution $\nu(\sigma(n,y))$ of equation (14) is uniformly compatible;*
(vii) *if the point $y \in Y$ is τ-periodic (respectively, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent), then the solution $\nu(\sigma(n,y))$ of equation (14) is also τ-periodic (respectively, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent).*

Proof. This statement follows from Theorem 2.5 and Corollary 2.21. □

3. Semi-linear difference equations

Definition 3.1. *A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be local Lipshchitzian with respect to variable $u \in \mathfrak{B}$ on $W \in \mathfrak{B}$ uniformly with respect to $y \in Y$, if there exists a nondecreasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that*
$$|F(y,u_1) - F(y,u_2)| \le L(r)|u_1 - u_2|$$
for all $u_1, u_2 \in B[0,r] \bigcap W$ and $y \in Y$, where $B[0,r] := \{u \in \mathfrak{B} : \ |u| \le r\}$.

Definition 3.2. *The smallest constant figuring in (16) is called Lipshchitz constant of function F on $Y \times B[0,r]$ (notation $Lip(r,F)$).*

Consider a difference equation
$$u(n+1) = A(\sigma(t,y))u(n) + F(\sigma(n,y),u) \quad (y \in Y) \tag{16}$$

in the Banach space $\mathfrak{B}$, where F is a nonlinear continuous mapping ("small" perturbation) acting from $Y \times \mathfrak{B}$ into $\mathfrak{B}$.

Denote by $CL_r(Y \times B[0,r], \mathfrak{B})$ the space of all locally Lipshchitzian functions $F \in C(Y \times B[0,r], \mathfrak{B})$ equipped with the distance

$$d_{CL_r}(F_1, F_2) := \max_{y \in Y} |F_1(y,0) - F_2(y,0)| + Lip(r, F_1 - F_2). \tag{17}$$

It is not difficult to check that the metric d, defined by (17), is complete.

Theorem 3.3. *Suppose that the following conditions are fulfilled:*

(i) *linear equation (15) is hyperbolic;*
(ii) *the function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is locally Lipshchitzian;*
(iii) *$Lip(r_0, F) \le \varepsilon_0$, where $\mathcal{N}$, $0 < \nu < 1$ are the positive constants from Theorem 2.5, $A := \max_{y \in Y} |F(y,0)|$, $0 < \varepsilon_0 < \frac{1-\nu}{(1+\nu)\mathcal{N}}$ and $r_0 := \mathcal{N}\frac{1+\nu}{1-\nu}A(1 - \varepsilon_0 \mathcal{N}\frac{1+\nu}{1-\nu})^{-1}$.*

Then

(i) *equation (16) admits a unique invariant section $\nu \in C(Y, B[0, r_0])$;*
(ii) *the mapping $F \to \nu_F$ from $CL_{r_0}(Y \times B[0, r_0], \mathfrak{B})$ into $C(Y, \mathfrak{B})$, where ν_F is a unique invariant section of (16), is continuous.*

Proof. Let equation (15) be hyperbolic and $G(n,y)$ be its Green's function, then by Theorem 2.5 there exist positive constants M and $0 < \nu < 1$ such that $||G(n,y)|| \le M\nu^{|n|}$ for all $y \in Y$ and $n \in \mathbb{Z}$. Denote by $C(Y, \mathfrak{B})$ the Banach space of all continuous mappings $\psi : Y \mapsto \mathfrak{B}$ equipped with the norm $||\psi|| := \max_{y \in Y} |\psi(y)|$. Consider the operator Φ defined on the space $C(Y, \mathfrak{B})$ by equality

$$(\Phi\psi)(y) = \sum_{m=-\infty}^{+\infty} G(-m, \sigma(-m,y)) F(\sigma(-m,y), \psi(\sigma(-m,y))). \tag{18}$$

Note that the operator Φ maps $C(Y, \mathfrak{B})$ into itself. In fact, since $F(\cdot, \psi(\cdot)) \in C(Y, \mathfrak{B})$ for all $\psi \in C(Y, \mathfrak{B})$, then by Theorem 2.5 the function $\Phi\psi$ defined by (18) belongs to $C(Y, \mathfrak{B})$. Note that $\Phi\psi = \mathbb{G}F(\cdot, \psi(\cdot))$, where $\mathbb{G}$ is the Green operator associate by $G(n,y)$ (see formula (10)). We will show that $\Phi(C(Y, B[0, r_0]) \subseteq C(Y, B[0, r_0])$. In fact. Let $\psi \in C(Y, B[0, r_0])$, then we have

$$|F(y, \psi(y))| \le A + Lip(r_0, F) r_0 \le A + \varepsilon_0 r_0$$

for all $y \in Y$ and according to choice of the number r_0 we obtain

$$||\Phi\psi|| \le \mathcal{N}\frac{1+\nu}{1-\nu} ||F(\cdot, \psi(\cdot))|| \le \mathcal{N}\frac{1+\nu}{1-\nu}(A + \varepsilon_0 r_0) \le r_0.$$

Now we will prove that $\Phi : C(Y, B[0, r_0]) \mapsto C(Y, [0, r_0])$ is a contraction. Indeed, we have

$$|(\Phi\psi_1)(y) - (\Phi\psi_1)(y)| \le \mathcal{N}\frac{1+\nu}{1-\nu} ||F(\cdot, \psi_1(\cdot)) - F(\cdot, \psi_2(\cdot))||. \tag{19}$$

On the other hand

$$|F(y,\psi_1(y)) - F(y,\psi_2(y))| \leq$$
$$Lip(r_0,F)|\psi_1(y)-\psi_2(y)| \leq \varepsilon_0||\psi_1-\psi_2||$$

for all $y \in Y$ and, consequently, we obtain

$$||F(\cdot,\psi_1(\cdot)) - F(\cdot,\psi_2(\cdot))|| \leq \varepsilon_0||\psi_1-\psi_2||. \quad (20)$$

From (19) and (20) it follows

$$||\Phi\psi_1 - \Phi\psi_2|| \leq \mathcal{N}\frac{1+\nu}{1-\nu}\varepsilon_0||\psi_1-\psi_2||.$$

Thus $\Phi : C(Y,B[0,r_0]) \mapsto C(Y,B[0,r_0])$ is a contraction, and consequently, Φ has a unique fixed point $\nu \in C(Y,B[0,r_0])$: $\Phi\nu = \nu$. It is easy to see that ν is a unique invariant section of perturbed equation (16) from $C(Y,B[0,r_0])$.

