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Linear Stochastic Differential Equations and Nonautonomous Dynamical Systems

David Cheban

Abstract

We prove that the linear stochastic equation $dx(t) = (Ax(t)+f(t))dt + g(t)dW(t)$ (*) with linear operator A generating a C_0-semigroup $\{U(t)\}_{t\geq 0}$ and Levitan almost periodic forcing terms f and g admits a unique Levitan almost periodic [3,ChIV] solution in distrution sense if it has at least one precompact solution on $\mathbb{R}_+$ and the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable.

Keywords: Levitan almost periodic solutions, linear stochastic differential equations.

1 Introduction

In this short communication we study the problem of existence of Levitan almost periodic solutions of equations (*), where A is generator of strongly asymptotically stable C_0-semigroup on a Banach space E and $f, g :\to E$ are some Levitan almost periodic functions.

In the deterministic case ($g = 0$) the problem of Bohr almost periodicity (respectively, almost automorphy) of solutions of equation (*) was studied in the works of S. Zaidman [2] (for Bohr almost periodic equations) and M. Zaki [3] (for almost automorphic equations).

2 Semigroup of operators

Let $(E, |\cdot|)$ be a Banach space with the norm $|\cdot|$ and $[E]$ be a Banach space of linear bounded operators A acting on the space A equipped with the norm $||A|| := \sup\{|Ax| : \ |x| \leq 1\}$.

A C_0-semigroup $\{U(t)\}_{t\geq 0}$ is said to be asymptotically stable if $\lim\limits_{t\to+\infty} U(t)x = 0$ for any $x \in E$.

Theorem 1.[1,ChI] *The following statements are equivalent:*

1. *the C_0-semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable;*
2. $\lim\limits_{t\to+\infty} \sup\limits_{x\in K} |U(t)x| = 0$ *for any compact subset $K \subset E$;*
3. *equation $x'(t) = Ax(t)$*
   (a) *admits a compact global attractor J;*
   (b) *does not admit any solution defined on $\mathbb{R}$ with precompact range, i.e., $J = \{0\}$.*

Let (X, ρ) be a compete metric space. Denote by $C(\mathbb{R}, X)$ the family of all continuous functions $f : \mathbb{R} \mapsto X$ equipped with the distance $d(f, g) := \sup\limits_{l>0} d_l(f, g)$, where $d_l(f, g) := \min\{\max\limits_{|t|\leq l} \rho(f(t), g(t)); l^{-1}\}$. The metric d is complete and it defines on $C(\mathbb{R}, X)$ the compact-open topology. Let $h \in \mathbb{R}$ denote by f_h the h-translation of f, that is, $f_h(s) := f(s + h)$ for all $s \in \mathbb{R}$.

Definition 1. *A function $f \in C(\mathbb{R}, X)$ is said to be Bohr almost periodic if for any $\varepsilon > 0$ there exists a positive number $L = L(\varepsilon)$ such that $\mathcal{T}(\varepsilon, f) \bigcap [a, a + L] \neq \emptyset$ for any $a \in \mathbb{R}$, where $\mathcal{T}(\varepsilon, f) := \{\tau \in \mathbb{R} : \rho(f(t + \tau), f((t))) < \varepsilon$ for any $t \in \mathbb{R}\}$.*

Definition 2. *Function $f \in C(\mathbb{R}, X)$ is called Levitan almost periodic if there exists a metric space Y and a Bohr almost periodic function $F \in C(\mathbb{R}, Y)$ such that for arbitrary $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\mathcal{T}(\delta, F) \subseteq \mathfrak{T}(\varepsilon, f)$, where $\mathfrak{T}(\varepsilon, f) := \{\tau \in \mathbb{R} : \max\limits_{|t|\leq 1/\varepsilon} \rho(f(t + \tau), f(t)) < \varepsilon\}$.*

Remark 1. 1.*Every Bohr almost periodic function is Levitan almost periodic.*

2.*The functions $f(t) = (2 + \cos t + \cos \sqrt{2}t)^{-1}$ and $g(t) = \cos(f(t))$ ($t \in \mathbb{R}$) are Levitan almost periodic, but not Bohr almost periodic [3,ChIV].*

3 Linear Stochastic Differential Equations

Let $(H, |\cdot|)$ be a real separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $L^2(\mathbb{P}, H)$ be the space of H-valued random variables x such that $\mathbb{E}|x|^2 := \int\limits_{\Omega} |x|^2 d\mathbb{P} < \infty$. Then $L^2(\mathbb{P}, H)$ is a Hilbert space equipped with the norm $||x||_2 := \Big(\int\limits_{\Omega} |x|^2 d\mathbb{P}\Big)^{1/2}$.

Consider the following linear stochastic differential equation

$$dx(t) = (Ax(t) + f(t)dt + g(t)dW(t), \tag{1}$$

where A is an infinitesimal generator which generates a C_0-semigroup $\{U(t)\}_{t\geq 0}$, $f, g \in C(\mathbb{R}, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{F}_t := \sigma\{W(u) : u \leq t\}$.

Recall that an $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in\mathbb{R}}$ is said to be a mild solution of equation (1) if it satisfies the stochastic integral equation

$$x(t) = U(t - t_0)x(t_0) + \int_{t_0}^{t} U(t - s)f(s)ds + \int_{t_0}^{t} U(t - s)g(s)dW(s),$$

for all $t \geq t_0$ and each $t_0 \in \mathbb{R}$.

Let $\mathcal{P}(H)$ be the space of all Borel probability measures on H endowed with the weak topology. It is well known that on the space $\mathcal{P}(H)$ there is a distance which defines this topology.

Definition 3. *Let $\varphi : \mathbb{R} \to E$ be a mild solution of equation* (1)*. Then φ is called Levitan almost periodic* in distribution *if the function $\phi \in C(\mathbb{R}, \mathcal{P}(H))$ is Levitan almost periodic, where $\phi(t) := \mathcal{L}(\varphi(t))$ for any $t \in \mathbb{R}$ and $\mathcal{L}(\varphi(t)) \in \mathcal{P}(H)$ is the law of random variable $\varphi(t)$.*

Theorem 2. *Suppose that the following conditions are fulfilled:*

a. *the C_0-semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable;*

b. *the functions $f, g \in C(\mathbb{R}, H)$ are Levitan almost periodic;*

c. *equation (1) admits a solution φ defined on $\mathbb{R}_+$ with precompact range, i.e., the set $Q := \overline{\varphi(\mathbb{R}_+)}$ is compact.*

Then equation (1) has a unique solution p defined on $\mathbb{R}$ *with precompact range which is Levitan almost periodic in distribution sense and* $\lim\limits_{t\to+\infty} |\varphi(t) - p(t)| = 0.$

To prove this statement we use some ideas, methods and results from the theory of nonautonomous (cocycle) dynamical systems [1].

References

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David Cheban

State University of Moldova
davidcheban@yahoo.com,
cheban@usm.md