Now we will prove the second statement of Theorem. Let $F_0, F_l \in CL(Y \times B[0,r_0],\mathfrak{B})$ with the condition $Lip(r_0,F_0), Lip(r_0,F_l) \leq \varepsilon_0$ (for all $l \in \mathbb{N}$) such that $F_l \to F_0$ as $l \to \infty$. Let $\nu_{F_0} \in C(Y,B[0,r_0])$ (respectively, $\nu_{F_l} \in C(Y,B[0,r_0])$) be the unique invariant section of equation

$$u(n+1) = A(\sigma(t,y))u(n) + F_0(\sigma(n,y),u) \quad (21)$$

(respectively, of equation

$$u(n+1) = A(\sigma(n,y))u(n) + F_l(\sigma(n,y),u)), \quad (22)$$

then $\nu_{F_l} - \nu_{F_0}$ is an invariant section of equation

$$u(n+1) = A(\sigma(n,y))u(n) +$$
$$F_0(\sigma(n,y),\nu_{F_0}(\sigma(n,y))) - F_l(\sigma(n,y),\nu_{F_l}(\sigma(t,y))). \quad (23)$$

Since equation (22) has a unique invariant section, then we obtain

$$\nu_{F_0}(y) - \nu_{F_l}(y) = \mathbb{G}[F_0(\cdot,\nu_{F_0}(\cdot)) - F_l(\cdot,\nu_{F_l}(\cdot))], \quad (24)$$

where $\mathbb{G}$ is a Green operator associate by $G(n,y)$. Note that

$$|F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| \leq |F_0(y,0) - F_l(y,0)| +$$
$$|(F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))) - (F_0(y,0) - F_l(y,0))| \leq$$
$$\max_{y\in Y}|F_0(y,0) - F_l(y,0)| + Lip(r_0,F_l-F_0)||\nu_{F_0}|| \leq$$
$$(1+||\nu_{F_0}||)d_{CL_{r_0}}(F_l,F_0) \quad (25)$$

and

$$|F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_l}(y))| \leq |F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| +$$
$$|F_l(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_l}(y))| \leq |F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| +$$
$$Lip(r_0,F_l)|\nu_{F_0}(y)) - \nu_{F_l}(y))| \quad (26)$$

for all $y \in Y$. From (24)–(26) we obtain

$$|\nu_{F_0}(y) - \nu_{F_l}(y)| \leq (1+||\nu_{F_0}||)||F_0 - F_l||_{CL} +$$
$$Lip(F_l)|\nu_{F_0}(y) - \nu_{F_l}(y)| \leq (1+||\nu_{F_0}||)d_{CL_{r_0}}(F_l,F_0) +$$
$$\varepsilon_0|\nu_{F_0}(y) - \nu_{F_l}(y)|$$

and, consequently,

$$|\nu_{F_0}(y) - \nu_{F_l}(y)| \leq (1-\varepsilon_0)^{-1}(1+||\nu_{F_0}||)d_{CL_{r_0}}(F_l,F_0),$$

i.e.,

$$||\nu_{F_0} - \nu_{F_l}|| \leq (1-\varepsilon_0)^{-1}(1+||\nu_{F_0}||)d_{CL_{r_0}}(F_l, F_0)$$

for all $n \in \mathbb{N}$. Passing into limit in the last inequality as $l \to \infty$ we obtain $\nu_{F_l} \to \nu_{F_0}$. Theorem is proved. □

Remark 3.4. *Under the conditions of Theorem 3.3 the following statements hold:*

(i) $\varphi(n, \nu_\lambda(y), y) = \nu_\lambda(\sigma(n,y))$ *(for all $n \in \mathbb{Z}$) is a bounded on $\mathbb{Z}$ solution of equation (16) with the values from $B[0, r_0]$.*
(ii) $\varphi(n, \nu_\lambda(y), y)$ *is a unique solution of equation (16) defined and bounded on $\mathbb{Z}$ with the values from $B[0, r_0]$.*

The first statement of Remark directly follows from Theorem 3.3. The second statement of Remark can be proved using the same ideas as in the proof of the first statement of Theorem 3.3.

Corollary 3.5. *Suppose that the following conditions are fulfilled:*

(i) *linear equation (15) is hyperbolic;*
(ii) *there exists a number $r_0 > 0$ such that the function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is locally Lipshchitzian on $B[0, r_0]$.*

Then there exists a positive number λ_0 so that:

(i) *for every $\lambda \in [-\lambda_0, \lambda_0]$ equation*

$$u(n+1) = A(\sigma(n,y))u(n) + f(\sigma(n,y)) + \lambda F(\sigma(n,y), u) \quad (y \in Y) \tag{27}$$

admits a unique invariant section $\gamma_\lambda \in C(Y, B[0, r_0])$;

(ii)

$$||\nu_\lambda - \nu_0|| \to 0 \tag{28}$$

as $\lambda \to 0$, where ν_0 is a unique invariant section of equation

$$u(n+1) = A(\sigma(t,y))u(n) + f(\sigma(t,y)) \quad (y \in Y).$$

Proof. This statement follows from Theorem 3.3. In fact. If we denote by $\mathcal{F}_\lambda(y,u) := f(y) + \lambda F(y,u)$ for all $(y,u) \in Y \times B[0,r_0]$, then $Lip(r_0, \mathcal{F}_\lambda) \leq |\lambda| Lip(r_0, F) \leq \varepsilon_0$ for all $|\lambda| \leq \lambda_0 \leq \varepsilon_0 / Lip(r_0, F)$ and $||\mathcal{F}_\lambda - f||_{CL_{r_0}} \to 0$ as $\lambda \to 0$ since $f = \mathcal{F}_0$. □

Remark 3.6. *Note that Corollary 3.5 assures existence and uniqueness of invariant section γ_λ of equation (27) for sufficiently small λ, but this equation can have on the space $\mathfrak{B}$ more than one invariant section. This fact we will confirm below by corresponding example.*

Example 3.7. *Let $\beta \geq \alpha > 0$ and $p \in C(Y, [\alpha, \beta])$. Consider difference equation*

$$x(n+1) = \frac{\sqrt{2}x(n)}{\sqrt{1 + 2\lambda p(\sigma(n,y))x^2(n)}}, \quad (y \in Y) \tag{29}$$

where $\lambda \in \mathbb{R}_+$. For $\lambda = 0$ equation (29) admits a unique invariant section $\nu_0(y) = 0$ for all $y \in Y$. If $\lambda > 0$, then equation (29) admits three invariant sections: $\nu_\lambda^1(y) = 0$, $\nu_\lambda^2(y) = q_\lambda(y)$ and $\nu_\lambda^3(y) = -q_\lambda(y)$ for all $y \in Y$, where

$$q_\lambda(y) = \lambda^{-1/2}(\Sigma_{-\infty}^{-1} 2^k p(\sigma(k,y)))^{-1/2} \quad (y \in Y).$$

Note that $||\nu_\lambda^1|| \to 0$ *as* $\lambda \to 0$. *Since* $0 < \alpha \le p(y) \le \beta$ *for all* $y \in Y$, *then* $(\lambda\beta)^{-1/2} \le q_\lambda(y) \le (\lambda\alpha)^{-1/2}$ *for all* $y \in Y$ *and, consequently,* $||\nu_\lambda^2|| \to \infty$ *and* $||\nu_\lambda^3|| \to \infty$ *as* λ *goes to* 0.

Definition 3.8. *A function* $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ *is said to be globally Lipshchitzian with respect to variable* $u \in \mathfrak{B}$ *uniformly with respect to* $y \in Y$ *if there exists a positive constant* L *such that*

$$|F(y,u_1) - F(y,u_2)| \le L|u_1 - u_2| \tag{30}$$

for all $u_1, u_2 \in \mathfrak{B}$ *and* $y \in Y$.

Definition 3.9. *The smallest constant* L *with the property (30) is called Lipshchitz constant of the function* F *(notation* $Lip(F)$*).*

Denote by $CL(Y \times \mathfrak{B}, \mathfrak{B})$ the Banach space of all globally Lipshchitzian functions $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ equipped with the norm

$$||F||_{CL} := \max_{y \in Y} |F(y,0)| + Lip(F).$$

Theorem 3.10. *Suppose that the following conditions are fulfilled:*

(i) *linear equation (15) is hyperbolic;*
(ii) *the function* $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ *is globally Lipshchitzian;*
(iii) *there exists a number* $\varepsilon_0 \in (0, \frac{1-\nu}{(1+\nu)\mathcal{N}})$ *such that* $Lip(F) \le \varepsilon_0$, *where* $\mathcal{N}$ *and* $0 < \nu < 1$ *are the positive constants from Theorem 2.5.*

Then

(i) *equation (16) admits a unique invariant section* $\nu \in C(Y, \mathfrak{B})$*;*
(ii) *the mapping* $F \to \nu_F$ *from* $CL(Y \times \mathfrak{B}, \mathfrak{B})$ *into* $C(Y, \mathfrak{B})$, *where* ν_F *is a unique invariant section of (16), is continuous.*

Proof. The first statement of Theorem 3.10 directly it follows from Theorem 3.3, because $Lip(r,F) \le Lip(F) \le \varepsilon_0$ for all $r > 0$ and, consequently equation (16) has a unique invariant section γ from $C(Y, B[0,r])$ for all $r \ge r_0 := \mathcal{N}\frac{1-\nu}{1+\nu}A(1 - \varepsilon_0\mathcal{N}\frac{1-\nu}{1+\nu})^{-1}$.

Now we will prove the second statement of Theorem. Let $F_0, F_l \in CL(Y \times \mathfrak{B}, \mathfrak{B})$ with the condition $Lip(F_0), Lip(F_l) \le \varepsilon_0$ (for all $n \in \mathbb{N}$) such that $F_l \to F_0$ as $n \to \infty$. Let ν_{F_0} (respectively, ν_{F_l}) be the unique invariant section of equation (21) (respectively, of equation (22)), then $\nu_{F_l} - \nu_{F_0}$ is an invariant section of equation (23) and, consequently,

$$\nu_{F_0}(y) - \nu_{F_l}(y) = \mathbb{G}[F_0(\cdot, \nu_{F_0}(\cdot)) - F_l(\cdot, \nu_{F_l}(\cdot))], \tag{31}$$

where $\mathbb{G}$ is a Green operator associate by $G(t,y)$. Note that

$$\begin{aligned} |F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| \le |F_0(y,0) - F_l(y,0)| + \\ |(F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))) - (F_0(y,0) - F_l(y,0))| \le \\ \max_{y \in Y} |F_0(y,0) - F_l(y,0)| + Lip(F_l - F_0)||\nu_{F_0}|| \le \\ (1 + ||\nu_{F_0}||)||F_l - F_0||_{CL} \end{aligned} \tag{32}$$

and

$$
\begin{aligned}
&|F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_l}(y))| \le |F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| + \\
&|F_l(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_l}(y))| \le |F_0(y,\nu_{F_0}(y)) - F_l(y,\nu_{F_0}(y))| + \\
&\qquad Lip(F_l)|\nu_{F_0}(y)) - \nu_{F_l}(y))| \qquad (33)
\end{aligned}
$$

for all $y \in Y$. From (31)–(33) we obtain

$$
\begin{aligned}
|\nu_{F_0}(y) - \nu_{F_l}(y)| &\le (1 + ||\nu_{F_0}||)||F_0 - F_l||_{CL} + \\
Lip(F_l)|\nu_{F_0}(y) - \nu_{F_l}(y)| &\le (1 + ||\nu_{F_0}||)||F_0 - F_l||_{CL} + \\
&\varepsilon_0|\nu_{F_0}(y) - \nu_{F_l}(y)|
\end{aligned}
$$

and, consequently,

$$|\nu_{F_0}(y) - \nu_{F_l}(y)| \le (1-\varepsilon_0)^{-1}(1 + ||\nu_{F_0}||)||F_0 - F_l||_{CL},$$

i.e.,

$$||\nu_{F_0} - \nu_{F_l}|| \le (1-\varepsilon_0)^{-1}(1 + ||\nu_{F_0}||)||F_0 - F_l||_{CL}$$

for all $n \in \mathbb{N}$. Passing into limit in the last inequality as $n \to \infty$ we obtain $\nu_{F_l} \to \nu_{F_0}$. Theorem is proved. □

Corollary 3.11. *Suppose that the following conditions are fulfilled:*

(i) *linear equation (15) is hyperbolic;*
(ii) *the function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is globally Lipshchitzian.*

Then there exists a positive number λ_0 so that:

(i) *for every $\lambda \in [-\lambda_0, \lambda_0]$ equation (27) admits a unique invariant section $\nu_\lambda \in C(Y, \mathfrak{B})$;*
(ii) *relation (28) holds as $\lambda \to 0$.*

Proof. This statement follows from Theorem 3.10. In fact. If we denote by $\mathcal{F}_\lambda(y,u) := f(y) + \lambda F(y,u)$ for all $(y,u) \in Y \times B[0,r_0]$, then $Lip(\mathcal{F}_\lambda) \le |\lambda| Lip(F) \le \varepsilon_0$ for all $|\lambda| \le \lambda_0 \le \varepsilon_0/Lip(F)$ and $||\mathcal{F}_\lambda - f||_{CL} \to 0$ as $\lambda \to 0$. □

4. Perturbation of gradient systems

Let $f \in C(\mathfrak{B}, \mathfrak{B})$ be a continuously differentiable function that is Lipschitz continuous in bounded subsets of $\mathfrak{B}$. In this section we consider a difference equation

$$x(n+1) = f(x(n)) \qquad (34)$$

and its non-autonomous perturbation

$$u(n+1) = f(u(n)) + F(\sigma(n,y), u(n), \lambda) \quad (y \in Y), \qquad (35)$$

where $(Y, \mathbb{Z}, \sigma)$ is a dynamical system on the metric space Y and λ is a small parameter (for example, $\lambda \in \Lambda := [-\lambda_0, \lambda_0]$ and λ_0 is a fixed positive number).

In this section we will use some conditions:

(**C1**): equation (34) has a compact global attractor I^0, that is, I^0 is a nonempty compact subset of $\mathfrak{B}$ possessing the following properties:

(a) the set I^0 is invariant, i.e., $\pi(n, I^0) = I^0$ for all $n \in \mathbb{Z}_+$, where $(\mathfrak{B}, \mathbb{Z}_+, \pi)$ is the dynamical system generated by equation (34);

(b) the set I^0 attracts every compact subset M from $\mathfrak{B}$, i.e.,

$$\lim_{n\to\infty} \beta(\pi(n, M), I^0) = 0,$$

where $\beta(A, B) := \sup_{a\in A} \rho(a, B)$, $\rho(a, B) := \inf_{b\in B} \rho(a, b)$ and $\rho(a, b) := |a - b|$.

(**C2**): for every $\lambda \in \Lambda$ equation (35) (the cocycle φ_λ) admits a compact global attractor $\mathbb{I}^\lambda := \{I^\lambda_y :\ y \in Y\}$, that is, $\mathbb{I}^\lambda$ possesses the following properties:

(a) $\mathbb{I}^\lambda$ is a nonempty, compact and invariant with respect to cocycle φ_λ, this means that the following conditions hold:

(i) for each $(\lambda, y) \in \Lambda \times Y$ the subset I^λ_y of $\mathfrak{B}$ is nonempty and compact;

(ii) the set $I^\lambda := \bigcup\{I^\lambda_y :\ y \in Y\}$ is relatively compact in $\mathfrak{B}$ (in fact by Theorems 7.1 and 7.2 [10, ChII] it is closed, and consequently, compact);

(iii) for each $\lambda \in \Lambda$ we have the equality $\varphi_\lambda(n, I^\lambda_y, y) = I^\lambda_{\sigma(n,y)}$ for all $(n, y) \in \mathbb{Z}_+ \times Y$;

(b) the compact set I^λ attracts (with respect to cocycle φ_λ) every nonempty compact subset K from $\mathfrak{B}$, that is,

$$\lim_{n\to\infty} \sup_{y\in Y} \beta(\varphi_\lambda(n, K, y), I^\lambda) = 0.$$

(**C3**): the functions f_0 and F are continuously differentiable with respect to spacial variable $u \in \mathfrak{B}$ and for any $r > 0$

$$\lim_{\lambda\to 0} \sup_{y\in Y, |u|\le r} \{|F(y, u, \lambda)| + |F'_u(y, u, \lambda)|\} = 0,$$

where by $F'_u(y, u, \lambda)$ is denoted the derivative of $F(y, \cdot, \lambda)$ with respect to $u \in \mathfrak{B}$.

Let $p \in \mathfrak{B}$ be a solution of equation

$$f(u) = u,$$

then $u(n) = p$ for all $n \in \mathbb{Z}$ is a stationary solution of equation (34). Denote by $\mathcal{A} := f'_u(p)$ and suppose that the stationary solution p is hyperbolic, that is, the linearized equation

$$v(n + 1) = \mathcal{A}v(n)$$

satisfies to the condition of exponential dichotomy. This means that there exists a projection $P \in [\mathfrak{B}]$ such that the following conditions hold:

(i) $V(n)P = PV(n)$ for all $t \in \mathbb{Z}_+$, where $\{V(n)\}_{n\in\mathbb{Z}_+}$ is the semigroup generated by $\mathcal{A}$, i.e., $V(n) = \mathcal{A}^n$;

(ii) there are positive constants $\mathcal{N}$ and $0 < \nu < 1$ such that

$$||V(n)P|| \le \mathcal{N}\nu^n$$

for all $n > 0$;

(iii) the operator $V(n)Q : Q\mathfrak{B} \mapsto Q\mathfrak{B}$, where $Q := Id_{\mathfrak{B}} - P$, is invertible and

$$||(V(n)Q)^{-1}|| \leq \mathcal{N}\nu^n$$

for all $n > 0$.

Definition 4.1. *A continuous function $V : X \mapsto \mathbb{R}$ is said to be a (global) Lyapunov function for $(X, \mathbb{Z}_+, \pi)$, if $V(\pi(n, x)) \leq V(x)$ for all $x \in X$ and $n \in \mathbb{Z}_+$.*

Definition 4.2. *A dynamical system $(X, \mathbb{Z}_+, \pi)$ with the Lyapunov function V is called a gradient system, if the equality $V(\pi(n, x)) = V(x)$ (for all $n \in \mathbb{Z}_+$) implies $x \in Fix(\pi)$, where $Fix(\pi)$ is the set of all fixed (stationary) point of $(X, \mathbb{Z}_+, \pi)$.*

Below we will give some examples of discrete gradient systems.

Example 4.3. *The simplest example of gradient dynamical system (with continuous time) is defined by differential equation*

$$x' = -\nabla V(x), \quad (x \in \mathbb{R}^m) \tag{36}$$

where $V : \mathbb{R}^m \mapsto \mathbb{R}$ is a continuously differentiable function and $\nabla := \{\partial_{x_1}, \ldots, \partial_{x_m}\}$. In fact. If we suppose that equation (36) admits a unique solution $\pi(t, x)$ passing through the point $x \in \mathbb{R}^m$ at the initial moment $t = 0$ and defined on $\mathbb{R}_+$, then

$$\frac{d}{dt}V(\pi(t, x)) = -|\nabla V(\pi(t, x)|^2 \leq 0 \tag{37}$$

for all $x \in \mathbb{R}^m$ and $t > 0$. From (37) we obtain

$$V(\pi(t, x)) \leq V(x) \tag{38}$$

for all $t \geq 0$, then from (36) we have $x \in Fix(\pi)$ if $V(\pi(t, x)) = V(x)$ for all $t \in \mathbb{R}_+$, i.e., $(\mathbb{R}^m, \mathbb{R}_+, \pi)$ (the dynamical system generated by equation (36)) is a gradient dynamical system.

Denote by $P(x) := \pi(1, x)$ for all $x \in \mathbb{R}^m$ and consider the discrete dynamical system $(\mathbb{R}^m, \mathbb{Z}_+, \tilde{\pi})$ generated by positive powers of the mapping P, i.e., $\tilde{\pi}(n, x) := P^n(x)$ and, consequently, we will have

$$V(\tilde{\pi}(n, x)) = V(\pi(n, x)) \tag{39}$$

for all $\mathbb{Z}_+ \times \mathbb{R}^m$. From (38) and (39) we obtain $V(\tilde{\pi}(n, x)) \leq V(x)$ for all $(n, x) \in \mathbb{Z}_+ \times \mathbb{R}^m$, i.e., $(\mathbb{R}^m, \mathbb{Z}_+, \tilde{\pi})$ is a discrete gradient dynamical system. In fact. Suppose that $V(\tilde{\pi}(n, x)) = V(x)$ for all $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$, then there exists $l \in \mathbb{Z}_+$ such that $t \in [l, l+1]$ and, consequently, $V(x) = V(\pi(l+1, x)) \leq V(\pi(t, x)) \leq V(\pi(l, x)) = V(x)$. From the last inequality we obtain $V(\pi(t, x)) = V(x)$ for all $t \in \mathbb{R}_+$. Since the dynamical system $(\mathbb{R}^m, \mathbb{R}_+, \pi)$ is gradient, then $x \in Fix(\pi)$ and, consequently, $\tilde{\pi}(n, x) = \pi(n, x) = x$ for all $n \in \mathbb{Z}_+$.

Example 4.4. *Consider the scalar gradient equation*

$$x' = \alpha x(K - x) \quad (\textit{logistic equation}) \tag{40}$$

with Lyapunov function $V(x) = \alpha(x^3/3 - Kx^2/2)$ $(\alpha, K > 0)$. By equation (40) is defined the dynamical system $(\mathbb{R}, \mathbb{R}, \pi)$ on the real axe $\mathbb{R}$, where

$$\pi(t, x) = \frac{Ke^{\alpha K t}x}{K + (e^{\alpha K t} - 1)x} \quad (\forall\ x, t \in \mathbb{R}).$$

Now we define the mapping $F : \mathbb{R} \mapsto \mathbb{R}$ by equality

$$F(x) := \pi(1, x) = \frac{\mu K x}{K + (\mu - 1)x} \quad (\forall\ x \in \mathbb{R}),$$

where $\mu := e^{\alpha K} > 1$.

According to Example 4.3 the difference equation

$$x_{n+1} = F(x_n) = \frac{\mu K x_n}{K + (\mu - 1)x_n} \quad \textit{(Beverton-Holt equation)}$$

is gradient with the Lyapunov function $V(x) = \alpha(x^3/3 - Kx^2/2)$ ($\alpha = K^{-1}\ln\mu$).

(**C4**): equation (34) is gradient (the dynamical system $(\mathfrak{B}, \mathbb{Z}_+, \pi)$, generated by (34), is gradient), it has a finite number of stationary solutions $p_1, p_2, \ldots, p_k$ and they are hyperbolic. Then by Lemma 4 [9] we will have

$$I^0 = \bigcup_{i=1}^{k} W^u(p_i),$$

where $W^u(p)$ is the unstable manifold of the stationary point p.

(**C5**): the set $I := \bigcup\{I^\lambda :\ \lambda \in \Lambda\}$ is relatively compact in $\mathfrak{B}$ and, consequently, by Lemma 7.3 [10, ChVII] the mapping $\lambda \mapsto I^\lambda$ is upper semi-continuous and, in particular, we will have

$$\lim_{\lambda \to 0} \beta(I^\lambda, I^0) = 0.$$

The aim of this section is a complete description of the structure of attractor $\mathbb{I}^\lambda$ for the sufficiently small λ ($\lambda \to 0$).

Theorem 4.5. *Assume that the conditions (**C1**)-(**C5**) are fulfilled. Then the following statements take place:*

(i) *there are positive numbers δ_0 and r_0 such that for every $\lambda \in \Lambda$ with $|\lambda| \leq \delta_0$ and every $i \in \{1, 2, \ldots, k\}$ in the r_0-neighborhood $B[p_i, r_0]$ of the stationary point p_i there exists a unique invariant section $\nu_\lambda^i \in C(Y, B[p_i, r_0])$ of equation (35);*

(ii) *for each $i \in \{1, 2, \ldots, k\}$ we have $\nu_\lambda^i \to p_i$ as $\lambda \to 0$, i.e., $\max_{y \in Y} |\nu_\lambda^i(y) - p_i| \to 0$ as $\lambda \to 0$;*

(iii) *$\varphi_\lambda(n, \nu_\lambda^i(y), y) = \nu_\lambda^i(\sigma(n, y))$ (for all $n \in \mathbb{Z}$) is a unique solution of equation (35) defined on $\mathbb{Z}$ with the values from $B[p_i, r_0]$.*

Proof. We will show that the first statement of Theorem 4.5 follows from Theorem 3.3. To this end in the neighborhood of the stationary point $p \in \{p_1, p_2, \ldots, p_k\}$ we will rewrite equation (35) in the following form

$$v(n+1) = \mathcal{A}v(n) + \mathcal{F}(\sigma(n, y), v, \lambda),$$

where $\mathcal{A} := f_u^{'}(p)$ and

$$\mathcal{F}(y, v, \lambda) := f(v + p) - f(p) - f_u^{'}(p)v + F(y, v, \lambda) \tag{41}$$

for all $(y, v, \lambda) \in Y \times B[0, r] \times \Lambda$. Then we obtain

$$\mathcal{F}_u^{'}(y, v, \lambda) = f_u^{'}(v + p) - f_u^{'}(p) + F_u^{'}(y, v + p, \lambda)$$

and taking into consideration condition (**C3**) we conclude that there exist positive numbers r_0 and $\delta_0 < \lambda_0$ such that for all $\lambda \in [-\delta_0, \delta_0]$ the function $\mathcal{F}$ defined by (41) satisfies all conditions of Theorem 3.3 and, consequently, the first statement of Theorem 4.5 is proved.

To prove the second statement of Theorem it is sufficient to establish that $\mathcal{F}_\lambda \to \mathcal{F}_0$ in the space $CL_{r_0}(Y \times B[p_i, r_0], \mathfrak{B})$ as $\lambda \to 0$, where $\mathcal{F}_\lambda := \mathcal{F}(\cdot, \cdot, \lambda)$ and $\mathcal{F}_0(v) := f(v + p_i) - f(p_i) - f_u^{'}(p_i)v$ for all $v \in B[0, r_0]$. Note that

$$d_{CL_{r_0}}(\mathcal{F}_\lambda, \mathcal{F}_0) = \max_{y \in Y} |\mathcal{F}_\lambda(y, 0) - \mathcal{F}_0(y, 0)| + Lip(r_0, \mathcal{F}_\lambda - \mathcal{F}_0) \leq$$

$$\max_{y \in Y} |F(y, p, \lambda)| + \sup_{y \in Y, |v| \leq r_0} |F_u'(y, v + p, \lambda)|$$

and taking into account condition (**C3**) we conclude that $\mathcal{F}_\lambda \to \mathcal{F}_0$ as $\lambda \to 0$ in the metric $d_{CL_{r_0}}$. Now to obtain the second statement of Theorem 4.5 it is sufficient to apply Theorem 3.3.

The third statement directly follows from Remark 3.4. $\square$

Denote by $(X, \mathbb{Z}_+, \pi_\lambda)$ (respectively, $(X, \mathbb{Z}_+, \pi_0)$) the dynamical system on the product space $X := \mathfrak{B} \times Y$ generated by cocycle φ_λ (respectively, by semigroup dynamical system $(\mathfrak{B}, \mathbb{Z}_+, \varphi_0)$), i.e., $\pi_\lambda = (\varphi_\lambda, \sigma)$ (respectively, $\pi_0 = (\varphi_0, \sigma)$).

Let $\Sigma \subseteq X$ be a compact positively invariant set and $\varepsilon > 0$.

Definition 4.6. *The collection $\{x = x_0, x_1, x_2, \dots, x_k = y; n_0, n_1, \dots, n_k\}$ of the points $x_i \in \Sigma$ and the numbers $n_i \in \mathbb{N}$ such that $\rho(\pi(n_i, x_i), x_{i+1}) < \varepsilon$ ($i = 0, 1, \dots, k-1$) is called (see, for example,* [4, 5], [13, 14] *and* [21]*) a (ε, π)–chain joining the points x and y.*

We denote by $P(\Sigma)$ the set $\{(x, y) : x, y \in \Sigma, \forall\ \varepsilon > 0\ \exists\ (\varepsilon, \pi)\text{-chain joining } x \text{ and } y\}$. The relation $P(\Sigma)$ is closed, invariant and transitive [4, 13, 18, 20, 21].

Definition 4.7. *The point $x \in \Sigma$ is called chain recurrent (in Σ) if $(x, x) \in P(\Sigma)$.*

We denote by $\mathfrak{R}(\Sigma)$ the set of all chain recurrent (in Σ) points from Σ.

Definition 4.8. *Let $A \subseteq X$ be a nonempty positively invariant set. The set A is called (see, for example,* [17]*) chain recurrent if $\mathfrak{R}(A) = A$, and chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ there is an (ε, π)-chain in A connecting a and b.*

Lemma 4.9. [17] *Let $x \in X$, $\gamma \in \Phi_x$ and the positive (respectively, negative) semi-trajectory of the point $x \in X$ is relatively compact. Then the ω (respectively, α)-limit set of the point x is chain transitive.*

Lemma 4.10. *Under the condition (**C4**) the following statements hold:*

(i) *the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi_0), (Y, \mathbb{Z}, \sigma), h \rangle$, where $h := pr_2 : X \mapsto Y$ is the second projection, is a gradient dynamical system admitting a compact global attractor $J^0 = I^0 \times Y$;*

(ii) $J^0 = \bigcup_{i=1}^k W^u(M_i)$, *where $M_i = \{p_i\} \times Y$ and $W^u(M_i) := W^u(p_i) \times Y$;*

(iii) *for each* $y \in Y$ *one has* $J^0_y = \bigcup_{i=1}^k W^u(\gamma^i_0(y))$, *where* $\gamma^i_0(y) := (p_i, y)$ *for all* $y \in Y$*;*

(iv)

$$\mathfrak{R}(J^0) = M_1 \coprod M_2 \ldots \coprod M_k. \tag{42}$$

Proof. This statement is straightforward. □

Definition 4.11. *A continuous mapping* $\varphi : \mathbb{R} \to X$ *with the properties:*

(i) $\varphi(0) = x$*;*
(ii) $\pi^t \varphi(s) = \varphi(t+s)$ *for all* $t \in \mathbb{R}_+$ *and* $s \in \mathbb{R}$

is called a whole (entire) trajectory *of one-sided dynamical system* $(X, \mathbb{R}_+, \pi)$.

Denote by $\mathcal{F}_x$ the family of all whole trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passing through the point $x \in X$ at the moment $t = 0$.

Definition 4.12. *The motion* $\pi(\cdot, x) : \mathbb{R}_+ \to X$ *of one-sided dynamical system* $(X, \mathbb{R}_+, \pi)$ *is called* extendable *on* $\mathbb{S}$*, if there exists a whole trajectory* φ *passing through point* x *such that* $\varphi(t) = \pi(t, x)$ *for all* $t \in \mathbb{R}_+$*.*

We denote by $\alpha_{\varphi_x} := \{y : \exists t_n \to -\infty,\ t_n \in \mathbb{S} \text{ and } \varphi(t_n) \to y\}$, where φ is an extension on $\mathbb{S}$ the motion $\pi(\cdot, x)$.

Let $\Lambda \subset X$ be an invariant subset of $(X, \mathbb{R}_+, \pi)$, i.e., $\pi(t, \Lambda) = \Lambda$ for all $t \in \mathbb{R}_+$.

Definition 4.13. *The set* $W^s(\Lambda)$ *(respectively,* $W^u(\Lambda)$*), defined by equality*

$$W^s(\Lambda) := \{x \in X | \lim_{t \to +\infty} \rho(xt, \Lambda) = 0\}$$

$$(\textit{respectively, } W^u(\Lambda) := \{x \in X | \exists\ \gamma \in \mathcal{F}_x \textit{ such that } \lim_{t \to -\infty} \rho(\gamma(t), \Lambda) = 0\}),$$

is called a stable (respectively, unstable) manifold *of the set* $\Lambda \subseteq X$*, where* $\rho(x, \Lambda) := \inf_{\lambda \in \Lambda} \rho(x, \lambda)$*.*

Definition 4.14. *By analogy with the work* [1] *in the collection of invariant subsets* $\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$ *we will introduce a relation of partial order as follows:* $\mathcal{R}_i < \mathcal{R}_j$*, if there exist* $i_1, i_2, \ldots, i_r$ *such that* $i_1 = i, i_r = j$ *and* $W^s(\mathcal{R}_{i_p}) \bigcap W^u(\mathcal{R}_{i_{p+1}}) \neq \emptyset$ *for all* $p = 1, 2, \ldots, r-1$*.*

Definition 4.15. *The ordered collection of* r $(r \geq 2)$ *different indexes* $\{i_1, i_2, \ldots, i_r\}$ *satisfying the condition* $\mathcal{R}_{i_1} < \mathcal{R}_{i_2} < \cdots < \mathcal{R}_{i_r} < \mathcal{R}_{i_1}$ *is called an* r*-cycle in the collection* $\{\mathcal{R}_1, \mathcal{R}_2 \ldots, \mathcal{R}_k\}$*. The 1-cycle is called such index* i *that* $W^s(\mathcal{R}_i) \bigcap W^u(\mathcal{R}_i) \setminus \mathcal{R}_i \neq \emptyset$*.*

Theorem 4.16. *Under conditions* (***C1***)$-$(***C5***) *the following statements take place:*

a) *for each* $i \in \{1, 2, \ldots, k\}$ *and* $\lambda \in [-\delta_0, \delta_0]$ *by equality* $\gamma^i_\lambda(y) := (\nu^i_\lambda(y), y)$ *(for all* $y \in Y$*) is defined a continuous invariant section of non-autonomous dynamical system* $\langle (X, \mathbb{Z}_+, \pi_\lambda), (Y, \mathbb{Z}, \sigma), h \rangle$*, where* $h := pr_2$*.*
b) *the dynamical system* $(X, \mathbb{Z}_+, \pi_\lambda))$ *is compact dissipative and its Levinson center* $J^\lambda = \bigcup_{y \in Y} I^\lambda_y \times \{y\}$*, where* $\{I^\lambda_y : y \in Y\}$ *is a compact global attractor of the cocycle* φ_λ*.*

c) *for each* $i \in \{1, 2, \dots, k\}$ *the set* $\mathfrak{R}_i^\lambda := \mathfrak{R}(J^\lambda) \bigcap B[M_i, r_0]$ *possesses the following properties:*
   c1) $\mathfrak{R}_i^\lambda = \mathfrak{R}(\gamma_\lambda^i(Y))$, *where by* $\mathfrak{R}(Y)$ *(respectively,* $\mathfrak{R}(\gamma_\lambda^i(Y))$*) is denoted the set of all chain recurrent points of dynamical system* $(Y, \mathbb{Z}, \sigma)$ *(respectively,* $(\gamma_\lambda^i(Y), \mathbb{Z}, \pi_\lambda)$*);*
   c2) $\mathfrak{R}_i^\lambda$ *is nonempty, closed and invariant;*
   c3) $\mathfrak{R}(J^\lambda) = \coprod_{i=1}^k \mathfrak{R}_i^\lambda$.

d) *if the set* $\mathfrak{R}(Y)$ *is chain transitive, then in the series of subsets* $\mathfrak{R}_1^\lambda, \mathfrak{R}_2^\lambda, \dots, \mathfrak{R}_k^\lambda$ *does not exit any* l*-cycle* ($l \in \mathbb{N}$)*.*

e)

$$J_y^\lambda = \bigcup_{i=1}^k W^u(\gamma_\lambda^i(y)) \tag{43}$$

for all $y \in Y$, *where* $W^u(\gamma_\lambda^i(y)) := \{x \in X :$ *such that* $h(x) = h(\gamma_\lambda^i(y))$ *and there exists* $\varphi \in \Phi_x$ *such that* $\lim_{n\to-\infty} \rho(\varphi(n), \gamma_\lambda^i(\sigma(n, y))) = 0\}$.

f) *if* $(Y, \mathbb{Z}, \sigma)$ *is chain recurrent (i.e,.* $Y = \mathfrak{R}(Y)$*), then for each* $y \in Y$ *and* $x \in \mathfrak{B}$ *there exists a unique number* $i \in \{1, 2, \dots, k\}$ *such that*

$$\lim_{n\to+\infty} |\varphi(n, x, y) - \nu_\lambda^i(\sigma(n, y))| = 0. \tag{44}$$

Proof. Item a) is evident.

Item b) follows from Theorem 2.24 [10, ChII].

To prove item c1) we note that $\mathfrak{R}_i^\lambda \subseteq \gamma_\lambda^i(Y)$. If we suppose that it is not so, then there exists a point $p = (u, y) \in \mathfrak{R}_i^\lambda \setminus \gamma_\lambda^i(Y)$, then we obtain a solution $\varphi_\lambda(n, u, y)$ of equation (35) defined on $\mathbb{Z}$ with values from $B[p_i, r_0]$. This contradicts to Remarks 3.4 (item 2). Now we will show that $\gamma_\lambda^i(\mathfrak{R}(Y)) \subseteq \mathfrak{R}_i$. Since the mapping $\gamma_\lambda^i : Y \mapsto \gamma_\lambda^i(Y)$ is a homeomorphism (and its inverse coincides with $pr_2\big|_{\gamma_\lambda^i(Y)}$), then $\gamma_\lambda^i(\mathfrak{R}(Y)) = \mathfrak{R}(\gamma_\lambda^i(\mathfrak{R}(Y))$ (see, for example, [4, Ch.I]), $\mathfrak{R}(\gamma_\lambda^i(\mathfrak{R}(Y)) \subseteq \mathfrak{R}(J^\lambda)$ and $\gamma_\lambda^i(\mathfrak{R}(Y)) \subseteq \gamma_\lambda^i(Y) \subseteq B[M_i, r_0]$, then we have $\gamma_\lambda^i(\mathfrak{R}(Y)) \subseteq B[M_i, r_0] \bigcap \mathfrak{R}(J^\lambda) = \mathfrak{R}_i^\lambda$ and, consequently, $\mathfrak{R}_i \neq \emptyset$. The closeness of the set $\mathfrak{R}_i$ is evident. Now we note that the sets $\mathfrak{R}_i$ are invariant. In fact, if we suppose that $\mathfrak{R}_{i_0}$ is not invariant, then there exist $p \in \mathfrak{R}_{i_0}$ and $n_0 > 0$ such that $\rho(\pi(n_0, p), M_{i_0}) \geq r_0$ and, consequently, $\pi(n_0, p) \notin \mathfrak{R}(J^\lambda)$. The obtained contradiction proves our statement.

Without loss of generality we can suppose that

$$r_0 < \frac{1}{3} \max\{\rho(p_i, p_j) : \ i \neq j, \ i, j = \overline{1, k}\}.$$

It easy to see that $\mathfrak{R}_i \bigcap \mathfrak{R}_j = \emptyset$ for all $i \neq j$ $(i, j = \overline{1, k})$.

By Theorem 3.11 [11] for sufficiently small δ_0 we have

$$\mathfrak{R}(J^\lambda) \subset B[\mathfrak{R}(J^0), r_0/2]. \tag{45}$$

Since for the sufficiently small r_0 we have

$$B[\mathfrak{R}(J^0), r_0] = \coprod_{i=1}^k B[M_i, r_0],$$

then from (42) and (45) we obtain

$$\mathfrak{R}(J^{\lambda}) = \coprod_{i=1}^{k} \mathfrak{R}_i. \tag{46}$$

Now we will establish the statement d). If we suppose that it is not true, then there exist $i_1, i_2, \ldots, i_s \in \{1, \ldots, k\}$ $(1 \le s \le k)$ such that $\mathfrak{R}_{i_1} \to \mathfrak{R}_{i_2} \to \ldots \to \mathfrak{R}_{i_s} \to \mathfrak{R}_{i_1}$. This means in particular, that there exists a point $x_0 \in \mathfrak{R}(J^{\lambda})$ such that $\alpha_{x_0} \subseteq \mathfrak{R}_{i_1}$, $\omega_{x_0} \subseteq \mathfrak{R}_{i_2}$ and for certain $n_0 \in \mathbb{Z}$ we will have $\rho(\pi_{\lambda}(n_0, x_0), p_0) \ge r_0$. From the last equality it follows that $\pi_{\lambda}(n_0, x_0) \notin \mathfrak{R}(J^{\lambda})$. Since the set $\mathfrak{R}(J^{\lambda})$ is invariant, then this fact contradicts to the choice of the point x_0. The obtained contradiction proves our statement.

Item e) follows from the fact that the set $\mathfrak{R}(Y)$ is chain transitive, by Lemma 4.3 [10, ChIV] and equality (46).

To establish equality (43) it is sufficient to prove that $J_y^{\lambda} \subseteq \bigcup_{i=1}^{k} W^u(\gamma_{\lambda}^i(y))$ because the reverse inclusion is evident. Let $x \in J_y^{\lambda}$, then $h(x) = y$ and there exists $\varphi \in \Phi_x$ such that $\varphi(\mathbb{R}) \subseteq J^{\lambda}$ and consequently its α-limit set α_{φ} is a nonempty, compact set and $\alpha_{\varphi} \subseteq \mathfrak{R}(J^{\lambda})$. Thus there exists a unique number $i = \overline{1,k}$ such that $\alpha_{\varphi} \subseteq \gamma_{\lambda}^i(Y)$. Now we will show that

$$\lim_{n \to -\infty} \rho(\varphi(n), \gamma_{\lambda}^i(\sigma(n, y))) = 0. \tag{47}$$

If we suppose that (47) is not true, then there are $\eta_0 > 0$ and $n_k \to -\infty$ as $k \to \infty$ such that

$$\rho(\varphi(n_k), \gamma_{\lambda}^i(\sigma(n_k, y))) \ge \eta_0. \tag{48}$$

Without loss of generality we can suppose that the sequences $\{\varphi(n_k)\}$ and $\{\sigma(n_k, y)\}$ are convergent. Denote their limits by $\bar{x}$ and $\bar{y}$ respectively. Now passing into limit in (48) we obtain

$$\rho(\bar{x}, \gamma_{\lambda}^i(\bar{y})) \ge \eta_0. \tag{49}$$

On the other hand $h(\varphi(n)) = \sigma(n, y)$ for all $n \in \mathbb{Z}$ and, consequently, $h(\bar{x}) = \bar{y}$, i.e., $\bar{x} \in J_{\bar{y}}$ and taking into account that $\bar{x} \in \alpha_{\varphi} \subseteq \gamma_{\lambda}^i(Y)$ we conclude that

$$\bar{x} = \gamma_{\lambda}(\bar{y}). \tag{50}$$

Equality (50) and inequality (49) are contradictory. The obtained contradiction proves this statement.

Finally, we will establish the statement f). Let $(x, y) \in \mathfrak{B} \times Y$ be an arbitrary point. Since the skew-product dynamical system $(X, \mathbb{R}_+, \pi_{\lambda})$ $(X = \mathfrak{B} \times Y$ and $\pi_{\lambda} = (\varphi_{\lambda}, \sigma))$ generated by equation (35) is compact dissipative, then the positive semi-trajectory $\Sigma_{(x,y)}^{\lambda} := \bigcup \{\pi_{\lambda}(n, (x, y)) : \ n \ge 0\}$ is relatively , compact, invariant and chain recurrent. From the last fact it follows that there exists a unique number $i \in \{1, 2, \ldots, k\}$ such that $\omega_{(x,y)} \subseteq \mathfrak{R}_i$. If we suppose that equality (44) is not true, then there are $y_0 \in Y$, $x_0 \in \mathfrak{B}$, $\varepsilon_0 > 0$ and $n_k \to +\infty$ such that

$$|\varphi(n_k, x_0, y_0) - \nu_{\lambda}^i(\sigma(n_k, y_0))| \ge \varepsilon_0, \tag{51}$$

where $\gamma_{\lambda}^i = (\nu_{\lambda}^i, Id_Y)$.

Without loss of generality we can suppose that the sequences $\{\sigma(n_k, y_0)\}$ and $\{\varphi(n_k, x_0, y_0)\}$ are convergent. Denote by $\bar{y} := \lim_{k\to\infty} \sigma(n_k, y_0)$ and $\bar{x} := \lim_{k\to\infty} \varphi(n_k, x_0, y_0)$, then passing into limit in (51) we obtain

$$|\bar{x} - \nu^i_\lambda(\bar{y})| \ge \varepsilon_0. \tag{52}$$

On the other hand we have $\omega_{(x_0,y_0)} \subseteq \mathfrak{R}_i$, $(\bar{x}, \bar{y}) \in \omega_{(x_0,y_0)}$ and, consequently, $(\bar{x}, \bar{y}) \in \omega_{(x_0,y_0)} \bigcap X_{\bar{y}} \subseteq \mathfrak{R}_i \bigcap X_{\bar{y}} = \{\gamma^i_\lambda(\bar{y})\}$. The last inclusion contradicts to (52). The obtained contradiction completes the proof of Theorem. □

Corollary 4.17. *Let Y be a compact minimal set consisting from τ-periodic (respectively quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) motions. Under the conditions (**C1**) – (**C5**) the following statements hold:*

(i) *there are positive numbers δ_0 and r_0 such that for every $\lambda \in \Lambda$ with $|\lambda| \le \delta_0$ and every $i = \overline{1,k}$ in the r_0-neighborhood $B[p_i, r_0]$ of the stationary point p_i there exists a unique τ-periodic (respectively quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff) $\nu^i_\lambda : \mathbb{Z} \mapsto B[p_i, r_0]$;*

(ii) *for every solution $\varphi(n, x, y)$ of equation (35) there exists a unique number $i \in \{1, 2, \dots, k\}$ such that*

$$\lim_{n\to+\infty} |\varphi(n, x, y) - \nu^i_\lambda(n)| = 0,$$

i.e., $\varphi(n, x, y)$ is asymptotically τ-periodic (respectively quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff);

(iii) *the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ generated by equation (35) is compact dissipative with the Levinson center $J := \bigcup\{J_y : y \in Y\}$, where $J_y := I_y \times \{y\}$, $I := \bigcup\{I^i_y : i = 1, 2, \dots, k\}$ and*

$$I^i_y := \{x \in \mathfrak{B} : \lim_{n\to-\infty} |\varphi(n, x, y) - \nu^i_\lambda(n)| = 0\}$$

.

Proof. This statement follows from Theorem 4.16 and Corollary 2.21. □

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